CHAPTER 2

SPECTRAL THEORY FOR POSITIVE SEMIGROUPS

In this chapter we are concerned with the remarkable spectral properties shown by positive semigroups on Banach lattices.
Throughout this chapter we suppose that $E \neq \{0\}$ is a complex Banach lattice.

2.1 STABILITY OF STRONGLY CONTINUOUS SEMIGROUPS

In this section we study the asymptotic behaviour of the solution of the abstract Cauchy problem

$$(ACP) \qquad \begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where A is the generator of a C_0–semigroup $T(\cdot)$ on a Banach space E.

To this purpose we define *the type of the trajectory* $T(\cdot)x$ by

$$\omega(x) := \inf\{\omega : \|T(t)x\| \leq Me^{\omega t} \text{ for a constant } M \text{ and all } t \geq 0\},$$

and the *growth bound* (or type) of $T(\cdot)$ by

$$\begin{aligned} \omega_0(A) &:= \sup\{\omega(x) : x \in E\} \\ &= \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq Me^{\omega t} \text{ for some constant } M \text{ and all } t \geq 0\}. \end{aligned}$$

The *type of the solutions* of (ACP) is

$$\omega_1(A) := \sup\{\omega(x) : x \in D(A)\}).$$

We now introduce different stability concepts.

Definition 2.1.1 *A C_0–semigroup $T(\cdot)$ with generator A is called*

(i) *uniformly exponentially stable if* $\omega_0(A) < 0$,

(ii) *exponentially stable if* $\omega_1(A) < 0$,

(iii) *strongly stable if* $\lim_{t\to\infty} \|T(t)x\| = 0$ *for every* $x \in E$,

(iv) *stable if* $\lim_{t\to\infty} \|T(t)x\| = 0$ *for every* $x \in D(A)$.

It is clear that

$$\begin{array}{ccc} (i) & \Longrightarrow & (ii) \\ \Downarrow & & \Downarrow \\ (iii) & \Longrightarrow & (iv). \end{array}$$

If $A \in \mathcal{L}(E)$, then $(i) \Longleftrightarrow (ii)$ and $(iii) \Longleftrightarrow (iv)$. In the case where A is unbounded the above concepts of stability may differ as one can see in the following examples.

Example 2.1.2 *1. On* $E := C_0(\mathbb{R}^n)$ *we consider the heat semigroup defined by*

$$(T(t)f)(x) := \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy \quad \text{for } t > 0 \text{ and}$$

$$T(0)f := f \in E.$$

Then $T(\cdot)$ *is a bounded holomorphic semigroup and it generator is the Laplacian* Δ *on* $C_0(\mathbb{R}^n)$. *Since* $T(t)f = k_t * f$, *where* $k_t(y) := \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{y^2}{4t}}$, $y \in \mathbb{R}^n$, *and since* $\|k_t\|_{L^1} = 1$, *it follows that*

$$\|T(t)\| \leq 1, \forall t \geq 0. \tag{2.1}$$

Take now $f \in C_c(\mathbb{R}^n)$. *Then,*

$$\|T(t)f\| \leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |f(y)|\,dy \to 0 \ \text{ as } t \to \infty.$$

Hence, it follows from the density of $C_c(\mathbb{R}^n)$ *in* $C_0(\mathbb{R}^n)$ *and (reflap) that* $\lim_{t\to\infty} T(t)f = 0$, *for every* $f \in E$. *This means that* $T(\cdot)$ *is strongly stable. On the other hand one can see that* $\mathrm{Im}\Delta \neq C_0(\mathbb{R}^n)$, *which implies that* $0 \in \sigma(\Delta)$. *Thus,* $T(\cdot)$ *is not uniformly exponentially stable, since* $s(\Delta) \leq \omega_0(\Delta)$. *For the definition of* $s(A)$ *see Section 2.3.*

2. We consider the translation semigroup

$$(T(t)f)(s) = f(s+t), \quad t, s \geq 0,$$

on $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^s ds)$. *Then* E *is a Banach lattice and* $T(\cdot)$ *is a* C_0*–semigroup with generator* A *given by*

$$Af = f' \text{ for } f \in D(A) = \{f \in E : f \in C^1(\mathbb{R}_+) \text{ and } f' \in E\}.$$

Moreover,

$$\rho(A) = \{\lambda \in \mathbb{C} : \Re(\lambda) > -1\}$$

and for $\Re(\lambda) > -1$,

$$R(\lambda,A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt \quad \text{exists for all } f \in E.$$

One can see that $\|T(t)\| = 1$ *and so* $\omega_0(A) = 0$. *On the other hand, for* $\Re(\lambda) > -1$, *we have*

$$T(t)f = e^{\lambda t} \left(f - \int_0^t e^{-\lambda s} T(s)(\lambda - A) f \, ds \right), \quad f \in D(A),$$

and since $\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)(\lambda - A) f \, ds$ *exists, it follows that*

$$\|T(t)f\| \le N e^{\lambda t}, \quad \text{for all } f \in D(A).$$

Hence,

$$\omega_1(A) \le -1 < 0 = \omega_0(A).$$

Consequently, $T(\cdot)$ *is exponentially stable but not uniformly exponentially stable. For more details see [9, Example V.1.4].*

The definition of the growth bound yields the following characterization of uniform exponential stability.

Proposition 2.1.3 *For the generator* A *of a* $\mathcal{C}_0$*–semigroup* $T(\cdot)$ *on a Banach space* E*, the following assertions are equivalent.*

(a) $\omega_0(A) < 0$, *i.e.,* $T(\cdot)$ *is uniformly exponentially stable.*

(b) $\lim_{t\to\infty} \|T(t)\| = 0$.

(c) $\|T(t_0)\| < 1$ *for some* $t_0 > 0$.

(d) $r(T(t_1)) < 1$ *for some* $t_1 > 0$.

Proof: The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy.

$(d) \Rightarrow (c)$: Since $r(T(t_1)) = \lim_{k\to\infty} \|T(t_1 k)\|^{\frac{1}{k}} < 1$, it follows that there is $k_0 \in \mathbb{N}$ with $\|T(k_0 t_1)\| < 1$.

$(c) \Rightarrow (a)$: For $\alpha := \|T(t_0)\| < 1$, $M := \sup_{0\le s\le t_0} \|T(s)\|$ and $t = kt_0 + s$ with $s \in [0,t_0)$, we have

$$\begin{aligned} \|T(t)\| &\le \|T(s)\| \|T(t_0 k)\| \\ &\le M\alpha^k = M e^{k \ln \alpha}. \end{aligned}$$

If we set $\varepsilon := \frac{-\ln \alpha}{t_0} > 0$ (because $\alpha < 1$), then

$$\|T(t)\| \le M e^{k \ln \alpha} \le \frac{M}{\alpha} e^{-\varepsilon t}.$$

□

It is clear that if $\omega_0(A) < 0$, then there are constants $\varepsilon > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\varepsilon t}, \quad t \geq 0.$$

Hence, for every $p \in [1,\infty)$, $\int_0^\infty \|T(t)x\|^p\,dt < \infty$ for all $x \in E$. The following result due to Datko [6] shows that the converse is also true.

Theorem 2.1.4 *A C_0–semigroup $T(\cdot)$ on a Banach space E is uniformly exponentially stable if and only if for some (and hence for every) $p \in [1,\infty)$,*

$$\int_0^\infty \|T(t)x\|^p\,dt < \infty$$

for all $x \in E$.

Proof: We have only to prove the converse. By Proposition 2.1.3 it suffices to prove that $\lim_{t\to\infty}\|T(t)\| = 0$. Since there are M, $\omega \in \mathbb{R}_+$ with $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$, we obtain

$$\begin{aligned} \frac{1-e^{-p\omega t}}{p\omega}\|T(t)x\|^p &= \int_0^t e^{-p\omega s}\|T(s)T(t-s)x\|^p\,ds \\ &\leq M^p\int_0^t \|T(t-s)x\|^p\,ds \\ &\leq M^pC^p\|x\|^p \end{aligned}$$

for all $x \in E$ and $t \geq 0$. Hence, $\|T(t)x\|^p \leq \frac{p\omega}{1-e^{-p\omega}}M^pC^p\|x\|^p$ for $x \in E$ and $t \geq 1$. Thus, there exists a constant $L > 0$ with $\|T(t)\| \leq L$ for all $t \geq 0$. Therefore,

$$\begin{aligned} t\|T(t)x\|^p &= \int_0^t \|T(t-s)T(s)x\|^p\,ds \\ &\leq L^p\int_0^t \|T(s)x\|^p\,ds \\ &\leq L^pC^p\|x\|^p \end{aligned}$$

for all $x \in E$ and $t \geq 0$. Thus,

$$\|T(t)\| \leq LCt^{-\frac{1}{p}}, \quad t > 0,$$

which implies $\lim_{t\to\infty}\|T(t)\| = 0$. □

In Hilbert spaces uniform exponential stability can be characterized in term of the generator as the following Gearhart-Prüss's result shows (see [11], [22, A-III.7], [25]).

Theorem 2.1.5 *Let $T(\cdot)$ be a C_0–semigroup on a Hilbert space H with generator A. Then $T(\cdot)$ is uniformly exponentially stable if and only if*

$$\{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A) \text{ and } M := \sup_{\Re(\lambda)>0}\|R(\lambda,A)\| < \infty.$$

Proof: Assume that $\omega_0(A) < 0$. Then $\int_0^\infty e^{-\lambda t} T(t)\,dt$ exists for all $\Re(\lambda) > 0$. So by [9, Theorem II.1.10], $\{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A)$ and $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\,dt$ and therefore

$$\sup_{\Re(\lambda)>0} \|R(\lambda, A)\| < \infty.$$

We now prove the converse. We know from the spectral theory for closed operators (cf. [9, Corollary IV.1.14]) that

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq M^{-1}, \quad \text{for all } \Re(\lambda) > 0.$$

Thus, $i\mathbb{R} \subseteq \rho(A)$ and $\sup_{\Re(\lambda)\geq 0} \|R(\lambda, A)\| < \infty$. Let $\omega > |\omega_0(A)| + 1$ and consider the $\mathcal{C}_0$–semigroup $T_{-\omega}(\cdot)$ defined by $T_{-\omega}(t) := e^{-\omega t} T(t)$, $t \geq 0$. By [9, Theorem II.1.10] we have

$$\begin{aligned} R(\omega + is, A)x &= R(is, A - \omega)x \\ &= \int_0^\infty e^{-ist} T_{-\omega}(t)x\,dt \\ &= \mathcal{F}\left(T_{-\omega}(\cdot)x\right)(s), \end{aligned}$$

where $\mathcal{F} f(s) := \int_{-\infty}^\infty e^{-ist} f(t)\,dt$ denotes de Fourier transform from $L^2(\mathbb{R}, H)$ into $L^2(\mathbb{R}, H)$. Here we extend $T_{-\omega}(\cdot)$ to $\mathbb{R}$ by taking $T_{-\omega}(t) = 0$ for $t < 0$. Since $T_{-\omega}(\cdot)$ is uniformly exponentially stable, we obtain $T_{-\omega}(\cdot)x \in L^2(\mathbb{R}, H)$. Then one can apply Plancherel's theorem, and we obtain

$$\int_{-\infty}^\infty \|R(\omega + is, A)x\|^2\,ds = 2\pi \int_0^\infty \|T_{-\omega}(t)x\|^2\,dt \leq L\|x\|^2$$

for some constant $L > 0$ and all $x \in H$. The resolvent identity gives

$$R(is, A) = R(\omega + is, A) + \omega R(is, A) R(\omega + is, A), \quad \text{for all } s \in \mathbb{R}.$$

Hence, $\|R(is, A)x\| \leq (1 + M\omega)\|R(\omega + is, A)x\|$ for $s \in \mathbb{R}$ and $x \in H$. This implies

$$\begin{aligned} \int_{-\infty}^\infty \|R(is, A)x\|^2\,ds &\leq (1 + \omega M)^2 \int_{-\infty}^\infty \|R(\omega + is, A)x\|^2\,ds \\ &\leq (1 + M\omega)^2 L\|x\|^2. \end{aligned}$$

On the other hand, by the inverse Laplace transform formula (cf. [9, Corollary III.5.16]) we know that

$$T(t)x = \frac{1}{2i\pi t} \lim_{n\to\infty} \int_{\omega - in}^{\omega + in} e^{\lambda t} R(\lambda, A)^2 x\,d\lambda, \quad t \geq 0, x \in D(A^2).$$

Then, by Cauchy's integral theorem,

$$\begin{aligned} (tT(t)x|y) &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{(\omega+is)t} \left(R(\omega + is, A)^2 x|y\right)\,ds \\ &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{ist} \left(R(is, A)^2 x|y\right)\,ds \\ &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{ist} \left(R(is, A)x|R(-is, A^*)y\right)\,ds \end{aligned}$$

for all $x \in D(A^2)$ and $y \in H$. As above one can see that

$$\int_{-\infty}^{\infty} \|R(is,A^*)y\|^2\, ds \leq (1+M\omega)^2 L \|y\|^2, \quad y \in H.$$

By applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |(tT(t)x|y)| &\leq \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|R(is,A)x\|^2\, ds\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \|R(is,A^*)y\|^2\, ds\right)^{\frac{1}{2}} \\ &\leq \frac{(1+M\omega)^2 L}{2\pi}\|x\|\|y\| \end{aligned}$$

for all $x \in D(A^2)$ and $y \in H$. Since $\overline{D(A^2)} = H$, it follows that

$$\begin{aligned} \|tT(t)\| &= \sup\left\{|(tT(t)x|y)|\,;\, x,y \in D(A^2),\, \|x\| = \|y\| = 1\right\} \\ &\leq \frac{(1+M\omega)^2}{2\pi} L. \end{aligned}$$

Hence, $\lim_{t\to\infty} \|T(t)\| = 0$ and therefore, $\omega_0(A) < 0$. □

2.2 THE ESSENTIAL SPECTRUM AND QUASI-COMPACT SEMIGROUPS

In this section we study the essential growth bound $\omega_{ess}(A)$ of the generator A of a C_0–semigroup $T(\cdot)$ on a Banach space E, in the case $\omega_{ess}(A) < 0$. Then we deduce important consequences for the asymptotic behaviour of $T(\cdot)$.

We start with some definitions. A bounded operator $S \in \mathcal{L}(E)$ is called a *Fredholm operator* if there is $T \in \mathcal{L}(E)$ such that $Id - TS$ and $Id - ST$ are compact. We denote by

$$\sigma_{ess}(S) = \mathbb{C} \setminus \rho_F(S)$$

the *essential spectrum* of S, where

$$\rho_F(S) := \{\lambda \in \mathbb{C} : (\lambda - S) \text{ is a Fredholm operator }\}.$$

The *Calkin algebra* $\mathcal{C}(E) := \mathcal{L}(E)/\mathcal{K}(E)$ equipped with the quotient norm

$$\|S\|_{ess} := \|S + \mathcal{K}(E)\| = \text{dist}(S, \mathcal{K}(E)) = \inf\{\|S - K\| : K \in \mathcal{K}(E)\}$$

is a Banach algebra with unit. The essential spectrum of $S \in \mathcal{L}(E)$ can also defined as the spectrum of $S + \mathcal{K}(E)$ in the Banach algebra $\mathcal{C}(E)$. This implies that, for $S \in \mathcal{L}(E)$, $\sigma_{ess}(S)$ is non-empty and compact.

For $S \in \mathcal{L}(E)$ we define the *essential spectral radius* by

$$r_{ess}(S) := r(S + \mathcal{K}(E)) = \max\{|\lambda| : \lambda \in \sigma_{ess}(S)\}.$$

Since $(S+\mathcal{K}(E))^n = S^n + \mathcal{K}(E)$ for $n \in \mathbb{N}$, we have $r_{ess}(S) = \lim_{n\to\infty} \|S^n\|_{ess}^{\frac{1}{n}}$ and consequently,

$$r_{ess}(S+K) = r_{ess}(S), \qquad \text{for every } K \in \mathcal{K}(E).$$

If we denote by

$$Pol(S) := \{\lambda \in \mathbb{C} : \lambda \text{ is a pole of finite algebraic multiplicity of } R(\cdot, S)\},$$

then one can prove that $Pol(S) \subseteq \rho_F(S)$ and an element of the unbounded connected component of $\rho_F(S)$ either is in $\rho(S)$ or a pole of finite algebraic multiplicity. For details concerning the essential spectrum we refer to [20, Sec. IV.5.6], [13, Chap. XVII] or [12, Sec. IV.2]. Thus we obtain the following characterization.

Proposition 2.2.1 *For $S \in \mathcal{L}(E)$ the essential spectral radius is given by*

$$r_{ess}(S) = \inf\{r > 0 : \lambda \in \sigma(S), |\lambda| > r \text{ and } \lambda \in Pol(S)\}.$$

Proof: If we set

$$a := \inf\{r > 0 : \lambda \in \sigma(S), |\lambda| > r \text{ and } \lambda \in Pol(S)\},$$

then for all $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that

$$\{\lambda \in \sigma(S) : |\lambda| > r_\varepsilon\} \subseteq Pol(S)$$

and $r_\varepsilon - \varepsilon \leq a$. On the other hand, we know that there is $\lambda_0 \in \sigma_{ess}(S)$ with $r_{ess}(S) = |\lambda_0|$. If we suppose that $r_{ess}(S) > r_\varepsilon$, then $\lambda_0 \in Pol(S)$. This implies that $\lambda_0 \in \rho_F(S)$ which is a contradiction. Hence, $r_{ess}(S) \leq r_\varepsilon \leq a + \varepsilon$. Thus, $r_{ess}(S) \leq a$.

To show the other inequality we know that

$$\{\lambda \in \sigma(S) : |\lambda| > r_{ess}(S)\} \subseteq \rho_F(S).$$

Therefore,

$$\{\lambda \in \sigma(S) : |\lambda| > r_{ess}(S)\} \subseteq Pol(S).$$

Consequently, $a \leq r_{ess}(S)$ and the proposition is proved. □

We define the *essential growth bound* $\omega_{ess}(A)$ of a $\mathcal{C}_0$–semigroup $T(\cdot)$ with generator A as the growth bound of the quotient semigroup $T(\cdot) + \mathcal{K}(E)$ on $\mathcal{C}(E)$, i.e.,

$$\omega_{ess}(A) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\|_{ess} \leq Me^{\omega t}, \forall t \geq 0\}.$$

Then, for all $t_0 > 0$, one can see that

$$\omega_{ess}(A) = \frac{\log r_{ess}(T(t_0))}{t_0} = \lim_{t\to\infty} \frac{\log \|T(t)\|_{ess}}{t}. \tag{2.2}$$

The following result gives the relationship between $\omega_{ess}(A)$ and $\omega_0(A)$.

Proposition 2.2.2 *Let $T(\cdot)$ be a $\mathcal{C}_0$–semigroup with generator A on a Banach space E. Then one has*

$$\omega_0(A) = \max\{s(A), \omega_{ess}(A)\}.$$

Proof: If $\omega_{ess}(A) < \omega_0(A)$, then $r_{ess}(T(1)) < r(T(1))$. Let $\lambda \in \sigma(T(1))$ such that $|\lambda| = r(T(1))$. So by Proposition 2.2.1, λ is an eigenvalue of $T(1)$ and by the spectral mapping theorem for the point spectrum (cf. [9, Theorem IV.3.7]) there is $\lambda_1 \in \sigma_p(A)$ with $e^{\lambda_1} = \lambda$. Therefore, $\Re(\lambda_1) = \omega_0(A)$ and thus $\omega_0(A) = s(A)$. □

By using the essential growth bound one can deduces important consequences for the asymptotic behaviour, the proof can be found in [9, Theorem V.3.1]

Theorem 2.2.3 *Let A be the generator of a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space E and $\lambda_1, \ldots, \lambda_m \in \sigma(A)$ with $\Re(\lambda_1), \ldots, \Re(\lambda_m) > \omega_{ess}(A)$. Then $\lambda_1, \ldots, \lambda_m$ are isolated spectral values of A with finite algebraic multiplicity. Furthermore, if $P_1, \ldots, P_m$ denote the corresponding spectral projections and $k_1, \ldots, k_m$ the corresponding orders of poles of $R(\cdot, A)$, then*

$$T(t) = T_1(t) + \ldots + T_m(t) + R_m(t),$$

where

$$T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!} (A - \lambda_n)^j P_n, \quad n = 1, \ldots, m.$$

Moreover, for every $\omega > \sup\{\omega_{ess}(A)\} \cup \{\Re(\lambda) : \lambda \in \sigma(A) \setminus \{\lambda_1, \ldots, \lambda_m\}\}$, there is $M > 0$ such that

$$\|R_m(t)\| \leq M e^{\omega t} \quad \text{for } t \geq 0.$$

We now introduce the concept of quasi-compact semigroups,

Definition 2.2.4 *A $\mathcal{C}_0$–semigroup $T(\cdot)$ with generator A on a Banach space E is called quasi-compact if $\omega_{ess}(A) < 0$.*

From (2.2) we deduce that any eventually compact $\mathcal{C}_0$–semigroup is quasi-compact.

The following description of the asymptotic behaviour of quasi-compact semigroups is an immediate consequence of Theorem 2.2.3.

Theorem 2.2.5 *Let A be the generator of a quasi-compact $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space E. Then the following assertions hold.*

(a) *The set $\{\lambda \in \sigma(A) : \Re(\lambda) \geq 0\}$ is finite (or empty) and consists of poles of $R(\cdot, A)$ of finite algebraic multiplicity.*
Denoting these poles by $\lambda_1, \ldots, \lambda_m$, the corresponding spectral projections $P_1, \ldots, P_m$ and the order of the poles $k_1, \ldots, k_m$, we have

(b) $T(t) = T_1(t) + \ldots + T_m(t) + R(t)$, *where*

$$T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!} (A - \lambda_n)^j P_n, \quad n = 1, \ldots, m,$$

and

$$\|R(t)\| \leq M e^{-\varepsilon t} \quad \text{for some } \varepsilon > 0, M \geq 1 \text{ and all } t \geq 0.$$

2.3 SPECTRAL BOUNDS FOR POSITIVE SEMIGROUPS

In this section we characterize the spectral bound

$$s(A) := \sup\{\Re(\lambda) : \lambda \in \sigma(A)\}$$

of the generator of a positive C_0–semigroup $T(\cdot)$ on a complex Banach lattice E. We will see that $s(A)$ is always contained in $\sigma(A)$ provided that $\sigma(A) \neq \emptyset$.
To that purpose the following result is essential.

Theorem 2.3.1 *Let A be the generator of a positive C_0–semigroup $T(\cdot)$ on E. For $\Re(\lambda) > s(A)$ we have*

$$R(\lambda,A)x = \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds, \quad x \in E.$$

Moreover, $\int_0^t e^{-\lambda s}T(s)\,ds$ converges to $R(\lambda,A)$ with respect to the operator norm as $t\to\infty$.

Proof: Let $\lambda_0 > \omega_0(A)$ be fixed. Since $R(\lambda_0,A)x = \int_0^\infty e^{-\lambda_0 t}T(t)x\,dt$ and by the resolvent identity we obtain

$$R(\lambda_0,A)^{n+1}x = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda_0 t}T(t)x\,dt$$

for $n\in\mathbb{N}$ and $x\in E$. Let $\mu \in (s(A),\lambda_0)$, $x\in E_+$ and $x^*\in E_+^*$. By the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]) one has $\frac{1}{\lambda_0-\mu} > r(R(\lambda_0,A))$ and hence,

$$\begin{aligned}
\langle R(\mu,A)x,x^*\rangle &= \sum_{n=0}^\infty (\lambda_0-\mu)^n\langle R(\lambda_0,A)^{n+1}x,x^*\rangle \\
&= \sum_{n=0}^\infty \int_0^\infty \frac{1}{n!}[(\lambda_0-\mu)s]^n e^{-\lambda_0 s}\langle T(s)x,x^*\rangle ds \\
&= \int_0^\infty \left(\sum_{n=0}^\infty \frac{1}{n!}[(\lambda_0-\mu)s]^n\right) e^{-\lambda_0 s}\langle T(s)x,x^*\rangle ds \\
&= \int_0^\infty e^{(\lambda_0-\mu)s}e^{-\lambda_0 s}\langle T(s)x,x^*\rangle ds \\
&= \int_0^\infty e^{-\mu s}\langle T(s)x,x^*\rangle ds \\
&= \lim_{t\to\infty}\langle\int_0^t e^{-\mu s}T(s)x\,ds,x^*\rangle.
\end{aligned}$$

Hence, $\left(\int_0^t e^{-\mu s}T(s)x\,ds\right)$ converges weakly to $R(\mu,A)x$ as $t\to\infty$. Since $x\in E_+$, it follows that $\left(\int_0^t e^{-\mu s}T(s)x\,ds\right)_{t\geq 0}$ is monotone increasing and so, by Proposition 1.1.13, we have strong convergence. Thus,

$$\lim_{t\to\infty}\int_0^t e^{-\mu s}T(s)x\,ds = R(\mu,A)x, \quad \text{for all } x\in E.$$

If $\lambda = \mu + i\gamma$ with $\mu, \gamma \in \mathbb{R}$ and $\mu > s(A)$, then for any $x \in E$ and $x^* \in E^*$, we have

$$\left| \langle \int_r^t e^{-\lambda s} T(s) x \, ds, x^* \rangle \right| \leq \int_r^t e^{-\mu s} \langle T(s)|x|, |x^*| \rangle ds.$$

Hence,

$$\left\| \int_r^t e^{-\lambda s} T(s) x \, ds \right\| \leq \left\| \int_r^t e^{-\mu s} T(s)|x| \, ds \right\|,$$

which implies that

$$\lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s) x \, ds \text{ exists for all } x \in E.$$

Then, by [9, Theorem II.1.10],

$$\lambda \in \rho(A) \text{ and } R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t) x \, dt \qquad \text{for all } x \in E.$$

It remains to prove that $\left(\int_0^t e^{-\lambda s} T(s) \, ds\right)$ converges in the operator norm as $t \to \infty$. We fix $\mu \in (s(A), \Re(\lambda))$. As we have seen above, the function

$$f_{x,x^*} : s \mapsto e^{-\mu s} \langle T(s)x, x^* \rangle \text{ belongs to } L^1(\mathbb{R}_+) \qquad \text{for all } x \in E, x^* \in E^*.$$

It follows from the closed graph theorem that the bilinear form

$$b : E \times E^* \to L^1(\mathbb{R}_+); \, (x, x^*) \mapsto f_{x,x^*}$$

is separately continuous and hence continuous. Thus, there exists $M > 0$ such that

$$\int_0^\infty e^{-\mu s} |\langle T(s)x, x^* \rangle| \, ds \leq M \|x\| \|x^*\|, \quad x \in E, \, x^* \in E^*.$$

For $0 \leq t < r$ and $\varepsilon := \Re(\lambda) - \mu$ we have

$$\begin{aligned} \left| \int_t^r e^{-\lambda s} \langle T(s)x, x^* \rangle ds \right| &\leq \int_t^r e^{-(\Re(\lambda) - \mu)s} e^{-\mu s} |\langle T(s)x, x^* \rangle| ds \\ &\leq e^{-\varepsilon t} \int_t^r e^{-\mu s} |\langle T(s)x, x^* \rangle| ds \\ &\leq e^{-\varepsilon t} M \|x\| \|x^*\|. \end{aligned}$$

Hence, $\left\| \int_t^r e^{-\lambda s} T(s) \, ds \right\| \leq M e^{-\varepsilon t}$ and this implies that $\left(\int_0^t e^{-\lambda s} T(s) \, ds\right)$ is a Cauchy sequence in $\mathcal{L}(E)$. □

As an immediate consequence we obtain the following corollary.

Corollary 2.3.2 *Let A be the generator of a positive C_0–semigroup $T(\cdot)$ on E. If $\Re(\lambda) > s(A)$, then*

$$|R(\lambda, A)x| \leq R(\Re(\lambda), A)|x| \qquad \textit{for all } x \in E.$$

An other interesting corollary is the following.

Corollary 2.3.3 *If A is the generator of a positive C_0–semigroup $T(\cdot)$ on E, then*

$$s(A) \in \sigma(A) \text{ or } s(A) = -\infty.$$

Proof: Assume that $s(A) > -\infty$ and $s(A) \notin \sigma(A)$. So it follows from Corollary 2.3.2 that

$$|R(\lambda,A)x| \leq R(\Re(\lambda),A)|x| \leq R(s(A),A)|x| \quad \text{for all } \Re(\lambda) > s(A),\, x \in E.$$

Hence the set $\{R(\lambda,A) : \Re(\lambda) > s(A)\}$ is uniformly bounded in $\mathcal{L}(E)$. Let $M := \sup_{\Re(\lambda)>s(A)} \|R(\lambda,A)\|$. Since $\|R(\lambda,A)\| \geq \frac{1}{\mathrm{dist}(\lambda,\sigma(A))}$ for $\lambda \in \rho(A)$ (cf. [9, Corollary IV.1.14]), it follows that

$$\{\lambda \in \mathbb{C} : \Re(\lambda) = s(A)\} \subseteq \rho(A) \text{ and } \|R(\lambda,A)\| \leq M, \forall \Re(\lambda) = s(A).$$

Thus,

$$\{\lambda \in \mathbb{C} : |\Re(\lambda) - s(A)| < M^{-1}\} \subseteq \rho(A).$$

This contradicts the definition of $s(A)$. □

The following consequence gives a relation between $s(A)$ and the positivity of the resolvent.

Corollary 2.3.4 *Suppose that A generates a positive on E and $\lambda_0 \in \rho(A)$. Then the following assertions hold.*

(i) *$R(\lambda_0,A)$ is positive if and only if $\lambda_0 > s(A)$.*

(ii) *If $\lambda > s(A)$, then $r(R(\lambda,A)) = \frac{1}{\lambda - s(A)}$.*

Proof: (ii) is a simple consequence from Corollary 2.3.3 and the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]).
(i) Assume first that $R(\lambda_0,A) \geq 0$. Since $Ag \in E_{\mathbb{R}}$ for all $0 \leq g \in D(A)$, we have $\lambda_0 \in \mathbb{R}$. On the other hand, Theorem 2.3.1 implies that $R(\lambda,A) \geq 0$ for all $\lambda > \max(\lambda_0, s(A))$ and hence

$$\begin{aligned} R(\lambda_0,A) &= R(\lambda,A) + (\lambda - \lambda_0)R(\lambda,A)R(\lambda_0,A) \\ &\geq R(\lambda,A) \geq 0 \end{aligned}$$

for all $\lambda > \max(\lambda_0, s(A))$. Therefore,

$$(\lambda - s(A))^{-1} = r(R(\lambda,A)) \leq \|R(\lambda,A)\| \leq \|R(\lambda_0,A)\|$$

for all $\lambda > \max(\lambda_0, s(A))$. But this is only true if $\lambda_0 > s(A)$.

The converse follows from Theorem 2.3.1. □

Remark 2.3.5 *(a) As an immediate consequence of Corollary 2.3.4 we obtain*

$$s(A) = \inf\{\lambda \in \rho(A) : R(\lambda, A) \geq 0\}$$

for the generator A of a positive C_0–semigroup on a Banach lattice E.

(b) If $E := C(K)$, K compact, then $s(A) > -\infty$. In fact: We know from the theory of C_0–semigroups that $\lim_{\lambda\to\infty} \lambda R(\lambda, A) f = f$ for all $f \in E$. In particular we find $\lambda_0 \in \mathbb{R}$ sufficiently large such that

$$\lambda_0 R(\lambda_0, A) \mathbb{1} \geq \frac{1}{2} \mathbb{1},$$

where $\mathbb{1}(x) := 1$ for all $x \in K$. Since $R(\lambda_0, A) \geq 0$, it follows that

$$R(\lambda_0, A)^n \mathbb{1} \geq \frac{1}{(2\lambda_0)^n} \mathbb{1} \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$r(R(\lambda_0, A)) = \lim_{n\to\infty} \|R(\lambda_0, A)^n\|^{\frac{1}{n}} \geq \frac{1}{2\lambda_0} > 0$$

and hence $\sigma(A) \neq \emptyset$.

The spectrum of a generator of a positive C_0–semigroup can be empty as the following examples show.

Example 2.3.6 (a) *On $E := C_0[0,1) := \{f \in C[0,1] : f(1) = 0\}$ we consider the nilpotent C_0–semigroup $T(\cdot)$ given by*

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t < 1 \\ 0 & \text{if } x+t \geq 1 \end{cases}$$

for $t \geq 0$, $x \in [0,1]$ and $f \in E$. Then, $T(t) = 0$ for $t \geq 1$ and hence $\sigma(T(t)) = \{0\}$. So by the spectral inclusion theorem (cf. [9, Theorem IV.3.6]), $\sigma(A) = \emptyset$.

(b) *Let $E := C_0[0,\infty) := \{f \in C(\mathbb{R}_+) : \lim_{t\to+\infty} f(t) = 0\}$. On E, we define the C_0–semigroup $T(\cdot)$ by*

$$(T(t)f)(x) := e^{-\frac{t^2}{2} - xt} f(x+t), \quad x, t \geq 0 \text{ and } f \in E.$$

Then, one can see that the generator A of $T(\cdot)$ on E is given by

$$(Af)(x) = f'(x) - xf(x),\ x \geq 0, \text{ and}$$
$$f \in D(A) = \{f \in E : f \in C^1(\mathbb{R}_+) \text{ and } Af \in E\}.$$

By a simple computation one obtains that $\sigma(A) = \emptyset$.

For generators of positive C_0–groups the spectrum is always nonempty. This is given by the following corollary.

Corollary 2.3.7 *If A generates a positive $\mathcal{C}_0$–group on a Banach lattice E, then $\sigma(A) \neq \emptyset$.*

Proof: Assume that $\sigma(A) = \emptyset$. By Theorem 2.3.1 we have $R(\lambda, A) \geq 0$ for all $\lambda \in \mathbb{R}$. Again, one can apply the same theorem to $-A$ and obtains $R(\lambda, -A) \geq 0$ for all $\lambda \in \mathbb{R}$. But $R(\lambda, -A) = -R(-\lambda, A) \leq 0$ for all $\lambda \in \mathbb{R}$, and hence, $R(\lambda, -A) = 0$ for all $\lambda \in \mathbb{R}$. This contradicts the fact that $E \neq \{0\}$. □

2.4 THE PROBLEM $\omega_0(A) = s(A)$ FOR POSITIVE SEMIGROUPS

In this section we study in detail the growth bound $\omega_0(A)$ of the generator A of a positive $\mathcal{C}_0$–semigroup on a Banach lattice E. In particular, we look for sufficient conditions implying the equality $\omega_0(A) = s(A)$ without supposing the spectral mapping theorem.

For a $\mathcal{C}_0$–semigroup $S(\cdot)$ with generator B on a Banach space X satisfying $||S(t)|| \leq Me^{\omega t}$, $t \geq 0$, for some constants $M, \omega \in \mathbb{R}$, it follows that $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\} \subseteq \rho(B)$. Thus,

$$s(B) \leq \omega_0(B)$$

is always satisfied.

By applying the Gearhardt-Pruess's theorem and Theorem 1.2.2 we obtain the first result on the opposite inequality.

Theorem 2.4.1 *Let A be the generator of a positive $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach lattice E. Then $\omega_0(A) = s(A)$ holds in the followings cases.*

(i) *E is a Hilbert space.*

(ii) *E is an AL-space.*

(iii) *$E := C_0(\Omega)$ or $E := C(K)$, where Ω is locally compact Hausdorff and K is compact Hausdorff.*

Proof: (i) Let $\mu > s(A)$ fixed. It follows from Corollary 2.3.2 that $\Lambda := \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A - \mu)$ and

$$||R(\lambda, A - \mu)|| \leq ||R(\Re(\lambda), A - \mu)|| \leq ||R(\mu, A)|| \quad \text{for all } \lambda \in \Lambda.$$

So, by Theorem 2.1.5, we have $\omega_0(A) - \mu < 0$ and hence,

$$\omega_0(A) \leq s(A).$$

(ii) For $\lambda > s(A)$ and $x \in E_+$ we obtain from Theorem 2.3.1 that

$$||R(\lambda, A)x|| = \left\| \int_0^\infty e^{-\lambda s} T(s)x\, ds \right\| = \int_0^\infty e^{-\lambda s} ||T(s)x||\, ds,$$

where the second equality follows from the fact that the norm is additive on the positive cone. Hence,

$$\int_0^\infty \|(e^{-\lambda s}T(s))x\|\,ds < \infty \qquad \text{for all } x \in E.$$

So, by Theorem 2.1.4, we have $\omega_0(A) - \lambda < 0$ and thus

$$\omega_0(A) \leq s(A).$$

(iii) It is easy to see that $\|f \vee g\| = \|f\| \vee \|g\|$ for all $f, g \in E_+$. Then, for $\gamma, \nu \in E_+^*$, we have

$$\begin{aligned} \langle f, \gamma\rangle + \langle g, \nu\rangle &\leq \langle f \vee g, \gamma + \nu\rangle \\ &\leq \|\gamma+\nu\|\|f \vee g\| \\ &= \|\gamma+\nu\|(\|f\| \vee \|g\|), \quad f, g \in E_+. \end{aligned}$$

Hence, $\langle f, \gamma\rangle + \langle g, \nu\rangle \leq \|\gamma+\nu\|$ for all $f, g \in E_+$ with $\|f\| = \|g\| = 1$. It follows from the Hahn-Banach theorem that $\|\gamma\| + \|\nu\| \leq \|\gamma+\nu\|$ and hence,

$$\|\gamma\| + \|\nu\| = \|\gamma+\nu\|, \quad \gamma, \nu \in E_+.$$

This implies that E^* is an AL-space. If we set $F := \overline{D(A^*)}$, then it follows from Theorem 1.2.2 that F is a closed ideal and hence also an AL-space. On F we consider the positive $\mathcal{C}_0$–semigroup $S(\cdot)$ given by

$$S(t) := T(t)^*_{|F} \qquad \text{for } t \geq 0,$$

and we denote by B its generator. Then B is the part of A^* in F, i.e.,

$$D(B) = \{\nu \in D(A^*) : A^*\nu \in F\} \text{ and } B\nu = A^*\nu \text{ for } \nu \in D(B).$$

Moreover, one can show that

$$\sigma(B) = \sigma(A^*) = \sigma(A).$$

Consequently, $s(B) = s(A)$ holds. Since B is the generator of the positive $\mathcal{C}_0$–semigroup $S(\cdot)$ on the AL-space F, it follows from (ii) that $s(B) = \omega_0(B)$. Now, it suffices to prove that $\omega_0(B) = \omega_0(A)$. The inequality $\omega_0(B) \leq \omega_0(A)$ is trivial. Let $\omega > \omega_0(B)$, $f \in E$ and $\nu \in F$. Then we have

$$|\langle T(t)f, \nu\rangle| = |\langle f, S(t)\nu\rangle| \leq M\|f\|e^{\omega t}\|\nu\|$$

for $t \geq 0$ and some constant $M \geq 1$. On the other hand, since $f = \lim_{\lambda\to\infty} \lambda R(\lambda, A)f$ for all $f \in E$, we have $c := \limsup_{\lambda\to\infty} \lambda\|R(\lambda, A)\| < \infty$. Therefore,

$$\begin{aligned} |\langle T(t)f, \gamma\rangle| &= \lim_{\lambda\to\infty} |\langle \lambda R(\lambda, A)T(t)f, \gamma\rangle| \\ &= \lim_{\lambda\to\infty} |\langle T(t)f, \lambda R(\lambda, A^*)\gamma\rangle| \\ &\leq M\|f\|e^{\omega t} \limsup_{\lambda\to\infty} \lambda\|R(\lambda, A)^*\gamma\| \\ &\leq Mce^{\omega t}\|f\|\|\gamma\|, \quad \gamma \in E^*. \end{aligned}$$

Consequently, $\|T(t)\| \leq Mce^{\omega t}$ for all $t \geq 0$ and hence $\omega_0(A) \leq \omega$ for all $\omega > \omega_0(B)$. Thus, we have shown that

$$\omega_0(B) = \omega_0(A).$$

□

The last result of this section is Weis's result concerning positive C_0–semigroups on $L^p(\Omega) := L^p(\Omega, \mu)$, $1 \leq p < \infty$, where (Ω, μ) a σ–finite measure space (see [33]). The proof presented here is due to W. Arendt (see [2, Theorem 5.3.6]).

We first need some preparations. We equip $\mathbb{R} \times \Omega$ with the product measure $\lambda_1 \otimes \mu$, where λ_1 is the Lebesgue measure on $\mathbb{R}$. We recall that $L^p(\mathbb{R} \times \Omega) \cong L^p(\mathbb{R}, L^p(\Omega))$. This allows us to identify the notations $g(t,\xi)$ and $g(t)(\xi)$ for $(t,\xi) \in \mathbb{R} \times \Omega$. Let us consider the non-linear map

$$\Phi : L^p(\mathbb{R}, L^p(\Omega)) \to L^p(\Omega);\, g \mapsto \Phi(g) := \left(\int_{\mathbb{R}} |g(t)|^p \, dt \right)^{\frac{1}{p}}.$$

It is clear that Φ is well-defined.

The following lemmas give some properties of the map Φ.

Lemma 2.4.2 *Let $g, h \in L^p(\mathbb{R}, L^p(\Omega))$, $f \in L^\infty(\Omega)$, and $s \in \mathbb{R}$. Then the following assertions hold.*

1. $\|\Phi(g)\|_{L^p(\Omega)} = \|g\|_{L^p(\mathbb{R}\times\Omega)}$.
2. $\Phi(g_s) = \Phi(g)$, *where* $g_s(t) := g(s+t)$, $t, s \in \mathbb{R}$.
3. $\Phi(f \cdot g) = |f|\Phi(g)$, *where* $(f \cdot g)(t,\xi) := f(\xi)g(t,\xi)$, $(t,\xi) \in \mathbb{R} \times \Omega$.
4. $\Phi(g+h) \leq \Phi(g) + \Phi(h)$.
5. Φ *is a continuous map.*

Proof: Assertions 1., 2. and 3. are simple to prove. For 4. we set $G_\xi(t) := g(t,\xi)$, $H_\xi(t) := h(t,\xi)$, $(t,\xi) \in \mathbb{R} \times \Omega$. For almost all $\xi \in \Omega$, we obtain $G_\xi, H_\xi \in L^p(\mathbb{R})$ and hence

$$\|G_\xi + H_\xi\|_{L^p(\mathbb{R})} \leq \|G_\xi\|_{L^p(\mathbb{R})} + \|H_\xi\|_{L^p(\mathbb{R})}.$$

Since $\|G_\xi\|_{L^p(\mathbb{R})} = (\int_{\mathbb{R}} |g(t,\xi)|^p \, dt)^{\frac{1}{p}} = \Phi(g)(\xi)$ and also $\|H_\xi\|_{L^p(\mathbb{R})} = \Phi(h)(\xi)$, it follows that

$$\Phi(g+h)(\xi) \leq \Phi(g)(\xi) + \Phi(h)(\xi), \quad \mu\text{-a.e. } \xi \in \Omega.$$

Thus, $\Phi(g+h) \leq \Phi(g) + \Phi(h)$.
By 4. we have

$$\Phi(g) \leq \Phi(g-h) + \Phi(h) \text{ and } \Phi(h) \leq \Phi(h-g) + \Phi(g).$$

This implies that $|\Phi(g)-\Phi(h)| \leq \Phi(g-h)$ and so by 1. we obtain

$$\|\Phi(g)-\Phi(h)\|_{L^p(\Omega)} \leq \|g-h\|_{L^p(\mathbb{R}\times\Omega)},$$

which proves 5.. □

Lemma 2.4.3 *For a continuous function $G:[a,b]\to L^p(\mathbb{R},L^p(\Omega))$ we have*

$$\Phi\left(\int_a^b G(s)\,ds\right) \leq \int_a^b \Phi(G(s))\,ds.$$

Proof: It follows from Lemma 2.4.2 that

$$\Phi\left(\frac{b-a}{2^n}\sum_{j=0}^{2^n-1} G\left(\frac{jb+(2^n-j)a}{2^n}\right)\right) \leq \frac{b-a}{2^n}\sum_{j=0}^{2^n-1}\Phi\left(G\left(\frac{jb+(2^n-j)a}{2^n}\right)\right).$$

Since Φ is continuous, we obtain the lemma by letting $n\to\infty$. □

Let $g\in L^p(\mathbb{R},L^p(\Omega))$ and $T\in\mathcal{L}(L^p(\Omega))$. We consider $T\circ g$ defined by

$$(T\circ g)(t) := T(g(t)), \quad t\in\mathbb{R}.$$

Lemma 2.4.4 *For $0\leq T\in\mathcal{L}(L^p(\Omega))$ and $0\leq g\in L^p(\mathbb{R},L^p(\Omega))$ the inequality*

$$\Phi(T\circ g)\leq T(\Phi(g))$$

holds.

Proof: By Lemma 2.4.2, it suffices to prove the lemma for simple functions. Let $g := \sum_{k=1}^n \chi_{A_k}\otimes g_k$, where $A_1,\dots,A_n$ are disjoint Borel subsets of $\mathbb{R}$, and $g_1,\dots,g_n\in L^p(\Omega)_+$. Setting $h_k := \lambda_1(A_k)^{\frac{1}{p}} g_k$ for $k\in\{1,\dots,n\}$. Since the sets (A_k) are disjoint, it follows that

$$\begin{aligned}\Phi(T\circ g) = \left(\sum_{k=1}^n \lambda_1(A_k)(Tg)^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^n (Th_k)^p\right)^{\frac{1}{p}},\\ T(\Phi(g)) = T\left(\sum_{k=1}^n \lambda_1(A_k)(g_k)^p\right)^{\frac{1}{p}} &= T\left(\sum_{k=1}^n (h_k)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Let $\alpha := (\alpha_k)_k\subset\mathbb{R}$ with $\|\alpha\|_{l^q}\leq 1$, where $\frac{1}{q}+\frac{1}{p}=1$. The Hölder inequality implies

$$\left(\sum_{k=1}^n \alpha_k h_k\right) \leq \left(\sum_{k=1}^n |h_k|^p\right)^{\frac{1}{p}} = \Phi(g),$$

hence

$$\left(\sum_{k=1}^n \alpha_k T h_k\right) = T\left(\sum_{k=1}^n \alpha_k h_k\right) \leq T(\Phi(g)).$$

Consequently,

$$\begin{aligned}\left(\sum_{k=1}^{n} |(Th_k)(\xi)|^p\right)^{\frac{1}{p}} &= \sup\left\{\left(\sum_{k=1}^{n} \alpha_k (Th_k)(\xi)\right) : \alpha_k \in \mathbb{R},\ \|(\alpha_k)\|_{l^q} \le 1\right\} \\ &\le T(\Phi(g))(\xi), \quad \mu\text{–a.e.}\, \xi \in \Omega,\end{aligned}$$

and $\Phi(T \circ g) \le T(\Phi(g))$. □

We are now ready to prove Weis's result.

Theorem 2.4.5 *Let (Ω, μ) be a σ–finite measure space, $1 \le p < \infty$, and $T(\cdot)$ a positive C_0–semigroup on $L^p(\Omega)$ with generator A. Then $\omega_0(A) = s(A)$.*

Proof: For $\xi > s(A)$ we set $T_\xi(t) := e^{-\xi t} T(t), t \ge 0$. We denote by $A_\xi := A - \xi$ the generator of the positive C_0–semigroup $T_\xi(\cdot)$ on $L^p(\Omega)$. Then $s(A_\xi) = s(A) - \xi < 0$. Let $\alpha > \max(0, \omega_0(A_\xi))$ fixed. Let $f \in L^p(\Omega)$ and consider the function $g \in L^p(\mathbb{R}, L^p(\Omega))$ defined by

$$g(t) = \begin{cases} e^{-\alpha t} T_\xi(t) f, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

We now introduce the function

$$G : \mathbb{R}_+ \to L^p(\mathbb{R}, L^p(\Omega)); s \mapsto G(s) := T_\xi(s) \circ g_{-s},$$

where $g_{-s}(t) := g(t-s), t \in \mathbb{R}$. Hence,

$$G(s)(t) = \begin{cases} e^{-\alpha(t-s)} T_\xi(t) f, & 0 \le s \le t, \\ 0, & t < s. \end{cases}$$

Thus,

$$\begin{aligned}\Phi\left(\int_0^m G(s)\, ds\right) &= \left(\int_0^\infty \left|\int_0^{\min(m,t)} e^{-\alpha(t-s)} T_\xi(t) f\, ds\right|^p dt\right)^{\frac{1}{p}} \\ &= \frac{1}{\alpha}\left(\int_0^\infty (e^{-\alpha \max(0,t-m)} - e^{-\alpha t})^p |T_\xi(t) f|^p\, dt\right)^{\frac{1}{p}}\end{aligned}$$

and hence

$$0 \le \frac{1}{\alpha}\left(\int_0^\infty (e^{-\alpha \max(0,t-m)} - e^{-\alpha t})^p |T_\xi(t) f|^p\, dt\right)^{\frac{1}{p}} = \Phi\left(\int_0^m G(s)\, ds\right). \quad (2.3)$$

So, by Lemmas 2.4.3, 2.4.4, and 2.4.2, it follows that

$$\begin{aligned} 0 &\leq \Phi\left(\int_0^m G(s)\,ds\right) \\ &\leq \int_0^m \Phi(G(s))\,ds \\ &= \int_0^m \Phi(T_\xi(s)\circ g_{-s})\,ds \\ &\leq \int_0^m T_\xi(s)(\Phi(g_{-s}))\,ds \\ &= \int_0^m T_\xi(s)(\Phi(g))\,ds. \end{aligned}$$

On the other hand, since $s(A_\xi) < 0$ and from Theorem 2.3.1, it follows that

$$\lim_{m\to\infty}\int_0^m T_\xi(s)(\Phi(g))\,ds = R(0,A_\xi)(\Phi(g)).$$

From (2.3) and the monotone convergence theorem we have

$$0 \leq \frac{1}{\alpha}\left(\int_0^\infty (1-e^{-\alpha t})^p |T_\xi(t) f|^p\,dt\right)^{\frac{1}{p}} \leq R(0,A_\xi)(\Phi(g)).$$

This implies

$$\left(\frac{1-e^{-\alpha}}{\alpha}\right)\left(\int_1^\infty |T_\xi(t) f|^p\,dt\right)^{\frac{1}{p}} \leq R(0,A_\xi)(\Phi(g))$$

and therefore

$$\int_\Omega\int_1^\infty |(T_\xi(t) f)(y)|^p\,dt\,d\mu(y) \leq \left(\frac{\alpha}{1-e^{-\alpha}}\right)^p \|R(0,A_\xi)\|^p \|\Phi(g)\|^p_{L^p(\Omega)},$$

which implies that

$$\int_1^\infty \|T_\xi(t) f\|^p_{L^p(\Omega)}\,dt < \infty.$$

So, by Theorem 2.1.4, we obtain $\omega_0(A_\xi) = \omega_0(A) - \xi < 0$. Consequently,

$$\omega_0(A) \leq s(A).$$

□

2.5 IRREDUCIBLE SEMIGROUPS

In many concrete examples the semigroup $T(\cdot)$ does not have exponential stability, however possesses an *asynchronous exponential growth*. This means that there is a rank one projection P and constants $\varepsilon > 0$, $M \geq 1$ such that

$$\|e^{-s(A)t}T(t) - P\| \leq Me^{-\varepsilon t} \qquad \text{for all } t \geq 0,$$

where A denotes the generator of $T(\cdot)$.

In order to study such kind of behaviour we introduce the concept of irreducibility for positive C_0–semigroups. For more details see [22] and the references therein.

Definition 2.5.1 *A positive C_0–semigroup $T(\cdot)$ on a Banach lattice E with generator A is called irreducible if one of the following equivalent properties is satisfied*

(i) *There is no $T(t)$–invariant closed ideal other than $\{0\}$ and E for all $t > 0$.*

(ii) *For $x \in E, x^* \in E^*$ with $x \gneqq 0$ and $x^* > 0$, there is $t_0 > 0$ such that*

$$\langle T(t_0)x, x^* \rangle > 0.$$

(iii) *For some (and then for every) $\lambda > s(A)$, there is no $R(\lambda, A)$–invariant closed ideal except $\{0\}$ and E.*

(iv) *For some (and then for every) $\lambda > s(A)$, $R(\lambda, A)x$ is a quasi-interior point of E_+ for every $x \gneqq 0$.*

Example 2.5.2 (a) *Let $E := L^p(\Omega, \mu)$, $1 \le p < \infty$, and $T(\cdot)$ be a positive C_0–semigroup on E with generator A. Then, it follows from Example 1.1.7 that $T(\cdot)$ is irreducible if and only if*

$$0 \lneqq f \in E \Longrightarrow (R(\lambda, A) f)(s) > 0 \text{ for a.e. } s \in \Omega \text{ and some } \lambda > s(A).$$

(b) *If $E := C_0(\Omega)$, where Ω is locally compact Hausdorff, and $T(\cdot)$ a positive C_0–semigroup on E with generator A, then, by Example 1.1.7, $T(\cdot)$ is irreducible if and only if*

$$0 \lneqq f \in E \Longrightarrow (R(\lambda, A) f)(s) > 0 \text{ for all } s \in \Omega \text{ and some } \lambda > s(A).$$

We now state some consequences of irreducibility.

Proposition 2.5.3 *Assume that A is the generator of an irreducible C_0–semigroup $T(\cdot)$ on a Banach lattice E. Then the following assertions hold.*

(a) *Every positive eigenvector of A is a quasi-interior point.*

(b) *Every positive eigenvector of A^* is strictly positive.*

(c) *If $\ker(s(A) - A^*)$ contains a positive element, then $\dim \ker(s(A) - A) \le 1$.*

(d) *If $s(A)$ is a pole of the resolvent, then it has algebraic (and geometric) multiplicity equal to 1. The corresponding residue has the form $P_{s(A)} = u^* \otimes x$, where $x \in E$ is a positive eigenvector of A, $u^* \in E^*$ is a positive eigenvector of A^* and $\langle x, u^* \rangle = 1$.*

Proof: (a) Let x be a positive eigenvector of A and $E_x := \cup_{n\in\mathbb{N}} n[-x,x]$ the ideal generated by x. If λ is such that $Ax = \lambda x$, then $\lambda \in \mathbb{R}$. This follows from

$$x \geq 0 \text{ and } Ax = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x).$$

Hence, $T(t)x = e^{\lambda t}x$ for $t \geq 0$. Thus, for $y \in E_x$,

$$|T(t)y| \leq T(t)|y| \leq nT(t)x = ne^{\lambda t}x, \quad t \geq 0.$$

Consequently, $T(t)E_x \subseteq E_x$ holds for all $t \geq 0$. Since $0 \neq x \in E_x$ and $T(\cdot)$ is irreducible, it follows that $\overline{E_x} = E$.

(b) Let x^* be a positive eigenvector of A^* and λ its corresponding eigenvalue. By the same argument we have $\lambda \in \mathbb{R}$ and $T(t)^*x^* = e^{\lambda t}x^*$ for $t \geq 0$. Hence,

$$\langle |T(t)u|, x^*\rangle \leq \langle T(t)|u|, x^*\rangle = \langle |u|, e^{\lambda t}x^*\rangle, \quad u \in E, t \geq 0.$$

Thus, $I := \{u \in E : \langle |u|, x^*\rangle = 0\}$ is a $T(t)$–invariant closed ideal for all $t \geq 0$. Since $x^* \neq 0$ we have $I \subsetneq E$ and so by the irreducibility we obtain $I = \{0\}$. Therefore, $x^* > 0$.

(c) Let $0 \lneq x^* \in \ker(s(A) - A^*)$. It follows from (b) that x^* is strictly positive. For $x \in \ker(s(A) - A)$ we have $T_{-s(A)}(t)x = x$ and hence,

$$|x| = |T_{-s(A)}(t)x| \leq T_{-s(A)}(t)|x|, \quad t \geq 0.$$

Thus, for $t \geq 0$,

$$\begin{aligned} \langle |x|, x^*\rangle &\leq \langle T_{-s(A)}(t)|x|, x^*\rangle \\ &= \langle |x|, x^*\rangle. \end{aligned}$$

This implies that $\langle T_{-s(A)}(t)|x| - |x|, x^*\rangle = 0$, and since $x^* > 0$, we obtain $T_{-s(A)}(t)|x| = |x|$ for $t \geq 0$. Therefore,

$$|x| \in \ker(s(A) - A).$$

Since $\left(T_{-s(A)}(t)x\right)^+ \leq T_{-s(A)}(t)x^+$, one can see by the same arguments as above that $x^+ \in \ker(s(A) - A)$ and $x^- \in \ker(s(A) - A)$. This implies that $F := E_{\mathbb{R}} \cap \ker(s(A) - A)$ is a real sublattice of E. For $x \in F$ we consider the ideal E_{x^+} (resp. E_{x^-}) generated by x^+ (resp. x^-). Then, E_{x^+} and E_{x^-} are $T_{-s(A)}(t)$–invariant for all $t \geq 0$. Since E_{x^+} and E_{x^-} are orthogonal, it follows from the irreducibility of $T_{-s(A)}(\cdot)$ that $x^+ = 0$ or $x^- = 0$. Consequently, F is totally ordered. So by Lemma 1.1.14 we have

$$\dim F = \dim \ker(s(A) - A) \leq 1.$$

(d) We claim that if $s(A)$ is a pole of the resolvent, then there is an eigenvector $0 \lneq x \in E$ of A corresponding to $s(A)$. Indeed, let k be the order of the pole $s(A)$ and $R_{-k} = \lim_{\lambda\to s(A)^+}(\lambda - s(A))^k R(\lambda, A)$ the corresponding residue. Then, $R_{-k} \neq 0$ and $R_{-(k+1)} = 0$. Moreover, by Corollary 2.3.4, we have $R_{-k} \geq 0$. Hence, there is

$0 \leq y \in E$ with $x := R_{-k}y \gneq 0$. By the relation $R_{-(k+1)} = (A - s(A))R_{-k} = 0$ we obtain $(A - s(A))x = 0$. This proves the claim.

We can now use (a) to obtain $\overline{E_x} = E$. By taking the adjoint $R^*_{-(k+1)}$ of $R_{-(k+1)}$ and by the same computation as before one has , if $s(A)$ is a pole of the resolvent, then there is $0 \lneq x^* \in \ker(s(A) - A^*)$. So by (c) we have $\dim\ker(s(A) - A) = 1$.

Now, assume that $k \geq 2$. Then we have

$$\begin{aligned} \langle x, x^* \rangle &= \langle R_{-k}y, x^* \rangle \\ &= \langle y, R^*_{-k}x^* \rangle \\ &= \langle y, R^*_{-(k-1)}(A^* - s(A))x^* \rangle \\ &= 0. \end{aligned}$$

Since $\overline{E_x} = E$, it follows that $\langle u, x^* \rangle = 0$ for all $u \in E_+$. This contradicts the assertion (b). Hence $k = 1$. From the inequality $m_g + k - 1 \leq m_a \leq m_g k$ (cf. [9] p. 247) we obtain

$$m_a = m_g = \dim P_{s(A)}E = \dim\ker(s(A) - A) = 1,$$

where we recall that $P_{s(A)} = R_{-1}$. Since $P_{s(A)}E \subseteq \ker(s(A) - A)$, it follows that

$$P_{s(A)}E = \ker(s(A) - A).$$

We now show the last part of Assertion (d). To this purpose let $0 \lneq x \in \ker(s(A) - A)$. Without loss of generality, we suppose that $||x|| = 1$. Then $P_{s(A)}E = \mathrm{Span}\{x\}$, i.e. $P_{s(A)}y = \lambda x$ for some $\lambda \in \mathbb{C}$ and every $y \in E$. By the Hahn-Banach theorem (see Proposition 1.1.12) there exists $0 \leq y^* \in (\ker(s(A) - A))^*$ with $||y^*|| = 1$ and $\langle x, y^* \rangle = ||x|| = 1$. Hence $\langle P_{s(A)}y, y^* \rangle = \lambda = \langle y, P^*_{s(A)}y^* \rangle$. If we put $u^* := P^*_{s(A)}y^* \geq 0$, then $P_{s(A)} = u^* \otimes x$ and $\langle x, u^* \rangle = \langle P_{s(A)}x, y^* \rangle = \langle x, y^* \rangle = 1$. This implies that $0 \lneq u^* \in P^*_{s(A)}E^* \subseteq \ker(s(A) - A^*)$. So $u^* > 0$ by (b). This ends the proof of the proposition. □

The following result describes the eigenvalues of an irreducible semigroup which are contained in the boundary spectrum $\sigma_b(A) := \{\lambda \in \sigma(A) : \Re(\lambda) = s(A)\}$, where A is the corresponding generator.

Theorem 2.5.4 *Let $T(\cdot)$ be an irreducible $\mathcal{C}_0$–semigroup with generator A on a Banach lattice E. Assume that $s(A) = 0$ and there is $0 \lneq x^* \in D(A^*)$ with $A^*x^* = 0$. If $\sigma_p(A) \cap i\mathbb{R} \neq \emptyset$, then the following assertions hold.*

(a) *For $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$ with $Ah = i\alpha h$, $|h|$ is a quasi-interior point and*

$$S_h(D(A)) = D(A) \text{ and } S_h^{-1}AS_h = A + i\alpha$$

hold, where S_h is the signum operator.

(b) $\dim\ker(\lambda - A) = 1$ *for every $\lambda \in \sigma_p(A) \cap i\mathbb{R}$.*

(c) *$\sigma_p(A) \cap i\mathbb{R}$ is an additive subgroup of $i\mathbb{R}$.*

(d) 0 *is the only eigenvalue of A admitting a positive eigenvector.*

Proof: We first remark that by Proposition 2.5.3.(b) we have $x^* > 0$ and $T(t)^*x^* = x^*$ for all $t \geq 0$.

(a) Assume that $Ah = i\alpha h$ for $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$. Then $T(t)h = e^{i\alpha t}h$ and hence $|h| = |T(t)h| \leq T(t)|h|$. This implies that

$$T(t)|h| - |h| \geq 0 \quad \text{for all } t \geq 0.$$

On the other hand,

$$\begin{aligned}\langle T(t)|h| - |h|, x^*\rangle &= \langle |h|, T(t)^*x^*\rangle - \langle |h|, x^*\rangle \\ &= 0 \quad \text{for all } t \geq 0.\end{aligned}$$

Since $x^* > 0$, we obtain $T(t)|h| = |h|$ for all $t \geq 0$, which implies that $A|h| = 0$. So, by Proposition 2.5.3.(a), $|h|$ is a quasi-interior point. If we set $T_\alpha(t) := e^{-i\alpha t}T(t)$, $t \geq 0$, then $T(t)$ and $T_\alpha(t)$ satisfy the assumptions of Lemma 1.2.5 and hence

$$T(t) = S_h^{-1}T_\alpha(t)S_h, \quad t \geq 0.$$

Therefore, $S_h(D(A)) = D(A)$ and $A = S_h^{-1}(A - i\alpha)S_h$ and (a) is proved.

(b) It follows from (a) that $S_h : \ker(i\alpha + A) \to \ker A$ for $i\alpha \in \sigma_p(A) \cap i\mathbb{R}$. On the other hand, the proof of (a) implies that $\ker A \neq \{0\}$. So, by Proposition 2.5.3.(c), $\dim\ker A = 1$ and hence $\dim\ker(i\alpha + A) = 1$.

(c): Let $0 \neq h, g \in D(A)$, $\alpha, \beta \in \mathbb{R}$ such that $Ah = i\alpha h$ and $Ag = i\beta g$. By (a) we have

$$S_g^{-1}AS_g = A + i\beta \text{ and } S_hAS_h^{-1} = A - i\alpha.$$

Thus $A + i(\beta - \alpha) = S_h(A + i\beta)S_h^{-1} = S_hS_g^{-1}AS_gS_h^{-1}$ which implies that $\ker(A + i(\beta - \alpha)) = S_hS_g^{-1}\ker A \neq \{0\}$. Therefore

$$i(\beta - \alpha) \in \sigma_p(A).$$

(d): If $Ax = \lambda x$, where $0 \lneqq x \in D(A)$, then

$$\lambda\langle x, x^*\rangle = \langle Ax, x^*\rangle = \langle x, A^*x^*\rangle = 0.$$

Since $x^* > 0$, it follows that $\langle x, x^*\rangle > 0$. Hence, $\lambda = 0$. □

For irreducible semigroups we obtain the following description of the boundary spectrum.

Theorem 2.5.5 *Let $T(\cdot)$ be an irreducible C_0–semigroup with generator A on a Banach lattice E and assume that $s(A)$ is a pole of the resolvent. Then there is* $\alpha \geq 0$ *such that*

$$\sigma_b(A) = s(A) + i\alpha\mathbb{Z}.$$

Moreover, $\sigma_b(A)$ contains only algebraically simple poles.

Proof: Without loss of generality we suppose that $s(A)=0$. It can be shown that $\sigma_b(A)\subseteq\sigma_p(A)$. The proof uses pseudo-resolvents on a suitable $\mathcal{F}$–product of E, where $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ which is finer than the Frechet filter (see [22], p. 314). Hence, $\sigma_b(A)=\sigma_p(A)\cap i\mathbb{R}$. By Proposition 2.5.3.(d) we obtain the existence of a positive eigenvector $x^*\in D(A^*)$ corresponding to the eigenvalue $s(A)=0$. It follows from Theorem 2.5.4.(c) that $\sigma_b(A)$ is a subgroup of $(i\mathbb{R},+)$. Since $\sigma_b(A)$ is closed and $s(A)=0$ is an isolated point, we have

$$\sigma_b(A)=i\alpha\mathbb{Z}\quad \text{for some } \alpha\geq 0.$$

Proposition 2.5.3.(d) implies that 0 is a simple pole and by Theorem 2.5.4.(a) we have, for $\lambda\in\rho(A)$,

$$R(\lambda+ik\alpha,A)=S_h^kR(\lambda,A)S_h^{-k}\quad \text{for all } k\in\mathbb{Z}.$$

Therefore, $ik\alpha$ is a simple pole for each $k\in\mathbb{Z}$. This ends the proof of the theorem. □

We now give sufficient conditions for a C_0–semigroup to possess an asynchronous exponential growth. This result will be very useful for many applications.

Theorem 2.5.6 *Let $T(\cdot)$ be an irreducible C_0–semigroup with generator A on a Banach lattice E. If $\omega_{ess}(A)<\omega_0(A)$, then there exists a quasi-interior point $0\leq x\in E$, $0<x^*\in E^*$ with $\langle x,x^*\rangle=1$ such that*

$$\|e^{-s(A)t}T(t)-x^*\otimes x\|\leq Me^{-\varepsilon t}\quad \textit{for all } t\geq 0,$$

and appropriate constants $M\geq 1$ and $\varepsilon>0$.

Proof: We first remark first that the rescaled semigroup $T_{-\omega_0}(t):=e^{-\omega_0(A)t}T(t)$, for $t\geq 0$, satisfies $\omega_{ess}(A_{-\omega_0})=\omega_{ess}(A)-\omega_0(A)<0$, where $A_{-\omega_0}:=A-\omega_0(A)$ denotes its generator. Thus, $T_{-\omega_0}(\cdot)$ is quasi-compact and, by Proposition 2.2.2, we have

$$s(A)=\omega_0(A).$$

On the other hand, since $\omega_{ess}(A)<\omega_0(A)$, it follows that $r_{ess}(T(1))<r(T(1))$. Hence, by Proposition 2.2.1, $r(T(1))$ is a pole of the resolvent of $T(1)$. This implies that $\omega_0(A)=s(A)$ is a pole of $R(\cdot,A)$. Thus, by Theorem 2.5.5, it follows that there exists $\alpha\geq 0$ such that $\sigma_b(A)=s(A)+i\alpha\mathbb{Z}$ and therefore $\sigma_b(A_{-\omega_0})=i\alpha\mathbb{Z}$. Since $T_{-\omega_0}(\cdot)$ is quasi-compact and $\omega_0(A_{-\omega_0})=0$, we have, by Theorem 2.2.5, that

$$\{\lambda\in\sigma(A_{-\omega_0}):\Re(\lambda)\geq 0\}=\{\lambda\in\sigma(A_{-\omega_0}):\Re(\lambda)=0\}=\sigma_b(A_{-\omega_0})$$

is finite. Therefore $\sigma_b(A_{-\omega_0})=\{0\}$. The theorem is now proved by applying Theorem 2.2.5 and Proposition 2.5.3 to the rescaled semigroup $T_{-\omega_0}(\cdot)$. □