In this chapter we are concerned with the remarkable spectral properties shown by positive semigroups on Banach lattices. Throughout this chapter we suppose that $E \neq \{0\}$ is a complex Banach lattice.

### 2.1 Stability of Strongly Continuous Semigroups

In this section we study the asymptotic behaviour of the solution of the abstract Cauchy problem

$$\begin{align*}
(ACP) \quad \begin{cases}
  u'(t) &= Au(t), \quad t \geq 0, \\
  u(0) &= x,
\end{cases}
\end{align*}$$

where $A$ is the generator of a $C_0$-semigroup $T(\cdot)$ on a Banach space $E$.

To this purpose we define the type of the trajectory $T(\cdot)x$ by

$$\omega(x) := \inf \{ \omega : \|T(t)x\| \leq Me^{\omega t} \text{ for a constant } M \text{ and all } t \geq 0 \},$$

and the growth bound (or type) of $T(\cdot)$ by

$$\omega_0(A) := \sup \{ \omega(x) : x \in E \} \quad \omega_0(A) := \inf \{ \omega \in \mathbb{R} : \|T(t)\| \leq Me^{\omega t} \text{ for some constant } M \text{ and all } t \geq 0 \}.$$

The type of the solutions of (ACP) is

$$\omega_1(A) := \sup \{ \omega(x) : x \in D(A) \}.$$

We now introduce different stability concepts.
Definition 2.1.1 A $C_0$–semigroup $T(\cdot)$ with generator $A$ is called

(i) uniformly exponentially stable if $\omega_0(A) < 0$,

(ii) exponentially stable if $\omega_1(A) < 0$,

(iii) strongly stable if $\lim_{t \to \infty} \|T(t)x\| = 0$ for every $x \in E$,

(iv) stable if $\lim_{t \to \infty} \|T(t)x\| = 0$ for every $x \in D(A)$.

It is clear that

$$
(i) \implies (ii) \downarrow \downarrow (iii) \implies (iv).
$$

If $A \in \mathcal{L}(E)$, then $(i) \iff (ii)$ and $(iii) \iff (iv)$. In the case where $A$ is unbounded
the above concepts of stability may differ as one can see in the following examples.

Example 2.1.2

1. On $E := C_0(\mathbb{R}^n)$ we consider the heat semigroup defined by

$$
(T(t)f)(x) := \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} f(y) dy \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^n.
$$

Then $T(\cdot)$ is a bounded holomorphic semigroup and it generator is the
Laplacian $\Delta$ on $C_0(\mathbb{R}^n)$. Since $T(t)f = k_t * f$, where $k_t(y) := \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{y^2}{4t}}$, $y \in \mathbb{R}^n$, and since $\|k_t\|_{L^1} = 1$, it follows that

$$
\|T(t)\| \leq 1, \forall t \geq 0. \quad (2.1)
$$

Take now $f \in C_c(\mathbb{R}^n)$. Then,

$$
\|T(t)f\| \leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |f(y)| dy \to 0 \text{ as } t \to \infty.
$$

Hence, it follows from the density of $C_c(\mathbb{R}^n)$ in $C_0(\mathbb{R}^n)$ and (reflap) that
$\lim_{t \to \infty} T(t)f = 0$, for every $f \in E$. This means that $T(\cdot)$ is strongly stable.

On the other hand one can see that $\text{Im} \Delta \neq C_0(\mathbb{R}^n)$, which implies that $0 \in \sigma(\Delta)$. Thus, $T(\cdot)$ is not uniformly exponentially stable, since $s(\Delta) \leq \omega_0(\Delta)$.

For the definition of $s(A)$ see Section 2.3.

2. We consider the translation semigroup

$$
(T(t)f)(s) = f(s+t), \quad t, s \geq 0,
$$

on $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^s ds)$. Then $E$ is a Banach lattice and $T(\cdot)$ is a
$C_0$–semigroup with generator $A$ given by

$$
Af = f' \text{ for } f \in D(A) = \{ f \in E : f \in C^1(\mathbb{R}_+) \text{ and } f' \in E \}.
$$
Moreover,\[
\rho(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) > -1 \}\]
and for $\Re(\lambda) > -1$,\[
R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt \quad \text{exists for all } f \in E.
\]

One can see that $\|T(t)\| = 1$ and so $\omega_0(A) = 0$. On the other hand, for $\Re(\lambda) > -1$, we have
\[
T(t)f = e^{\lambda t} \left( f - \int_0^t e^{-\lambda s} T(s)(\lambda - A)f \, ds \right), \quad f \in D(A),
\]
and since $\lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s)(\lambda - A)f \, ds$ exists, it follows that
\[
\|T(t)f\| \leq Ne^{\lambda t}, \quad \text{for all } f \in D(A).
\]

Hence,\[
\omega_1(A) \leq -1 < 0 = \omega_0(A).
\]

Consequently, $T(\cdot)$ is exponentially stable but not uniformly exponentially stable. For more details see [9, Example V.1.4].

The definition of the growth bound yields the following characterization of uniform exponential stability.

**Proposition 2.1.3** For the generator $A$ of a $C_0$–semigroup $T(\cdot)$ on a Banach space $E$, the following assertions are equivalent.

(a) $\omega_0(A) < 0$, i.e., $T(\cdot)$ is uniformly exponentially stable.

(b) $\lim_{t \to \infty} \|T(t)\| = 0$.

(c) $\|T(t_0)\| < 1$ for some $t_0 > 0$.

(d) $r(T(t_1)) < 1$ for some $t_1 > 0$.

**Proof:** The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy.

$(d) \Rightarrow (c)$: Since $r(T(t_1)) = \lim_{k \to \infty} \|T(t_{1k})\|^{1/k} < 1$, it follows that there is $k_0 \in \mathbb{N}$ with $\|T(k_0t_1)\| < 1$.

$(c) \Rightarrow (a)$: For $\alpha := \|T(t_0)\| < 1$, $M := \sup_{0 \leq s \leq t_0} \|T(s)\|$ and $t = kt_0 + s$ with $s \in [0, t_0]$, we have
\[
\|T(t)\| \leq \|T(s)\|\|T(t_0)\| \leq M\alpha^k = Me^{k\ln\alpha}.
\]

If we set $\varepsilon := \frac{-\ln\alpha}{k_0} > 0$ (because $\alpha < 1$), then
\[
\|T(t)\| \leq Me^{k\ln\alpha} \leq \frac{M}{\alpha} e^{-\varepsilon t}.
\]
It is clear that if $\omega_0(A) < 0$, then there are constants $\varepsilon > 0$ and $M \geq 1$ such that
\[ \|T(t)\| \leq Me^{-\varepsilon t}, \quad t \geq 0. \]
Hence, for every $p \in [1, \infty)$, \( \int_0^\infty \|T(t)x\|^p dt < \infty \) for all $x \in E$. The following result due to Datko [6] shows that the converse is also true.

**Theorem 2.1.4** A $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space $E$ is uniformly exponentially stable if and only if for some (and hence for every) $p \in [1, \infty)$,
\[ \int_0^\infty \|T(t)x\|^p dt < \infty \]
for all $x \in E$.

**Proof:** We have only to prove the converse. By Proposition 2.1.3 it suffices to prove that $\lim_{t \to \infty} \|T(t)\| = 0$. Since there are $M, \omega \in \mathbb{R}_+$ with $\|T(t)\| \leq Me^{\omega t}, t \geq 0$, we obtain
\[
\frac{1 - e^{-\rho\omega}}{p\omega} \|T(t)x\|^p = \int_0^t e^{-\rho\omega} \|T(s)T(t-s)x\|^p ds \\
\leq M^p \int_0^t \|T(t-s)x\|^p ds \\
\leq M^p C^p \|x\|^p
\]
for all $x \in E$ and $t \geq 0$. Hence, $\|T(t)x\|^p \leq \frac{\rho\omega}{1-e^{-\rho\omega}} M^p C^p \|x\|^p$ for $x \in E$ and $t \geq 1$. Thus, there exists a constant $L > 0$ with $\|T(t)\| \leq L$ for all $t \geq 0$. Therefore,
\[
t\|T(t)x\|^p = \int_0^t \|T(t-s)T(s)x\|^p ds \\
\leq L^p \int_0^t \|T(s)x\|^p ds \\
\leq L^p C^p \|x\|^p
\]
for all $x \in E$ and $t \geq 0$. Thus,
\[ \|T(t)\| \leq L C t^{-\frac{1}{p}}, \quad t > 0, \]
which implies $\lim_{t \to \infty} \|T(t)\| = 0.$ \(\square\)

In Hilbert spaces uniform exponential stability can be characterized in term of the generator as the following Gearhart-Prüss’s result shows (see [11], [22, A-III.7], [25]).

**Theorem 2.1.5** Let $T(\cdot)$ be a $\mathcal{C}_0$–semigroup on a Hilbert space $H$ with generator $A$. Then $T(\cdot)$ is uniformly exponentially stable if and only if
\[ \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \subseteq \rho(A) \text{ and } M := \sup_{\Re(\lambda) > 0} \|R(\lambda,A)\| < \infty. \]
2.1 Stability of strongly continuous semigroups

Proof: Assume that $\omega_0(A) < 0$. Then $\int_0^\infty e^{-\lambda t} T(t) \, dt$ exists for all $\Re(\lambda) > 0$. So by [9, Theorem II.1.10], $\{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \subseteq \rho(A)$ and $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) \, dt$ and therefore

$$\sup_{\Re(\lambda) > 0} \| R(\lambda, A) \| < \infty.$$  

We now prove the converse. We know from the spectral theory for closed operators (cf. [9, Corollary IV.1.14]) that

$$\text{dist}(\lambda, \sigma(A)) \geq \frac{1}{\| R(\lambda, A) \|} \geq M^{-1}, \quad \text{for all } \Re(\lambda) > 0.$$  

Thus, $\mathbb{R} \subseteq \rho(A)$ and $\sup_{\Re(\lambda) \geq 0} \| R(\lambda, A) \| < \infty$. Let $\omega > |\omega_0(A)| + 1$ and consider the $C_0$-semigroup $T_{-\omega}(\cdot)$ defined by $T_{-\omega}(t) := e^{-\omega t} T(t), \ t \geq 0$. By [9, Theorem II.1.10] we have

$$R(\omega + is, A) = R(is, A - \omega)$$  

$$= \int_0^\infty e^{-ist} T_{-\omega}(t) x \, dt$$  

$$= \mathcal{F} \left( (T_{-\omega}(\cdot) x)(s) \right),$$

where $\mathcal{F}(f) := \int_\mathbb{R} e^{-ist} f(t) \, dt$ denotes de Fourier transform from $L^2(\mathbb{R}, H)$ into $L^2(\mathbb{R}, H)$. Here we extend $T_{-\omega}(\cdot)$ to $\mathbb{R}$ by taking $T_{-\omega}(t) = 0$ for $t < 0$. Since $T_{-\omega}(\cdot)$ is uniformly exponentially stable, we obtain $T_{-\omega}(\cdot) x \in L^2(\mathbb{R}, H)$. Then one can apply Plancherel’s theorem, and we obtain

$$\int_{-\infty}^{\infty} \| R(\omega + is, A) x \|^2 \, ds = 2\pi \int_0^\infty \| T_{-\omega}(t) x \|^2 \, dt \leq L \| x \|^2$$

for some constant $L > 0$ and all $x \in H$. The resolvent identity gives

$$R(is, A) = R(\omega + is, A) + \omega R(is, A) R(\omega + is, A), \quad \text{for all } s \in \mathbb{R}.$$  

Hence, $\| R(is, A) x \| \leq (1 + M\omega) \| R(\omega + is, A) x \|$ for $s \in \mathbb{R}$ and $x \in H$. This implies

$$\int_{-\infty}^{\infty} \| R(is, A) x \|^2 \, ds \leq \left( 1 + \omega M \right)^2 \int_{-\infty}^{\infty} \| R(\omega + is, A) x \|^2 \, ds$$  

$$\leq \left( 1 + \omega M \right)^2 L \| x \|^2.$$  

On the other hand, by the inverse Laplace transform formula (cf. [9, Corollary III.5.16]) we know that

$$T(t) x = \frac{1}{2\pi i} \lim_{\rho \to \infty} \int_{\alpha - \rho i}^{\alpha + \rho i} e^{\lambda t} R(\lambda, A)^2 x \, d\lambda, \quad t \geq 0, x \in D(A^2).$$

Then, by Cauchy’s integral theorem,

$$\langle iT(t) x, y \rangle = \frac{1}{2\pi i} \int_{\alpha - \rho i}^{\alpha + \rho i} e^{(\omega + is)t} (R(\omega + is, A)^2 x) \, d\lambda$$  

$$= \int_{-\infty}^{\infty} e^{ist} (R(is, A)^2 x) \, ds$$  

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} (R(is, A) x) \, ds$$  

and

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} (R(is, A) x) \, ds.$$
for all \( x \in D(A^2) \) and \( y \in H \). As above one can see that
\[
\int_{-\infty}^{\infty} |R(is, A^*) y|^2 \, ds \leq (1 + M\omega)^2 L ||y||^2, \quad y \in H.
\]
By applying the Cauchy-Schwarz inequality we obtain
\[
|\langle tT(t)x, y \rangle| \leq \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} ||R(is, A)x||^2 \, ds \right)^{1/2} \left( \int_{-\infty}^{\infty} ||R(is, A^*) y||^2 \, ds \right)^{1/2} \leq \frac{(1 + M\omega)^2 L}{2\pi} ||x|| ||y||
\]
for all \( x \in D(A^2) \) and \( y \in H \). Since \( D(A^2) = H \), it follows that
\[
||tT(t)|| = \sup \left\{ |\langle tT(t)x, y \rangle| : x, y \in D(A^2), ||x|| = ||y|| = 1 \right\} 
\leq \frac{(1 + M\omega)^2 L}{2\pi}.
\]
Hence, \( \lim_{t \to \infty} ||T(t)|| = 0 \) and therefore, \( \omega_0(A) < 0 \).

\section{The Essential Spectrum and Quasi-Compact Semigroups}

In this section we study the essential growth bound \( \omega_{\text{ess}}(A) \) of the generator \( A \) of a \( C_0 \)-semigroup \( T(\cdot) \) on a Banach space \( E \), in the case \( \omega_{\text{ess}}(A) < 0 \). Then we deduce important consequences for the asymptotic behaviour of \( T(\cdot) \).

We start with some definitions. A bounded operator \( S \in \mathcal{L}(E) \) is called a Fredholm operator if there is \( T \in \mathcal{L}(E) \) such that \( \text{Id} - TS \) and \( \text{Id} - ST \) are compact.

We denote by \( \sigma_{\text{ess}}(S) = \mathbb{C} \setminus \rho_F(S) \)
the essential spectrum of \( S \), where
\[
\rho_F(S) := \{ \lambda \in \mathbb{C} : (\lambda - S) \text{ is a Fredholm operator } \}.
\]

The Calkin algebra \( \mathcal{C}(E) := \mathcal{L}(E) / \mathcal{K}(E) \) equipped with the quotient norm
\[
||S||_{\text{ess}} := ||S + \mathcal{K}(E)|| = \text{dist}(S, \mathcal{K}(E)) = \inf \{ ||S - K|| : K \in \mathcal{K}(E) \}
\]
is a Banach algebra with unit. The essential spectrum of \( S \in \mathcal{L}(E) \) can also defined as the spectrum of \( S + \mathcal{K}(E) \) in the Banach algebra \( \mathcal{C}(E) \). This implies that, for \( S \in \mathcal{L}(E) \), \( \sigma_{\text{ess}}(S) \) is non-empty and compact.

For \( S \in \mathcal{L}(E) \) we define the essential spectral radius by
\[
r_{\text{ess}}(S) := r(S + \mathcal{K}(E)) = \max \{ |\lambda| : \lambda \in \sigma_{\text{ess}}(S) \}.
\]
2.2 The essential spectrum and quasi-compact semigroups

Since \( (S + \mathcal{K}(E))^n = S^n + \mathcal{K}(E) \) for \( n \in \mathbb{N} \), we have \( r_{\text{ess}}(S) = \lim_{n \to \infty} \|S^n\|^{\frac{1}{n}} \) and consequently,

\[
r_{\text{ess}}(S + K) = r_{\text{ess}}(S), \quad \text{for every } K \in \mathcal{K}(E).
\]

If we denote by

\[
Pol(S) := \{ \lambda \in \mathbb{C} : \lambda \text{ is a pole of finite algebraic multiplicity of } R(\cdot, S) \},
\]

then one can prove that \( Pol(S) \subseteq \rho_F(S) \) and an element of the unbounded connected component of \( \rho_F(S) \) either is in \( \rho(S) \) or a pole of finite algebraic multiplicity. For details concerning the essential spectrum we refer to [20, Sec. IV.5.6], [13, Chap. XVII] or [12, Sec. IV.2]. Thus we obtain the following characterization.

**Proposition 2.2.1** For \( S \in \mathcal{L}(E) \) the essential spectral radius is given by

\[
r_{\text{ess}}(S) = \inf \{ r > 0 : \lambda \in \sigma(S), |\lambda| > r \text{ and } \lambda \in Pol(S) \}.
\]

**Proof:** If we set

\[
a := \inf \{ r > 0 : \lambda \in \sigma(S), |\lambda| > r \text{ and } \lambda \in Pol(S) \},
\]

then for all \( \varepsilon > 0 \) there is \( r_\varepsilon > 0 \) such that

\[
\{ \lambda \in \sigma(S) : |\lambda| > r_\varepsilon \} \subseteq Pol(S)
\]

and \( r_\varepsilon - \varepsilon \leq a \). On the other hand, we know that there is \( \lambda_0 \in \sigma_{\text{ess}}(S) \) with \( r_{\text{ess}}(S) = |\lambda_0| \). If we suppose that \( r_{\text{ess}}(S) > r_\varepsilon \), then \( \lambda_0 \in Pol(S) \). This implies that \( \lambda_0 \in \rho_F(S) \) which is a contradiction. Hence, \( r_{\text{ess}}(S) \leq r_\varepsilon \leq a + \varepsilon \). Thus, \( r_{\text{ess}}(S) \leq a \).

To show the other inequality we know that

\[
\{ \lambda \in \sigma(S) : |\lambda| > r_{\text{ess}}(S) \} \subseteq \rho_F(S).
\]

Therefore,

\[
\{ \lambda \in \sigma(S) : |\lambda| > r_{\text{ess}}(S) \} \subseteq Pol(S).
\]

Consequently, \( a \leq r_{\text{ess}}(S) \) and the proposition is proved. \( \square \)

We define the essential growth bound \( \omega_{\text{ess}}(A) \) of a \( C_0 \)-semigroup \( T(\cdot) \) with generator \( A \) as the growth bound of the quotient semigroup \( T(\cdot) + \mathcal{K}(E) \) on \( C(E) \), i.e.,

\[
\omega_{\text{ess}}(A) := \inf \{ \omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\|_{\text{ess}} \leq Me^{\omega t}, \forall t \geq 0 \}.
\]

Then, for all \( t_0 > 0 \), one can see that

\[
\omega_{\text{ess}}(A) = \frac{\log r_{\text{ess}}(T(t_0))}{t_0} = \lim_{t \to \infty} \frac{\log \|T(t)\|_{\text{ess}}}{t}.
\]

(2.2)

The following result gives the relationship between \( \omega_{\text{ess}}(A) \) and \( \omega_0(A) \).
Proposition 2.2.2 Let \( T(\cdot) \) be a \( C_0 \)-semigroup with generator \( A \) on a Banach space \( E \). Then one has
\[
\omega_0(A) = \max \{ s(A), \omega_{\text{ess}}(A) \}.
\]

Proof: If \( \omega_{\text{ess}}(A) < \omega_0(A) \), then \( r_{\text{ess}}(T(1)) < r(T(1)) \). Let \( \lambda \in \sigma(T(1)) \) such that \( |\lambda| = r(T(1)) \). So by Proposition 2.2.1, \( \lambda \) is an eigenvalue of \( T(1) \) and by the spectral mapping theorem for the point spectrum (cf. [9, Theorem IV.3.7]) there is \( \lambda_1 \in \sigma_p(A) \) with \( e^{\lambda_1} = \lambda \). Therefore, \( \Re(\lambda_1) = \omega_0(A) \) and thus \( \omega_0(A) = s(A) \). \( \square \)

By using the essential growth bound one can deduces important consequences for the asymptotic behaviour, the proof can be found in [9, Theorem V.3.1]

Theorem 2.2.3 Let \( A \) be the generator of a \( C_0 \)-semigroup \( T(\cdot) \) on a Banach space \( E \) and \( \lambda_1, \ldots, \lambda_m \in \sigma(A) \) with \( \Re(\lambda_1), \ldots, \Re(\lambda_m) > \omega_{\text{ess}}(A) \). Then \( \lambda_1, \ldots, \lambda_m \) are isolated spectral values of \( A \) with finite algebraic multiplicity. Furthermore, if \( P_1, \ldots, P_m \) denote the corresponding spectral projections and \( k_1, \ldots, k_m \) the corresponding orders of poles of \( R(t, A) \), then
\[
T(t) = T_1(t) + \ldots + T_m(t) + R_m(t),
\]
where
\[
T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!} (A - \lambda_n)^j P_n, \quad n = 1, \ldots, m.
\]
Moreover, for every \( \omega > \sup \{ \omega_{\text{ess}}(A) \} \cup \{ \Re(\lambda) : \lambda \in \sigma(A) \setminus \{ \lambda_1, \ldots, \lambda_m \} \} \), there is \( M > 0 \) such that
\[
\|R_m(t)\| \leq Me^{\omega t} \quad \text{for } t \geq 0.
\]

We now introduce the concept of quasi-compact semigroups.

Definition 2.2.4 A \( C_0 \)-semigroup \( T(\cdot) \) with generator \( A \) on a Banach space \( E \) is called quasi-compact if \( \omega_{\text{ess}}(A) < 0 \).

From (2.2) we deduce that any eventually compact \( C_0 \)-semigroup is quasi-compact.

The following description of the asymptotic behaviour of quasi-compact semigroups is an immediate consequence of Theorem 2.2.3.

Theorem 2.2.5 Let \( A \) be the generator of a quasi-compact \( C_0 \)-semigroup \( T(\cdot) \) on a Banach space \( E \). Then the following assertions hold.

(a) The set \( \{ \lambda \in \sigma(A) : \Re(\lambda) \geq 0 \} \) is finite (or empty) and consists of poles of \( R(t, A) \) of finite algebraic multiplicity.

Denoting these poles by \( \lambda_1, \ldots, \lambda_m \), the corresponding spectral projections \( P_1, \ldots, P_m \) and the order of the poles \( k_1, \ldots, k_m \), we have

(b) \( T(t) = T_1(t) + \ldots + T_m(t) + R(t) \), where
\[
T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!} (A - \lambda_n)^j P_n, \quad n = 1, \ldots, m,
\]
and
\[
\|R(t)\| \leq Me^{-\varepsilon t} \quad \text{for some } \varepsilon > 0, M \geq 1 \text{ and all } t \geq 0.
\]
2.3 Spectral bounds for positive semigroups

In this section we characterize the spectral bound

\[ s(A) := \sup \{ \Re(\lambda) : \lambda \in \sigma(A) \} \]

of the generator of a positive \( C_0 \)-semigroup \( T(\cdot) \) on a complex Banach lattice \( E \).

We will see that \( s(A) \) is always contained in \( \sigma(A) \) provided that \( \sigma(A) \neq \emptyset \).

To that purpose the following result is essential.

**Theorem 2.3.1** Let \( A \) be the generator of a positive \( C_0 \)-semigroup \( T(\cdot) \) on \( E \). For \( \Re(\lambda) > s(A) \) we have

\[ R(\lambda, A)x = \lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s)xds, \quad x \in E. \]

Moreover, \( \int_0^\infty e^{-\lambda t} T(s)ds \) converges to \( R(\lambda, A) \) with respect to the operator norm as \( t \to \infty \).

**Proof:** Let \( \lambda_0 > \omega_0(A) \) be fixed. Since \( R(\lambda_0, A)x = \int_0^\infty e^{-\lambda_0 t} T(t)xdt \) and by the resolvent identity we obtain

\[ R(\lambda_0, A)^{n+1} x = \frac{1}{n!} \int_0^\infty t^n e^{-\lambda_0 t} T(t)xdt \]

for \( n \in \mathbb{N} \) and \( x \in E \). Let \( \mu \in (s(A), \lambda_0) \), \( x \in E_+ \) and \( x^* \in E_+^* \). By the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]) one has \( \frac{1}{\lambda_0 - \mu} > r(R(\lambda_0, A)) \) and hence,

\[
\langle R(\mu, A)x, x^* \rangle = \sum_{n=0}^\infty (\lambda_0 - \mu)^n \langle R(\lambda_0, A)^{n+1} x, x^* \rangle
\]

\[ = \sum_{n=0}^\infty \int_0^\infty \frac{1}{n!} [\lambda_0 - \mu]^{n} e^{-\lambda_0 s} \langle T(s)x, x^* \rangle ds \]

\[ = \int_0^\infty \left( \sum_{n=0}^\infty \frac{1}{n!} [\lambda_0 - \mu]^{n} \right) e^{-\lambda_0 s} \langle T(s)x, x^* \rangle ds \]

\[ = \int_0^\infty e^{(\lambda_0 - \mu)s} e^{-\lambda_0 s} \langle T(s)x, x^* \rangle ds \]

\[ = \int_0^\infty e^{-\mu s} \langle T(s)x, x^* \rangle ds \]

\[ = \lim_{t \to \infty} \int_0^t e^{-\mu s} \langle T(s)x, x^* \rangle ds. \]

Hence, \( \left( \int_0^\infty e^{-\mu s} T(s)xds \right) \) converges weakly to \( R(\mu, A)x \) as \( t \to \infty \). Since \( x \in E_+ \), it follows that \( \left( \int_0^\infty e^{-\mu s} T(s)xds \right) \) is monotone increasing and so, by Proposition 1.1.13, we have strong convergence. Thus,

\[
\lim_{t \to \infty} \int_0^t e^{-\mu s} T(s)xds = R(\mu, A)x, \quad \text{for all } x \in E.
\]
If $\lambda = \mu + i\gamma$ with $\mu, \gamma \in \mathbb{R}$ and $\mu > s(A)$, then for any $x \in E$ and $x^* \in E^*$, we have

$$\left| \int_{t}^{\infty} e^{-\lambda s} T(s)x ds, x^* \right| \leq \int_{t}^{\infty} e^{-\mu s} \|T(s)|x|, |x^*|\| ds.$$ 

Hence,

$$\left\| \int_{t}^{\infty} e^{-\lambda s} T(s)x ds \right\| \leq \left( \int_{t}^{\infty} e^{-\mu s} T(s) ds \right) \|x\|,$$

which implies that

$$\lim_{t \to \infty} \int_{0}^{t} e^{-\lambda s} T(s)x ds$$

exists for all $x \in E$. Then, by [9, Theorem II.1.10],

$$\lambda \in \rho(A) \text{ and } R(\lambda, A)x = \int_{0}^{\infty} e^{-\mu s} T(s) x dt \quad \text{for all } x \in E.$$

It remains to prove that $(\int_{0}^{\infty} e^{-\mu s} T(s) ds)$ converges in the operator norm as $t \to \infty$. We fix $\mu \in (s(A), \Re(\lambda))$. As we have seen above, the function

$$f_{x,x^*}: s \mapsto e^{-\mu s} \langle T(s)x, x^* \rangle$$

belongs to $L^1(\mathbb{R}_+)$ for all $x \in E, x^* \in E^*$. It follows from the closed graph theorem that the bilinear form

$$b: E \times E^* \to L^1(\mathbb{R}_+): (x, x^*) \mapsto f_{x,x^*}$$

is separately continuous and hence continuous. Thus, there exists $M > 0$ such that

$$\int_{0}^{\infty} e^{-\mu s} |\langle T(s)x, x^* \rangle| ds \leq M \|x\| \||x^*||, \quad x \in E, x^* \in E^*.$$ 

For $0 \leq t < r$ and $\varepsilon := \Re(\lambda) - \mu$ we have

$$\left| \int_{t}^{r} e^{-\lambda s} \langle T(s)x, x^* \rangle ds \right| \leq \int_{t}^{r} e^{-\Re(\lambda) s} e^{-\mu s} |\langle T(s)x, x^* \rangle| ds$$

$$\leq e^{-\mu t} \int_{t}^{r} e^{-\mu s} |\langle T(s)x, x^* \rangle| ds$$

$$\leq e^{-\varepsilon t} M \|x\| \||x^*||.$$

Hence, $\|\int_{0}^{r} e^{-\lambda s} T(s) ds\| \leq M e^{-\varepsilon t}$ and this implies that $(\int_{0}^{r} e^{-\lambda s} T(s) ds)$ is a Cauchy sequence in $L(E)$. □

As an immediate consequence we obtain the following corollary.

**Corollary 2.3.2** Let $A$ be the generator of a positive $C_0$–semigroup $T(\cdot)$ on $E$. If $\Re(\lambda) > s(A)$, then

$$|R(\lambda, A)x| \leq R(\Re(\lambda), A)|x| \quad \text{for all } x \in E.$$
An other interesting corollary is the following.

**Corollary 2.3.3** If $A$ is the generator of a positive $C_0$–semigroup $T(\cdot)$ on $E$, then

$$s(A) \in \sigma(A) \text{ or } s(A) = -\infty.$$ 

**Proof:** Assume that $s(A) > -\infty$ and $s(A) \notin \sigma(A)$. So it follows from Corollary 2.3.2 that

$$|R(\lambda, A)x| \leq R(\Re(\lambda), A)|x| \leq R(s(A), A)|x| \quad \text{for all } \Re(\lambda) > s(A), x \in E.$$ 

Hence the set $\{R(\lambda, A) : \Re(\lambda) > s(A)\}$ is uniformly bounded in $L(E)$. Let $M := \sup_{\Re(\lambda) > s(A)} |R(\lambda, A)|$. Since $|R(\lambda, A)| \geq \frac{1}{\Re(\lambda) - s(A)}$ for $\lambda \in \rho(A)$ (cf. [9, Corollary IV.1.14]), it follows that

$$\{\lambda \in \mathbb{C} : \Re(\lambda) = s(A)\} \subseteq \rho(A) \text{ and } |R(\lambda, A)| \leq M, \forall \Re(\lambda) = s(A).$$ 

Thus,

$$\{\lambda \in \mathbb{C} : |\Re(\lambda) - s(A)| < M^{-1}\} \subseteq \rho(A).$$ 

This contradicts the definition of $s(A)$. \qed

The following consequence gives a relation between $s(A)$ and the positivity of the resolvent.

**Corollary 2.3.4** Suppose that $A$ generates a positive on $E$ and $\lambda_0 \in \rho(A)$. Then the following assertions hold.

(i) $R(\lambda_0, A)$ is positive if and only if $\lambda_0 > s(A)$.

(ii) If $\lambda > s(A)$, then $r(R(\lambda, A)) = \frac{1}{\lambda - s(A)}$.

**Proof:** (ii) is a simple consequence from Corollary 2.3.3 and the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]).

(i) Assume first that $R(\lambda_0, A) \geq 0$. Since $Ag \in E_\mathbb{R}$ for all $0 \leq g \in D(A)$, we have $\lambda_0 \in \mathbb{R}$. On the other hand, Theorem 2.3.1 implies that $R(\lambda, A) \geq 0$ for all $\lambda > \max(\lambda_0, s(A))$ and hence

$$R(\lambda_0, A) = R(\lambda, A)/(\lambda - \lambda_0)R(\lambda, A)R(\lambda_0, A) \geq R(\lambda, A) \geq 0$$

for all $\lambda > \max(\lambda_0, s(A))$. Therefore,

$$(\lambda - s(A))^{-1} \leq r(R(\lambda, A)) \leq |R(\lambda, A)| \leq |R(\lambda_0, A)|$$

for all $\lambda > \max(\lambda_0, s(A))$. But this is only true if $\lambda_0 > s(A)$.

The converse follows from Theorem 2.3.1. \qed
Remark 2.3.5 (a) As an immediate consequence of Corollary 2.3.4 we obtain

\[ s(A) = \inf \{ \lambda \in \mathbb{R} : R(\lambda, A) \geq 0 \} \]

for the generator \( A \) of a positive \( C_0 \)-semigroup on a Banach lattice \( E \).

(b) If \( E := C(K), K \) compact, then \( s(A) > -\infty \). In fact: We know from the theory of \( C_0 \)-semigroups that \( \lim_{\lambda \to -\infty} \lambda R(\lambda, A)f = f \) for all \( f \in E \). In particular we find \( \lambda_0 \in \mathbb{R} \) sufficiently large such that

\[ \lambda_0 R(\lambda_0, A) \geq \frac{1}{2} \mathbb{1}, \]

where \( \mathbb{1}(x) := 1 \) for all \( x \in K \). Since \( R(\lambda_0, A) \geq 0 \), it follows that

\[ R(\lambda_0, A)^n \mathbb{1} \geq \frac{1}{(2\lambda_0)^n} \mathbb{1} \quad \text{for all } n \in \mathbb{N}. \]

Thus,

\[ r(R(\lambda_0, A)) = \lim_{n \to \infty} \| R(\lambda_0, A)^n \|^\frac{1}{n} \geq \frac{1}{2\lambda_0} > 0 \]

and hence \( \sigma(A) \neq \emptyset \).

The spectrum of a generator of a positive \( C_0 \)-semigroup can be empty as the following examples show.

Example 2.3.6 (a) On \( E := C_0(0,1) := \{ f \in C[0,1] : f(1) = 0 \} \) we consider the nilpotent \( C_0 \)-semigroup \( T(\cdot) \) given by

\[ (T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t < 1 \\ 0 & \text{if } x+t \geq 1 \end{cases} \]

for \( t \geq 0, x \in [0,1] \) and \( f \in E \). Then, \( T(t) = 0 \) for \( t \geq 1 \) and hence \( \sigma(T(t)) = \{0\} \). So by the spectral inclusion theorem (cf. [9, Theorem IV.3.6]), \( \sigma(A) = \emptyset \).

(b) Let \( E := C_0(0,\infty) := \{ f \in C(\mathbb{R}_+) : \lim_{t \to +\infty} f(t) = 0 \} \). On \( E \), we define the \( C_0 \)-semigroup \( T(\cdot) \) by

\[ (T(t)f)(x) := e^{-\frac{2}{t} - x}f(x+t), \quad x,t \geq 0 \text{ and } f \in E. \]

Then, one can see that the generator \( A \) of \( T(\cdot) \) on \( E \) is given by

\[ (Af)(x) = f'(x) - x f(x), \quad x \geq 0, \text{ and } \]

\[ f \in D(A) = \{ f \in E : f \in C^1(\mathbb{R}_+) \text{ and } Af \in E \}. \]

By a simple computation one obtains that \( \sigma(A) = \emptyset \).

For generators of positive \( C_0 \)-groups the spectrum is always nonempty. This is given by the following corollary.
Corollary 2.3.7 If $A$ generates a positive $C_0$-group on a Banach lattice $E$, then $\sigma(A) \neq \emptyset$.

Proof: Assume that $\sigma(A) = \emptyset$. By Theorem 2.3.1 we have $R(\lambda, A) \geq 0$ for all $\lambda \in \mathbb{R}$. Again, one can apply the same theorem to $-A$ and obtains $R(\lambda, -A) \geq 0$ for all $\lambda \in \mathbb{R}$. But $R(\lambda, -A) = -R(-\lambda, A) \leq 0$ for all $\lambda \in \mathbb{R}$, and hence, $R(\lambda, -A) = 0$ for all $\lambda \in \mathbb{R}$. This contradicts the fact that $E \neq \{0\}$.

2.4 THE PROBLEM $\omega_0(A) = s(A)$ FOR POSITIVE SEMIGROUPS

In this section we study in detail the growth bound $\omega_0(A)$ of the generator $A$ of a positive $C_0$-semigroup on a Banach lattice $E$. In particular, we look for sufficient conditions implying the equality $\omega_0(A) = s(A)$ without supposing the spectral mapping theorem.

For a $C_0$-semigroup $S(\cdot)$ with generator $B$ on a Banach space $X$ satisfying $\|S(t)\| \leq M e^{\omega t}$, $t \geq 0$, for some constants $M$, $\omega \in \mathbb{R}$, it follows that $\{\lambda \in \mathbb{C} : \Re \lambda > \omega\} \subset p(B)$. Thus,

$$s(B) \leq \omega_0(B)$$

is always satisfied.

By applying the Gearhardt-Pruess’s theorem and Theorem 1.2.2 we obtain the first result on the opposite inequality.

Theorem 2.4.1 Let $A$ be the generator of a positive $C_0$-semigroup $T(\cdot)$ on a Banach lattice $E$. Then $\omega_0(A) = s(A)$ holds in the followings cases.

(i) $E$ is a Hilbert space.
(ii) $E$ is an AL-space.
(iii) $E := C_0(\Omega)$ or $E := C(K)$, where $\Omega$ is locally compact Hausdorff and $K$ is compact Hausdorff.

Proof: (i) Let $\mu > s(A)$ fixed. It follows from Corollary 2.3.2 that $\Lambda := \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subset p(A - \mu)$ and

$$\|R(\lambda, A - \mu)\| \leq \|R(\Re(\lambda), A - \mu)\| \leq \|R(\mu, A)\|$$

for all $\lambda \in \Lambda$.

So, by Theorem 2.1.5, we have $\omega_0(A) - \mu < 0$ and hence,

$$\omega_0(A) \leq s(A).$$

(ii) For $\lambda > s(A)$ and $x \in E_+$ we obtain from Theorem 2.3.1 that

$$\|R(\lambda, A)x\| = \left\| \int_0^\infty e^{-\lambda s}T(s)x ds \right\| = \int_0^\infty e^{-\lambda s}\|T(s)x\| ds,$$
where the second equality follows from the fact that the norm is additive on the positive cone. Hence,
\[
\int_0^\infty \|(e^{-\lambda s}T(s)x)\| ds < \infty \quad \text{for all } x \in E.
\]

So, by Theorem 2.1.4, we have \(\omega_0(A) - \lambda < 0\) and thus\[\omega_0(A) \leq s(A)\].

(iii) It is easy to see that \(\|f \vee g\| = \|f\| \|g\|\) for all \(f, g \in E^+\). Then, for \(\gamma, \nu \in E^+_+\), we have
\[
\langle f, \gamma \rangle + \langle g, \nu \rangle \leq \langle f \vee g, \gamma + \nu \rangle \leq \|\gamma + \nu\| \|f \vee g\|
\]
\[
= \|\gamma + \nu\| (\|f\| \|\nu\| + \|g\|), \quad f, g \in E^+.
\]

Hence, \(\langle f, \gamma \rangle + \langle g, \nu \rangle \leq \|\gamma + \nu\|\) for all \(f, g \in E^+\) with \(\|f\| = \|g\| = 1\). It follows from the Hahn-Banach theorem that \(\|\gamma\| + \|\nu\| \leq \|\gamma + \nu\|\) and hence,
\[
\|\gamma\| + \|\nu\| = \|\gamma + \nu\|, \quad \gamma, \nu \in E^+_+.
\]

This implies that \(E^*\) is an AL-space. If we set \(F := D(A\cap)\), then it follows from Theorem 1.2.2 that \(F\) is a closed ideal and hence also an AL-space. On \(F\) we consider the positive \(C_0\)-semigroup \(S(\cdot)\) given by
\[
S(t) := (T(t))_{B}^* \quad \text{for } t \geq 0,
\]
and we denote by \(B\) its generator. Then \(B\) is the part of \(A^*\) in \(F\), i.e.,
\[
D(B) = \{v \in D(A^*) : A^*v \in F\} \quad \text{and } Bv = A^*v \text{ for } v \in D(B).
\]

Moreover, one can show that
\[
\sigma(B) = \sigma(A^*) = \sigma(A).
\]

Consequently, \(s(B) = s(A)\) holds. Since \(B\) is the generator of the positive \(C_0\)-semigroup \(S(\cdot)\) on the AL-space \(F\), it follows from (ii) that \(s(B) = \omega_0(B)\). Now, it suffices to prove that \(\omega_0(B) = \omega_0(A)\). The inequality \(\omega_0(B) \leq \omega_0(A)\) is trivial. Let \(\omega > \omega_0(B), f \in E\) and \(v \in F\). Then we have
\[
|\langle T(t)f, \nu \rangle| = |\langle f, S(t)v \rangle| \leq M\|f\| e^{\omega t}\|\nu\|
\]
for \(t \geq 0\) and some constant \(M \geq 1\). On the other hand, since \(f = \lim_{\lambda \to \infty} \lambda R(\lambda, A)f\) for all \(f \in E\), we have \(c := \lim \sup_{\lambda \to \infty} \lambda \|R(\lambda, A)\| < \infty\). Therefore,
\[
|\langle T(t)f, \gamma \rangle| = \lim_{\lambda \to \infty} |\langle \lambda R(\lambda, A)T(t)f, \gamma \rangle|
\]
\[
= \lim_{\lambda \to \infty} |\langle T(t)f, \lambda R(\lambda, A^*)\gamma \rangle|
\]
\[
\leq M\|f\| e^{\omega t} \lim \sup_{\lambda \to \infty} \lambda \|R(\lambda, A)^*\gamma\|
\]
\[
\leq Mce^{\omega t}\|\gamma\|, \quad \gamma \in E^*.
\]
Consequently, \( \|T(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \) and hence \( \omega_0(A) \leq \omega \) for all \( \omega > \omega_0(B) \). Thus, we have shown that

\[
\omega_0(B) = \omega_0(A).
\]

\( \square \)

The last result of this section is Weis’s result concerning positive \( C_0 \)-semigroups on \( L^p(\Omega) := L^p(\Omega, \mu), 1 \leq p < \infty \), where \( (\Omega, \mu) \) a \( \sigma \)-finite measure space (see [33]). The proof presented here is due to W. Arendt (see [2, Theorem 5.3.6]).

We first need some preparations. We equip \( \mathbb{R} \times \Omega \) with the product measure \( \lambda_1 \otimes \mu \), where \( \lambda_1 \) is the Lebesgue measure on \( \mathbb{R} \). We recall that \( L^p(\mathbb{R} \times \Omega) \cong L^p(\mathbb{R}, L^p(\Omega)) \). This allows us to identify the notations \( g(t, \xi) \) and \( g(t)(\xi) \) for \( (t, \xi) \in \mathbb{R} \times \Omega \). Let us consider the non-linear map

\[
\Phi : L^p(\mathbb{R}, L^p(\Omega)) \to L^p(\Omega) ; g \mapsto \Phi(g) := \left( \int_{\mathbb{R}} |g(t)|^p \, dt \right)^{\frac{1}{p}}.
\]

It is clear that \( \Phi \) is well-defined.

The following lemmas give some properties of the map \( \Phi \).

**Lemma 2.4.2** Let \( g, h \in L^p(\mathbb{R}, L^p(\Omega)), f \in L^\infty(\Omega), \) and \( s \in \mathbb{R} \). Then the following assertions hold.

1. \( \|\Phi(g)\|_{L^p(\Omega)} = \|g\|_{L^p(\mathbb{R} \times \Omega)} \).
2. \( \Phi(g_s) = \Phi(g), \) where \( g_s(t) := g(s + t), t, s \in \mathbb{R} \).
3. \( \Phi(f \cdot g) = |f|\Phi(g), \) where \( (f \cdot g)(t, \xi) := f(\xi)g(t, \xi) \), \( (t, \xi) \in \mathbb{R} \times \Omega \).
4. \( \Phi(g + h) \leq \Phi(g) + \Phi(h) \).
5. \( \Phi \) is a continuous map.

**Proof:** Assertions 1., 2. and 3. are simple to prove. For 4. we set \( G_\xi(t) := g(t, \xi), H_\xi(t) := h(t, \xi) \), \( (t, \xi) \in \mathbb{R} \times \Omega \). For almost all \( \xi \in \Omega \), we obtain \( G_\xi, H_\xi \in L^p(\mathbb{R}) \) and hence

\[
\|G_\xi + H_\xi\|_{L^p(\mathbb{R})} \leq \|G_\xi\|_{L^p(\mathbb{R})} + \|H_\xi\|_{L^p(\mathbb{R})}.
\]

Since \( \|G_\xi\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |g(t, \xi)|^p \, dt \right)^{\frac{1}{p}} = \Phi(g)(\xi) \) and also \( \|H_\xi\|_{L^p(\mathbb{R})} = \Phi(h)(\xi) \), it follows that

\[
\Phi(g + h)(\xi) \leq \Phi(g)(\xi) + \Phi(h)(\xi), \quad \mu\text{-a.e. } \xi \in \Omega.
\]

Thus, \( \Phi(g + h) \leq \Phi(g) + \Phi(h) \).

By 4. we have

\[
\Phi(g) \leq \Phi(g - h) + \Phi(h) \quad \text{and} \quad \Phi(h) \leq \Phi(h - g) + \Phi(g).
\]
This implies that $|\Phi(g) - \Phi(h)| \leq \Phi(g - h)$ and so by 1. we obtain

$$||\Phi(g) - \Phi(h)||_{L^p(\Omega)} \leq ||g - h||_{L^p(\mathbb{R} \times \Omega)}.$$ 

which proves 5. \qed

Lemma 2.4.3 For a continuous function $G : [a, b] \rightarrow L^p(\mathbb{R}, L^p(\Omega))$ we have

$$\Phi \left( \int_a^b G(s) \, ds \right) \leq \int_a^b \Phi(G(s)) \, ds.$$ 

Proof: It follows from Lemma 2.4.2 that

$$\Phi \left( \frac{b-a}{2^n} \sum_{j=0}^{2^n-1} G \left( \frac{jb + (2^n - j)a}{2^n} \right) \right) \leq \frac{b-a}{2^n} \sum_{j=0}^{2^n-1} \Phi \left( G \left( \frac{jb + (2^n - j)a}{2^n} \right) \right).$$

Since $\Phi$ is continuous, we obtain the lemma by letting $n \to \infty$. \qed

Let $g \in L^p(\mathbb{R}, L^p(\Omega))$ and $T \in L(L^p(\Omega))$. We consider $T \circ g$ defined by

$$(T \circ g)(t) := T(g(t)), \quad t \in \mathbb{R}.$$ 

Lemma 2.4.4 For $0 \leq T \in L(L^p(\Omega))$ and $0 \leq g \in L^p(\mathbb{R}, L^p(\Omega))$ the inequality

$$\Phi(T \circ g) \leq T(\Phi(g))$$

holds.

Proof: By Lemma 2.4.2, it suffices to prove the lemma for simple functions. Let $g := \sum_{k=1}^{n} \chi_{A_k} \otimes g_k$, where $A_1, \ldots, A_n$ are disjoint Borel subsets of $\mathbb{R}$, and $g_1, \ldots, g_n \in L^p(\Omega)$. Setting $h_k := \lambda_1(A_k)^{\frac{1}{p}} g_k$ for $k \in \{1, \ldots, n\}$. Since the sets $(A_k)$ are disjoint, it follows that

$$\Phi(T \circ g) = \left( \sum_{k=1}^{n} \lambda_1(A_k) (Tg_k)^p \right)^{\frac{1}{p}} = \left( \sum_{k=1}^{n} (Th_k)^p \right)^{\frac{1}{p}},$$

$$T(\Phi(g)) = T \left( \sum_{k=1}^{n} \lambda_1(A_k) (g_k)^p \right)^{\frac{1}{p}} = T \left( \sum_{k=1}^{n} (h_k)^p \right)^{\frac{1}{p}}.$$ 

Let $\alpha := (\alpha_k)_k \subset \mathbb{R}$ with $\|\alpha\|_\infty \leq 1$, where $\frac{1}{q} + \frac{1}{p} = 1$. The Hölder inequality implies

$$\left( \sum_{k=1}^{n} \alpha_k h_k \right)^p \leq \left( \sum_{k=1}^{n} |h_k|^p \right)^{\frac{1}{p}} = \Phi(g),$$

hence

$$\left( \sum_{k=1}^{n} \alpha_k Th_k \right)^p \leq T(\Phi(g)).$$
2.4 The problem $\omega_0(A) = s(A)$ for positive semigroups

Consequently,

$$\left( \sum_{k=1}^{n} |(Th_k)(\xi)|^p \right)^{\frac{1}{p}} \leq \sup \left\{ \left( \sum_{k=1}^{n} \alpha_k |(Th_k)(\xi)| \right) : \alpha_k \in \mathbb{R}, \|\alpha_k\|_{\infty} \leq 1 \right\}$$

and $\Phi(T \circ g) \leq T(\Phi(g))$. □

We are now ready to prove Weis’s result.

**Theorem 2.4.5** Let $(\Omega, \mu)$ be a $\sigma$–finite measure space, $1 \leq p < \infty$, and $T(\cdot)$ a positive $C_0$–semigroup on $L^p(\Omega)$ with generator $A$. Then $\omega_0(A) = s(A)$.

**Proof:** For $\xi > s(A)$ we set $T_{\xi}(t) := e^{-\xi t} T(t)$, $t \geq 0$. We denote by $A_{\xi} := A - \xi$ the generator of the positive $C_0$–semigroup $T_{\xi}(\cdot)$ on $L^p(\Omega)$. Then $s(A_{\xi}) = s(A) - \xi < 0$. Let $\alpha > \max(0, \omega_0(A_{\xi}))$ fixed. Let $f \in L^p(\Omega)$ and consider the function $g \in L^p(\mathbb{R}, L^p(\Omega))$ defined by

$$g(t) = \begin{cases} e^{\alpha t} T_{\xi}(t)f, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

We now introduce the function

$$G : \mathbb{R}_+ \rightarrow L^p(\mathbb{R}, L^p(\Omega)) : s \mapsto G(s) := T_{\xi}(s) \circ g_{-s},$$

where $g_{-s}(r) := g(t-s)$, $t \in \mathbb{R}$. Hence,

$$G(s)(t) = \begin{cases} e^{-\alpha(t-s)} T_{\xi}(t)f, & 0 \leq s \leq t, \\ 0, & t < s. \end{cases}$$

Thus,

$$\Phi \left( \int_0^m G(s) \, ds \right) = \left( \int_0^m \left( \int_0^{\min(m,t)} e^{-\alpha(t-s)} T_{\xi}(t)f \, ds \right)^p \, dt \right)^{\frac{1}{p}}$$

$$= \frac{1}{\alpha} \left( \int_0^m (e^{-\alpha \max(0,t-m)} - e^{-\alpha})^p |T_{\xi}(t)f|^p \, dt \right)^{\frac{1}{p}}$$

and hence

$$0 \leq \frac{1}{\alpha} \left( \int_0^m (e^{-\alpha \max(0,t-m)} - e^{-\alpha})^p |T_{\xi}(t)f|^p \, dt \right)^{\frac{1}{p}} = \Phi \left( \int_0^m G(s) \, ds \right). \quad (2.3)$$
So, by Lemmas 2.4.3, 2.4.4, and 2.4.2, it follows that

\[ 0 \leq \Phi \left( \int_0^m G(s) \, ds \right) \leq \int_0^m \Phi(G(s)) \, ds = \int_0^m \Phi(T_\xi(s) \circ g_{-\lambda}) \, ds \leq \int_0^m T_\xi(s)(\Phi(g_{-\lambda})) \, ds = \int_0^m T_\xi(s)(\Phi(g)) \, ds. \]

On the other hand, since \( s(A_\xi) < 0 \) and from Theorem 2.3.1, it follows that

\[ \lim_{m \to \infty} \int_0^m T_\xi(s)(\Phi(g)) \, ds = R(0, A_\xi)(\Phi(g)). \]

From (2.3) and the monotone convergence theorem we have

\[ 0 \leq \frac{1}{\alpha} \left( \int_0^\infty (1 - e^{-\alpha t})^p |T_\xi(t) f|^p \, dt \right)^{\frac{1}{p}} \leq R(0, A_\xi)(\Phi(g)). \]

This implies

\[ \left( \frac{1 - e^{-\alpha}}{\alpha} \right) \left( \int_1^\infty |T_\xi(t) f|^p \, dt \right)^{\frac{1}{p}} \leq R(0, A_\xi)(\Phi(g)) \]

and therefore

\[ \int \int_\Omega \left| (T_\xi(t) f)(y) \right|^p \, d\mu(y) \leq \left( \frac{\alpha}{1 - e^{-\alpha}} \right)^p \| R(0, A_\xi) \| \| \Phi(g) \|_{L^p(\Omega)}^p, \]

which implies that

\[ \int_1^m \| T_\xi(t) f \|_{L^p(\Omega)} \, dt < \infty. \]

So, by Theorem 2.1.4, we obtain \( \omega_0(A_\xi) = \omega_0(A) - \xi < 0 \). Consequently, \( \omega_0(A) \leq s(A) \).

\[ \square \]

### 2.5 Irreducible Semigroups

In many concrete examples the semigroup \( T(\cdot) \) does not have exponential stability, however possesses an asynchronous exponential growth. This means that there is a rank one projection \( P \) and constants \( \varepsilon > 0, \, M \geq 1 \) such that

\[ \| e^{-sA} T(t) - P \| \leq M e^{-\varepsilon t} \quad \text{for all } t \geq 0, \]
where $A$ denotes the generator of $T(\cdot)$.

In order to study such kind of behaviour we introduce the concept of irreducibility for positive $C_0$-semigroups. For more details see [22] and the references therein.

**Definition 2.5.1** A positive $C_0$-semigroup $T(\cdot)$ on a Banach lattice $E$ with generator $A$ is called irreducible if one of the following equivalent properties is satisfied

(i) There is no $T(t)$-invariant closed ideal other than $\{0\}$ and $E$ for all $t > 0$.

(ii) For $x \in E$, $x^* \in E^*$ with $x \geq 0$ and $x^* > 0$, there is $t_0 > 0$ such that

$$\langle T(t_0)x, x^* \rangle > 0.$$

(iii) For some (and then for every) $\lambda > s(A)$, there is no $R(\lambda, A)$-invariant closed ideal except $\{0\}$ and $E$.

(iv) For some (and then for every) $\lambda > s(A)$, $R(\lambda, A)x$ is a quasi-interior point of $E_+$ for every $x \geq 0$.

**Example 2.5.2** (a) Let $E := L^p(\Omega, \mu)$, $1 \leq p < \infty$, and $T(\cdot)$ be a positive $C_0$-semigroup on $E$ with generator $A$. Then, it follows from Example 1.1.7 that $T(\cdot)$ is irreducible if and only if

$$0 \leq f \in E \implies (R(\lambda, A)f)(s) > 0$$

for a.e. $s \in \Omega$ and some $\lambda > s(A)$.

(b) If $E := C_0(\Omega)$, where $\Omega$ is locally compact Hausdorff, and $T(\cdot)$ a positive $C_0$-semigroup on $E$ with generator $A$, then, by Example 1.1.7, $T(\cdot)$ is irreducible if and only if

$$0 \leq f \in E \implies (R(\lambda, A)f)(s) > 0$$

for all $s \in \Omega$ and some $\lambda > s(A)$.

We now state some consequences of irreducibility.

**Proposition 2.5.3** Assume that $A$ is the generator of an irreducible $C_0$-semigroup $T(\cdot)$ on a Banach lattice $E$. Then the following assertions hold.

(a) Every positive eigenvector of $A$ is a quasi-interior point.

(b) Every positive eigenvector of $A^*$ is strictly positive.

(c) If $\ker(s(A) - A^*)$ contains a positive element, then $\dim \ker(s(A) - A) \leq 1$.

(d) If $s(A)$ is a pole of the resolvent, then it has algebraic (and geometric) multiplicity equal to 1. The corresponding residue has the form $P_{s(A)} = u^* \otimes x$, where $x \in E$ is a positive eigenvector of $A$, $u^* \in E^*$ is a positive eigenvector of $A^*$ and $\langle x, u^* \rangle = 1$. 
Proof: (a) Let \( x \) be a positive eigenvector of \( A \) and \( E_x := \bigcup_{n \in \mathbb{N}} [0, x] \) the ideal generated by \( x \). If \( \lambda \) is such that \( Ax = \lambda x \), then \( \lambda \in \mathbb{R} \). This follows from
\[
x \geq 0 \text{ and } Ax = \lim_{t \to 0^+} \frac{1}{t}(T(t)x - x).
\]
Hence, \( T(t)x = e^{\lambda t}x \) for \( t \geq 0 \). Thus, for \( y \in E_x \),
\[
|T(t)y| \leq T(t)|y| \leq nT(t)x = ne^{\lambda t}x, \quad t \geq 0.
\]
Consequently, \( T(t)E_x \subseteq E_x \) holds for all \( t \geq 0 \). Since \( 0 \neq x \in E_x \) and \( T(\cdot) \) is irreducible, it follows that \( E_x = E \).

(b) Let \( x^\star \) be a positive eigenvector of \( A^* \) and \( \lambda \) its corresponding eigenvalue. By the same argument we have \( \lambda \in \mathbb{R} \) and \( T(t)^*x^\star = e^{\lambda t}x^\star \) for \( t \geq 0 \). Hence,
\[
(\langle T(t)u, x^\star \rangle) \leq (\langle T(t)|u|, x^\star \rangle) = (\langle |u|, s e^{\lambda t}x^\star \rangle), \quad u \in E, t \geq 0.
\]
Thus, \( I := \{u \in E : \langle |u|, x^\star \rangle = 0\} \) is a \( T(t) \)-invariant closed ideal for all \( t \geq 0 \). Since \( x^\star \neq 0 \) we have \( I \subsetneq E \) and so by the irreducibility we obtain \( I = \{0\} \). Therefore, \( x^\star > 0 \).

(c) Let \( 0 \leq x^\star \in \ker(s(A) - A^*) \). It follows from (b) that \( x^\star \) is strictly positive. For \( x \in \ker(s(A) - A) \) we have \( T_{-s(A)}(t)x = x \) and hence,
\[
|x| = |T_{-s(A)}(t)x| \leq T_{-s(A)}(t)|x|, \quad t \geq 0.
\]
Thus, for \( t \geq 0 \),
\[
|\langle |x|, x^\star \rangle \rangle \leq \langle T_{-s(A)}(t)|x|, x^\star \rangle = \langle |x|, x^\star \rangle.
\]
This implies that \( T_{-s(A)}(t)|x| - |x|, x^\star \rangle = 0 \), and since \( x^\star > 0 \), we obtain \( T_{-s(A)}(t)|x| = |x| \) for \( t \geq 0 \). Therefore,
\[
|x| \in \ker(s(A) - A).
\]
Since \( \langle T_{-s(A)}(t)x^\star, x^\star \rangle \leq T_{-s(A)}(t)|x^\star| \), one can see by the same arguments as above that \( x^\star \in \ker(s(A) - A) \) and \( x^- \in \ker(s(A) - A) \). This implies that \( F := E_x \cap \ker(s(A) - A) \) is a real sublattice of \( E \). For \( x \in F \) we consider the ideal \( E_x^+ \) (resp. \( E_x^- \)) generated by \( x^+ \) (resp. \( x^- \)). Then, \( E_x^+ \) and \( E_x^- \) are \( T_{-s(A)}(t) \)-invariant for all \( t \geq 0 \). Since \( E_x^+ \) and \( E_x^- \) are orthogonal, it follows from the irreducibility of \( T_{-s(A)}(\cdot) \) that \( x^+ = 0 \) or \( x^- = 0 \). Consequently, \( F \) is totally ordered. So by Lemma 1.1.14 we have
\[
dim F = \dim \ker(s(A) - A) \leq 1.
\]

(d) We claim that if \( s(A) \) is a pole of the resolvent, then there is an eigenvector \( 0 \leq x \in E \) of \( A \) corresponding to \( s(A) \). Indeed, let \( k \) be the order of the pole \( s(A) \) and \( R_{-k} = \lim_{\lambda \to s(A)^+}(\lambda - s(A))^k R(\lambda, A) \) the corresponding residue. Then, \( R_{-k} \neq 0 \) and \( R_{-(k+1)} = 0 \). Moreover, by Corollary 2.3.4, we have \( R_{-k} \geq 0 \). Hence, there is
we obtain

0 \leq x \in E \text{ with } x := R_{-(k+1)} y \geq 0. \text{ By the relation } R_{-(k+1)} = (A - s(A)) R_{-k} = 0 \text{ we obtain } (A - s(A)) x = 0. \text{ This proves the claim.}

We can now use (a) to obtain $\overline{E}_x = E$. By taking the adjoint $R^s_{-(k+1)}$ of $R_{-(k+1)}$ and the same computation as before one has, if $s(A)$ is a pole of the resolvent, then there is $0 \leq x^* \in \ker(s(A) - A^*)$. So by (c) we have $\dim \ker(s(A) - A) = 1$.

Now, assume that $k \geq 2$. Then we have

\[
\langle x, x^* \rangle = \langle R_{-(k+1)} y, x^* \rangle \\
= \langle y, R_{-(k+1)}^* x^* \rangle \\
= \langle y, R_{-(k+1)}^* (A^* - s(A)) x^* \rangle \\
= 0.
\]

Since $\overline{E}_x = E$, it follows that $\langle u, x^* \rangle = 0$ for all $u \in E_+$. This contradicts the assertion (b). Hence $k = 1$. From the inequality $m_x + k - 1 \leq m_u \leq m_y k$ (cf. [9] p. 247) we obtain

\[
m_u = m_e = \dim P_{s(A)} E = \dim \ker(s(A) - A) = 1,
\]

where we recall that $P_{s(A)} = R_{-1}$. Since $P_{s(A)} E \subseteq \ker(s(A) - A)$, it follows that

\[
P_{s(A)} E = \ker(s(A) - A).
\]

We now show the last part of Assertion (d). To this purpose let $0 \leq x \in \ker(s(A) - A)$. Without loss of generality, we suppose that $\|x\| = 1$. Then $P_{s(A)} E = \text{Span}\{x\}$, i.e. $P_{s(A)} y = \lambda x$ for some $\lambda \in \mathbb{C}$ and every $y \in E$. By the Hahn-Banach theorem (see Proposition 1.1.12) there exists $0 \leq y^* \in (\ker(s(A) - A))^\ast$ with $\|y^*\| = 1$ and $\langle x, y^* \rangle = \|x\| = 1$. Hence $\langle P_{s(A)} y, y^* \rangle = \lambda = \langle y, P_{s(A)}^* y^* \rangle$. If we put $u^* := P_{s(A)}^* y^* \geq 0$, then $P_{s(A)} = u^* \otimes x$ and $\langle x, u^* \rangle = \langle P_{s(A)} x, y^* \rangle = \langle x, y^* \rangle = 1$. This implies that $0 \leq u^* \in P_{s(A)}^* E^* \subseteq \ker(s(A) - A^*)$. So $u^* > 0$ by (b). This ends the proof of the proposition.

The following result describes the eigenvalues of an irreducible semigroup which are contained in the boundary spectrum $\sigma_b(A) := \{\lambda \in \sigma(A) : \Re(\lambda) = s(A)\}$, where $A$ is the corresponding generator.

**Theorem 2.5.4** Let $T(\cdot)$ be an irreducible $C_0$–semigroup with generator $A$ on a Banach lattice $E$. Assume that $s(A) = 0$ and there is $0 \leq x^* \in D(A^*)$ with $A^* x^* = 0$. If $\sigma_p(A) \cap i\mathbb{R} \neq \emptyset$, then the following assertions hold.

(a) For $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$ with $Ah = i\alpha h$, $|h|$ is a quasi-interior point and

\[
S_h(D(A)) = D(A) \text{ and } S_h^{-1} A S_h = A + i\alpha
\]

hold, where $S_h$ is the signum operator.

(b) $\dim \ker(\lambda - A) = 1$ for every $\lambda \in \sigma_p(A) \cap i\mathbb{R}$.

(c) $\sigma_p(A) \cap i\mathbb{R}$ is an additive subgroup of $i\mathbb{R}$.
(d) \(0\) is the only eigenvalue of \(A\) admitting a positive eigenvector.

**Proof:** We first remark that by Proposition 2.5.3.(b) we have \(x^* > 0\) and \(T(t)^*x^* = x^*\) for all \(t \geq 0\).

(a) Assume that \(Ah = it\alpha h\) for \(0 \neq h \in D(A)\) and \(\alpha \in \mathbb{R}\). Then \(T(t)h = e^{it\alpha}h\) and hence \(|h| = |T(t)h| \leq T(t)|h|\). This implies that

\[
T(t)|h| - |h| \geq 0 \quad \text{for all } t \geq 0.
\]

On the other hand,

\[
\langle T(t)|h| - |h|, x^* \rangle = \langle |h|, T(t)^*x^* \rangle - \langle |h|, x^* \rangle = 0 \quad \text{for all } t \geq 0.
\]

Since \(x^* > 0\), we obtain \(T(t)|h| = |h|\) for all \(t \geq 0\), which implies that \(A|h| = 0\). So, by Proposition 2.5.3.(a), \(|h|\) is a quasi-interior point. If we set \(T_a(t) := e^{-it\alpha}T(t), t \geq 0\), then \(T(t)\) and \(T_a(t)\) satisfy the assumptions of Lemma 1.2.5 and hence

\[
T(t) = S_h^{-1}T_a(t)S_h, \quad t \geq 0.
\]

Therefore, \(S_h(D(A)) = D(A)\) and \(A = S_h^{-1}(A - i\alpha)S_h\) and (a) is proved.

(b) It follows from (a) that \(S_h : \ker(i\alpha + A) \to \ker A\) for \(i\alpha \in \sigma_p(A) \cap i\mathbb{R}\). On the other hand, the proof of (a) implies that \(\ker A \neq \{0\}\). So, by Proposition 2.5.3.(c), \(\dim \ker A = 1\) and hence \(\dim \ker(i\alpha + A) = 1\).

(c) Let \(0 \neq h, g \in D(A), \alpha, \beta \in \mathbb{R}\) such that \(Ah = it\alpha h\) and \(Ag = i\beta g\). By (a) we have

\[
S_h^{-1}AS_h = A + i\beta\text{ and } S_hAS_h^{-1} = A - i\alpha.
\]

Thus \(A + (\beta - \alpha) = S_h(A + i\beta)S_h^{-1} = S_hS_h^{-1}AS_hS_h^{-1} = S_hS_h^{-1}\ker A \neq \{0\}\). Therefore

\[
i(\beta - \alpha) \in \sigma_p(A).
\]

(d): If \(Ax = \lambda x\), where \(0 \lessgtr x \in D(A)\), then

\[
\lambda \langle x, x^* \rangle = \langle Ax, x^* \rangle = \langle x, A^*x^* \rangle = 0.
\]

Since \(x^* > 0\), it follows that \(\langle x, x^* \rangle > 0\). Hence, \(\lambda = 0\). \(\square\)

For irreducible semigroups we obtain the following description of the boundary spectrum.

**Theorem 2.5.5** Let \(T(\cdot)\) be an irreducible \(C_0\)-semigroup with generator \(A\) on a Banach lattice \(E\) and assume that \(s(A)\) is a pole of the resolvent. Then there is \(\alpha \geq 0\) such that

\[
\sigma_h(A) = s(A) + i\alpha \mathbb{Z}.
\]

Moreover, \(\sigma_h(A)\) contains only algebraically simple poles.
Proof: Without loss of generality we suppose that $s(A) = 0$. It can be shown that $\sigma_b(A) \subseteq \sigma_p(A)$. The proof uses pseudo-resolvents on a suitable $f$–product of $E$, where $f$ is an ultrafilter on $\mathbb{N}$ which is finer than the Fréchet filter (see [22], p. 314). Hence, $\sigma_b(A) = \sigma_p(A) \cap i\mathbb{R}$. By Proposition 2.5.3.(d) we obtain the existence of a positive eigenvector $x^* \in D(A^*)$ corresponding to the eigenvalue $s(A) = 0$. It follows from Theorem 2.5.4.(c) that $\sigma_b(A)$ is a subgroup of $(i\mathbb{R}_+, +)$. Since $\sigma_b(A)$ is closed and $s(A) = 0$ is an isolated point, we have

$$\sigma_b(A) = i\alpha\mathbb{Z} \quad \text{for some } \alpha \geq 0.$$ 

Proposition 2.5.3.(d) implies that 0 is a simple pole and by Theorem 2.5.4.(a) we have, for $\lambda \in \rho(A)$,

$$R(\lambda + ik\alpha, A) = S_b^kR(\lambda, A)S_b^{-k} \quad \text{for all } k \in \mathbb{Z}.$$ 

Therefore, $i\kappa\alpha$ is a simple pole for each $k \in \mathbb{Z}$. This ends the proof of the theorem.

We now give sufficient conditions for a $C_0$–semigroup to possess an asynchronous exponential growth. This result will be very useful for many applications.

Theorem 2.5.6 Let $T(\cdot)$ be an irreducible $C_0$–semigroup with generator $A$ on a Banach lattice $E$. If $\omega_{ess}(A) < \omega_0(A)$, then there exists a quasi-interior point $0 \leq x \in E, 0 < x^* \in E^*$ with $\langle x, x^* \rangle = 1$ such that

$$\|e^{-(s(A)t)} T(t) - x^* \otimes x\| \leq Me^{-\alpha t} \quad \text{for all } t \geq 0,$$

and appropriate constants $M \geq 1$ and $\varepsilon > 0$.

Proof: We first remark first that the rescaled semigroup $T_{-\omega_0}(t) := e^{-\omega_0(A)t}T(t)$, for $t \geq 0$, satisfies $\omega_{ess}(A_{-\omega_0}) = \omega_{ess}(A) - \omega_0(A) < 0$, where $A_{-\omega_0} := A - \omega_0(A)$ denotes its generator. Thus, $T_{-\omega_0}(\cdot)$ is quasi-compact and, by Proposition 2.2.2, we have

$$s(A) = \omega_0(A).$$

On the other hand, since $\omega_{ess}(A) < \omega_0(A)$, it follows that $r_{ess}(T(1)) < r(T(1))$. Hence, by Proposition 2.2.1, $r(T(1))$ is a pole of the resolvent of $T(1)$. This implies that $\omega_0(A) = s(A)$ is a pole of $R(\cdot, A)$. Thus, by Theorem 2.5.5, it follows that there exists $\alpha > 0$ such that $\sigma_b(A) = s(A) + i\alpha\mathbb{Z}$ and therefore $\sigma_b(A_{-\omega_0}) = i\alpha\mathbb{Z}$. Since $T_{-\omega_0}(\cdot)$ is quasi-compact and $\omega_0(A_{-\omega_0}) = 0$, we have, by Theorem 2.2.5, that

$$\{\lambda \in \sigma(A_{-\omega_0}) : \Re(\lambda) \geq 0\} = \{\lambda \in \sigma(A_{-\omega_0}) : \Re(\lambda) = 0\} = \sigma_b(A_{-\omega_0})$$

is finite. Therefore $\sigma_b(A_{-\omega_0}) = \{0\}$. The theorem is now proved by applying Theorem 2.2.5 and Proposition 2.5.3 to the rescaled semigroup $T_{-\omega_0}(\cdot)$. □