PREFACE

The main subject of this work is the analysis of the asymptotic behaviour of the solution of the Cauchy problem:

$$u'(t) = Au(t) \text{ with } u(0) = x \in D(A),$$

where A generates a C_0–semigroup on a Banach space E. We are particularly interested in *positive* C_0–semigroups on Banach *lattices*.

The theory of generation of semigroups of linear contractions, which is at the basis of a theory of evolution equations, was developed by Hille and Yosida in 1948. W. Feller (1952) and R.S. Phillips (1962) obtained first results concerning the characterization of the generators of special positive semigroups.

On the other hand, in the 60's and 70's the theory of ordered Banach spaces and positive operators was developed and is well documented in the monographs by H.H. Schaefer [26], A.C. Zaanen [37], Aliprantis and Burkinshaw [1], Meyer-Nirberg [21] and many others.

In the 80's, applications of positivity to Cauchy problems and specially to concrete evolution equations from transport theory, mathematical biology, and physics, has attracted much interest and was the subject of many papers. Most results of what was known around 1985 about this subject can be found in the book written by the functional analysis group in Tübingen, see [22]. This led to remarkable progress during the last decade.

We have organized these notes as follows.
We concentrate our attention on the asymptotic behaviour of positive C_0–semigroups of linear operators on Banach lattices and applications to transport theory.

In Chapter 1 we recall some basic and useful results on Banach lattices and positive operators. In Chapter 2 we discuss the uniform exponential stability of C_0–semigroups and present the Perron-Frobenius theory and its application to the asymptotic behaviour of irreducible C_0–semigroups. The last Chapter is dedicated to the application of our results to transport equations.

We have assumed that the reader is already familiar with basic functional analysis and the theory of C_0–semigroups on Banach spaces.

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CHAPTER 1

A SHORT INTRODUCTION TO BANACH LATTICES AND POSITIVE OPERATORS

In this chapter we give a brief introduction to Banach lattices and positive operators. Most results of this chapter can be found, e.g., in [26], [1] or [21].

1.1 BANACH LATTICES

A non empty set M with a relation $\leq$ is said to be an *ordered set* if the following conditions are satisfied.

i) $x \leq x$ for every $x \in M$,

ii) $x \leq y$ and $y \leq x$ implies $x = y$, and

iii) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Let A be a subset of an ordered set M. The element $x \in M$ (resp. $z \in M$) is called *an upper bound* (*lower bound* resp.) of A if $y \leq x$ for all $y \in A$ (resp. $z \leq y$ for all $y \in A$). Moreover, if there is an upper bound (resp. lower bound) of A, then A is said *bounded from above* (*bounded from below* resp.). If A is bounded from above and from below, then A is called *order bounded*. Let $x,y \in M$ such that $x \leq y$. We denote by

$$[x,y] := \{z \in M : x \leq z \leq y\}$$

the *order interval* between x and y. It is obvious that a subset A is order bounded if and only if it is contained in some order interval.

Definition 1.1.1 *A real vector space E which is ordered by some order relation $\leq$ is called a vector lattice if any two elements $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup(x,y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x,y)$ and the following properties are satisfied.*

(L1) $x \leq y$ *implies* $x+z \leq y+z$ *for all* $x,y,z \in E$,

(L2) $0 \leq x$ *implies* $0 \leq tx$ *for all* $x \in E$ *and* $t \in \mathbb{R}_+$.

Let E be a vector lattice. We denote by $E_+ := \{x \in E : 0 \leq x\}$ the *positive cone* of E. For $x \in E$ let

$$x^+ := x \vee 0, x^- := (-x) \vee 0, \text{ and } |x| := x \vee (-x)$$

be the *positive part*, the *negative part*, and the *absolute value* of x, respectively. Two elements $x,y \in E$ are called *orthogonal* (or *lattice disjoint*) (denoted by $x \perp y$) if $|x| \wedge |y| = 0$.

For a vector lattice E we have the following properties (cf. [26, Proposition II.1.4, Corollary II.1.1 and II.1.2] or [21, Theorem 1.1.1]).

Proposition 1.1.2 *For all $x,y,z \in E$ and $a \in \mathbb{R}$ the following assertions are satisfied.*

(i) $x+y = (x \vee y) + (x \wedge y)$,
$x \vee y = -(-x) \wedge (-y)$,
$(x \vee y) + z = (x+z) \vee (y+z)$,
and $(x \wedge y) + z = (x+z) \wedge (y+z)$.

(ii) $x = x^+ - x^-$.

(iii) $|x| = x^+ + x^-$, $|ax| = |a||x|$, *and* $|x+y| \leq |x| + |y|$.

(iv) $x^+ \perp x^-$ *and the decomposition of x into the difference of two orthogonal positive elements in unique.*

(v) $x \leq y$ *is equivalent to* $x^+ \leq y^+$ *and* $y^- \leq x^-$.

(vi) $x \perp y$ *is equivalent to* $|x| \vee |y| = |x| + |y|$. *In this case we have* $|x+y| = |x| + |y|$.

(vii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

(viii) *For all* $x,y,z \in E_+$ *we have* $(x+y) \wedge z \leq (x \wedge z) + (y \wedge z)$.

(ix) $|x-y| = (x \vee y) - (x \wedge y)$, *and* $|x-y| = |(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)|$.

A norm on a vector lattice E is called a *lattice norm* if

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\| \quad \text{for } x,y \in E.$$

Definition 1.1.3 *A Banach lattice is a real Banach space E endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.*

For a Banach lattice E the following properties hold (cf. [26, Proposition II.5.2] or [21, Proposition 1.1.6]).

Proposition 1.1.4 *Let E be a Banach lattice. Then,*

(a) the lattice operations are continuous,

(b) the positive cone E_+ is closed, and

(c) order intervals are closed and bounded.

• Sublattices, solids, bands and ideals

A vector subspace F of a vector lattice E is a *vector sublattice* if and only if the following are satisfied.

(1) $|x| \in F$ for all $x \in F$,

(2) $x^+ \in F$ or $x^- \in F$ for all $x \in F$.

A subset S of a vector lattice E is called *solid* if $x \in S$, $|y| \leq |x|$ implies $y \in S$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|x \vee y| \leq |x| + |y|$. One can see that a subspace I of a Banach lattice E is an ideal if and only if

$$x \in I \text{ implies } |x| \in I \text{ and } 0 \leq y \leq x \in I \text{ implies } y \in I.$$

Consequently, a vector sublattice F is an ideal in E if $x \in F$ and $0 \leq y \leq x$ imply $y \in F$. A subspace $B \subseteq E$ is a *band* in E if B is an ideal in E and $\sup(M)$ is contained in B whenever M is contained in B and has an upper bound (supremum) in E. Since the notion of sublattice, ideal, band are invariant under the formation of arbitrary intersections, there exists, for any subset M of E, a uniquely determined smallest sublattice (ideal, band) of E containing M. This will be called the sublattice (ideal, band) generated by M.

Next, we summarize all properties which we will need in the sequel (cf. [21, Proposition 1.1.5, 1.2.3 and 1.2.5]).

Proposition 1.1.5 *If E is a Banach lattice, then the following properties hold.*

(i) If I_1, I_2 are ideals of E, then $I_1 + I_2$ is an ideal and if furthermore I_1 and I_2 are closed, then $I_1 + I_2$ is also a closed ideal.

(ii) The closure of every solid subset of E is solid.

(iii) The closure of every sublattice of E is a sublattice.

(iv) The closure of every ideal of E is an ideal.

(v) Every band in E is closed.

(vi) For every non-empty subset $A \subset E$, the ideal generated by A is given by

$$I(A) = \bigcup \{n[-y,y] : n \in \mathbb{N},\ y = |x_1| \vee \ldots \vee |x_r|, x_1, \ldots, x_r \in A\}.$$

(vii) For every $x \in E_+$, the ideal generated by $\{x\}$ is

$$E_x = \bigcup \{n[-x,x] : n \in \mathbb{N}\}.$$

Example 1.1.6 **1.** *If $E = L^p(\Omega,\mu)$, $1 \le p < \infty$, where μ is σ-finite, then the closed ideals in E are characterized as follows: A subspace I of E is a closed ideal if and only if there exists a measurable subset Y of Ω such that*

$$I = \{\psi \in E : \psi(x) = 0 \text{ a.e. } x \in Y\}.$$

2. *If $E = C_0(X)$, where X is a locally compact topological space, then a subspace J of E is a closed ideal if and only if there is a closed subset A of X such that*

$$J = \{\varphi \in E : \varphi(x) = 0 \text{ for all } x \in A\}.$$

Let E be a Banach lattice. If $E_e = E$ holds for some $e \in E_+$, then e is called an *order unit*. If $\overline{E_e} = E$, then $e \in E_+$ is called a *quasi interior point* of E_+.
It follows that e is an order unit of E if and only if e is an interior point of E_+. Quasi interior points of the positive cone exist, for example, in every separable Banach lattice.

Example 1.1.7 **1.** *If $E = C(K)$, K compact, then the function constant $1\!\!1_K$ equal to 1 is an order unit. In fact, for every $f \in E$, there is $n \in \mathbb{N}$ such that $\|f\|_\infty \le n$. Hence, $|f(s)| \le n 1\!\!1_K(s)$ for all $s \in K$. This implies $f \in n[-1\!\!1_K, 1\!\!1_K]$.*

2. *If $E = L^p(\mu)$ with σ-finite measure μ and $1 \le p < \infty$, then the quasi interior points of E_+ coincide with the μ–a.e. strictly positive functions, while E_+ does not contain any interior point.*

• Spaces with order continuous norm

If the norm on E satisfies

$$\|x \vee y\| = \sup(\|x\|, \|y\|) \text{ for } x, y \in E_+$$

then E is called an *AM-space*. The above condition implies that the dual norm satisfies

$$\|x^* + y^*\| = \|x^*\| + \|y^*\| \text{ for } x^*, y^* \in E^*_+.$$

Such spaces are called *AL-spaces*.

Definition 1.1.8 *The norm of a Banach lattice E is called order continuous if every monotone order bounded sequence of E is convergent.*

One can prove the following result (cf. [21, Theorem 2.4.2]).

Proposition 1.1.9 *A Banach lattice E has order continuous norm if and only if every order interval of E is weakly compact.*

As a consequence one obtains the following examples.

Example 1.1.10 *Every reflexive Banach lattice and every L^1-space has order continuous norm.*

The Banach space dual E^* of a Banach lattice E is a Banach lattice with respect to the ordering $\leq$ defined by

$$0 \leq x^* \text{ if and only if } \langle x, x^* \rangle \geq 0 \text{ for all } x \in E_+.$$

A linear form $x^* \in E^*$ is called *strictly positive* if $\langle x, x^* \rangle > 0$ (notation: $x^* > 0$) for all $0 \lneqq x$ (means $0 \leq x$ and $x \neq 0$). The absolute value of $x^* \in E^*$ being given by

$$\langle x, |x^*| \rangle = \sup\{\langle y, x^* \rangle : |y| \leq x\}, \quad x \in E_+.$$

• Hahn-Banach's theorem

The following results are consequences of the Hahn-Banach theorem.

Proposition 1.1.11 *Let E be a Banach lattice . Then $0 \leq x$ is equivalent to $\langle x, x^* \rangle \geq 0$ for all $x^* \in E^*_+$.*

Proposition 1.1.12 *Let E be a Banach lattice. For each $0 \lneqq x \in E$ there exists $x^* \in E^*_+$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Proposition 1.1.13 *In a Banach lattice E every weakly convergent increasing sequence (x_n) is norm-convergent.*

Proof: Let $A := \{\sum_{i=1}^n a_i x_i : n \in \mathbb{N},\, a_i \geq 0,\, a_1 + \ldots + a_n = 1\}$ be the convex hull of $\{x_n : n \in \mathbb{N}\}$. By the Hahn-Banach theorem, the norm-closure of A coincide with the weak closure. This implies that $x \in \overline{A}$, where $x := \text{weak} - \lim_{n\to\infty} x_n$. Thus, for $\varepsilon > 0$ there exist

$$y = a_1 x_1 + \ldots + a_n x_n \in A,\, a_1, \ldots, a_n \geq 0,\, a_1 + \ldots + a_n = 1,$$

such that $\|y - x\| < \varepsilon$. Since $x_k \leq x$, it follows that $\|x - x_k\| \leq \|x - y\| < \varepsilon$ for all $k \geq n$. □

The following lemma will be useful in the proof of Proposition 2.5.3.

Lemma 1.1.14 *Let E be a totally ordered (this means $x \in E \Rightarrow 0 \leq x$ or $x \leq 0$) real Banach lattice. Then $\dim E \leq 1$.*

Proof: Let $e \in E_+$ and $x \in E$. We consider the closed subsets $C_+ := \{\alpha \in \mathbb{R} : \alpha e \geq x\}$ and $C_- := \{\alpha \in \mathbb{R} : \alpha e \leq x\}$ of $\mathbb{R}$. It is obvious that $C_+ \cup C_- = \mathbb{R}$. Since $\mathbb{R}$ is connected, it follows that $C_+ \cap C_- \neq \emptyset$. Hence there is $\alpha \in \mathbb{R}$ such that $x = \alpha e$. $\square$

• **Complexification of real Banach lattices** (cf. [26, II.11])

It is often necessary to consider complex vector spaces (for instance in spectral theory). Therefore, we introduce the concept of a complex Banach lattice.

The complexification of a real Banach lattice E is the complex Banach space $E_{\mathbb{C}}$ whose elements are pairs $(x,y) \in E \times E$, with addition and scalar multiplication defined by $(x_0,y_0)+(x_1,y_1) := (x_0+x_1, y_0+y_1)$ and $(a+ib)(x,y) := (ax-by, ay+bx)$, and norm

$$\|(x,y)\| := \| \sup_{0 \leq \theta \leq 2\pi} (x \sin\theta + y \cos\theta)\|.$$

One can show that the above supremum exists in E (cf. [26], p. 134). By identifying $(x,0) \in E_{\mathbb{C}}$ with $x \in E$, E is isometrically isomorphic to a real linear subspace of $E_{\mathbb{C}}$. We write $0 \leq x \in E_{\mathbb{C}}$ if and only if $x \in E_+$.

A complex Banach lattice is an ordered complex Banach space $(E_{\mathbb{C}}, \leq)$ that arises as the complexification of a real Banach lattice E. The underlying real Banach lattice E is called the real part of $E_{\mathbb{C}}$ and is uniquely determined as the closed linear span of all $x \in (E_{\mathbb{C}})_+$.

Instead of the notation (x,y) for elements of $E_{\mathbb{C}}$, we usually write $x+iy$. The complex conjugate of an element $z = x+iy \in E_{\mathbb{C}}$ is the element $\overline{z} = x - iy$. we use also the notation $\Re(z) := x$ for $z = x+iy \in E_{\mathbb{C}}$. The modulus $|\cdot|$ in E extends to $E_{\mathbb{C}}$ by

$$|x+iy| := \sup_{0 \leq \theta \leq 2\pi} (x \sin\theta + y \cos\theta).$$

All concepts first introduced for real Banach lattices have a natural extension to complex Banach lattices. A complex Banach lattice has order continuous norm if its real part has.

1.2 Positive operators

This section is concerned with positive operators and their properties. Let E,F be two complex Banach lattices. A linear operator T from E into F is called *positive* (notation: $T \geq 0$) if $TE_+ \subset F_+$, which is equivalent to

$$|Tx| \leq T|x| \quad \text{for all } x \in E.$$

Every positive linear operator $T : E \to F$ is continuous (cf. [21, Proposition 1.3.5]). Furthermore,

$$\|T\| = \sup\{\|Tx\| : x \in E_+,\ \|x\| \leq 1\}.$$

We denote by $\mathcal{L}(E,F)_+$ the set of all positive linear operators from E into F. For positive operators one can prove the following properties.

Proposition 1.2.1 *Let $T \in \mathcal{L}(E,F)_+$. Then the following properties hold.*

(i) $(Tx)^+ \leq Tx^+$ *and* $(Tx)^- \leq Tx^-$ *for all* $x \in E_{\mathbb{R}}$.

(ii) If $S \in \mathcal{L}(E,F)$ *such that* $0 \leq S \leq T$ *(this means that* $0 \leq Sx \leq Tx$ *for all* $x \in E_+$*), then* $\|S\| \leq \|T\|$.

Let $(A,D(A))$ be a linear operator on a Banach lattice E. It is a *resolvent positive* operator if there is $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subseteq \rho(A)$ and $0 \leq R(\lambda,A)$ for all $\lambda > \omega$. A C_0-semigroup on E is called positive if $0 \leq T(t)$ for all $t \geq 0$. Since

$$R(\lambda,A) = \int_0^\infty e^{\lambda t} T(t)\, dt \text{ for } \lambda > \omega_0(A) \text{ and}$$

$$T(t)x = \lim_{n\to\infty} (\frac{n}{t} R(\frac{n}{t},A))^n x$$

for all $x \in E$ and $t \geq 0$ (cf. [2, Corollary 3.3.6]), it follows that a C_0-semigroup on a Banach lattice E is positive if and only if its generator is resolvent positive operator.
For resolvent positive operators one has the following result (see [2, Theorem 3.11.8]).

Theorem 1.2.2 *Let E be a Banach lattice with order continuous norm. If A is a resolvent positive operator, then $\overline{D(A)}$ is an ideal in E.*

Proof: Since E is the complexification of a real Banach lattice $E_{\mathbb{R}}$ and $R(\lambda,A)E_{\mathbb{R}} \subseteq E_{\mathbb{R}}$, $\lambda > \omega$, we have $\Re(z) \in \overline{D(A)}$ for $z \in \overline{D(A)}$. Remark that if I is a closed ideal of $E_{\mathbb{R}}$, then $I \oplus iI$ is a closed ideal of E. Therefore we can suppose, without loss of generality, that E is a real Banach lattice. Moreover, we assume $s(A) < 0$, by considering $A - \omega$ instead of A otherwise.

a) Let $0 \leq y \leq R(0,A)x$, $x \in E_+$. We claim that $y \in \overline{D(A)}$. In fact, for $\lambda > 0$ we have

$$0 \leq \lambda R(\lambda,A)y \leq \lambda R(\lambda,A)R(0,A)x = R(0,A)x - R(\lambda,A)x \leq R(0,A)x.$$

From Proposition 1.1.3 it follows that $[0,R(0,A)x]$ is weakly compact. Hence, there is $z \in E$ such that $z = \text{weak} - \lim_{\lambda\to\infty} \lambda R(\lambda,A)y$. In particular, $z \in \overline{D(A)}$ (because $\overline{D(A)} = \overline{D(A)}^{\text{weak}}$). Therefore,

$$\begin{aligned} \text{weak} - \lim_{\lambda\to\infty} (R(0,A)y - R(\lambda,A)y) &= \text{weak} - \lim_{\lambda\to\infty} \lambda R(\lambda,A)R(0,A)y \\ &= R(0,A)z. \end{aligned}$$

Since $0 \leq R(\lambda,A)y \leq \frac{1}{\lambda}R(0,A)x$, we have $R(0,A)y = R(0,A)z$ and hence $y = z$.

b) Let $y \in \overline{D(A)}$. Then there is $(y_n) \subseteq D(A)$ such that $\lim_{n\to\infty} y_n = y$. Moreover, there exists $x_n \in E$ with $y_n = R(0,A)x_n$ and then $0 \leq |y_n| \leq R(0,A)|x_n|$. Now **a)** implies that $|y_n| \in \overline{D(A)}$ and hence $|y| \in \overline{D(A)}$.

c) Let $0 \le y \le x \in \overline{D(A)}$. Let $(x_n) \in D(A)$ with $\lim_{n\to\infty} x_n = x$. From **b)** we have $|x_n| \in \overline{D(A)}$. On the other hand,

$$y \wedge |x_n| \le |x_n| = |R(0,A)Ax_n| \le R(0,A)|Ax_n|$$

and **a)** implies that $y \wedge |x_n| \in \overline{D(A)}$. Hence,

$$y = \lim_{n\to\infty} y \wedge |x_n| \in \overline{D(A)}.$$

□

Positive operators on $C(K)$ with $T1\!\!\mathrm{I}_K = 1\!\!\mathrm{I}_K$ are contraction operators (cf. [22, B.III. Lemma 2.1]).

Lemma 1.2.3 *Suppose that K is compact and $T : C(K) \to C(K)$ is a linear operator satisfying $T1\!\!\mathrm{I}_K = 1\!\!\mathrm{I}_K$. Then $0 \le T$ if and only if $\|T\| \le 1$.*

Proof: If $0 \le T$, then

$$|Tf| \le T|f| \le T(\|f\|_\infty 1\!\!\mathrm{I}_K) = \|f\|_\infty 1\!\!\mathrm{I}_K.$$

Hence $\|T\| \le 1$.
To prove the converse, we first observe that

$$-1\!\!\mathrm{I}_K \le f \le 1\!\!\mathrm{I}_K \Leftrightarrow \|f - ir1\!\!\mathrm{I}_K\|_\infty \le \rho_r := \sqrt{1+r^2} \text{ for all } r \in \mathbb{R}. \qquad (1.1)$$

Let $f \in C(K)$ with $0 \le f \le 21\!\!\mathrm{I}_K$. Then $-1\!\!\mathrm{I}_K \le f - 1\!\!\mathrm{I}_K \le 1\!\!\mathrm{I}_K$. By (1.1) we have $\|f - 1\!\!\mathrm{I}_K - ir1\!\!\mathrm{I}_K\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. Since $T1\!\!\mathrm{I}_k = 1\!\!\mathrm{I}_K$ and $\|T\| \le 1$, $\|Tf - 1\!\!\mathrm{I}_K - ir1\!\!\mathrm{I}_K\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. So by (1.1) we obtain $-1\!\!\mathrm{I}_K \le Tf - 1\!\!\mathrm{I}_K \le 1\!\!\mathrm{I}_K$. This implies $0 \le Tf \le 21\!\!\mathrm{I}_K$. □

• Lattice homomorphism and signum operators

Let E, F be two Banach lattices and $T \in \mathcal{L}(E,F)$. It is called *lattice homomorphism* if one of the following equivalent conditions is satisfied (cf. [21, Proposition 1.3.11]).

(a) $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.

(b) $|Tx| = T|x|, x \in E$.

(c) $Tx^+ \wedge Tx^- = 0, x \in E$.

The following result, due to Kakutani, shows that for every $e \in E_+$ the generated ideal satisfies $E_e \cong C(K)$ for some compact K. Here, E_e is equipped with the norm $\|x\|_e := \inf\{\lambda > 0 : x \in \lambda[-e,e]\}, x \in E_e$ (cf. [21, Theorem 2.1.3]).

Theorem 1.2.4 *Let $e \in E_+$ and take E_e the ideal generated by e. Let $B := \{x^* \in (E_e)^*_+ : \langle e, x^*\rangle = 1\}$ and $K = \mathrm{ex}(B)$ the set of all extreme points of B. Then K is $\sigma(E^*,E)$-compact and the mapping $U_e : E_e \ni x \mapsto f_x \in C(K)$; $f_x(x^*) = \langle x, x^*\rangle$, $x^* \in K$, is an isometric lattice isomorphism.*

If $|h|$ is a quasi interior point of E_+, then $E_{|h|}$ is a dense subspace of E isomorphic to a space $C(K)$. Consider the lattice isomorphism $U_{|h|}$ from Kakutani's theorem. Let $\widetilde{h} := U_{|h|}h$. Then, $|\widetilde{h}| = U_{|h|}|h| = 1\!\!1_K$. Consider the operator

$$\widetilde{S}_0 : C(K) \to C(K); f \mapsto (\text{sign}\widetilde{h})f := \frac{\widetilde{h}}{|\widetilde{h}|}f = \widetilde{h}f,$$

and put $S_h := U_{|h|}^{-1}\widetilde{S}_0 U_{|h|}$. Then S_h is a linear mapping from $E_{|h|}$ into itself satisfying

(i) $S_h\overline{h} = |h|$,

(ii) $|S_h x| \le |x|$ for every $x \in E_{|h|}$,

(iii) $S_h x = 0$ for every $x \in E_{|h|}$ orthogonal to h.

Since (ii) implies the continuity of S_h for the norm induced by E and $\overline{E_{|h|}} = E$, S_h can be uniquely extended to E. This extension will be also denoted by S_h and is called *signum operator* with respect to h.
We now give the following auxiliary result which we need in Section 2.5. See [22, B.III. Lemma 2.3] for a similar result.

Lemma 1.2.5 *Let $T, R \in \mathcal{L}(E)$ and assume that $|h|$ is a quasi interior point of E_+. Suppose we have $Rh = h$, $T|h| = |h|$, and $|Rx| \le T|x|$ for all $x \in E$. Then $T = S_h^{-1}RS_h$.*

Proof: It follows from $|Rx| \le T|x|, x \in E$, that T is a positive operator. Since $T|h| = |h|$, $E_{|h|}$ is T–and R–invariant. Consider the operators $\widetilde{T} := U_{|h|}TU_{|h|}^{-1}$, $\widetilde{R} := U_{|h|}RU_{|h|}^{-1}$, and put $\widetilde{h} := U_{|h|}h$. We then have

$$\widetilde{R}\widetilde{h} = \widetilde{h},\ \widetilde{T}1\!\!1_K = 1\!\!1_K,\ |\widetilde{R}f| \le \widetilde{T}|f| \text{ for all } f \in C(K). \tag{1.2}$$

Define $T_1 := M_{\widetilde{h}}^{-1}\widetilde{R}M_{\widetilde{h}}$, where $M_{\widetilde{h}}$ is the multiplication operator by $\widetilde{h}$ on $C(K)$. By (1.2) we have

$$\begin{aligned} T_1 1\!\!1_K &= 1\!\!1_K \quad \text{and} \\ |T_1 f| &= |M_{\widetilde{h}}^{-1}\widetilde{R}M_{\widetilde{h}}f| = |\widetilde{R}M_{\widetilde{h}}f| \le \widetilde{T}|M_{\widetilde{h}}f| = \widetilde{T}|f| \end{aligned} \tag{1.3}$$

for all $f \in C(K)$. Hence $\|T_1\| \le \|\widetilde{T}\| = \|\widetilde{T}1\!\!1_K\|_\infty = 1$. So by Lemma 1.2.3, T_1 is a positive operator and (1.3) implies that $0 \le T_1 \le \widetilde{T}$. Therefore, $\|\widetilde{T} - T_1\| = \|(\widetilde{T} - T_1)1\!\!1_K\|_\infty = 0$. Since $|\widetilde{h}| = |U_{|h|}h| = U_{|h|}|h| = 1\!\!1_K$, it follows that $\widetilde{S}_0 = M_{\widetilde{h}}$. Thus, $S_h = U_{|h|}^{-1}M_{\widetilde{h}}U_{|h|}$ and $T_1 = \widetilde{T}$ implies that $T = S_h^{-1}RS_h$. □

CHAPTER 2

SPECTRAL THEORY FOR POSITIVE SEMIGROUPS

In this chapter we are concerned with the remarkable spectral properties shown by positive semigroups on Banach lattices.
Throughout this chapter we suppose that $E \neq \{0\}$ is a complex Banach lattice.

2.1 STABILITY OF STRONGLY CONTINUOUS SEMIGROUPS

In this section we study the asymptotic behaviour of the solution of the abstract Cauchy problem

$$(ACP) \qquad \begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where A is the generator of a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space E.

To this purpose we define *the type of the trajectory* $T(\cdot)x$ by

$$\omega(x) := \inf\{\omega : \|T(t)x\| \leq Me^{\omega t} \text{ for a constant } M \text{ and all } t \geq 0\},$$

and the *growth bound* (or type) of $T(\cdot)$ by

$$\begin{aligned} \omega_0(A) &:= \sup\{\omega(x) : x \in E\} \\ &= \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq Me^{\omega t} \text{ for some constant } M \text{ and all } t \geq 0\}. \end{aligned}$$

The *type of the solutions* of (ACP) is

$$\omega_1(A) := \sup\{\omega(x) : x \in D(A)\}).$$

We now introduce different stability concepts.

Definition 2.1.1 *A C_0–semigroup $T(\cdot)$ with generator A is called*

(i) uniformly exponentially stable if $\omega_0(A) < 0$,

(ii) exponentially stable if $\omega_1(A) < 0$,

(iii) strongly stable if $\lim_{t\to\infty} \|T(t)x\| = 0$ *for every* $x \in E$,

(iv) stable if $\lim_{t\to\infty} \|T(t)x\| = 0$ *for every* $x \in D(A)$.

It is clear that

$$\begin{array}{ccc} (i) & \Longrightarrow & (ii) \\ \Downarrow & & \Downarrow \\ (iii) & \Longrightarrow & (iv). \end{array}$$

If $A \in \mathcal{L}(E)$, then $(i) \Longleftrightarrow (ii)$ and $(iii) \Longleftrightarrow (iv)$. In the case where A is unbounded the above concepts of stability may differ as one can see in the following examples.

Example 2.1.2 *1. On $E := C_0(\mathbb{R}^n)$ we consider the heat semigroup defined by*

$$(T(t)f)(x) := \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy \quad \text{for } t > 0 \text{ and}$$

$$T(0)f := f \in E.$$

Then $T(\cdot)$ is a bounded holomorphic semigroup and it generator is the Laplacian Δ on $C_0(\mathbb{R}^n)$. Since $T(t)f = k_t * f$, where $k_t(y) := \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{y^2}{4t}}$, $y \in \mathbb{R}^n$, and since $\|k_t\|_{L^1} = 1$, it follows that

$$\|T(t)\| \leq 1, \forall t \geq 0. \tag{2.1}$$

Take now $f \in C_c(\mathbb{R}^n)$. Then,

$$\|T(t)f\| \leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |f(y)|\,dy \to 0 \;\text{ as } t \to \infty.$$

Hence, it follows from the density of $C_c(\mathbb{R}^n)$ in $C_0(\mathbb{R}^n)$ and (reflap) that $\lim_{t\to\infty} T(t)f = 0$, for every $f \in E$. This means that $T(\cdot)$ is strongly stable. On the other hand one can see that $\mathrm{Im}\Delta \neq C_0(\mathbb{R}^n)$, which implies that $0 \in \sigma(\Delta)$. Thus, $T(\cdot)$ is not uniformly exponentially stable, since $s(\Delta) \leq \omega_0(\Delta)$. For the definition of $s(A)$ see Section 2.3.

2. We consider the translation semigroup

$$(T(t)f)(s) = f(s+t), \quad t, s \geq 0,$$

on $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^s ds)$. Then E is a Banach lattice and $T(\cdot)$ is a C_0–semigroup with generator A given by

$$Af = f' \text{ for } f \in D(A) = \{f \in E : f \in C^1(\mathbb{R}_+) \text{ and } f' \in E\}.$$

Moreover,

$$\rho(A) = \{\lambda \in \mathbb{C} : \Re(\lambda) > -1\}$$

and for $\Re(\lambda) > -1$,

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt \quad \text{exists for all } f \in E.$$

One can see that $\|T(t)\| = 1$ *and so* $\omega_0(A) = 0$. *On the other hand, for* $\Re(\lambda) > -1$, *we have*

$$T(t)f = e^{\lambda t}\left(f - \int_0^t e^{-\lambda s} T(s)(\lambda - A) f \, ds\right), \quad f \in D(A),$$

and since $\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)(\lambda - A) f \, ds$ *exists, it follows that*

$$\|T(t)f\| \le N e^{\lambda t}, \quad \text{for all } f \in D(A).$$

Hence,

$$\omega_1(A) \le -1 < 0 = \omega_0(A).$$

Consequently, $T(\cdot)$ *is exponentially stable but not uniformly exponentially stable. For more details see [9, Example V.1.4].*

The definition of the growth bound yields the following characterization of uniform exponential stability.

Proposition 2.1.3 *For the generator* A *of a* C_0*–semigroup* $T(\cdot)$ *on a Banach space* E*, the following assertions are equivalent.*

(a) $\omega_0(A) < 0$, *i.e.,* $T(\cdot)$ *is uniformly exponentially stable.*

(b) $\lim_{t\to\infty} \|T(t)\| = 0$.

(c) $\|T(t_0)\| < 1$ *for some* $t_0 > 0$.

(d) $r(T(t_1)) < 1$ *for some* $t_1 > 0$.

Proof: The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy.

$(d) \Rightarrow (c)$: Since $r(T(t_1)) = \lim_{k\to\infty} \|T(t_1 k)\|^{\frac{1}{k}} < 1$, it follows that there is $k_0 \in \mathbb{N}$ with $\|T(k_0 t_1)\| < 1$.

$(c) \Rightarrow (a)$: For $\alpha := \|T(t_0)\| < 1$, $M := \sup_{0\le s\le t_0} \|T(s)\|$ and $t = kt_0 + s$ with $s \in [0, t_0)$, we have

$$\begin{aligned} \|T(t)\| &\le \|T(s)\|\|T(t_0 k)\| \\ &\le M\alpha^k = M e^{k \ln \alpha}. \end{aligned}$$

If we set $\varepsilon := \frac{-\ln \alpha}{t_0} > 0$ (because $\alpha < 1$), then

$$\|T(t)\| \le M e^{k \ln \alpha} \le \frac{M}{\alpha} e^{-\varepsilon t}.$$

□

It is clear that if $\omega_0(A) < 0$, then there are constants $\varepsilon > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\varepsilon t}, \quad t \geq 0.$$

Hence, for every $p \in [1, \infty)$, $\int_0^\infty \|T(t)x\|^p \, dt < \infty$ for all $x \in E$. The following result due to Datko [6] shows that the converse is also true.

Theorem 2.1.4 *A $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space E is uniformly exponentially stable if and only if for some (and hence for every) $p \in [1, \infty)$,*

$$\int_0^\infty \|T(t)x\|^p \, dt < \infty$$

for all $x \in E$.

Proof: We have only to prove the converse. By Proposition 2.1.3 it suffices to prove that $\lim_{t\to\infty} \|T(t)\| = 0$. Since there are $M, \omega \in \mathbb{R}_+$ with $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$, we obtain

$$\begin{aligned} \frac{1 - e^{-p\omega t}}{p\omega} \|T(t)x\|^p &= \int_0^t e^{-p\omega s} \|T(s)T(t-s)x\|^p \, ds \\ &\leq M^p \int_0^t \|T(t-s)x\|^p \, ds \\ &\leq M^p C^p \|x\|^p \end{aligned}$$

for all $x \in E$ and $t \geq 0$. Hence, $\|T(t)x\|^p \leq \frac{p\omega}{1-e^{-p\omega}} M^p C^p \|x\|^p$ for $x \in E$ and $t \geq 1$. Thus, there exists a constant $L > 0$ with $\|T(t)\| \leq L$ for all $t \geq 0$. Therefore,

$$\begin{aligned} t\|T(t)x\|^p &= \int_0^t \|T(t-s)T(s)x\|^p \, ds \\ &\leq L^p \int_0^t \|T(s)x\|^p \, ds \\ &\leq L^p C^p \|x\|^p \end{aligned}$$

for all $x \in E$ and $t \geq 0$. Thus,

$$\|T(t)\| \leq LCt^{-\frac{1}{p}}, \quad t > 0,$$

which implies $\lim_{t\to\infty} \|T(t)\| = 0$. □

In Hilbert spaces uniform exponential stability can be characterized in term of the generator as the following Gearhart-Prüss's result shows (see [11], [22, A-III.7], [25]).

Theorem 2.1.5 *Let $T(\cdot)$ be a $\mathcal{C}_0$–semigroup on a Hilbert space H with generator A. Then $T(\cdot)$ is uniformly exponentially stable if and only if*

$$\{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A) \text{ and } M := \sup_{\Re(\lambda)>0} \|R(\lambda, A)\| < \infty.$$

Proof: Assume that $\omega_0(A) < 0$. Then $\int_0^\infty e^{-\lambda t} T(t)\,dt$ exists for all $\Re(\lambda) > 0$. So by [9, Theorem II.1.10], $\{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A)$ and $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\,dt$ and therefore

$$\sup_{\Re(\lambda)>0} \|R(\lambda, A)\| < \infty.$$

We now prove the converse. We know from the spectral theory for closed operators (cf. [9, Corollary IV.1.14]) that

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq M^{-1}, \quad \text{for all } \Re(\lambda) > 0.$$

Thus, $i\mathbb{R} \subseteq \rho(A)$ and $\sup_{\Re(\lambda)\geq 0} \|R(\lambda, A)\| < \infty$. Let $\omega > |\omega_0(A)| + 1$ and consider the C_0–semigroup $T_{-\omega}(\cdot)$ defined by $T_{-\omega}(t) := e^{-\omega t} T(t)$, $t \geq 0$. By [9, Theorem II.1.10] we have

$$\begin{aligned} R(\omega + is, A)x &= R(is, A - \omega)x \\ &= \int_0^\infty e^{-ist} T_{-\omega}(t) x\,dt \\ &= \mathcal{F}\left(T_{-\omega}(\cdot)x\right)(s), \end{aligned}$$

where $\mathcal{F} f(s) := \int_{-\infty}^\infty e^{-ist} f(t)\,dt$ denotes de Fourier transform from $L^2(\mathbb{R}, H)$ into $L^2(\mathbb{R}, H)$. Here we extend $T_{-\omega}(\cdot)$ to $\mathbb{R}$ by taking $T_{-\omega}(t) = 0$ for $t < 0$. Since $T_{-\omega}(\cdot)$ is uniformly exponentially stable, we obtain $T_{-\omega}(\cdot)x \in L^2(\mathbb{R}, H)$. Then one can apply Plancherel's theorem, and we obtain

$$\int_{-\infty}^\infty \|R(\omega + is, A)x\|^2\,ds = 2\pi \int_0^\infty \|T_{-\omega}(t)x\|^2\,dt \leq L\|x\|^2$$

for some constant $L > 0$ and all $x \in H$. The resolvent identity gives

$$R(is, A) = R(\omega + is, A) + \omega R(is, A) R(\omega + is, A), \quad \text{for all } s \in \mathbb{R}.$$

Hence, $\|R(is, A)x\| \leq (1 + M\omega)\|R(\omega + is, A)x\|$ for $s \in \mathbb{R}$ and $x \in H$. This implies

$$\begin{aligned} \int_{-\infty}^\infty \|R(is, A)x\|^2\,ds &\leq (1 + \omega M)^2 \int_{-\infty}^\infty \|R(\omega + is, A)x\|^2\,ds \\ &\leq (1 + M\omega)^2 L\|x\|^2. \end{aligned}$$

On the other hand, by the inverse Laplace transform formula (cf. [9, Corollary III.5.16]) we know that

$$T(t)x = \frac{1}{2i\pi t} \lim_{n\to\infty} \int_{\omega - in}^{\omega + in} e^{\lambda t} R(\lambda, A)^2 x\,d\lambda, \quad t \geq 0, x \in D(A^2).$$

Then, by Cauchy's integral theorem,

$$\begin{aligned} (tT(t)x|y) &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{(\omega+is)t} \left(R(\omega + is, A)^2 x|y\right) ds \\ &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{ist} \left(R(is, A)^2 x|y\right) ds \\ &= \frac{1}{2i\pi} \int_{-\infty}^\infty e^{ist} \left(R(is, A)x|R(-is, A^*)y\right) ds \end{aligned}$$

for all $x \in D(A^2)$ and $y \in H$. As above one can see that

$$\int_{-\infty}^{\infty} \|R(is,A^*)y\|^2\,ds \le (1+M\omega)^2 L\|y\|^2, \quad y \in H.$$

By applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |(tT(t)x|y)| &\le \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|R(is,A)x\|^2\,ds\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \|R(is,A^*)y\|^2\,ds\right)^{\frac{1}{2}} \\ &\le \frac{(1+M\omega)^2 L}{2\pi}\|x\|\|y\| \end{aligned}$$

for all $x \in D(A^2)$ and $y \in H$. Since $\overline{D(A^2)} = H$, it follows that

$$\begin{aligned} \|tT(t)\| &= \sup\left\{|(tT(t)x|y)|\,; x,y \in D(A^2),\ \|x\| = \|y\| = 1\right\} \\ &\le \frac{(1+M\omega)^2}{2\pi} L. \end{aligned}$$

Hence, $\lim_{t\to\infty} \|T(t)\| = 0$ and therefore, $\omega_0(A) < 0$. □

2.2 THE ESSENTIAL SPECTRUM AND QUASI-COMPACT SEMIGROUPS

In this section we study the essential growth bound $\omega_{ess}(A)$ of the generator A of a C_0–semigroup $T(\cdot)$ on a Banach space E, in the case $\omega_{ess}(A) < 0$. Then we deduce important consequences for the asymptotic behaviour of $T(\cdot)$.

We start with some definitions. A bounded operator $S \in \mathcal{L}(E)$ is called a *Fredholm operator* if there is $T \in \mathcal{L}(E)$ such that $Id - TS$ and $Id - ST$ are compact. We denote by

$$\sigma_{ess}(S) = \mathbb{C} \setminus \rho_F(S)$$

the *essential spectrum* of S, where

$$\rho_F(S) := \{\lambda \in \mathbb{C} : (\lambda - S) \text{ is a Fredholm operator }\}.$$

The *Calkin algebra* $\mathcal{C}(E) := \mathcal{L}(E)/\mathcal{K}(E)$ equipped with the quotient norm

$$\|S\|_{ess} := \|S + \mathcal{K}(E)\| = \operatorname{dist}(S, \mathcal{K}(E)) = \inf\{\|S - K\| : K \in \mathcal{K}(E)\}$$

is a Banach algebra with unit. The essential spectrum of $S \in \mathcal{L}(E)$ can also defined as the spectrum of $S + \mathcal{K}(E)$ in the Banach algebra $\mathcal{C}(E)$. This implies that, for $S \in \mathcal{L}(E)$, $\sigma_{ess}(S)$ is non-empty and compact.

For $S \in \mathcal{L}(E)$ we define the *essential spectral radius* by

$$r_{ess}(S) := r(S + \mathcal{K}(E)) = \max\{|\lambda| : \lambda \in \sigma_{ess}(S)\}.$$

Since $(S+\mathcal{K}(E))^n = S^n + \mathcal{K}(E)$ for $n \in \mathbb{N}$, we have $r_{ess}(S) = \lim_{n\to\infty} \|S^n\|_{ess}^{\frac{1}{n}}$ and consequently,

$$r_{ess}(S+K) = r_{ess}(S), \quad \text{for every } K \in \mathcal{K}(E).$$

If we denote by

$$Pol(S) := \{\lambda \in \mathbb{C} : \lambda \text{ is a pole of finite algebraic multiplicity of } R(\cdot, S)\},$$

then one can prove that $Pol(S) \subseteq \rho_F(S)$ and an element of the unbounded connected component of $\rho_F(S)$ either is in $\rho(S)$ or a pole of finite algebraic multiplicity. For details concerning the essential spectrum we refer to [20, Sec. IV.5.6], [13, Chap. XVII] or [12, Sec. IV.2]. Thus we obtain the following characterization.

Proposition 2.2.1 *For $S \in \mathcal{L}(E)$ the essential spectral radius is given by*

$$r_{ess}(S) = \inf\{r > 0 : \lambda \in \sigma(S),\, |\lambda| > r \text{ and } \lambda \in Pol(S)\}.$$

Proof: If we set

$$a := \inf\{r > 0 : \lambda \in \sigma(S),\, |\lambda| > r \text{ and } \lambda \in Pol(S)\},$$

then for all $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that

$$\{\lambda \in \sigma(S) : |\lambda| > r_\varepsilon\} \subseteq Pol(S)$$

and $r_\varepsilon - \varepsilon \le a$. On the other hand, we know that there is $\lambda_0 \in \sigma_{ess}(S)$ with $r_{ess}(S) = |\lambda_0|$. If we suppose that $r_{ess}(S) > r_\varepsilon$, then $\lambda_0 \in Pol(S)$. This implies that $\lambda_0 \in \rho_F(S)$ which is a contradiction. Hence, $r_{ess}(S) \le r_\varepsilon \le a + \varepsilon$. Thus, $r_{ess}(S) \le a$.

To show the other inequality we know that

$$\{\lambda \in \sigma(S) : |\lambda| > r_{ess}(S)\} \subseteq \rho_F(S).$$

Therefore,

$$\{\lambda \in \sigma(S) : |\lambda| > r_{ess}(S)\} \subseteq Pol(S).$$

Consequently, $a \le r_{ess}(S)$ and the proposition is proved. □

We define the *essential growth bound* $\omega_{ess}(A)$ of a C_0–semigroup $T(\cdot)$ with generator A as the growth bound of the quotient semigroup $T(\cdot) + \mathcal{K}(E)$ on $\mathcal{C}(E)$, i.e.,

$$\omega_{ess}(A) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\|_{ess} \le Me^{\omega t}, \forall t \ge 0\}.$$

Then, for all $t_0 > 0$, one can see that

$$\omega_{ess}(A) = \frac{\log r_{ess}(T(t_0))}{t_0} = \lim_{t\to\infty} \frac{\log \|T(t)\|_{ess}}{t}. \tag{2.2}$$

The following result gives the relationship between $\omega_{ess}(A)$ and $\omega_0(A)$.

Proposition 2.2.2 *Let $T(\cdot)$ be a C_0–semigroup with generator A on a Banach space E. Then one has*

$$\omega_0(A) = \max\{s(A), \omega_{ess}(A)\}.$$

Proof: If $\omega_{ess}(A) < \omega_0(A)$, then $r_{ess}(T(1)) < r(T(1))$. Let $\lambda \in \sigma(T(1))$ such that $|\lambda| = r(T(1))$. So by Proposition 2.2.1, λ is an eigenvalue of $T(1)$ and by the spectral mapping theorem for the point spectrum (cf. [9, Theorem IV.3.7]) there is $\lambda_1 \in \sigma_p(A)$ with $e^{\lambda_1} = \lambda$. Therefore, $\Re(\lambda_1) = \omega_0(A)$ and thus $\omega_0(A) = s(A)$. □

By using the essential growth bound one can deduces important consequences for the asymptotic behaviour, the proof can be found in [9, Theorem V.3.1]

Theorem 2.2.3 *Let A be the generator of a C_0–semigroup $T(\cdot)$ on a Banach space E and $\lambda_1,\ldots,\lambda_m \in \sigma(A)$ with $\Re(\lambda_1),\ldots,\Re(\lambda_m) > \omega_{ess}(A)$. Then $\lambda_1,\ldots,\lambda_m$ are isolated spectral values of A with finite algebraic multiplicity. Furthermore, if $P_1,\ldots,P_m$ denote the corresponding spectral projections and $k_1,\ldots,k_m$ the corresponding orders of poles of $R(\cdot,A)$, then*

$$T(t) = T_1(t) + \ldots + T_m(t) + R_m(t),$$

where

$$T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!}(A-\lambda_n)^j P_n, \quad n = 1,\ldots,m.$$

Moreover, for every $\omega > \sup\{\omega_{ess}(A)\} \cup \{\Re(\lambda) : \lambda \in \sigma(A) \setminus \{\lambda_1,\ldots,\lambda_m\}\}$, there is $M > 0$ such that

$$\|R_m(t)\| \le Me^{\omega t} \quad \text{for } t \ge 0.$$

We now introduce the concept of quasi-compact semigroups,

Definition 2.2.4 *A C_0–semigroup $T(\cdot)$ with generator A on a Banach space E is called quasi-compact if $\omega_{ess}(A) < 0$.*

From (2.2) we deduce that any eventually compact C_0–semigroup is quasi-compact.

The following description of the asymptotic behaviour of quasi-compact semigroups is an immediate consequence of Theorem 2.2.3.

Theorem 2.2.5 *Let A be the generator of a quasi-compact C_0–semigroup $T(\cdot)$ on a Banach space E. Then the following assertions hold.*

(a) *The set $\{\lambda \in \sigma(A) : \Re(\lambda) \ge 0\}$ is finite (or empty) and consists of poles of $R(\cdot,A)$ of finite algebraic multiplicity.*
Denoting these poles by $\lambda_1,\ldots,\lambda_m$, the corresponding spectral projections $P_1,\ldots,P_m$ and the order of the poles $k_1,\ldots,k_m$, we have

(b) *$T(t) = T_1(t) + \ldots + T_m(t) + R(t)$, where*

$$T_n(t) := e^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!}(A-\lambda_n)^j P_n, \quad n = 1,\ldots,m,$$

and

$$\|R(t)\| \le Me^{-\varepsilon t} \quad \text{for some } \varepsilon > 0, M \ge 1 \text{ and all } t \ge 0.$$

2.3 Spectral bounds for positive semigroups

In this section we characterize the spectral bound

$$s(A) := \sup\{\Re(\lambda) : \lambda \in \sigma(A)\}$$

of the generator of a positive C_0–semigroup $T(\cdot)$ on a complex Banach lattice E. We will see that $s(A)$ is always contained in $\sigma(A)$ provided that $\sigma(A) \neq \emptyset$.
To that purpose the following result is essential.

Theorem 2.3.1 *Let A be the generator of a positive C_0–semigroup $T(\cdot)$ on E. For $\Re(\lambda) > s(A)$ we have*

$$R(\lambda, A)x = \lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)x\, ds, \quad x \in E.$$

Moreover, $\int_0^t e^{-\lambda s} T(s)\, ds$ converges to $R(\lambda, A)$ with respect to the operator norm as $t \to \infty$.

Proof: Let $\lambda_0 > \omega_0(A)$ be fixed. Since $R(\lambda_0, A)x = \int_0^\infty e^{-\lambda_0 t} T(t)x\, dt$ and by the resolvent identity we obtain

$$R(\lambda_0, A)^{n+1}x = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda_0 t} T(t)x\, dt$$

for $n \in \mathbb{N}$ and $x \in E$. Let $\mu \in (s(A), \lambda_0)$, $x \in E_+$ and $x^* \in E_+^*$. By the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]) one has $\frac{1}{\lambda_0 - \mu} > r(R(\lambda_0, A))$ and hence,

$$\begin{aligned}
\langle R(\mu, A)x, x^*\rangle &= \sum_{n=0}^\infty (\lambda_0 - \mu)^n \langle R(\lambda_0, A)^{n+1}x, x^*\rangle \\
&= \sum_{n=0}^\infty \int_0^\infty \frac{1}{n!}[(\lambda_0 - \mu)s]^n e^{-\lambda_0 s}\langle T(s)x, x^*\rangle ds \\
&= \int_0^\infty \left(\sum_{n=0}^\infty \frac{1}{n!}[(\lambda_0 - \mu)s]^n\right) e^{-\lambda_0 s}\langle T(s)x, x^*\rangle ds \\
&= \int_0^\infty e^{(\lambda_0 - \mu)s} e^{-\lambda_0 s}\langle T(s)x, x^*\rangle ds \\
&= \int_0^\infty e^{-\mu s}\langle T(s)x, x^*\rangle ds \\
&= \lim_{t\to\infty}\langle \int_0^t e^{-\mu s} T(s)x\, ds, x^*\rangle.
\end{aligned}$$

Hence, $\left(\int_0^t e^{-\mu s} T(s)x\, ds\right)$ converges weakly to $R(\mu, A)x$ as $t \to \infty$. Since $x \in E_+$, it follows that $\left(\int_0^t e^{-\mu s} T(s)x\, ds\right)_{t\geq 0}$ is monotone increasing and so, by Proposition 1.1.13, we have strong convergence. Thus,

$$\lim_{t\to\infty}\int_0^t e^{-\mu s} T(s)x\, ds = R(\mu, A)x, \quad \text{for all } x \in E.$$

If $\lambda = \mu + i\gamma$ with $\mu, \gamma \in \mathbb{R}$ and $\mu > s(A)$, then for any $x \in E$ and $x^* \in E^*$, we have

$$\left| \langle \int_r^t e^{-\lambda s} T(s) x \, ds, x^* \rangle \right| \leq \int_r^t e^{-\mu s} \langle T(s)|x|, |x^*| \rangle ds.$$

Hence,

$$\left\| \int_r^t e^{-\lambda s} T(s) x \, ds \right\| \leq \left\| \int_r^t e^{-\mu s} T(s)|x| \, ds \right\|,$$

which implies that

$$\lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s) x \, ds \text{ exists for all } x \in E.$$

Then, by [9, Theorem II.1.10],

$$\lambda \in \rho(A) \text{ and } R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t) x \, dt \quad \text{for all } x \in E.$$

It remains to prove that $\left(\int_0^t e^{-\lambda s} T(s)\, ds\right)$ converges in the operator norm as $t \to \infty$. We fix $\mu \in (s(A), \Re(\lambda))$. As we have seen above, the function

$$f_{x,x^*} : s \mapsto e^{-\mu s} \langle T(s)x, x^* \rangle \text{ belongs to } L^1(\mathbb{R}_+) \quad \text{for all } x \in E, x^* \in E^*.$$

It follows from the closed graph theorem that the bilinear form

$$b : E \times E^* \to L^1(\mathbb{R}_+);\ (x, x^*) \mapsto f_{x,x^*}$$

is separately continuous and hence continuous. Thus, there exists $M > 0$ such that

$$\int_0^\infty e^{-\mu s} |\langle T(s)x, x^* \rangle| \, ds \leq M \|x\| \|x^*\|, \quad x \in E, x^* \in E^*.$$

For $0 \leq t < r$ and $\varepsilon := \Re(\lambda) - \mu$ we have

$$\begin{aligned} \left| \int_t^r e^{-\lambda s} \langle T(s)x, x^* \rangle ds \right| &\leq \int_t^r e^{-(\Re(\lambda)-\mu)s} e^{-\mu s} |\langle T(s)x, x^* \rangle| ds \\ &\leq e^{-\varepsilon t} \int_t^r e^{-\mu s} |\langle T(s)x, x^* \rangle| ds \\ &\leq e^{-\varepsilon t} M \|x\| \|x^*\|. \end{aligned}$$

Hence, $\left\| \int_t^r e^{-\lambda s} T(s)\, ds \right\| \leq M e^{-\varepsilon t}$ and this implies that $\left(\int_0^t e^{-\lambda s} T(s)\, ds\right)$ is a Cauchy sequence in $\mathcal{L}(E)$. □

As an immediate consequence we obtain the following corollary.

Corollary 2.3.2 *Let A be the generator of a positive C_0–semigroup $T(\cdot)$ on E. If $\Re(\lambda) > s(A)$, then*

$$|R(\lambda, A)x| \leq R(\Re(\lambda), A)|x| \quad \textit{for all } x \in E.$$

An other interesting corollary is the following.

Corollary 2.3.3 *If A is the generator of a positive C_0–semigroup $T(\cdot)$ on E, then*

$$s(A) \in \sigma(A) \text{ or } s(A) = -\infty.$$

Proof: Assume that $s(A) > -\infty$ and $s(A) \notin \sigma(A)$. So it follows from Corollary 2.3.2 that

$$|R(\lambda,A)x| \leq R(\Re(\lambda),A)|x| \leq R(s(A),A)|x| \quad \text{for all } \Re(\lambda) > s(A), x \in E.$$

Hence the set $\{R(\lambda,A) : \Re(\lambda) > s(A)\}$ is uniformly bounded in $\mathcal{L}(E)$. Let $M := \sup_{\Re(\lambda)>s(A)} \|R(\lambda,A)\|$. Since $\|R(\lambda,A)\| \geq \frac{1}{\operatorname{dist}(\lambda,\sigma(A))}$ for $\lambda \in \rho(A)$ (cf. [9, Corollary IV.1.14]), it follows that

$$\{\lambda \in \mathbb{C} : \Re(\lambda) = s(A)\} \subseteq \rho(A) \text{ and } \|R(\lambda,A)\| \leq M, \forall \Re(\lambda) = s(A).$$

Thus,

$$\{\lambda \in \mathbb{C} : |\Re(\lambda) - s(A)| < M^{-1}\} \subseteq \rho(A).$$

This contradicts the definition of $s(A)$. □

The following consequence gives a relation between $s(A)$ and the positivity of the resolvent.

Corollary 2.3.4 *Suppose that A generates a positive on E and $\lambda_0 \in \rho(A)$. Then the following assertions hold.*

(i) *$R(\lambda_0,A)$ is positive if and only if $\lambda_0 > s(A)$.*

(ii) *If $\lambda > s(A)$, then $r(R(\lambda,A)) = \frac{1}{\lambda - s(A)}$.*

Proof: (ii) is a simple consequence from Corollary 2.3.3 and the spectral mapping theorem for the resolvent (cf. [9, Theorem IV.1.13]).
(i) Assume first that $R(\lambda_0,A) \geq 0$. Since $Ag \in E_{\mathbb{R}}$ for all $0 \leq g \in D(A)$, we have $\lambda_0 \in \mathbb{R}$. On the other hand, Theorem 2.3.1 implies that $R(\lambda,A) \geq 0$ for all $\lambda > \max(\lambda_0, s(A))$ and hence

$$\begin{aligned} R(\lambda_0,A) &= R(\lambda,A) + (\lambda - \lambda_0)R(\lambda,A)R(\lambda_0,A) \\ &\geq R(\lambda,A) \geq 0 \end{aligned}$$

for all $\lambda > \max(\lambda_0, s(A))$. Therefore,

$$(\lambda - s(A))^{-1} = r(R(\lambda,A)) \leq \|R(\lambda,A)\| \leq \|R(\lambda_0,A)\|$$

for all $\lambda > \max(\lambda_0, s(A))$. But this is only true if $\lambda_0 > s(A)$.
The converse follows from Theorem 2.3.1. □

Remark 2.3.5 *(a) As an immediate consequence of Corollary 2.3.4 we obtain*

$$s(A) = \inf\{\lambda \in \rho(A) : R(\lambda, A) \geq 0\}$$

for the generator A of a positive C_0–semigroup on a Banach lattice E.

(b) If $E := C(K)$, K compact, then $s(A) > -\infty$. In fact: We know from the theory of C_0–semigroups that $\lim_{\lambda\to\infty} \lambda R(\lambda, A) f = f$ for all $f \in E$. In particular we find $\lambda_0 \in \mathbb{R}$ sufficiently large such that

$$\lambda_0 R(\lambda_0, A)\,\mathbb{1} \geq \frac{1}{2}\mathbb{1},$$

where $\mathbb{1}(x) := 1$ for all $x \in K$. Since $R(\lambda_0, A) \geq 0$, it follows that

$$R(\lambda_0, A)^n\,\mathbb{1} \geq \frac{1}{(2\lambda_0)^n}\mathbb{1} \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$r(R(\lambda_0, A)) = \lim_{n\to\infty} \|R(\lambda_0, A)^n\|^{\frac{1}{n}} \geq \frac{1}{2\lambda_0} > 0$$

and hence $\sigma(A) \neq \emptyset$.

The spectrum of a generator of a positive C_0–semigroup can be empty as the following examples show.

Example 2.3.6 (a) *On $E := C_0[0,1) := \{f \in C[0,1] : f(1) = 0\}$ we consider the nilpotent C_0–semigroup $T(\cdot)$ given by*

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t < 1 \\ 0 & \text{if } x+t \geq 1 \end{cases}$$

for $t \geq 0$, $x \in [0,1]$ and $f \in E$. Then, $T(t) = 0$ for $t \geq 1$ and hence $\sigma(T(t)) = \{0\}$. So by the spectral inclusion theorem (cf. [9, Theorem IV.3.6]), $\sigma(A) = \emptyset$.

(b) *Let $E := C_0[0,\infty) := \{f \in C(\mathbb{R}_+) : \lim_{t\to+\infty} f(t) = 0\}$. On E, we define the C_0–semigroup $T(\cdot)$ by*

$$(T(t)f)(x) := e^{-\frac{t^2}{2} - xt} f(x+t), \quad x, t \geq 0 \text{ and } f \in E.$$

Then, one can see that the generator A of $T(\cdot)$ on E is given by

$$(Af)(x) = f'(x) - xf(x),\ x \geq 0, \text{ and}$$
$$f \in D(A) = \{f \in E : f \in C^1(\mathbb{R}_+) \text{ and } Af \in E\}.$$

By a simple computation one obtains that $\sigma(A) = \emptyset$.

For generators of positive C_0–groups the spectrum is always nonempty. This is given by the following corollary.

Corollary 2.3.7 *If A generates a positive C_0–group on a Banach lattice E, then $\sigma(A) \neq \emptyset$.*

Proof: Assume that $\sigma(A) = \emptyset$. By Theorem 2.3.1 we have $R(\lambda, A) \geq 0$ for all $\lambda \in \mathbb{R}$. Again, one can apply the same theorem to $-A$ and obtains $R(\lambda, -A) \geq 0$ for all $\lambda \in \mathbb{R}$. But $R(\lambda, -A) = -R(-\lambda, A) \leq 0$ for all $\lambda \in \mathbb{R}$, and hence, $R(\lambda, -A) = 0$ for all $\lambda \in \mathbb{R}$. This contradicts the fact that $E \neq \{0\}$. □

2.4 The problem $\omega_0(A) = s(A)$ for positive semigroups

In this section we study in detail the growth bound $\omega_0(A)$ of the generator A of a positive C_0–semigroup on a Banach lattice E. In particular, we look for sufficient conditions implying the equality $\omega_0(A) = s(A)$ without supposing the spectral mapping theorem.

For a C_0–semigroup $S(\cdot)$ with generator B on a Banach space X satisfying $||S(t)|| \leq Me^{\omega t}$, $t \geq 0$, for some constants $M, \omega \in \mathbb{R}$, it follows that $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\} \subseteq \rho(B)$. Thus,

$$s(B) \leq \omega_0(B)$$

is always satisfied.

By applying the Gearhardt-Pruess's theorem and Theorem 1.2.2 we obtain the first result on the opposite inequality.

Theorem 2.4.1 *Let A be the generator of a positive C_0–semigroup $T(\cdot)$ on a Banach lattice E. Then $\omega_0(A) = s(A)$ holds in the followings cases.*

(i) *E is a Hilbert space.*

(ii) *E is an AL-space.*

(iii) *$E := C_0(\Omega)$ or $E := C(K)$, where Ω is locally compact Hausdorff and K is compact Hausdorff.*

Proof: (i) Let $\mu > s(A)$ fixed. It follows from Corollary 2.3.2 that $\Lambda := \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \subseteq \rho(A - \mu)$ and

$$||R(\lambda, A - \mu)|| \leq ||R(\Re(\lambda), A - \mu)|| \leq ||R(\mu, A)|| \quad \text{for all } \lambda \in \Lambda.$$

So, by Theorem 2.1.5, we have $\omega_0(A) - \mu < 0$ and hence,

$$\omega_0(A) \leq s(A).$$

(ii) For $\lambda > s(A)$ and $x \in E_+$ we obtain from Theorem 2.3.1 that

$$||R(\lambda, A)x|| = \left\|\int_0^\infty e^{-\lambda s} T(s)x\, ds\right\| = \int_0^\infty e^{-\lambda s} ||T(s)x||\, ds,$$

where the second equality follows from the fact that the norm is additive on the positive cone. Hence,

$$\int_0^\infty ||(e^{-\lambda s}T(s))x||\,ds < \infty \quad \text{for all } x \in E.$$

So, by Theorem 2.1.4, we have $\omega_0(A) - \lambda < 0$ and thus

$$\omega_0(A) \leq s(A).$$

(iii) It is easy to see that $||f \vee g|| = ||f|| \vee ||g||$ for all $f, g \in E_+$. Then, for $\gamma, \nu \in E_+^*$, we have

$$\begin{aligned} \langle f,\gamma\rangle + \langle g,\nu\rangle &\leq \langle f\vee g, \gamma+\nu\rangle \\ &\leq ||\gamma+\nu||\,||f\vee g|| \\ &= ||\gamma+\nu||(||f||\vee||g||), \quad f,g \in E_+. \end{aligned}$$

Hence, $\langle f,\gamma\rangle + \langle g,\nu\rangle \leq ||\gamma+\nu||$ for all $f, g \in E_+$ with $||f|| = ||g|| = 1$. It follows from the Hahn-Banach theorem that $||\gamma|| + ||\nu|| \leq ||\gamma+\nu||$ and hence,

$$||\gamma|| + ||\nu|| = ||\gamma+\nu||, \quad \gamma, \nu \in E_+.$$

This implies that E^* is an AL-space. If we set $F := \overline{D(A^*)}$, then it follows from Theorem 1.2.2 that F is a closed ideal and hence also an AL-space. On F we consider the positive C_0–semigroup $S(\cdot)$ given by

$$S(t) := T(t)^*_{|F} \quad \text{for } t \geq 0,$$

and we denote by B its generator. Then B is the part of A^* in F, i.e.,

$$D(B) = \{\nu \in D(A^*) : A^*\nu \in F\} \text{ and } B\nu = A^*\nu \text{ for } \nu \in D(B).$$

Moreover, one can show that

$$\sigma(B) = \sigma(A^*) = \sigma(A).$$

Consequently, $s(B) = s(A)$ holds. Since B is the generator of the positive C_0–semigroup $S(\cdot)$ on the AL-space F, it follows from (ii) that $s(B) = \omega_0(B)$. Now, it suffices to prove that $\omega_0(B) = \omega_0(A)$. The inequality $\omega_0(B) \leq \omega_0(A)$ is trivial. Let $\omega > \omega_0(B)$, $f \in E$ and $\nu \in F$. Then we have

$$|\langle T(t)f,\nu\rangle| = |\langle f, S(t)\nu\rangle| \leq M||f||e^{\omega t}||\nu||$$

for $t \geq 0$ and some constant $M \geq 1$. On the other hand, since $f = \lim_{\lambda\to\infty} \lambda R(\lambda,A)f$ for all $f \in E$, we have $c := \limsup_{\lambda\to\infty} \lambda||R(\lambda,A)|| < \infty$. Therefore,

$$\begin{aligned} |\langle T(t)f,\gamma\rangle| &= \lim_{\lambda\to\infty} |\langle \lambda R(\lambda,A)T(t)f,\gamma\rangle| \\ &= \lim_{\lambda\to\infty} |\langle T(t)f, \lambda R(\lambda,A^*)\gamma\rangle| \\ &\leq M||f||e^{\omega t} \limsup_{\lambda\to\infty} \lambda ||R(\lambda,A)^*\gamma|| \\ &\leq Mce^{\omega t}||f||\,||\gamma||, \quad \gamma \in E^*. \end{aligned}$$

Consequently, $\|T(t)\| \leq Mce^{\omega t}$ for all $t \geq 0$ and hence $\omega_0(A) \leq \omega$ for all $\omega > \omega_0(B)$. Thus, we have shown that

$$\omega_0(B) = \omega_0(A).$$

□

The last result of this section is Weis's result concerning positive C_0–semigroups on $L^p(\Omega) := L^p(\Omega,\mu)$, $1 \leq p < \infty$, where (Ω,μ) a σ–finite measure space (see [33]). The proof presented here is due to W. Arendt (see [2, Theorem 5.3.6]).

We first need some preparations. We equip $\mathbb{R} \times \Omega$ with the product measure $\lambda_1 \otimes \mu$, where λ_1 is the Lebesgue measure on $\mathbb{R}$. We recall that $L^p(\mathbb{R} \times \Omega) \cong L^p(\mathbb{R}, L^p(\Omega)$. This allows us to identify the notations $g(t,\xi)$ and $g(t)(\xi)$ for $(t,\xi) \in \mathbb{R} \times \Omega$. Let us consider the non-linear map

$$\Phi : L^p(\mathbb{R}, L^p(\Omega)) \to L^p(\Omega); g \mapsto \Phi(g) := \left(\int_{\mathbb{R}} |g(t)|^p \, dt \right)^{\frac{1}{p}}.$$

It is clear that Φ is well-defined.

The following lemmas give some properties of the map Φ.

Lemma 2.4.2 *Let $g,h \in L^p(\mathbb{R}, L^p(\Omega))$, $f \in L^\infty(\Omega)$, and $s \in \mathbb{R}$. Then the following assertions hold.*

1. $\|\Phi(g)\|_{L^p(\Omega)} = \|g\|_{L^p(\mathbb{R}\times\Omega)}$.
2. $\Phi(g_s) = \Phi(g)$, *where* $g_s(t) := g(s+t)$, $t,s \in \mathbb{R}$.
3. $\Phi(f \cdot g) = |f|\Phi(g)$, *where* $(f \cdot g)(t,\xi) := f(\xi)g(t,\xi)$, $(t,\xi) \in \mathbb{R} \times \Omega$.
4. $\Phi(g+h) \leq \Phi(g) + \Phi(h)$.
5. Φ *is a continuous map.*

Proof: Assertions 1., 2. and 3. are simple to prove. For 4. we set $G_\xi(t) := g(t,\xi)$, $H_\xi(t) := h(t,\xi)$, $(t,\xi) \in \mathbb{R} \times \Omega$. For almost all $\xi \in \Omega$, we obtain $G_\xi, H_\xi \in L^p(\mathbb{R})$ and hence

$$\|G_\xi + H_\xi\|_{L^p(\mathbb{R})} \leq \|G_\xi\|_{L^p(\mathbb{R})} + \|H_\xi\|_{L^p(\mathbb{R})}.$$

Since $\|G_\xi\|_{L^p(\mathbb{R})} = (\int_{\mathbb{R}} |g(t,\xi)|^p \, dt)^{\frac{1}{p}} = \Phi(g)(\xi)$ and also $\|H_\xi\|_{L^p(\mathbb{R})} = \Phi(h)(\xi)$, it follows that

$$\Phi(g+h)(\xi) \leq \Phi(g)(\xi) + \Phi(h)(\xi), \quad \mu\text{-a.e. } \xi \in \Omega.$$

Thus, $\Phi(g+h) \leq \Phi(g) + \Phi(h)$.
By 4. we have

$$\Phi(g) \leq \Phi(g-h) + \Phi(h) \text{ and } \Phi(h) \leq \Phi(h-g) + \Phi(g).$$

This implies that $|\Phi(g)-\Phi(h)| \leq \Phi(g-h)$ and so by 1. we obtain

$$||\Phi(g)-\Phi(h)||_{L^p(\Omega)} \leq ||g-h||_{L^p(\mathbb{R}\times\Omega)},$$

which proves 5.. □

Lemma 2.4.3 *For a continuous function $G : [a,b] \to L^p(\mathbb{R}, L^p(\Omega))$ we have*

$$\Phi\left(\int_a^b G(s)\, ds\right) \leq \int_a^b \Phi(G(s))\, ds.$$

Proof: It follows from Lemma 2.4.2 that

$$\Phi\left(\frac{b-a}{2^n}\sum_{j=0}^{2^n-1} G\left(\frac{jb+(2^n-j)a}{2^n}\right)\right) \leq \frac{b-a}{2^n}\sum_{j=0}^{2^n-1}\Phi\left(G\left(\frac{jb+(2^n-j)a}{2^n}\right)\right).$$

Since Φ is continuous, we obtain the lemma by letting $n \to \infty$. □

Let $g \in L^p(\mathbb{R}, L^p(\Omega))$ and $T \in \mathcal{L}(L^p(\Omega))$. We consider $T\circ g$ defined by

$$(T\circ g)(t) := T(g(t)), \quad t \in \mathbb{R}.$$

Lemma 2.4.4 *For $0 \leq T \in \mathcal{L}(L^p(\Omega))$ and $0 \leq g \in L^p(\mathbb{R}, L^p(\Omega))$ the inequality*

$$\Phi(T\circ g) \leq T(\Phi(g))$$

holds.

Proof: By Lemma 2.4.2, it suffices to prove the lemma for simple functions. Let $g := \sum_{k=1}^n \chi_{A_k} \otimes g_k$, where $A_1,\ldots,A_n$ are disjoint Borel subsets of $\mathbb{R}$, and $g_1,\ldots,g_n \in L^p(\Omega)_+$. Setting $h_k := \lambda_1(A_k)^{\frac{1}{p}} g_k$ for $k \in \{1,\ldots,n\}$. Since the sets (A_k) are disjoint, it follows that

$$\Phi(T\circ g) = \left(\sum_{k=1}^n \lambda_1(A_k)(Tg)^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^n (Th_k)^p\right)^{\frac{1}{p}},$$

$$T(\Phi(g)) = T\left(\sum_{k=1}^n \lambda_1(A_k)(g_k)^p\right)^{\frac{1}{p}} = T\left(\sum_{k=1}^n (h_k)^p\right)^{\frac{1}{p}}.$$

Let $\alpha := (\alpha_k)_k \subset \mathbb{R}$ with $||\alpha||_{l^q} \leq 1$, where $\frac{1}{q}+\frac{1}{p}=1$. The Hölder inequality implies

$$\left(\sum_{k=1}^n \alpha_k h_k\right) \leq \left(\sum_{k=1}^n |h_k|^p\right)^{\frac{1}{p}} = \Phi(g),$$

hence

$$\left(\sum_{k=1}^n \alpha_k T h_k\right) = T\left(\sum_{k=1}^n \alpha_k h_k\right) \leq T(\Phi(g)).$$

Consequently,

$$\begin{aligned}\left(\sum_{k=1}^{n} |(Th_k)(\xi)|^p\right)^{\frac{1}{p}} &= \sup\left\{\left(\sum_{k=1}^{n} \alpha_k (Th_k)(\xi)\right) : \alpha_k \in \mathbb{R},\ \|(\alpha_k)\|_{l^q} \le 1\right\} \\ &\le T(\Phi(g))(\xi), \quad \mu\text{–a.e.}\ \xi \in \Omega,\end{aligned}$$

and $\Phi(T \circ g) \le T(\Phi(g))$. □

We are now ready to prove Weis's result.

Theorem 2.4.5 *Let (Ω,μ) be a σ–finite measure space, $1 \le p < \infty$, and $T(\cdot)$ a positive C_0–semigroup on $L^p(\Omega)$ with generator A. Then $\omega_0(A) = s(A)$.*

Proof: For $\xi > s(A)$ we set $T_\xi(t) := e^{-\xi t}T(t),\ t \ge 0$. We denote by $A_\xi := A - \xi$ the generator of the positive C_0–semigroup $T_\xi(\cdot)$ on $L^p(\Omega)$. Then $s(A_\xi) = s(A) - \xi < 0$. Let $\alpha > \max(0, \omega_0(A_\xi))$ fixed. Let $f \in L^p(\Omega)$ and consider the function $g \in L^p(\mathbb{R}, L^p(\Omega))$ defined by

$$g(t) = \begin{cases} e^{-\alpha t}T_\xi(t)f, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

We now introduce the function

$$G : \mathbb{R}_+ \to L^p(\mathbb{R}, L^p(\Omega));\ s \mapsto G(s) := T_\xi(s) \circ g_{-s},$$

where $g_{-s}(t) := g(t-s),\ t \in \mathbb{R}$. Hence,

$$G(s)(t) = \begin{cases} e^{-\alpha(t-s)}T_\xi(t)f, & 0 \le s \le t, \\ 0, & t < s. \end{cases}$$

Thus,

$$\begin{aligned}\Phi\left(\int_0^m G(s)\,ds\right) &= \left(\int_0^\infty \left|\int_0^{\min(m,t)} e^{-\alpha(t-s)}T_\xi(t)f\,ds\right|^p dt\right)^{\frac{1}{p}} \\ &= \frac{1}{\alpha}\left(\int_0^\infty (e^{-\alpha\max(0,t-m)} - e^{-\alpha t})^p |T_\xi(t)f|^p\,dt\right)^{\frac{1}{p}}\end{aligned}$$

and hence

$$0 \le \frac{1}{\alpha}\left(\int_0^\infty (e^{-\alpha\max(0,t-m)} - e^{-\alpha t})^p |T_\xi(t)f|^p\,dt\right)^{\frac{1}{p}} = \Phi\left(\int_0^m G(s)\,ds\right). \quad (2.3)$$

So, by Lemmas 2.4.3, 2.4.4, and 2.4.2, it follows that

$$\begin{aligned} 0 &\le \Phi\left(\int_0^m G(s)\,ds\right) \\ &\le \int_0^m \Phi(G(s))\,ds \\ &= \int_0^m \Phi(T_\xi(s)\circ g_{-s})\,ds \\ &\le \int_0^m T_\xi(s)(\Phi(g_{-s}))\,ds \\ &= \int_0^m T_\xi(s)(\Phi(g))\,ds. \end{aligned}$$

On the other hand, since $s(A_\xi) < 0$ and from Theorem 2.3.1, it follows that

$$\lim_{m\to\infty}\int_0^m T_\xi(s)(\Phi(g))\,ds = R(0,A_\xi)(\Phi(g)).$$

From (2.3) and the monotone convergence theorem we have

$$0 \le \frac{1}{\alpha}\left(\int_0^\infty (1-e^{-\alpha t})^p |T_\xi(t)f|^p\,dt\right)^{\frac{1}{p}} \le R(0,A_\xi)(\Phi(g)).$$

This implies

$$\left(\frac{1-e^{-\alpha}}{\alpha}\right)\left(\int_1^\infty |T_\xi(t)f|^p\,dt\right)^{\frac{1}{p}} \le R(0,A_\xi)(\Phi(g))$$

and therefore

$$\int_\Omega\int_1^\infty |(T_\xi(t)f)(y)|^p\,dt\,d\mu(y) \le \left(\frac{\alpha}{1-e^{-\alpha}}\right)^p \|R(0,A_\xi)\|^p\|\Phi(g)\|^p_{L^p(\Omega)},$$

which implies that

$$\int_1^\infty \|T_\xi(t)f\|^p_{L^p(\Omega)}\,dt < \infty.$$

So, by Theorem 2.1.4, we obtain $\omega_0(A_\xi) = \omega_0(A) - \xi < 0$. Consequently,

$$\omega_0(A) \le s(A).$$

□

2.5 IRREDUCIBLE SEMIGROUPS

In many concrete examples the semigroup $T(\cdot)$ does not have exponential stability, however possesses an *asynchronous exponential growth*. This means that there is a rank one projection P and constants $\varepsilon > 0$, $M \ge 1$ such that

$$\|e^{-s(A)t}T(t) - P\| \le Me^{-\varepsilon t} \qquad \text{for all } t \ge 0,$$

where A denotes the generator of $T(\cdot)$.

In order to study such kind of behaviour we introduce the concept of irreducibility for positive C_0–semigroups. For more details see [22] and the references therein.

Definition 2.5.1 *A positive C_0–semigroup $T(\cdot)$ on a Banach lattice E with generator A is called irreducible if one of the following equivalent properties is satisfied*

(i) *There is no $T(t)$–invariant closed ideal other than $\{0\}$ and E for all $t > 0$.*

(ii) *For $x \in E, x^* \in E^*$ with $x \gneq 0$ and $x^* > 0$, there is $t_0 > 0$ such that*

$$\langle T(t_0)x, x^* \rangle > 0.$$

(iii) *For some (and then for every) $\lambda > s(A)$, there is no $R(\lambda, A)$–invariant closed ideal except $\{0\}$ and E.*

(iv) *For some (and then for every) $\lambda > s(A)$, $R(\lambda, A)x$ is a quasi-interior point of E_+ for every $x \gneq 0$.*

Example 2.5.2 (a) *Let $E := L^p(\Omega, \mu)$, $1 \le p < \infty$, and $T(\cdot)$ be a positive C_0–semigroup on E with generator A. Then, it follows from Example 1.1.7 that $T(\cdot)$ is irreducible if and only if*

$$0 \lneq f \in E \Longrightarrow (R(\lambda, A)f)(s) > 0 \text{ for a.e. } s \in \Omega \text{ and some } \lambda > s(A).$$

(b) *If $E := C_0(\Omega)$, where Ω is locally compact Hausdorff, and $T(\cdot)$ a positive C_0–semigroup on E with generator A, then, by Example 1.1.7, $T(\cdot)$ is irreducible if and only if*

$$0 \lneq f \in E \Longrightarrow (R(\lambda, A)f)(s) > 0 \text{ for all } s \in \Omega \text{ and some } \lambda > s(A).$$

We now state some consequences of irreducibility.

Proposition 2.5.3 *Assume that A is the generator of an irreducible C_0–semigroup $T(\cdot)$ on a Banach lattice E. Then the following assertions hold.*

(a) *Every positive eigenvector of A is a quasi-interior point.*

(b) *Every positive eigenvector of A^* is strictly positive.*

(c) *If $\ker(s(A) - A^*)$ contains a positive element, then $\dim\ker(s(A) - A) \le 1$.*

(d) *If $s(A)$ is a pole of the resolvent, then it has algebraic (and geometric) multiplicity equal to 1. The corresponding residue has the form $P_{s(A)} = u^* \otimes x$, where $x \in E$ is a positive eigenvector of A, $u^* \in E^*$ is a positive eigenvector of A^* and $\langle x, u^* \rangle = 1$.*

Proof: (a) Let x be a positive eigenvector of A and $E_x := \cup_{n\in\mathbb{N}} n[-x,x]$ the ideal generated by x. If λ is such that $Ax = \lambda x$, then $\lambda \in \mathbb{R}$. This follows from

$$x \geq 0 \text{ and } Ax = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x).$$

Hence, $T(t)x = e^{\lambda t}x$ for $t \geq 0$. Thus, for $y \in E_x$,

$$|T(t)y| \leq T(t)|y| \leq nT(t)x = ne^{\lambda t}x, \quad t \geq 0.$$

Consequently, $T(t)E_x \subseteq E_x$ holds for all $t \geq 0$. Since $0 \neq x \in E_x$ and $T(\cdot)$ is irreducible, it follows that $\overline{E_x} = E$.

(b) Let x^* be a positive eigenvector of A^* and λ its corresponding eigenvalue. By the same argument we have $\lambda \in \mathbb{R}$ and $T(t)^*x^* = e^{\lambda t}x^*$ for $t \geq 0$. Hence,

$$\langle |T(t)u|, x^*\rangle \leq \langle T(t)|u|, x^*\rangle = \langle |u|, e^{\lambda t}x^*\rangle, \quad u \in E, t \geq 0.$$

Thus, $I := \{u \in E : \langle |u|, x^*\rangle = 0\}$ is a $T(t)$–invariant closed ideal for all $t \geq 0$. Since $x^* \neq 0$ we have $I \subsetneq E$ and so by the irreducibility we obtain $I = \{0\}$. Therefore, $x^* > 0$.

(c) Let $0 \lneqq x^* \in \ker(s(A) - A^*)$. It follows from (b) that x^* is strictly positive. For $x \in \ker(s(A) - A)$ we have $T_{-s(A)}(t)x = x$ and hence,

$$|x| = |T_{-s(A)}(t)x| \leq T_{-s(A)}(t)|x|, \quad t \geq 0.$$

Thus, for $t \geq 0$,

$$\begin{aligned} \langle |x|, x^*\rangle &\leq \langle T_{-s(A)}(t)|x|, x^*\rangle \\ &= \langle |x|, x^*\rangle. \end{aligned}$$

This implies that $\langle T_{-s(A)}(t)|x| - |x|, x^*\rangle = 0$, and since $x^* > 0$, we obtain $T_{-s(A)}(t)|x| = |x|$ for $t \geq 0$. Therefore,

$$|x| \in \ker(s(A) - A).$$

Since $\left(T_{-s(A)}(t)x\right)^+ \leq T_{-s(A)}(t)x^+$, one can see by the same arguments as above that $x^+ \in \ker(s(A) - A)$ and $x^- \in \ker(s(A) - A)$. This implies that $F := E_{\mathbb{R}} \cap \ker(s(A) - A)$ is a real sublattice of E. For $x \in F$ we consider the ideal E_{x^+} (resp. E_{x^-}) generated by x^+ (resp. x^-). Then, E_{x^+} and E_{x^-} are $T_{-s(A)}(t)$–invariant for all $t \geq 0$. Since E_{x^+} and E_{x^-} are orthogonal, it follows from the irreducibility of $T_{-s(A)}(\cdot)$ that $x^+ = 0$ or $x^- = 0$. Consequently, F is totally ordered. So by Lemma 1.1.14 we have

$$\dim F = \dim\ker(s(A) - A) \leq 1.$$

(d) We claim that if $s(A)$ is a pole of the resolvent, then there is an eigenvector $0 \lneqq x \in E$ of A corresponding to $s(A)$. Indeed, let k be the order of the pole $s(A)$ and $R_{-k} = \lim_{\lambda\to s(A)^+}(\lambda - s(A))^k R(\lambda, A)$ the corresponding residue. Then, $R_{-k} \neq 0$ and $R_{-(k+1)} = 0$. Moreover, by Corollary 2.3.4, we have $R_{-k} \geq 0$. Hence, there is

$0 \le y \in E$ with $x := R_{-k}y \gneq 0$. By the relation $R_{-(k+1)} = (A - s(A))R_{-k} = 0$ we obtain $(A - s(A))x = 0$. This proves the claim.

We can now use (a) to obtain $\overline{E_x} = E$. By taking the adjoint $R^*_{-(k+1)}$ of $R_{-(k+1)}$ and by the same computation as before one has , if $s(A)$ is a pole of the resolvent, then there is $0 \lneq x^* \in \ker(s(A) - A^*)$. So by (c) we have $\dim\ker(s(A) - A) = 1$.

Now, assume that $k \ge 2$. Then we have

$$\begin{aligned}\langle x, x^*\rangle &= \langle R_{-k}y, x^*\rangle \\ &= \langle y, R^*_{-k}x^*\rangle \\ &= \langle y, R^*_{-(k-1)}(A^* - s(A))x^*\rangle \\ &= 0.\end{aligned}$$

Since $\overline{E_x} = E$, it follows that $\langle u, x^*\rangle = 0$ for all $u \in E_+$. This contradicts the assertion (b). Hence $k = 1$. From the inequality $m_g + k - 1 \le m_a \le m_g k$ (cf. [9] p. 247) we obtain

$$m_a = m_g = \dim P_{s(A)}E = \dim\ker(s(A) - A) = 1,$$

where we recall that $P_{s(A)} = R_{-1}$. Since $P_{s(A)}E \subseteq \ker(s(A) - A)$, it follows that

$$P_{s(A)}E = \ker(s(A) - A).$$

We now show the last part of Assertion (d). To this purpose let $0 \lneq x \in \ker(s(A) - A)$. Without loss of generality, we suppose that $\|x\| = 1$. Then $P_{s(A)}E = \mathrm{Span}\{x\}$, i.e. $P_{s(A)}y = \lambda x$ for some $\lambda \in \mathbb{C}$ and every $y \in E$. By the Hahn-Banach theorem (see Proposition 1.1.12) there exists $0 \le y^* \in (\ker(s(A) - A))^*$ with $\|y^*\| = 1$ and $\langle x, y^*\rangle = \|x\| = 1$. Hence $\langle P_{s(A)}y, y^*\rangle = \lambda = \langle y, P^*_{s(A)}y^*\rangle$. If we put $u^* := P^*_{s(A)}y^* \ge 0$, then $P_{s(A)} = u^* \otimes x$ and $\langle x, u^*\rangle = \langle P_{s(A)}x, y^*\rangle = \langle x, y^*\rangle = 1$. This implies that $0 \lneq u^* \in P^*_{s(A)}E^* \subseteq \ker(s(A) - A^*)$. So $u^* > 0$ by (b). This ends the proof of the proposition. ☐

The following result describes the eigenvalues of an irreducible semigroup which are contained in the boundary spectrum $\sigma_b(A) := \{\lambda \in \sigma(A) : \Re(\lambda) = s(A)\}$, where A is the corresponding generator.

Theorem 2.5.4 *Let $T(\cdot)$ be an irreducible C_0–semigroup with generator A on a Banach lattice E. Assume that $s(A) = 0$ and there is $0 \lneq x^* \in D(A^*)$ with $A^*x^* = 0$. If $\sigma_p(A) \cap i\mathbb{R} \neq \emptyset$, then the following assertions hold.*

(a) *For $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$ with $Ah = i\alpha h$, $|h|$ is a quasi-interior point and*

$$S_h(D(A)) = D(A) \text{ and } S_h^{-1}AS_h = A + i\alpha$$

hold, where S_h is the signum operator.

(b) $\dim\ker(\lambda - A) = 1$ *for every $\lambda \in \sigma_p(A) \cap i\mathbb{R}$.*

(c) $\sigma_p(A) \cap i\mathbb{R}$ *is an additive subgroup of $i\mathbb{R}$.*

(d) 0 *is the only eigenvalue of A admitting a positive eigenvector.*

Proof: We first remark that by Proposition 2.5.3.(b) we have $x^* > 0$ and $T(t)^* x^* = x^*$ for all $t \geq 0$.

(a) Assume that $Ah = i\alpha h$ for $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$. Then $T(t)h = e^{i\alpha t}h$ and hence $|h| = |T(t)h| \leq T(t)|h|$. This implies that

$$T(t)|h| - |h| \geq 0 \quad \text{for all } t \geq 0.$$

On the other hand,

$$\begin{aligned}\langle T(t)|h| - |h|, x^* \rangle &= \langle |h|, T(t)^* x^* \rangle - \langle |h|, x^* \rangle \\ &= 0 \quad \text{for all } t \geq 0.\end{aligned}$$

Since $x^* > 0$, we obtain $T(t)|h| = |h|$ for all $t \geq 0$, which implies that $A|h| = 0$. So, by Proposition 2.5.3.(a), $|h|$ is a quasi-interior point. If we set $T_\alpha(t) := e^{-i\alpha t}T(t), t \geq 0$, then $T(t)$ and $T_\alpha(t)$ satisfy the assumptions of Lemma 1.2.5 and hence

$$T(t) = S_h^{-1} T_\alpha(t) S_h, \quad t \geq 0.$$

Therefore, $S_h(D(A)) = D(A)$ and $A = S_h^{-1}(A - i\alpha)S_h$ and (a) is proved.

(b) It follows from (a) that $S_h : \ker(i\alpha + A) \to \ker A$ for $i\alpha \in \sigma_p(A) \cap i\mathbb{R}$. On the other hand, the proof of (a) implies that $\ker A \neq \{0\}$. So, by Proposition 2.5.3.(c), $\dim\ker A = 1$ and hence $\dim\ker(i\alpha + A) = 1$.

(c): Let $0 \neq h, g \in D(A)$, $\alpha, \beta \in \mathbb{R}$ such that $Ah = i\alpha h$ and $Ag = i\beta g$. By (a) we have

$$S_g^{-1} A S_g = A + i\beta \text{ and } S_h A S_h^{-1} = A - i\alpha.$$

Thus $A + i(\beta - \alpha) = S_h(A + i\beta)S_h^{-1} = S_h S_g^{-1} A S_g S_h^{-1}$ which implies that $\ker(A + i(\beta - \alpha)) = S_h S_g^{-1} \ker A \neq \{0\}$. Therefore

$$i(\beta - \alpha) \in \sigma_p(A).$$

(d): If $Ax = \lambda x$, where $0 \lneq x \in D(A)$, then

$$\lambda \langle x, x^* \rangle = \langle Ax, x^* \rangle = \langle x, A^* x^* \rangle = 0.$$

Since $x^* > 0$, it follows that $\langle x, x^* \rangle > 0$. Hence, $\lambda = 0$. □

For irreducible semigroups we obtain the following description of the boundary spectrum.

Theorem 2.5.5 *Let $T(\cdot)$ be an irreducible C_0–semigroup with generator A on a Banach lattice E and assume that $s(A)$ is a pole of the resolvent. Then there is $\alpha \geq 0$ such that*

$$\sigma_b(A) = s(A) + i\alpha\mathbb{Z}.$$

Moreover, $\sigma_b(A)$ contains only algebraically simple poles.

Proof: Without loss of generality we suppose that $s(A) = 0$. It can be shown that $\sigma_b(A) \subseteq \sigma_p(A)$. The proof uses pseudo-resolvents on a suitable $\mathcal{F}$–product of E, where $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ which is finer than the Frechet filter (see [22], p. 314). Hence, $\sigma_b(A) = \sigma_p(A) \cap i\mathbb{R}$. By Proposition 2.5.3.(d) we obtain the existence of a positive eigenvector $x^* \in D(A^*)$ corresponding to the eigenvalue $s(A) = 0$. It follows from Theorem 2.5.4.(c) that $\sigma_b(A)$ is a subgroup of $(i\mathbb{R}, +)$. Since $\sigma_b(A)$ is closed and $s(A) = 0$ is an isolated point, we have

$$\sigma_b(A) = i\alpha\mathbb{Z} \quad \text{for some } \alpha \geq 0.$$

Proposition 2.5.3.(d) implies that 0 is a simple pole and by Theorem 2.5.4.(a) we have, for $\lambda \in \rho(A)$,

$$R(\lambda + ik\alpha, A) = S_h^k R(\lambda, A) S_h^{-k} \quad \text{for all } k \in \mathbb{Z}.$$

Therefore, $ik\alpha$ is a simple pole for each $k \in \mathbb{Z}$. This ends the proof of the theorem. □

We now give sufficient conditions for a C_0–semigroup to possess an asynchronous exponential growth. This result will be very useful for many applications.

Theorem 2.5.6 *Let $T(\cdot)$ be an irreducible C_0–semigroup with generator A on a Banach lattice E. If $\omega_{ess}(A) < \omega_0(A)$, then there exists a quasi-interior point $0 \leq x \in E$, $0 < x^* \in E^*$ with $\langle x, x^* \rangle = 1$ such that*

$$\|e^{-s(A)t} T(t) - x^* \otimes x\| \leq M e^{-\varepsilon t} \quad \textit{for all } t \geq 0,$$

and appropriate constants $M \geq 1$ and $\varepsilon > 0$.

Proof: We first remark first that the rescaled semigroup $T_{-\omega_0}(t) := e^{-\omega_0(A)t} T(t)$, for $t \geq 0$, satisfies $\omega_{ess}(A_{-\omega_0}) = \omega_{ess}(A) - \omega_0(A) < 0$, where $A_{-\omega_0} := A - \omega_0(A)$ denotes its generator. Thus, $T_{-\omega_0}(\cdot)$ is quasi-compact and, by Proposition 2.2.2, we have

$$s(A) = \omega_0(A).$$

On the other hand, since $\omega_{ess}(A) < \omega_0(A)$, it follows that $r_{ess}(T(1)) < r(T(1))$. Hence, by Proposition 2.2.1, $r(T(1))$ is a pole of the resolvent of $T(1)$. This implies that $\omega_0(A) = s(A)$ is a pole of $R(\cdot, A)$. Thus, by Theorem 2.5.5, it follows that there exists $\alpha \geq 0$ such that $\sigma_b(A) = s(A) + i\alpha\mathbb{Z}$ and therefore $\sigma_b(A_{-\omega_0}) = i\alpha\mathbb{Z}$. Since $T_{-\omega_0}(\cdot)$ is quasi-compact and $\omega_0(A_{-\omega_0}) = 0$, we have, by Theorem 2.2.5, that

$$\{\lambda \in \sigma(A_{-\omega_0}) : \Re(\lambda) \geq 0\} = \{\lambda \in \sigma(A_{-\omega_0}) : \Re(\lambda) = 0\} = \sigma_b(A_{-\omega_0})$$

is finite. Therefore $\sigma_b(A_{-\omega_0}) = \{0\}$. The theorem is now proved by applying Theorem 2.2.5 and Proposition 2.5.3 to the rescaled semigroup $T_{-\omega_0}(\cdot)$. □

CHAPTER 3

POSITIVE SEMIGROUPS FOR TRANSPORT EQUATIONS

The time evolution describing the motion of neutrons in an absorbing and scattering homogeneous medium is given by the following integrodifferential equation

$$\frac{\partial}{\partial t}u(t,x,v) = -v\cdot\nabla_x u(t,x,v) - \sigma(x,v)u(t,x,v) + \int_V \kappa(x,v,v')u(t,x,v')\,dv', \quad (3.1)$$

where $u(t,x,v)$ represents the density distribution of the neutrons in terms of the variables of space $x \in D \subseteq \mathbb{R}^n$ and velocity $v \in V \subseteq \mathbb{R}^n$, at time t. Here D denotes the set describing the interior of the vessel in which neutron transport takes place. The medium D is to be thought surrounded by a total absorber (or by a vacuum if D is convex), and neutrons migrate in this volume, are scattered and absorbed by this material. We suppose that neutrons do not interact with each other.

The *free streaming term* $-v\cdot \operatorname{grad}_x u$ in (3.1) is responsible for the motion for the particles between collisions with the background material. The second term of the right-hand side of (3.1) corresponds to collisions including absorption, and the third term to scattering of neutrons: particles at the position x with the incoming speed v' generate particles at x with the outgoing speed v and the transition is governed by a scattering kernel $\kappa(x,v,v')$.

The fact that $u(t,\cdot,\cdot)$ should describe a density suggests to require that $u(t,\cdot,\cdot)$ is an element of $L^1(D\times V)$ for all $t \geq 0$. Following this line and introducing the vector-valued function $u(t) := u(t,\cdot,\cdot)$, (3.1) is equivalent to the following abstract Cauchy problem

$$\begin{cases} u'(t) = (A+K_\kappa)u(t) := (A_0 - M_\sigma)u(t) + K_\kappa u(t), \quad t \geq 0, \\ u(0) \in D(A+K_\kappa). \end{cases}$$

Here $u(t)$, $t \geq 0$, is an element of $L^1(D\times V)$ and A_0 denotes the *free streaming opertor* $-v\cdot\operatorname{grad}_x$ on a suitable domain. We refer to [29, Theorem 1.11, p. 36] for a precise description of the domain of A_0. Moreover M_σ is the multiplication operator by σ and is called the *absorption operator*. The *scattering operator* K_κ is defined by

$$(K_\kappa f)(x,v) := \int_V \kappa(x,v,v')f(x,v')\,dv', \quad (x,v) \in D\times V, f\in L^1(D\times V).$$

For more details and information concerning the physical meaning of Equation (3.1) we refer to [34, Chapter 8], (see also [5, Sect. 1.3], [19], [3], [4]).

By using the abstract results given in Section 2.5 we propose to study the asymptotic behaviour of the solution of the transport equation (3.1) (cf. [9, VI.2], [15], [16]). In the first section we present the one-dimensional case and in the second the more general one.

3.1 The one-dimensional reactor problem

In this section we prove the existence of the semigroup solution of the following transport equation

$$(TE) \qquad \begin{cases} \frac{\partial u}{\partial t}u(t,x,v) = -v\frac{\partial u}{\partial x}(t,x,v) - \sigma(x,v)u(t,x,v) + \\ \qquad + \int_V \kappa(x,v,v')u(t,x,v')\,dv', \\ \qquad\qquad t\geq 0,\ (x,v)\in J\times V, \\ u(t,0,v)=0 \text{ if } v>0 \text{ and } u(t,1,v)=0 \text{ if } v<0, \quad t\geq 0, \\ u(0,x,v)=f(x,v), \quad (x,v)\in J\times V, \end{cases}$$

where $0\leq\sigma\in L^\infty(J\times V)$, $0\leq\kappa\in L^\infty(J\times V\times V)$, and $J:=[0,1]$, $V:=\{v\in\mathbb{R} : v_{min}\leq |v|\leq v_{max}\}$ for given constants $0<v_{min}<v_{max}<\infty$.
If we suppose that the scattering kernel κ satisfies

$$\kappa(x,v,v')>0 \quad \text{for all } (x,v,v')\in J\times V\times V, \tag{3.2}$$

then one can apply Theorem 2.5.6 and deduce the asymptotic behaviour of the semigroup solution of (TE).
To do so, we recall some results from perturbation theory of C_0–semigroups on Banach spaces.

Let A with domain $D(A)$ be the generator of a C_0–semigroup $T(\cdot)$ on a Banach space E and $B\in\mathcal{L}(E)$. Then $A+B$ generates a C_0–semigroup $S(\cdot)$ given by the Dyson-Phillips expansion

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{3.3}$$

where

$$\begin{aligned} S_0(t) &:= T(t) \text{ and} \\ S_{n+1}(t) &:= \int_0^t T(t-s)BS_n(s)\,ds \quad \text{for } x\in E, n\in\mathbb{N} \text{ and } t\geq 0. \end{aligned}$$

The series converges in the operator norm uniformly on bounded intervals of $\mathbb{R}_+$. Some times it is also possible to express the perturbed semigroup $S(\cdot)$ by the Chernoff product formula

$$S(t)x = \lim_{n\to\infty} \left(T(\frac{t}{n})e^{\frac{t}{n}B}\right)^n x, \quad t \geq 0, x \in E. \tag{3.4}$$

For these results we refer to [7, III.1], [23, III], [14, I.6] or [9, III].

Recall that an operator $B \in \mathcal{L}(E)$ is called *strictly power compact* if there is $n \in \mathbb{N}$ such that $(BT)^n$ is compact for all $T \in \mathcal{L}(E)$. In particular, if E is an L^1–space, then every weakly compact operator is strictly power compact (cf. [8, Corollary VI.8.13]). The following theorem gives the relationship between the essential spectrum of the perturbed and the unperturbed semigroups (see [28] or [9, Theorem IV.4.4]).

Theorem 3.1.1 *Let A be the generator of a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space E and $B \in \mathcal{L}(E)$. Let $S(\cdot)$ the $\mathcal{C}_0$–semigroup generated by $A+B$. Assume that there exists $n \in \mathbb{N}$ and a sequence $(t_k) \subset \mathbb{R}_+, t_k \to \infty$, such that the remainder $R_n(t_k) := \sum_{p=n} S_p(t_k)$ of the Dyson-Phillips (3.3) at t_k is strictly power compact for all $k \in \mathbb{N}$. Then*

$$r_{ess}(S(t)) \leq r_{ess}(T(t)), \quad t \geq 0.$$

We now give a short description of a special class of regular operators. We denote The *center* of E by

$$\mathcal{Z}(E) := \{M \in \mathcal{L}(E) : MI \subset I \text{ for every closed ideal } I \subset E\},$$

where E is a Banach lattice. It is known that

$$M \in \mathcal{Z}(E) \Longleftrightarrow \pm M \leq \|M\| Id. \tag{3.5}$$

From (3.5) one can see that $(e^{\pm tM})_{t\geq 0}$ is a positive $\mathcal{C}_0$–semigroup whenever $M \in \mathcal{Z}(E)$.

If (Ω, Σ, μ) is a σ–finite measure space, then the center $\mathcal{Z}(L^p(\mu))$ is isomorphic to $L^\infty(\mu)$ with the isomorphism

$$L^\infty(\mu) \ni \varphi \mapsto T_\varphi f = \varphi f.$$

To check the irreducibility of the solution semigroup of (TE) we need the following result.

Proposition 3.1.2 *Let A_0 with domain $D(A_0)$ be the generator of a positive $\mathcal{C}_0$–semigroup $T_0(\cdot)$ on a Banach lattice E and $0 \leq K \in \mathcal{L}(E)$. Assume that $0 \leq M \in \mathcal{Z}(E)$. Let $S(\cdot)$ (resp. $T(\cdot)$) be the positive $\mathcal{C}_0$–semigroup generated by $A_0 - M + K$ (resp. $A_0 - M$). If $I \subseteq E$ is a closed ideal, then the following assertion are equivalent.*

(a) *I is $S(\cdot)$–invariant.*

(b) *I is invariant both under $T_0(\cdot)$ and K.*

Proof: (a) $\Longrightarrow$ (b) Suppose that I is $S(\cdot)$–invariant. Since $0 \le T(t) \le S(t)$, $t \ge 0$, it follows that I is $T(\cdot)$–invariant. On the other hand the assumption on M, the closedness of I and the formula $e^{tM} = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n$ imply that I is e^{tM}–invariant for all $t \ge 0$. Now, from the product formula (3.4)

$$T_0(t)x = \lim_{n\to\infty} \left(T(\frac{t}{n}) e^{\frac{t}{n}M} \right)^n, \quad t \ge 0, x \in E,$$

we obtain that I is $T_0(\cdot)$–invariant. By (3.3) we have

$$\lim_{t\downarrow 0} \frac{1}{t} \left(S(t)x - T(t)x \right) = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t T(t-s) K S(s) x \, ds = Kx$$

for $x \in E$. Since I is closed and invariant both under $S(\cdot)$ and $T(\cdot)$, we obtain that I is K–invariant.

(b) $\Longrightarrow$ (a) It is easy to see that $0 \le T(t) \le T_0(t)$, $t \ge 0$. Thus, I is also $T(\cdot)$–invariant. Now, by applying the product formulas (3.4) to $T(t)$ and e^{tK}, $t \ge 0$, and using the closedness of I, we obtain (a). □

We now return to the transport equation (TE) and define the *free streaming* operator A_0 by

$$(A_0 f)(x,v) := -v \frac{\partial f}{\partial x}(x,v) \text{ with}$$

$$D(A_0) \quad := \quad \left\{ f \in L^1(J \times V) : v\frac{\partial f}{\partial x} \in L^1(J \times V), \begin{array}{ll} f(0,v) = 0 & \text{if } v > 0 \\ f(1,v) = 0 & \text{if } v < 0 \end{array} \right\},$$

the *absorption* operator

$$(M_\sigma f)(x,v) := \sigma(x,v) f(x,v), \quad (x,v) \in J \times V, f \in L^1(J \times V),$$

and the *scattering* operator

$$(K_\kappa f)(x,v) := \int_V \kappa(x,v,v') f(x,v') \, dv', \quad (x,v) \in J \times V, f \in L^1(J \times V).$$

Let us study first the free streaming operator. By an easy computation one can see that $(0,\infty) \subseteq \rho(A_0)$ and

$$\left(R(\lambda,A_0) f\right)(x,v) = \begin{cases} \frac{1}{v} \int_0^x e^{-\frac{\lambda}{v}(x-x')} f(x',v) \, dx' & \text{if } v > 0, \\ -\frac{1}{v} \int_x^1 e^{-\frac{\lambda}{v}(x-x')} f(x',v) \, dx' & \text{if } v < 0, \end{cases} \tag{3.6}$$

for $(x,v) \in J \times V$ and $f \in L^1(J \times V)$. Hence,

$$(0,\infty) \subseteq \rho(A_0) \text{ and } \|R(\lambda,A_0)\| \le \frac{1}{\lambda} \quad \text{for all } \lambda > 0.$$

Therefore, by the Hille-Yosida generation theorem (cf. [9, Theorem II.3.5]), A_0 with domain $D(A_0)$ generates a C_0–semigroup $T_0(\cdot)$ of contractions on $L^1(J \times V)$. Moreover, $T_0(\cdot)$ is positive since $R(\lambda, A_0) \geq 0$ for all $\lambda > 0$. On the other hand, one deduces that

$$(R(\lambda, A_0)f)(x,v) = \int_0^\infty e^{-\lambda t} \chi_J(x - vt) f(x - vt)\, dt$$

for $(x,v) \in J \times V$, $f \in L^1(J \times V)$, where $\chi_J(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \notin J. \end{cases}$
So, by the uniqueness of the Laplace transform, we obtain

$$(T_0(t)f)(x,v) = \chi_J(x - tv) f(x - tv, v), \quad (x,v) \in J \times V, f \in L^1(J \times V). \tag{3.7}$$

Moreover, since the absorption operator M_σ is bounded, it follows that

$$A := A_0 - M_\sigma \text{ with } D(A) = D(A_0)$$

generates the positive C_0–semigroup $T(\cdot)$ given by

$$(T(t)f)(x,v) = e^{-\int_0^t \sigma(x - \tau v, v)\, d\tau} (T_0(t)f)(x,v), \tag{3.8}$$

for $(x,v) \in J \times V$, $f \in L^1(J \times V)$. The boundedness and the positivity of the scattering operator K_κ implies that the transport operator $A + K_\kappa$ with domain $D(A_0)$ generates the positive C_0–semigroup $S(\cdot)$ given by the Dyson-Phillips expansion (3.3). This semigroup will be called the *transport semigroup* and satisfies the following properties.

Proposition 3.1.3 *The streaming semigroup $T(\cdot)$ and the transport semigroup $S(\cdot)$ satisfy*

$$0 \leq T(t) \leq S(t) \quad \text{for all } t \geq 0 \text{ and} \tag{3.9}$$
$$\omega_0(A + K_\kappa) = s(A + K_\kappa).$$

Proof: The first assertion follows from the positivity of K_κ and the Dyson-Phillips expansion (3.3). The second is a consequence from Theorem 2.4.1.(ii). □

For the study of the asymptotic behaviour of the transport semigroup we need some properties of weakly compact operators on L^1–spaces (see [15, Proposition 2.1] and the references therein).

Proposition 3.1.4 *Let (Ω, Σ, μ) be a σ–finite, positive measure space and S, T be two bounded linear operator on $L^1(\Omega, \mu)$. Then the following assertions hold.*

(a) *The set of all weakly compact operators is a norm-closed subset of $\mathcal{L}(L^1(\Omega, \mu))$.*

(b) *If T is weakly compact and $0 \leq S \leq T$, then S is also weakly compact.*

(c) *If S and T are weakly compact, then ST is compact.*

We now show the weak compactness of the remainder $R_2(t)$ of the Dyson-Phillips series (3.3) and the irreducibility of the transport semigroup $S(\cdot)$.

Lemma 3.1.5 *For the transport semigroup $S(\cdot)$ defined above the following properties hold.*

(i) *The remainder $R_2(t) := \sum_{n=2}^{\infty} S_n(t)$, $t \geq 0$, of the Dyson-Phillips expansion (3.3) is a weakly compact operator on $L^1(J \times V)$.*

(ii) *If the scattering kernel satisfies (3.2), then the transport semigroup $S(\cdot)$ is irreducible.*

Proof: For $0 \leq f \in L^1(J \times V)$ and $t > 0$ we have

$$\begin{aligned}
(K_\kappa T(t) K_\kappa f)(x,v) &\leq (K_\kappa T_0(t) K_\kappa f)(x,v) \\
&\leq \|\kappa\|_\infty^2 \int_V \int_V \chi_J(x - tv'') f(x - tv'', v')\, dv''\, dv' \\
&\leq t^{-1} \|\kappa\|_\infty^2 \int_V \int_J f(x', v')\, dx'\, dv'.
\end{aligned}$$

Hence

$$K_\kappa T(t) K_\kappa \leq \frac{\|\kappa\|_\infty^2}{t} (1\mathrm{I} \otimes 1\mathrm{I}), \tag{3.10}$$

where $1\mathrm{I} \otimes 1\mathrm{I}$ is the bounded linear operator defined by

$$(1\mathrm{I} \otimes 1\mathrm{I}) f := \left(\int_J \int_V f(x,v)\, dv\, dx \right) 1\mathrm{I}, \quad f \in L^1(J \times V).$$

By using the definition of the terms $S_n(t)$ in the Dyson-phillips series (3.3) one can see that

$$R_{n+1}(t) := \sum_{k=n+1}^{\infty} S_k(t) = \int_0^t T(t-s) K_\kappa R_n(s)\, ds, \quad t \geq 0, n \in \mathbb{N}.$$

In particular, $R_2(t) = \int_0^t \int_0^{t-s_2} T(s_1) K_\kappa T(s_2) K_\kappa S(t - s_1 - s_2)\, ds_1 ds_2$ for $t \geq 0$. Take $t > \varepsilon > 0$ and consider

$$R_{2,\varepsilon}(t) := \int_\varepsilon^t \int_0^{t-s_2} T(s_1) \left(K_\kappa T(s_2) K_\kappa \right) S(t - s_1 - s_2)\, ds_1 ds_2.$$

Then it is easy to verify that

$$\lim_{\varepsilon \to 0} \|R_{2,\varepsilon}(t) - R_2(t)\| = 0 \quad \text{for all } t > 0.$$

On the other hand, it follows from (3.10) that

$$R_{2,\varepsilon}(t) \leq \|\kappa\|_\infty^2 \int_\varepsilon^t \int_0^{t-s_2} \frac{1}{s_2} T(s_1) \circ (1\mathrm{I} \otimes 1\mathrm{I}) S(t - s_1 - s_2)\, ds_1 ds_2.$$

From the definition of $T_0(\cdot)$ and since $0 \le T(t) \le T_0(t)$, one can see that $T(t) \circ (1\!\!1 \otimes 1\!\!1) \le (1\!\!1 \otimes 1\!\!1)$ for the order in $\mathcal{L}(L^1(J \times V))$. Now, for $0 \le f \in L^1(J \times V)$, and $s_1 + s_2 \le t$, we obtain

$$\begin{aligned}(1\!\!1 \otimes 1\!\!1)S(t-s_1-s_2)f &= \left(\int_J\int_V (S(t-s_1-s_2)f)(x,v)\,dv\,dx\right) 1\!\!1 \\ &\le Me^{\omega(t-s_1-s_2)}\left(\int_J\int_V f(x,v)\,dv\,dx\right) 1\!\!1 \\ &= Me^{\omega(t-s_1-s_2)}(1\!\!1 \otimes 1\!\!1)f,\end{aligned}$$

where $M \ge 1$ and $\omega \in \mathbb{R}$ are such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Consequently,

$$\begin{aligned}R_{2,\varepsilon}(t) &\le M\|\kappa\|_\infty^2 \left(\int_\varepsilon^t \frac{1}{s_2}\int_0^{t-s_2} e^{\omega(t-s_1-s_2)}\,ds_1 ds_2\right)(1\!\!1 \otimes 1\!\!1) \\ &= \frac{M\|\kappa\|_\infty^2}{\omega}\left(\int_\varepsilon^t \frac{e^{\omega(t-s_2)}-1}{s_2}\,ds_2\right)(1\!\!1 \otimes 1\!\!1).\end{aligned}$$

This implies that $R_{2,\varepsilon}(t)$ is dominated by a one-dimensional operator. So, by Proposition 3.1.4, we obtain that $R_{2,\varepsilon}(t)$ is weakly compact and therefore $R_2(t)$ is weakly compact for all $t \ge 0$. This proves (i).
We recall that every closed ideal in $L^1(J \times V)$ has the form

$$I = \{f \in L^1(J \times V) : f \text{ vanish a.e. on } \Omega\}$$

for some measurable subset $\Omega \subseteq J \times V$. We suppose that I is $S(\cdot)$–invariant. Then, by Proposition 3.1.2, I is K_κ–invariant. Assume that $\Omega \neq \emptyset$. Since $\chi_{J\times V\setminus\Omega} \in I$, we obtain

$$\begin{aligned}\left(K_\kappa \chi_{J\times V\setminus\Omega}\right)(x,v) &= \int_V \kappa(x,v,v')\chi_{J\times V\setminus\Omega}(x,v')\,dv' \\ &= \int_{V\setminus\Omega_x} \kappa(x,v,v')\,dv' = 0\end{aligned}$$

for $(x,v) \in \Omega$ and $\Omega_x := \{v \in V : (x,v) \in \Omega\}$. Since κ is strictly positive, it follows that $\Omega_x = V$. Hence, $\Omega = Y \times V$ for some measurable subset Y of J.
On the other hand, again by Proposition 3.1.2, I is $T_0(\cdot)$–invariant. Thus, I is $R(\lambda, A_0)$–invariant for all $\lambda > 0$. Hence, $\left(R(\lambda, A_0)\chi_{J\times V\setminus\Omega}\right)(x,v) = 0$ for a.e. $(x,v) \in \Omega$. So, by using (3.6), one can see that

$$\int_0^x \chi_{J\setminus Y}(s)\,ds = 0 \text{ and } \int_x^1 \chi_{J\setminus Y}(s)\,ds = 0.$$

Therefore, $\int_0^1 \chi_{J\setminus Y}(s)\,ds = 0$ and this implies that $Y = J$. Consequently, $I = \{0\}$ or $I = L^1(J \times V)$ and (ii) is proved. □

We can now describe the asymptotic behaviour of the transport semigroup.

Theorem 3.1.6 *Assume that κ satisfies (3.2). Then the transport semigroup $S(\cdot)$ has balanced exponential growth. More precisely, there are two strictly positive functions $\varphi \in L^1(J \times V)$ and $\psi \in L^\infty(J \times V)$ satisfying $\int_{J\times V} \varphi(x,v)\psi(x,v)\,dv\,dx = 1$ such that*

$$\|e^{-s(A+K_\kappa)t}S(t) - \psi \otimes \varphi\| \le Me^{-\varepsilon t}$$

for all $t \ge 0$ and some constants $M \ge 0$ and $\varepsilon > 0$.

Proof: Since $v_{min} > 0$, it follows that $T(\cdot)$ is a nilpotent semigroup, i.e., there is $t_0 > 0$ such that

$$T(t) = 0 \quad \text{for all } t \ge t_0. \tag{3.11}$$

Hence, $r(T(t)) = r_{ess}(T(t)) = 0$ for all $t > 0$. So, by Lemma 3.1.5.(i) and Theorem 3.1.1, we have

$$\omega_{ess}(A + K_\kappa) = -\infty.$$

On the other hand, it follows from (3.11) that

$$S_1(t) = \int_0^t T(s)K_\kappa T(t-s)\,ds = 0 \quad \text{for all } t \ge 2t_0$$

and therefore

$$R_2(t) = S(t) \quad \text{for all } t \ge 2t_0.$$

So, by Lemma 3.1.5.(ii), we obtain that $R_2(t)$ is irreducible for all $t \ge 2t_0$. Now, one can apply [27, Theorem A.(iii)] to obtain that $r(S(t)) = r(R_2(t)) > 0$ for all $t \ge 2t_0$. Therefore,

$$-\infty = \omega_{ess}(A + K_\kappa) < \omega_0(A + K_\kappa).$$

Then one can apply Theorem 2.5.6 to the transport semigroup $S(\cdot)$ and obtains the assertions. □

3.2 THE N-DIMENSIONAL REACTOR PROBLEM

The second example is concerned with the n-dimensional transport equation (see [30] and [31])

$$(nTE) \quad \begin{cases} \frac{\partial u}{\partial t}u(t,x,v) = -v \cdot \nabla_x u(t,x,v) - \sigma(x,v)u(t,x,v) + \\ + \qquad\qquad + \int_V \kappa(x,v,v')u(t,x,v')\,dv', \\ \qquad\qquad\qquad\qquad t \ge 0,\ (x,v) \in D \times V, \\ u(t,\cdot,\cdot)|_{\Gamma_-} = 0, \quad t \ge 0, \\ u(0,x,v) = f(x,v), \quad (x,v) \in D \times V \end{cases}$$

on $L^1(D \times V)$, where $\Gamma_- := \{(x,v) \in \partial D \times V : v \cdot n(x) < 0\}$ and $n(x)$ is the outward normal at $x \in \partial D$.

We propose again to apply the theory developed in Section 2.5 to study the asymptotic behaviour of the solution of the transport equation (nTE). Proving the irreducibility of the transport semigroup in this case is not so easy.

We suppose that D is a smooth open subset of $\mathbb{R}^n$ and V is an open subset of $\mathbb{R}^n$. The collision σ and the scattering kernel κ are nonnegative and measurable functions satisfying

$$\sigma \in L^\infty(D\times V) \text{ and } \sup_{(x,v)\in D\times V}\left(\int_V \kappa(x,v',v)\,dv'\right) < \infty. \tag{3.12}$$

Condition (3.12) implies that the absorption operator M_σ and the scattering operator K_κ are both bounded on $L^1(D\times V,\mu)$, where μ is the $2n$-dimensional Lebesgue measure. As in the previous section, we define the free streaming semigroup, the absorption semigroup and the transport semigroup respectively by

$$\begin{aligned}
(T_0(t)f)(x,v) &:= f(x-tv,v)\chi_t(x,v)\\
(T(t)f)(x,v) &:= \exp\left(-\int_{-t}^0 \sigma(x+sv,v)\,ds\right)(T_0(t)f)(x,v)\\
S(t) &:= \sum_{n=0}^\infty S_n(t),
\end{aligned}$$

where $\chi_t(x,v) := \begin{cases} 1 & \text{if } t_-(x,v) > t\\ 0 & \text{if } t_-(x,v)\le t\end{cases}$ and $t_-(x,v) := \inf\{s>0 : x-sv\notin D\}$, $(x,v)\in D\times V$, $S_0(t) = T(t)$ and

$$S_{n+1}(t) = \int_0^t T(t-s)K_\kappa S_n(s)\,ds \text{ for } t\ge 0 \text{ and } (x,v)\in D\times V.$$

If we denote by A_0 the generator of $(T_0(t))_{t\ge 0}$, then $A = A_0 - M_\sigma$ and $A+K_\kappa$ are the generator of $T(\cdot)$ and $S(\cdot)$ respectively. We note that those semigroups are positive and strongly continuous on $L^1(D\times V,\mu)$.

In order to illustrate the theory given in Section 2.5, let us consider the special case where

$$\begin{cases} D \text{ is bounded and connected and } \{v\in\mathbb{R}^n : \xi_1 < |v| < \xi_2\} =: V_0 \subset\\ \qquad V\subset V_1 := \{v\in\mathbb{R}^n : |v| > v_{\min}\}\end{cases} \tag{3.13}$$

for some constants $v_{\min} > 0$ and $0\le \xi_1 < \xi_2 \le \infty$.

Without loss of generality one can suppose that $\xi_2 < \infty$.

As in the previous section, the second order remainder

$$R_2(t) := \sum_{n=2}^\infty S_n(t), \quad t\ge 0,$$

of the Dyson-Phillips expansion (3.3) will be of particular importance. If we denote $S_t := \{(s_1,s_2) : s_1,s_2\ge 0 \text{ and } s_1+s_2\le t\}$, one can see that

$$R_2(t) = \int_{S_t} T(s_1)K_\kappa T(s_2)K_\kappa S(t-s_1-s_2)\,ds_1\,ds_2$$

holds for $t \geq 0$. In particular, we have

$$\begin{aligned}
& (T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) \\
= \; & \sigma_{s_1}(x,v)\int_V \kappa(x-s_1v,v,v'')\sigma_{s_2}(x-s_1v,v'') \\
& \cdot \int_V \kappa(x-s_1v-s_2v'',v'',v')f(x-s_1v-s_2v'',v')\,dv'\,dv''
\end{aligned}$$

for $f \in L^1(D\times V)$, where

$$\sigma_s(x,v) := \chi_s(x,v)\exp\left(-\int_{-s}^0 \sigma(x+\tau v,v)\,d\tau\right)$$

for $(x,v) \in D\times V$. By taking the new variable $x' := x - s_1v - s_2v''$ we obtain

$$(T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) = \int_{D\times V} \tilde{\kappa}_{s_1,s_2}(x,v,x',v')f(x',v')\,dx'dv',$$

where

$$\begin{aligned}
& \tilde{\kappa}_{s_1,s_2}(x,v,x',v') && (3.14) \\
:= \; & \sigma_{s_1}(x,v)s_2^{-n}\kappa\left(x-s_1v,v,\frac{x-x'-s_1v}{s_2}\right) \\
& \cdot\sigma_{s_2}\left(x-s_1v,\frac{x-x'-s_1v}{s_2}\right)\kappa\left(x',\frac{x-x'-s_1v}{s_2},v'\right). && (3.15)
\end{aligned}$$

Here and in the sequel we use the convention that all functions defined on $D\times V$ ($D\times V\times V$ resp.) are extended by zero to $\mathbb{R}^n\times\mathbb{R}^n$ (resp. $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$). If we suppose that κ satisfies the conditions

$$\exists\gamma \in L^1(V)/\,\kappa(x,v,v') \leq \gamma(v) \text{ for all } (x,v,v') \in D\times V\times V \qquad (3.16)$$

and

$$V_0 \subset V \text{ and } \kappa(\cdot,\cdot,\cdot) > 0 \text{ on } (D\times V_0\times V)\cup(D\times V\times V_0), \qquad (3.17)$$

then we have the main result of this section.

Theorem 3.2.1 *Suppose that* (3.12) *and* (3.13) *hold. If* κ *satisfies the conditions* (3.16) *and* (3.17)*, then there exist* $0 < \varphi \in L^1(D\times V)$, $0 < \psi \in L^\infty(D\times V)$ *with* $\int_D\int_V \varphi(x,v)\psi(x,v)\,dv\,dx = 1$ *such that*

$$\|e^{-s(A+B)t}S(t) - \psi\otimes\varphi\| \leq Me^{-\varepsilon t}$$

for all $t \geq 0$ *and some constants* $M \geq 1$ *and* $\varepsilon > 0$.

The proof is split into two lemmas.

Lemma 3.2.2 *Suppose that D is bounded and* (3.12)*,* (3.16) *are satisfied. Then the second order remainder* $R_2(t)$ *is weakly compact for all* $t > 0$. *Therefore,*

$$r_{ess}(S(t)) \leq r_{ess}(T(t) \quad \textit{for } t \geq 0.$$

Proof: Since

$$R_2(t) = \int_{S_t} T(s_1)K_\kappa T(s_2)K_\kappa S(t-s_1-s_2)\,ds_1\,ds_2$$

and by [32, Theorem 1.3], it suffices to show that the operators $T(s_1)K_\kappa T(s_2)K_\kappa$ are weakly compact for all $(s_1,s_2) \in S_t$ with $s_2 > 0$. Let us note that we have

$$(T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) = \int_{D\times V} \tilde{\kappa}_{s_1,s_2}(x,v,x',v')f(x',v')\,dx'\,dv'$$

with $\tilde{\kappa}_{s_1,s_2}$ from (3.14). It follows from the Dunford-Pettis theorem (cf. [8, Theorem 10, p. 507]) that it suffices to prove that the set

$$M := \{\tilde{\kappa}_{s_1,s_2}(\cdot,\cdot,x',v');\,(x',v') \in D\times V\}$$

is contained in a weakly compact subset of $L^1(D\times V,\mu)$. We note that the function

$$D \ni x' \mapsto g(x') \in L^1(D\times V,\mu)$$

defined by

$$g(x')(x,v) := s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1v}{s_2}\right), \quad (x,v)\in D\times V,$$

is continuous. This statement follows from a simple estimate by approximating $\gamma \in L^1(V)$ by continuous functions with compact support. So, since D is bounded, it follows that the set

$$\tilde{M} := \{s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1v}{s_2}\right) : x' \in \overline{D}\}$$

is relatively compact in $L^1(D\times V,\mu)$. By (3.16) we now have

$$0 \le \tilde{\kappa}_{s_1,s_2}(x,v,x',v') \le s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1v}{s_2}\right)$$

for $(x,v,x',v') \in (D\times V)\times(D\times V)$. Therefore, the Dunford-Pettis theorem (cf. [21, Theorem 2.5.4.(iv)]) implies that M is relatively weakly compact in $L^1(D\times V,\mu)$. The last assertion follows from Theorem 3.1.1. □

Lemma 3.2.3 *Assume that D is connected and* (3.12) *is satisfied. Let V_0 be the set given in* (3.13)*. If* (3.17) *holds, then $(S(t))_{t\ge 0}$ is irreducible.*

Proof: **1.** Let us prove first that, for $x_0 \in D$ and $r > 0$ such that

$$B(x_0,3r) := \{x \in \mathbb{R}^n : |x_0 - x| < 3r\} \subset D,$$

we have for each $0 \le f \in L^1(D\times V,\mu)$ with $f|_{B(x_0,r)\times V} \neq 0$

$$\left(S(\frac{2r}{\xi_2})f\right)(x,v) > 0 \text{ for a.e. } (x,v)\in B(x_0,r)\times V. \tag{3.18}$$

To this purpose let us consider the second order term $S_2(\cdot)$ of the Dyson-Phillips series (3.3) and put $t_0 := \frac{2r}{\xi_2}$. Then by a simple calculation one can see that

$$\begin{aligned}(S_2(t_0)f)(x,v) &= \left(\int_{S_{t_0}} T(s_1)K_\kappa T(t_0-s_1-s_2)K_\kappa T(s_2)f\,ds_1 ds_2\right)(x,v)\\ &= \int_{D\times V}\beta(x,v,x',v')f(x',v')\,dx'dv',\end{aligned}$$

where

$$\begin{aligned}&\beta(x,v,x',v')\\ :=\ &\int_{S_{t_0}}\sigma_{s_1}(x,v)\sigma_{t_0-s_1-s_2}\left(x-s_1v,\frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\cdot\sigma_{s_2}(x'+s_2v',v')(t_0-s_1-s_2)^{-n}\kappa\left(x-s_1v,v,\frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\qquad\cdot\kappa\left(x'+s_2v',\frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2},v'\right)ds_2\,ds_1,\\ =\ &\int_{S_{t_0}}\sigma_{s_1}(x,v)\sigma_{t_0-s_1-s_2}\left(x-s_1v,\frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\cdot\sigma_{s_2}(x'+s_2v',v')(t_0-s_1-s_2)^{-n}\\ &\cdot\tilde\beta\left(x-s_1v,x'+s_2v',v,\frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2},v'\right)ds_2\,ds_1,\end{aligned}$$

with $\tilde\beta : D\times D\times V\times V\times V \to [0,\infty)$ given by

$$\tilde\beta(x,x',v,v',v'') := \kappa(x,v,v')\kappa(x',v',v'').$$

From (3.17) we know that $\tilde\beta(\cdot,\cdot,\cdot,\cdot,\cdot) > 0$ on $B(x_0,3r)\times B(x_0,3r)\times V\times V_0\times V$. Now, for a.e. $(x,v,x',v')\in B(x_0,r)\times V\times B(x_0,r)\times V$, it follows from Exercise 3.2.4 below that $\beta(x,v,x',v') > 0$. Therefore, since $0\le S_2(t_0)\le S(t_0)$, we obtain the first assertion.

2. The claim given in (3.18) holds for all $t \ge t_0$. In fact, choose $m\in\mathbb{N}$ such that $\frac{t-t_0}{m}\le t_0$ and instead of r we take $r' := \frac{\xi_2(t-t_0)}{2m}\ (\le r)$. Then, (3.18) can be applied m times to each ball $B(x_0,r')$ contained in $B(x_0,r)$ and we obtain

$$(S(t)f)(x,v) = S\left(\frac{t-t_0}{m}\right)^m (S(t_0)f)(x,v) > 0$$

for a.e. $(x,v)\in B(x_0,r')\times V$ and for all $0\le f\in L^1(D\times V,\mu)$ such that $f|_{B(x_0,r)\times V}\neq 0$. Consequently, $(S(t)f)(x,v)>0$ for a.e. $(x,v)\in B(x_0,r)\times V$.

3. Finally we show that $(S(t))_{t\geq 0}$ is irreducible. Let $0 \lneqq f \in L^1(D \times V, \mu)$. Then there is $x_0 \in D$ such that for all $\varepsilon > 0$ with $B(x_0, \varepsilon) \subset D$ and $f|_{B(x_0,\varepsilon)\times V} \neq 0$. Let $t > 0$ and consider $x' \in D$ such that there exists a polygonal path C of length $< \xi_2 t$ connecting x_0 with x'. There exists a covering of C by balls $(B(x_i, r_i))_{i=0,\ldots,m}$ such that $x_m = x'$, $B(x_i, r_i) \cap B(x_{i-1}, r_{i-1}) \neq \emptyset$ for $i = 1, \ldots, m$, $B(x_i, 3r_i) \subset D$ for $i = 0, \ldots, m$ and $2\sum_{i=0}^m r_i \leq \xi_2 t$. If we repeat this procedure we have

$$(S(t)f)(x,v) > 0 \quad \text{for a.e. } (x,v) \in B(x', r_m) \times V,$$

and the lemma is proved. □

Proof of Theorem 3.2.1 From (3.13) we have in particular

$$V \subset V_1 := \{v \in \mathbb{R}^n; |v| > v_{min}\}.$$

This and the boundedness of D imply that there is $t_0 > 0$ such that $T_0(t) = T(t) = 0$ for all $t \geq t_0$ and therefore, $\omega_0(A_0) = \omega_0(A) = -\infty$. Thus

$$\int_0^t T(t-s)K_\kappa T(s)\,ds = 0 \quad \text{for all } t \geq 2t_0.$$

This implies,

$$S(t) = R_2(t) \quad \text{for all } t \geq 2t_0.$$

So, by Lemma 3.2.2 and Lemma 3.2.3, $(S(t))_{t\geq 0}$ is irreducible and consists of weakly compact operators for all $t \geq 2t_0$. Hence, it follows from [27, Theorem A] that $\omega_0(A + K_\kappa) > \omega_{ess}(A + K_\kappa)\,(= -\infty)$. Now, the result follows from Theorem 2.5.6. □

Exercise 3.2.4 *Use the notation from the proof of Lemma 3.2.3 and define the function* $\alpha : D \times V \times D \times V \times S_{t_0} \to \mathbb{R}^{5n}$,

$$\alpha(x,v,x',v',s_1,s_2) := \left(x - s_1 v, x' + s_2 v', v, \frac{x - s_1 v - x' - s_2 v'}{t_0 - s_1 - s_2}, v'\right).$$

Show that, for a.e. $(x,v,x',v') \in B(x_0,r) \times V \times B(x_0,r) \times V$, *the set*

$$\{(s_1,s_2) \in S_{t_0}; \alpha(x,v,x',v',s_1,s_2) \in B(x_0,3r) \times B(x_0,3r) \times V \times V_0 \times V\}$$

is open and nonempty.

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