UNIVERSITÀ DEGLI STUDI DI LECCE
DIPARTIMENTO DI MATEMATICA
"Ennio De Giorgi"

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Gaussian Measures on Separable Hilbert Spaces and Applications

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"Ennio De *Giorgi"*

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PREFACE

The aim of these Iecture notes is to present some basic facts and ideas of the theory of Gaussian measures on infinite dimensionaI Hilbert spaces and to show to the reader how this theory can be applied to solve the infinite dimensionaI heat equation and, more generally, its perturbations by a linear drift term.

In particular, the Cameron-Martin theorem will be useful to obtain regularity properties of the semigroup generated by the Gross-Laplacian and the Ornstein-Ublenbeck semigroup.

These notes originated from a course given by the second author at the University of Lecce in May 2002 and at the University of Halle-Wittenberg in May 2003.

We have organized these notes as follows.

In Chapter I we present a self consistent and relatively complete introduction to the theory of Gaussian measures on separable Hilbert spaces.

Gaussian measures and the Cameron-Martin theorem are used, in Chapter II, to study the infinite dimensionaI heat equation. Regularity results and the spectrum of the solution semigroup are also obtained.

Chapter III is concemed with the Ornstein-Uhlenbeck semigroup, first on the space of bounded continuous functions, and second on *LP-spaces* with invariant measure. Regularity results and characterization of the domain of the generator are also obtained.

In the appendix we recall in the first part the classical Bochner theorem in $\mathbb{R}^N$ including, for the sake of completeness, a proof. In the second part we recall some basic and useful results of the theory of Co-semigroups on Banach spaces.

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Stefania Maniglia
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CONTENTS

CHAPTER 1

GAUSSIAN MEASURES ON HILBERT SPACES

The aim of this chapter is to show the Minlos-Sazanov theorem and deduce a characterization of Gaussian measures on separable Hilbert spaces by its Fourier transform. By using the notion of the Hellinger integral we prove the Kakutani theorem on infinite product measures. As a consequence we obtain the Cameron-Martin theorem.

For Gaussian measures on Banach spaces and their relationship with parabolic equations with many infinitely variables we refer to [22] and [12] and the references therein.

1.1 BOREL MEASURES ON HILBERT SPACES

Let H be a real separable Hilbert space, $\mathcal{B}(H)$ the Borel σ-algebra on H. Then $\mathcal{B}(H)$ is a separable σ-algebra. A measure on the measurable space $(H, \mathcal{B}(H))$ is called a **Borel measure** on H. Here we only investigate finite Borel measures.

Definition 1.1.1 *Let μ be a finite Borel measure on H. The* **Fourier transform** *of μ is defined by*

$$\widehat{\mu}(x) := \int_H e^{i<x,y>} \mu(dy), \qquad x \in H.$$

Clearly $\widehat{\mu}$ possesses the following properties.

Proposition 1.1.2 *The Fourier transform of a finite Borel measure satisfies the following properties*

(1) $\widehat{\mu}(0) = \mu(H)$.

(2) $\widehat{\mu}$ *is continuous on* H.

(3) $\widehat{\mu}$ *is positive definite in the sense that*

$$\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\overline{\alpha_k} \geq 0. \tag{1.1}$$

for any $n \geq 1,\ x_1, x_2, \cdots, x_n \in H$, *and* $\alpha_1, \alpha_2, \cdots, \alpha_n \ \in \mathbb{C}$.

Proof: We have only to prove the third assertion. For $n \geq 1,\ x_1, x_2, \dots, x_n \in H$, and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ we have

$$\begin{aligned}
\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\overline{\alpha_k} &= \sum_{l,k=1}^{n} \int_H e^{i<x_l,y>} e^{-i<x_k,y>} \alpha_l\overline{\alpha_k}\mu(dy) \\
&= \sum_{l,k=1}^{n} \int_H \left(e^{i<x_l,y>}\alpha_l\right) \overline{\left(e^{i<x_k,y>}\alpha_k\right)}\mu(dy) \\
&= \langle \sum_{l=1}^{n} e^{i<x_l,\cdot>}\alpha_l, \sum_{k=1}^{n} e^{i<x_k,\cdot>}\alpha_k \rangle_{L^2(H,\mu)} \\
&= \int_H \left| \sum_{k=1}^{n} e^{i<x_k,y>}\alpha_k \right|^2 \mu(dy) \ \geq 0.
\end{aligned}$$

Here $L^2(H,\mu)$ denotes the space of all measurable functions $f : H \to \mathbb{R}$ satisfying

$$\int_H |f(x)|^2 \, \mu(dx) < \infty.$$

□

A natural question arises. Is any positive definite continuous functional on H the Fourier transform of some finite Borel measure?
The answer is affirmative if $\dim H < \infty$. This is exactly the classical Bochner theorem (see Theorem A.1.3). But in the infinite dimensional case the answer is negative. Take, for example,

$$\phi(x) := \exp\left(-\frac{1}{2}|x|^2\right), \qquad x \in H.$$

Then it is easy to see that ϕ is a positive definite functional on H. But ϕ is not the Fourier transform of any finite Borel measure on H as we will see later (see Proposition 1.2.11).
To this end let us prove some auxiliary results.

Lemma 1.1.3 *Let* ϕ *be a positive definite functional on* H. *Then, for any* $x,\ y \in H$,

(1) $|\phi(x)| \leq \phi(0)$, $\overline{\phi(x)} = \phi(-x)$.

(2) $|\phi(x) - \phi(y)| \leq 2\sqrt{\phi(0)}\sqrt{\phi(0) - \phi(x-y)}$.

(3) $|\phi(0) - \phi(x)| \leq \sqrt{2\phi(0)(\phi(0) - \Re(\phi)(x))}$.

Proof: For $x, y \in H$, set

$$A := \begin{pmatrix} \phi(0) & \phi(x) \\ \phi(-x) & \phi(0) \end{pmatrix}$$

$$B := \begin{pmatrix} \phi(0) & \phi(x) & \phi(y) \\ \phi(-x) & \phi(0) & \phi(y-x) \\ \phi(-y) & \phi(x-y) & \phi(0) \end{pmatrix}$$

Since ϕ is positive definite, one can see that both A and B are positive definite matrices. In particular $\overline{A}^t = A$. Hence, $\overline{\phi(x)} = \phi(-x)$ for all $x \in H$. From $\det(A) \geq 0$, it follows that $|\phi(x)| \leq \phi(0)$.
On the other hand, we have

$$\begin{aligned} \det B &= \phi(0)^3 - \phi(0)|\phi(x-y)|^2 - \phi(x)[\phi(0)\overline{\phi(x)} - \overline{\phi(x-y)\phi(y)}] + \\ &\quad \phi(y)[\overline{\phi(x)}\phi(x-y) - \phi(0)\overline{\phi(y)}] \\ &= \phi(0)^3 - \phi(0)|\phi(x-y)|^2 - \phi(0)|\phi(x) - \phi(y)|^2 + \\ &\quad 2\Re[\phi(y)\overline{\phi(x)}(\phi(x-y) - \phi(0))]. \end{aligned}$$

Using the inequality $a^3 - ab^2 \leq 2a^2|a-b|$ for $|b| < a$, we find

$$\phi(0)^3 - \phi(0)|\phi(x-y)|^2 \leq 2\phi(0)^2|\phi(0) - \phi(x-y)|.$$

Therefore,

$$0 \leq \det B \leq 4\phi(0)^2|\phi(0) - \phi(x-y)| - \phi(0)|\phi(x) - \phi(y)|^2$$

This proves (2).
Finally (3) follows from

$$\begin{aligned} |\phi(0) - \phi(x)|^2 &= (\phi(0) - \phi(x))\left(\phi(0) - \overline{\phi(x)}\right) \\ &= \phi(0)^2 - 2\Re(\phi(0)\phi(x)) + |\phi(x)|^2 \\ &\leq 2\phi(0)^2 - 2\phi(0)\Re(\phi)(x). \end{aligned}$$

□

The following lemma will be useful for the proof of the Minlos-Sazanov theorem.

Lemma 1.1.4 *Let μ be a finite Borel measure on H. Then the following assertions are equivalent.*

(i) $\int_H |x|^2\mu(dx) < \infty$.

(ii) *There exists a positive, symmetric, trace class operator Q such that for $x, y \in H$*

$$\langle Qx, y\rangle = \int_H \langle x, z\rangle\langle y, z\rangle\mu(dz). \tag{1.2}$$

If (ii) holds, then $\mathrm{Tr}Q = \int_H |x|^2\mu(dx)$.

Proof: Suppose that (ii) holds. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Then

$$\int_H |x|^2\mu(dx) = \sum_{n=1}^{\infty}\int_H |\langle x, e_n\rangle|^2\mu(dx) = \sum_{n=1}^{\infty}\langle Qe_n, e_n\rangle = \mathrm{Tr}Q < \infty. \tag{1.3}$$

Conversely, assume that (i) is satisfied. Thus,

$$\int_H |\langle x, z\rangle\langle y, z\rangle|\mu(dz) \le |x||y|\int_H |z|^2\mu(dz).$$

By the Riesz representation theorem there exists $Q \in \mathcal{L}(H)$ such that (1.2) is satisfied. Obviously, Q is positive and symmetric. Furthermore, by (1.3),

$$\mathrm{Tr}Q = \int_H |x|^2\mu(dx) < \infty.$$

Hence Q is of trace class. □

Let show now the Minlos-Sazanov theorem.

Theorem 1.1.5 *Let ϕ be a positive definite functional on a separable real Hilbert space H. Then the following assertions are equivalent.*

(1) *ϕ is the Fourier transform of a finite Borel measure on H.*

(2) *For every $\varepsilon > 0$ there is a symmetric positive operator of trace class Q_ε such that*

$$\langle Q_\varepsilon x, x\rangle < 1 \implies \Re(\phi(0) - \phi(x)) < \varepsilon.$$

(3) *There exists a positive symmetric operator of trace class Q on H such that ϕ is continuous (or, equivalently, continuous at $x = 0$) with respect to the semi-norm $|\cdot|_Q$, where*

$$|x|_Q := \sqrt{\langle Qx, x\rangle} = |Q^{1/2}x|, \quad x \in H.$$

Proof: (1) $\implies$ (2): Let $\phi = \widehat{\mu}$. By applying the inequality

$$2(1 - \cos\vartheta) \le \vartheta^2, \qquad \forall\, \vartheta \in \mathbb{R},$$

we obtain, for any $\gamma > 0$,

$$\begin{aligned}\Re(\phi(0) - \phi(x)) &= \int_H (1 - \cos\langle x, z\rangle)\ \mu(dz) \\ &\le \frac{1}{2}\int_{\{|z|\le\gamma\}} \langle x, z\rangle^2\ \mu(dz) + 2\mu\left(\{z : |z| > \gamma\}\right).\end{aligned}$$

Set $\mu_1(A) := \mu(A \cap \{|z| \le \gamma\})$ for $A \in \mathcal{B}(H)$, and apply Lemma 1.1.4 to μ_1. Thus there is a positive symmetric operator of trace class B_γ such that

$$\langle B_\gamma z_1, z_2\rangle = \int_{\{|z|\le\gamma\}} \langle z_1, z\rangle\langle z_2, z\rangle\ \mu(dz).$$

On the other hand, for every $\varepsilon > 0$ there is $\gamma > 0$ such that $\mu(\{z : |z| > \gamma\}) \le \frac{\varepsilon}{4}$. Put $Q_\varepsilon := \frac{1}{\varepsilon}B_\gamma$, then

$$\Re\left(\phi(0) - \phi(x)\right) \le \frac{\varepsilon}{2}\langle Q_\varepsilon x, x\rangle + \frac{\varepsilon}{2}.$$

(2) $\Longrightarrow$ (1): Assume that (2) holds. Then $\Re(\phi)(x)$ is continuous at $x = 0$. So, by Lemma 1.1.3-(2), ϕ is continuous on H.
Now, take any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and for $n \ge 1$ put

$$f_{i_1,\cdots,i_n}(\omega_1, \cdots, \omega_n) : \phi(\omega_1 e_1 + \cdots + \omega_n e_n), \qquad \omega_j \in \mathbb{R},\ 1 \le j \le n.$$

Then $f_{i_1,\cdots,i_n}$ is a positive definite function on $\mathbb{R}^n$. By the classical Bochner theorem (see Theorem A.1.3) there exists a finite Borel measure $\mu_{i_1,\cdots,i_n}$ on $\mathbb{R}^n$ such that

$$f_{i_1,\cdots,i_n} = \widehat{\mu}_{i_1,\cdots,i_n}.$$

The family $\{\mu_{i_1,\cdots,i_n}\}$ satisfies the *consistency conditions of Kolmogorov's extension theorem* for measures (cf. [30], p. 144). Hence there is a unique finite Borel measure γ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that

$$\mu_{i_1,\cdots,i_n} = \gamma \circ (X_{i_1}, \cdots, X_{i_n})^{-1},$$

where $\gamma \circ (X_{i_1}, \cdots, X_{i_n})^{-1}$ is defined by

$$\gamma \circ (X_{i_1}, \cdots, X_{i_n})^{-1}(A) = \gamma((X_{i_1}, \cdots, X_{i_n})^{-1}(A)) \quad \text{for } A \in \mathcal{B}(H),$$

and $X_j(\omega) = \omega_j$, $\omega = (\omega_1, \cdots, \omega_n, \cdots) \in \mathbb{R}^\infty$, $j \in \mathbb{N}$.
Claim: $\sum_{k=1}^\infty X_k^2 < \infty \quad \gamma$-a.e..
Let $\mathbb{P}_n$ be the standard Gaussian measure on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} e^{i(a_1 y_1 + \cdots + a_n y_n)}\mathbb{P}_n(dy) = \exp\left(-\frac{1}{2}\sum_{j=1}^n a_j^2\right).$$

By assumption, we know that for every $\varepsilon > 0$ there is a positive symmetric operator Q_ε of trace class such that

$$\langle Q_\varepsilon x, x\rangle < 1 \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon.$$

Hence, by Lemma 1.1.3-(1),

$$\phi(0) - \Re(\phi)(x) \leq \varepsilon + 2\phi(0)\langle Q_\varepsilon x, x\rangle \quad \text{for } x \in H.$$

By Fubini's theorem we obtain

$$\begin{aligned}
&\phi(0) - \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=1}^n X_{k+j}^2\right)\gamma(d\omega) \\
= \;& \phi(0) - \int_{\mathbb{R}^\infty} \gamma(d\omega)\int_{\mathbb{R}^n} \exp\left(i\sum_{j=1}^n y_j X_{k+j}\right)\mathbb{P}_n(dy) \\
= \;& \phi(0) - \int_{\mathbb{R}^n} \mathbb{P}_n(dy)\int_{\mathbb{R}^\infty} \exp\left(i\sum_{j=1}^n y_j X_{k+j}\right)\gamma(d\omega) \\
= \;& \phi(0) - \int_{\mathbb{R}^n} \mathbb{P}_n(dy)\phi\left(\sum_{j=1}^n y_j e_{k+j}\right) \\
= \;& \int_{\mathbb{R}^n} [\phi(0) - \Re(\phi)(\sum_{j=1}^n y_j e_{k+j})]\mathbb{P}_n(dy) \\
\leq \;& \varepsilon + 2\phi(0)\int_{\mathbb{R}^n} \langle Q_\varepsilon \sum_{j=1}^n y_j e_{k+j}, \sum_{l=1}^n y_l e_{k+l}\rangle \mathbb{P}_n(dy) \\
= \;& \varepsilon + 2\phi(0)\sum_{l,j=1}^n \langle Q_\varepsilon e_{k+j}, e_{l+j}\rangle \int_{\mathbb{R}^n} y_j y_l \mathbb{P}_n(dy) \\
= \;& \varepsilon + 2\phi(0)\sum_{j=1}^n \langle Q_\varepsilon e_{k+j}, e_{k+j}\rangle \underbrace{\int_{\mathbb{R}^n} y_j^2 \mathbb{P}_n(dy)}_{=1} \\
= \;& \varepsilon + 2\phi(0)\sum_{j=1}^n \langle Q_\varepsilon e_{k+j}, e_{k+j}\rangle.
\end{aligned}$$

Hence,

$$\phi(0) - \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=1}^n X_{k+j}^2\right)\gamma(d\omega) \leq \varepsilon + 2\phi(0)\sum_{j=k+1}^\infty \langle Q_\varepsilon e_j, e_j\rangle.$$

Now, let $k \longrightarrow \infty$, and $\varepsilon \longrightarrow 0$, so we get

$$\lim_{k\to+\infty} \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2\right)\gamma(d\omega) = \phi(0)\,(=\gamma(\mathbb{R}^\infty)\neq 0).$$

This means that the function $\exp(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2)$ converges in $L^1(\mathbb{R}^\infty,\gamma)$ to the constant function 1. Thus there is a subsequence of

$$\exp(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2)$$

converging to 1 γ–a.e., which implies that

$$\sum_{j=1}^{\infty} X_j^2 < \infty \ \gamma - \text{a.e.},$$

and the claim is proved.
Finally, let

$$X(\omega) := \sum_{j=1}^{\infty} X_j(\omega)e_j, \quad \omega \in \mathbb{R}^\infty.$$

Then X is defined on $\mathbb{R}^\infty$ γ-a.e., and X is an H-valued measurable function. Put

$$\mu := \gamma \circ X^{-1}.$$

Then μ is a finite Borel measure on H and since $\mu_{i_1,\cdots,i_n} = \gamma \circ (X_{i_1},\cdots,X_{i_n})^{-1}$ we obtain

$$\begin{aligned}\widehat{\mu}\left(\sum_{j=1}^{n}\langle x,e_j\rangle e_j\right) &= f_{1,\cdots,n}\left(\langle x,e_1\rangle,\cdots,\langle x,e_n\rangle\right)\\ &= \phi\left(\sum_{j=1}^{n}\langle x,e_j\rangle e_j\right).\end{aligned}$$

By letting $n \longrightarrow \infty$ we obtain $\widehat{\mu} = \phi$ and the equivalence (1)$\Longleftrightarrow$(2) is proved.
(2) $\Longrightarrow$ (3): Assume that (2) holds. In (2) take $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$ and $\lambda_k > 0$ such that $\sum_{k=1}^{\infty}\lambda_k \mathrm{Tr} Q_{\frac{1}{k}} < \infty$. Set $Q := \sum_{k=1}^{\infty}\lambda_k Q_{\frac{1}{k}}$. It is obvious that Q is a positive symmetric operator of trace class on H. Moreover Q satisfies

$$\begin{aligned}\langle Qx,x\rangle < \lambda_k \quad &\Rightarrow \quad \langle Q_{\frac{1}{k}}x,x\rangle < 1\\ &\Rightarrow \quad \Re\left(\phi(0)-\phi(x)\right) < \frac{1}{k}.\end{aligned}$$

So, by Lemma 1.1.3, ϕ is continuous on H with respect to $\|\cdot\|_Q$ and hence (3) is proved.
(3) $\Longrightarrow$ (2): Conversely, suppose (3) is satisfied. So for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x|_Q < \delta \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon.$$

Set $Q_\varepsilon := \delta^{-1} Q$. Then,

$$\langle Q_\varepsilon x, x\rangle < 1 \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon$$

and Q_ε satisfies (2). □

1.2 Gaussian measures on Hilbert spaces

We will study a special class of Borel probability measures on H. We first introduce the notions of **mean vectors** and **covariance operators** for general Borel probability measures.

Definition 1.2.1 *Let μ be a Borel probability measure on H. If for any $x \in H$ the function $z \mapsto \langle x, z\rangle$ is integrable with respect to μ, and there exists an element $m \in H$ such that*

$$\langle m, x\rangle = \int_H \langle x, z\rangle\, \mu(dz), \quad x \in H,$$

then m is called the **mean vector** *of μ. If furthermore there is a positive symmetric linear operator B on H such that*

$$\langle Bx, y\rangle = \int_H \langle z - m, x\rangle \langle z - m, y\rangle\, \mu(dz), \quad x, y \in H,$$

then B is called the **covariance operator** *of μ.*

Mean vectors and covariance operators do not necessarily exist in general. But if $\int_H |x|\mu(dx) < \infty$, then by Riesz' representation theorem, the mean vector m exists, and $|m| \le \int_H |x|\mu(dx)$. If furthermore, $\int_H |x|^2\mu(dx) < \infty$, then by Lemma 1.1.4, there is a positive symmetric trace class operator Q such that

$$\langle Qx, y\rangle = \int_H \langle x, z\rangle \langle y, z\rangle\, \mu(dz) \quad x, y \in H.$$

Set $Bx = Qx - \langle m, x\rangle m,\ x \in H$. Then it is easy to verify that B is the covariance operator of μ. Note that B is also a positive symmetric trace class operator.

We introduce now Gaussian measures.

Definition 1.2.2 *Let μ be a Borel probability measure on H. If for any $x \in H$ the random variable $\langle x, \cdot \rangle$ has a Gaussian distribution, then μ is called a* **Gaussian measure***.*

Remark 1.2.3 *The scalar function $\langle x, \cdot \rangle$ has a Gaussian distribution means that there exists a real number m_x and a positive number σ_x such that*

$$\widehat{\mu}(x) = \int_H e^{i\langle x, z\rangle} \mu(dz) = \exp\left(i m_x - \frac{1}{2}\sigma_x^2 \right), \quad x \in H.$$

In the sequel we will characterize Gaussian measures by means of Fourier transform.

Lemma 1.2.4 *Let $(\alpha_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\sum_{j=1}^{\infty} \alpha_j^2 = \infty$. Then there exists a sequence of real numbers (β_j) such that*

$$\alpha_j \beta_j \geq 0 \textit{ for all } j \geq 1, \quad \sum_{j=1}^{\infty} \beta_j^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \alpha_j \beta_j = \infty.$$

Proof: Set $n_0 = 0$ and define n_k inductively as follows

$$n_k := \inf\{ l : \sum_{j=n_{k-1}+1}^{l} \alpha_j^2 \geq 1\}, \quad k \geq 1.$$

Then, $n_k \nearrow \infty$. Put

$$\beta_j := \frac{\alpha_j}{k+1} \left(\sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2 \right)^{-\frac{1}{2}}, \quad n_k + 1 \leq j \leq n_{k+1}, \; k = 0, 1, \ldots.$$

Then, for all $j \geq 1$, $\alpha_j \beta_j \geq 0$, and

$$\sum_{j=1}^{\infty} \beta_j^2 = \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \beta_j^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty,$$

$$\begin{aligned}
\sum_{j=1}^{\infty} \alpha_j \beta_j &= \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \alpha_j \beta_j \\
&= \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2 \right)^{\frac{1}{2}} \\
&\geq \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty.
\end{aligned}$$

□

The following result gives a characterization of Gaussian measures on separable Hilbert spaces.

Theorem 1.2.5 *A Borel probability measure μ on H is a Gaussian measure if and only if its Fourier transform is given by*

$$\widehat{\mu}(x) = \exp\left(i < m, x > -\frac{1}{2} < Bx, x >\right),$$

where $m \in H$, B is a positive symmetric trace class operator on H. In this case, m and B are the mean vector and covariance operator of μ respectively. Moreover,

$$\int_H |x|^2 \mu(dx) = \mathrm{Tr} B + |m|^2.$$

Proof: Let μ be a Gaussian measure on H.
Claim: $\int_H |x|^2 \mu(dx) < \infty$.
By assumption, for any $x \in H$, $\langle x, \cdot \rangle$ has a Gaussian distribution. Thus there are $m_x \in \mathbb{R}$, and $\sigma_x > 0$ such that

$$\widehat{\mu}(x) = \int_H e^{i<x,z>} \mu(dz) = \exp\left(i m_x - \frac{1}{2}\sigma_x^2\right). \tag{1.4}$$

Let (e_j) be an orthonormal basis of H. Since $\int_{\mathbb{R}} (\xi - m)^2 \mathcal{N}(m, \sigma^2)(d\xi) = \sigma^2$ and
$\int_{\mathbb{R}} \xi \mathcal{N}(m, \sigma^2)(d\xi) = m$, we have

$$\begin{aligned} \int_H |x|^2 \mu(dx) &= \sum_{j=1}^{\infty} \int_H \langle x, e_j \rangle^2 \mu(dx) \\ &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} x_j^2 \mu(dx_j) \\ &= \sum_{j=1}^{\infty} (\sigma_{e_j}^2 + m_{e_j}^2). \end{aligned}$$

Let $(\beta_j) \subseteq \mathbb{R}$ such that $\beta_j m_{e_j} \geq 0$ and $\sum_{j=1}^{\infty} \beta_j^2 < \infty$. Set

$$\xi(x) := \sum_{j=1}^{\infty} \beta_j \langle e_j, x \rangle$$

By Schwarz'inequality, the above series converges absolutely and

$$|\xi(x)| \leq \left(\sum_{j=1}^{\infty} \beta_j^2\right)^{\frac{1}{2}} |x|, \quad x \in H.$$

Moreover, ξ is linear. So by Riesz'representation theorem there is $z \in H$ such that $\xi(x) = \langle z, x \rangle$, $x \in H$. By assumption $\xi = \langle z, \cdot \rangle$ is a Gaussian variable with a finite mean, i.e., $\sum_{j=1}^{\infty} \beta_j m_{e_j} < \infty$. Now, by Lemma 1.2.4,

$\sum_{j=1}^{\infty} m_{e_j}^2 < \infty$. Thus, in order to prove $\int_H |x|^2 \mu(dx) < \infty$, it suffices to check $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$.
By Theorem 1.1.5, there is a positive, symmetric, trace class operator Q such that

$$\langle Qx, x\rangle \ < 1 \Rightarrow 1 - \Re\widehat{\mu}(x) < \frac{1}{6}.$$

Hence,

$$1 - \exp\left(-\frac{1}{2}\sigma_x^2\right) \leq 1 - \Re\widehat{\mu}(x) \leq 2\langle Qx, x\rangle + \frac{1}{6}, \ \ \forall\, x \in H. \tag{1.5}$$

Without loss of generality we may assume that the kernel of Q is $\{0\}$.
For $x \in H \setminus \{0\}$, set $y := \frac{1}{\sqrt{3<Qx,x>}} x$. Then

$$\sigma_y^2 = \frac{1}{3\langle Qx, x\rangle}\sigma_x^2, \ \text{ and } \langle Qy, y\rangle = \frac{1}{3}.$$

Replacing x by y in (1.5), we obtain

$$1 - \exp\left(-\frac{\sigma_x^2}{6\langle Qx, x\rangle}\right) \leq \frac{2}{3} + \frac{1}{6}.$$

This implies that

$$\sigma_x^2 \leq (6 \log 6)\langle Qx, x\rangle, \quad x \in H.$$

Thus,

$$\sum_{j=1}^{\infty} \sigma_{e_j}^2 \leq (6 \log 6)\mathrm{Tr}Q < \infty.$$

Hence, $\int_H |x|^2 \mu(dx) < \infty$ and the claim is proved. So by the remark following Definition 1.2.1 the mean vector m and the covariance operator B of μ exist. The above notation gives

$$m_x = \int_H \langle x, z\rangle \mu(dz) = \langle m, x\rangle \ \text{ and}$$

$$\begin{aligned} \sigma_x^2 &= \int_H \langle x, z\rangle^2 \mu(dz) - m_x^2 \\ &= \int_H [\langle x, z\rangle^2 - \langle m, x\rangle^2]\mu(dz) \\ &= \int_H \langle x, z - m\rangle^2 \mu(dz) = \langle Bx, x\rangle. \end{aligned}$$

From (1.4) we obtain

$$\widehat{\mu}(x) = \exp\left(i\langle m, x\rangle - \frac{1}{2}\langle Bx, x\rangle\right), \quad x \in H.$$

Moreover,

$$\int_H |x|^2 \mu(dx) = \sum_{j=1}^{\infty} (\sigma_{e_j}^2 + m_{e_j}^2) = \mathrm{Tr}B + |m|^2$$

which proves the first implication.
Conversely, let $m \in H$ and B be a positive, symmetric, trace class operator, and consider the positive definite functional

$$\phi(x) = \exp\left(i\langle m, x\rangle - \frac{1}{2}\langle Bx, x\rangle\right), \quad x \in H.$$

Set $Qx := Bx + \langle m, x\rangle m$, $x \in H$. Then Q is a positive, symmetric, trace class operator on H. Define $|\cdot|_Q$ on H as follows

$$|x|_Q = |Q^{1/2}x| = \langle Qx, x\rangle^{1/2} = \left(\langle Bx, x\rangle + \langle m, x\rangle^2\right)^{1/2}.$$

Then $\phi(x)$ is continuous at $x = 0$ with respect to $|\cdot|_Q$. So by Theorem 1.1.5, ϕ is the Fourier transform of some Borel probability measure μ on H. Clearly for any $x \in H$, $\langle x, \cdot\rangle$ is a Gaussian random variable with mean $\langle m, x\rangle$ and covariance $\langle Bx, x\rangle$ under μ. Thus, μ is a Gaussian measure. $\square$

A Gaussian measure with mean vector m and covariance operator B will be denoted by $\mathcal{N}(m, B)$. We propose now to compute some Gaussian integrals.

Proposition 1.2.6 *Let $\mathcal{N}(0, B)$ be a Gaussian measure on H. Then there is an orthonormal basis (e_n) of H such that $Be_n = \lambda_n e_n$, $\lambda_n \geq 0$, $n \in \mathbb{N}$. Moreover, for any $\alpha < \alpha_0 := \inf_n \frac{1}{\lambda_n}$, we have*

$$\int_H e^{\frac{\alpha}{2}|x|^2} \mathcal{N}(0, B)(dx) = \left(\prod_{k=1}^{\infty}(1 - \alpha\lambda_k)\right)^{-\frac{1}{2}} = (\det(I - \alpha B))^{-\frac{1}{2}}.$$

Proof: The first assertion follows from the fact that B is symmetric and positive. Since $\mathrm{Tr}B = \sum_{k=1}^{\infty} \lambda_k < \infty$, it follows that

$$0 \neq \prod_{k=1}^{\infty}(1 - \alpha\lambda_k)^{-\frac{1}{2}} < \infty \quad \text{for } \alpha < \alpha_0.$$

Furthermore,

$$\begin{aligned}
\int_H e^{\frac{\alpha}{2}|<x, e_1>|^2} \mathcal{N}(0, B)(dx) &= \int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2} \mathcal{N}(0, \lambda_1)(d\xi) \\
&= \frac{1}{\sqrt{2\pi\lambda_1}} \int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2} e^{-\frac{\xi^2}{2\lambda_1}}\, d\xi \\
&= (1 - \alpha\lambda_1)^{-\frac{1}{2}}.
\end{aligned}$$

In similar way we have

$$\int_H e^{\frac{\alpha}{2}\sum_{k=1}^n |<x,e_k>|^2} \mathcal{N}(0,B)(dx) = \left(\prod_{k=1}^n (1-\alpha\lambda_k)\right)^{-\frac{1}{2}}$$

and the result follows from the monotone convergence theorem. □

Before proving a more general result we propose first to study the transformation of a Gaussian measure by an affine mapping.

Lemma 1.2.7 *Let H and $\widetilde{H}$ be two separable Hilbert spaces. Consider the affine transformation $F : H \to \widetilde{H}$ defined by $F(x) = Qx + z$, where $Q \in \mathcal{L}(H, \widetilde{H})$ and $z \in \widetilde{H}$. If we set $\mu_F := \mathcal{N}(m,B) \circ F^{-1}$, the measure defined by $\mu_F(A) = \mathcal{N}(m,B)(F^{-1}(A))$, $A \in \mathcal{B}(\widetilde{H})$, then*

$$\mu_F = \mathcal{N}(Qm + z, QBQ^*).$$

Proof: Let compute the Fourier transform of μ_F. From Theorem 1.2.5 we obtain

$$\begin{aligned}
\widehat{\mu_F}(x) &= \int_{\widetilde{H}} e^{i\langle x,\widetilde{y}\rangle} \mu_F(d\widetilde{y}) \\
&= \int_H e^{i\langle x,Qy+z\rangle} \mu(dy) \\
&= e^{i\langle x,z\rangle} \int_H e^{i\langle Q^*x,y\rangle} \mu(dy) \\
&= e^{i\langle x,Qm+z\rangle} e^{-\frac{1}{2}\langle QBQ^*x,x\rangle} \\
&= \mathcal{N}(Qm \widehat{+ z, QB}Q^*)(x)
\end{aligned}$$

for $x \in H$. So the lemma follows from Theorem 1.2.5. □

From the above lemma follows the following result.

Proposition 1.2.8 *Let $\alpha_0 := \inf_k \frac{1}{\lambda_k}$. Then, for any $\alpha < \alpha_0$,*

$$\int_H e^{\frac{\alpha}{2}|x|^2} \mathcal{N}(m,B)(dx) = (\det(I-\alpha B))^{-\frac{1}{2}} \exp\left(\frac{\alpha}{2}\langle (I-\alpha B)^{-1}m, m\rangle\right).$$

Proof: From Lemma 1.2.7 we have

$$
\begin{aligned}
\int_H e^{\frac{\alpha}{2}|x|^2}\mathcal{N}(m,B)(dx) &= \int_H e^{\frac{\alpha}{2}|x+m|^2}\mathcal{N}(0,B)(dx)\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2+\alpha m_k\xi}e^{-\frac{\xi^2}{2\lambda_k}}d\xi\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}} e^{-\left[\frac{1-\alpha\lambda_k}{2\lambda_k}\xi^2-\alpha m_k\xi\right]}d\xi\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}e^{\frac{\lambda_k\alpha^2m_k^2}{2(1-\alpha\lambda_k)}}\int_{\mathbb{R}} e^{-\frac{(1-\alpha\lambda_k)}{2\lambda_k}(\xi-\frac{\lambda_k\alpha m_k}{1-\alpha\lambda_k})^2}d\xi\\
&= \prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}e^{\frac{\alpha}{2}m_k^2}e^{\frac{\lambda_k\alpha^2m_k^2}{2(1-\alpha\lambda_k)}}\left(\int_{\mathbb{R}} e^{-\xi^2}d\xi\right)\left(\frac{2\lambda_k}{1-\alpha\lambda_k}\right)^{\frac12}\\
&= \prod_{k=1}^{\infty}(1-\alpha\lambda_k)^{-\frac12}e^{\frac{\alpha m_k^2}{2(1-\alpha\lambda_k)}}\\
&= (\det(I-\alpha B))^{-\frac12}\,e^{\frac{\alpha}{2}\langle (I-\alpha B)^{-1}m,m\rangle}.
\end{aligned}
$$

□

Example 1.2.9 *Let compute the integrals*

(a)

$$\int_H |x|^{2m}\mathcal{N}(0,B)(dx),\quad m\in\mathbb{N},$$

(b)

$$\int_H |My|^2\,\mathcal{N}(0,B)(dy),\quad \textit{where } M\in\mathcal{L}(H).$$

(a) For the integral in (a) we consider the function

$$f(\alpha) := \int_H e^{\frac{\alpha}{2}|x|^2}\mathcal{N}(0,B)(dx) = (\det(I-\alpha B))^{-\frac12}\quad \textit{for } \alpha<\alpha_0.$$

Now, it is easy to see that $(-\infty,\alpha_0)\ni\alpha\mapsto\det(I-\alpha B)$ *is* C^∞ *and*

$$\frac{d}{d\alpha}\det(I-\alpha B) = \mathrm{Tr}(B(I-\alpha B)^{-1})\det(I-\alpha B),\quad \alpha<\alpha_0.$$

Furthermore we can differentiate under the integral sign. Hence,

$$\int_H |x|^{2m}\mathcal{N}(0,B)(dx) = 2^m\frac{d^m}{d\alpha^m}\left(\det(I-\alpha B)\right)^{-\frac12}_{|\alpha=0}.$$

This implies that

$$\int_H |x|^2 \mathcal{N}(0,B)(dx) = \mathrm{Tr}B$$

and

$$\int_H |x|^4 \mathcal{N}(0,B)(dx) = 2\mathrm{Tr}B^2 + (\mathrm{Tr}B)^2.$$

(b) It follows from Lemma 1.2.7 that

$$\int_H |My|^2\, \mathcal{N}(0,B)(dy) = \int_H |y|^2\, \mathcal{N}(0, MBM^*)(dy).$$

So by Theorem 1.2.5 we have

$$\int_H |My|^2\, \mathcal{N}(0,B)(dy) = \mathrm{Tr}(MBM^*) = \mathrm{Tr}(M^*MB). \tag{1.6}$$

By a same computation as above one has

Proposition 1.2.10 *For any $\alpha,\ m \in H$, we have*

$$\int_H e^{\langle \alpha, x\rangle} \mathcal{N}(m,B)(dx) = e^{\langle \alpha, m\rangle} e^{\frac{1}{2}\langle B\alpha,\alpha\rangle}.$$

We end this section by proving that the positive definite functional on H defined by $\varphi(x) = e^{-\frac{1}{2}|x|^2}$, $x \in H$, is not the Fourier transform of any Borel measures provided that $\dim H = \infty$.

Proposition 1.2.11 *Let Q be a positive, symmetric operator on a separable Hilbert space H. Then the functional*

$$\phi(x) = \exp\left(-\frac{1}{2} < Qx, x >\right), \quad x \in H,$$

is the Fourier transform of a probability measure on H if and only if Tr$Q < \infty$.

Proof: Suppose that Tr$Q < \infty$. Then $\phi(0) = 1$ and ϕ is $|\cdot|_Q$-continuous positive functional on H. So by Theorem 1.1.5 there exists a probability measure μ such that $\widehat{\mu}(x) = \phi(x),\ x \in H$.
To show the converse, assume that there is a probability measure μ such that

$$\int_H e^{i<x,y>} \mu(dy) = \exp\left(-\frac{1}{2} < Qx, x >\right).$$

Then by Theorem 1.1.5, for any $\varepsilon \in (0, \frac{1}{3})$, there exists a positive, symmetric operator Q_ε of trace class such that

$$\begin{aligned} < Q_\varepsilon x, x > < 1 \quad &\Rightarrow \quad \phi(0) - \mathrm{Re}\phi(x) < \varepsilon \\ &\Rightarrow \quad < Qx, x > < 3\varepsilon. \end{aligned}$$

Let now $y_0 \in H$ and $< Q_\varepsilon y_0, y_0 >=: c^2$, with $c > 0$. Let $d > c$ arbitrary. Then
$< Q_\varepsilon \frac{y_0}{d}, \frac{y_0}{d} >= \frac{c^2}{d^2} < 1$.
Hence, $< Q\frac{y_0}{d}, \frac{y_0}{d} > < \varepsilon$, i.e. $< Qy_0, y_0 > < \varepsilon d^2$. Letting $d \to c$, we have $< Qy_0, y_0 >\leq \varepsilon < Q_\varepsilon y_0, y_0 >$. Since y_0 is arbitrary, we obtain

$$< Qy, y > \leq \varepsilon < Q_\varepsilon y, y >$$

for all $y \in H$. In particular, for an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, we obtain

$$\mathrm{Tr}Q = \sum_n < Qe_n, e_n > \leq \varepsilon \sum_n < Q_\varepsilon e_n, e_n >= \varepsilon \mathrm{Tr}Q_\varepsilon < \infty.$$

□

As an immediate consequence we obtain that the functional

$$\phi(x) = \exp\left(-\frac{1}{2}|x|^2\right), \quad x \in H,$$

is not the Fourier transform of any probability measure on H if $\dim H = \infty$.

1.3 THE HELLINGER INTEGRAL AND THE CAMERON-MARTIN THEOREM

The **Cameron-Martin formula** permits us to differentiate under the integral sign with respect to Gaussian measures in infinite dimensional Hilbert spaces. This allows us to obtain some regularity results for parabolic equations with many infinitely variables.
First we need some preparations.
We denote by $\mathcal{L}_1^+(H)$ the space of all positive, symmetric operators of trace class on a separable Hilbert space H. Let $B \in \mathcal{L}_1^+(H)$ and consider an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $Be_n = \lambda_n e_n, \ n \in \mathbb{N}$. Suppose also that $\ker B = \{0\}$.
If we denote by $x_n :=< x, e_n >$, then

$$Bx = \sum_{n=1}^{\infty} \lambda_n x_n e_n \quad \text{and} \quad B^{\frac{1}{2}}x = \sum_{n=1}^{\infty} \lambda^{\frac{1}{2}} x_n e_n, \quad x \in H.$$

We set also

$$B_n x := \sum_{k=1}^{n} \lambda_k x_k e_k \quad \text{and} \quad B_n^{-\frac{1}{2}}x := \sum_{k=1}^{n} \lambda_k^{-\frac{1}{2}} x_k e_k.$$

Let consider, for $a \in H$ and $n \in \mathbb{N}$, the function

$$g_{a,n}(x) := \langle a, B_n^{-\frac{1}{2}}x\rangle = \sum_{k=1}^{n} \lambda_k^{-\frac{1}{2}} x_k a_k.$$

If $a \in B^{\frac{1}{2}}(H)$ then one can define the function

$$g_a(x) := \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} x_k a_k, \quad x \in H.$$

The following proposition shows that it is always possible to define g_a as an $L^2(H,\mu)$-function even if $a \notin B^{\frac{1}{2}}(H)$.

Proposition 1.3.1 *Let $B \in \mathcal{L}_1^+(H)$ with $\ker B = \{0\}$ and $\mu := \mathcal{N}(0,B)$ its corresponding Gaussian measure on H. Then the limit*

$$\lim_{n\to+\infty} g_{a,n} =: g_a$$

exists in $L^2(H,\mu)$. Moreover,

$$\int_H |g_a(x)|^2 \mu(dx) = |a|^2$$

for a given $a \in H$.

Proof: We have

$$\begin{aligned}
\int_H |g_{a,n+p}(x) - g_{a,n}(x)|^2 \mu(dx) &= \int_H \left| \sum_{k=n+1}^{n+p} \lambda_k^{-\frac{1}{2}} x_k a_k \right|^2 \mu(dx) \\
&= \sum_{h,k=n+1}^{n+p} (\lambda_h \lambda_k)^{-\frac{1}{2}} a_h a_k \int_H x_h x_k \mu(dx) \\
&= \sum_{k=n+1}^{n+p} \lambda_k^{-1} a_k^2 \int_H x_k^2 \mu(dx) \\
&= \sum_{k=n+1}^{n+p} a_k^2.
\end{aligned}$$

Hence $(g_{a,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(H,\mu)$. Moreover,

$$\int_H |g_{a,n}|^2 \mu(dx) \sum_{k=1}^{n} \frac{1}{\lambda_k} a_k^2 \int_H x_k^2 \mu(dx) \sum_{k=1}^{n} a_k^2$$

and the theorem is proved by letting $n \to \infty$. □

Remark 1.3.2 *Suppose that $\ker B = \{0\}$ and take $x \in H$ such that $\langle B^{\frac{1}{2}} a, x\rangle = 0$ for all $a \in H$. Hence, $B^{\frac{1}{2}} x = 0$ and so $Bx = 0$, which implies that $x = 0$. This proves that $B^{\frac{1}{2}}(H)$ is dense in H. For the converse, let $x \in H$ with $Bx = 0$. Thus, $B^{\frac{1}{2}} x = 0$ and hence, $\langle B^{\frac{1}{2}} x, y\rangle = \langle x, B^{\frac{1}{2}} y\rangle = 0$ for*

all $y \in H$. Since $\overline{B^{\frac{1}{2}}(H)} = H$, it follows that $x = 0$.
By the same arguments as in the proof of Proposition 1.3.1 one can show that g_a is well defined as an $L^2(H,\mu)$–function and

$$\|g_a\|_{L^2(H,\mu)} = |a| \quad \text{for } a \in \overline{B^{\frac{1}{2}}(H)}.$$

In the sequel we will use the notation

$$g_a(x) := \langle a, B^{-\frac{1}{2}}x\rangle, \quad x \in H.$$

Proposition 1.3.3 *Let $B \in \mathcal{L}_1^+(H)$ with $\ker B = \{0\}$ and $\mu := \mathcal{N}(0,B)$ its corresponding Gaussian measure on H. Then the limit*

$$\lim_{n\to\infty} e^{g_{a,n}} =: e^{g_a}$$

exists in $L^2(H,\mu)$ for a given $a \in H$. Moreover, for any $a \in H$,

$$\int_H e^{\langle a, B^{-\frac{1}{2}}x\rangle}\mathcal{N}(0,B)(dx) = e^{\frac{1}{2}|a|^2}.$$

Proof: By applying Proposition 1.2.10 we obtain

$$\begin{aligned}
&\int_H |e^{g_{a,n}} - e^{g_{a,m}}|^2\mu(dx) \\
=\ &\int_H \left(e^{2\langle B_n^{-\frac{1}{2}}a,x\rangle} - 2e^{\langle B_n^{-\frac{1}{2}}a,x\rangle + \langle B_m^{-\frac{1}{2}}a,x\rangle} + e^{2\langle B_m^{-\frac{1}{2}}a,x\rangle}\right)\mu(dx) \\
=\ &e^{2\sum_{k=1}^n a_k^2} + e^{2\sum_{k=1}^m a_k^2} - 2\int_H e^{\langle (B_n^{-\frac{1}{2}} + B_m^{-\frac{1}{2}})a,x\rangle}\mu(dx) \\
=\ &e^{2\sum_{k=1}^n a_k^2} + e^{2\sum_{k=1}^m a_k^2} - 2e^{2\sum_{k=1}^n a_k^2 + \frac{1}{2}\sum_{k=n+1}^m a_k^2} \\
=\ &e^{2\sum_{k=1}^n a_k^2}\left(1 + e^{2\sum_{k=n+1}^m a_k^2} - 2e^{\frac{1}{2}\sum_{k=n+1}^m a_k^2}\right) \longrightarrow 0 \ \ (n,m\to\infty).
\end{aligned}$$

This proves that $(e^{g_{a,n}})$ is a Cauchy sequence in $L^2(H,\mu)$ and one can see that

$$\int_H e^{\langle a, B^{-\frac{1}{2}}x\rangle}\mathcal{N}(0,B)(dx) = e^{\frac{1}{2}|a|^2}$$

is satisfied for every $a \in H$. □

We propose now to recall the definition of the **Hellinger** integral.
Let μ, ν be two probability measures on a measurable space (Ω,Σ). We say that μ and ν are **singular** (notation: $\mu\perp\nu$) if there is a set $B \in \Sigma$ such that $\mu(B) = 0$ and $\nu(\Omega \setminus B) = 0$. It is easy to see that two probability measures μ and ν are singular if and only if for any $\varepsilon > 0$ there is $B \in \Sigma$ such that $\mu(B) < \varepsilon$ and $\nu(\Omega \setminus B) < \varepsilon$. Further, μ is called ν-**absolutely continuous** (notation: $\mu \prec \nu$) if $\nu(B) = 0$ implies $\mu(B) = 0$ for any $B \in \Sigma$. So by the theorem of Radon-Nikodym we know that if μ is ν-absolutely continuous,

then there is a non-negative measurable function φ defined on Ω, called the **density function** of μ, such that

$$\mu(B) = \int_B \varphi(\omega)\nu(d\omega)$$

for any $B \in \Sigma$. The density φ is denoted by

$$\varphi(\omega) := \frac{d\mu}{d\nu}(\omega), \quad \omega \in \Omega.$$

If $\mu \prec \nu$ and $\nu \prec \mu$ are satisfied then μ and ν are called **equivalent** (notation: $\mu \sim \nu$). If $\mu \sim \nu$, then the two density functions $\varphi = \frac{d\mu}{d\nu}$ and $\psi = \frac{d\nu}{d\mu}$ satisfy $\varphi(\omega)\psi(\omega) = 1$, a.e. $\omega \in \Omega$. Hence, $\varphi(\omega) > 0$ μ-a.e. $\omega \in \Omega$.

Let now μ and ν two arbitrary probability measures on (Ω, Σ). Let γ be a probability measure on (Ω, Σ) such that $\mu \prec \gamma$ and $\nu \prec \gamma$. Such a measure exists, we have to take for example $\gamma = \frac{1}{2}(\mu + \nu)$. Thus, the following integral is well-defined

$$H(\mu, \nu) := \int_\Omega \sqrt{\frac{d\mu}{d\gamma}(\omega)\frac{d\nu}{d\gamma}(\omega)}\,\gamma(d\omega).$$

This integral will be called the **Hellinger** integral.

Let now consider the measurable space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$, where $\mathcal{B}(\mathbb{R}^\infty)$ is the Borel field of subsets B of $\mathbb{R}^\infty$. On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we consider two sequences of measures (μ_n) and $\nu_n)$ with

$$\mu_n \sim \nu_n, \quad \forall n \in \mathbb{N}. \tag{1.7}$$

Then one has

$$H(\mu_n, \nu_n) = \int_{\mathbb{R}} \sqrt{\frac{d\nu_n}{d\mu_n}(x_n)}\,\mu_n(dx_n).$$

Let us consider two infinite product measures

$$\mu := \prod_{n=1}^\infty \mu_n \text{ and } \nu := \prod_{n=1}^\infty \nu_n$$

defined on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. The following result is du to S. Kakutani [21] and gives a condition under which these two measures μ and ν are equivalent.

Theorem 1.3.4 *Assume that (1.7) is satisfied. Then the following assertions hold.*

(i) *If* $\prod_{n=1}^\infty H(\mu_n, \nu_n) > 0$ *then* $\mu \sim \nu$ *and*

$$\frac{d\nu}{d\mu}(x) = \prod_{k=1}^\infty \frac{d\nu_k}{d\mu_k}(x), \quad \text{a.e. } x \in \mathbb{R}^\infty.$$

(ii) *If* $\prod_{n=1}^{\infty} H(\mu_n, \nu_n) = 0$ *then* $\mu \perp \nu$.

Moreover,

$$H(\mu, \nu) = \prod_{n=1}^{\infty} H(\mu_n, \nu_n). \tag{1.8}$$

Proof: If we set $\psi_n(x) := \prod_{k=1}^{n} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}$ for $x \in \mathbb{R}^\infty$ and $n \in \mathbb{N}$, then

$$\|\psi_n\|^2_{L^2(\mathbb{R}^\infty, \mu)} = \int_{\mathbb{R}^\infty} \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k)\, \mu(dx) = \prod_{k=1}^{n} \int_{\mathbb{R}} \nu_k(dx_k) = 1 \text{ and}$$

$$\begin{aligned}
\|\psi_n - \psi_m\|^2_{L^2(\mathbb{R}^\infty, \mu)} &= \int_{\mathbb{R}^\infty} \left(\prod_{k=1}^{n} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} - \prod_{k=1}^{m} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \right)^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) \left(1 - \prod_{k=n+1}^{m} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \right)^2 \mu(dx) \\
&= 2\left(1 - \prod_{k=n+1}^{m} \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)\, \mu_k(dx_k)} \right) \\
&= 2\left(1 - \prod_{k=n+1}^{m} H(\mu_k, \nu_k) \right) \qquad (1.9)
\end{aligned}$$

for any positive integers n and m with $n < m$.
(i) If $\prod_{n=1}^{\infty} H(\mu_n, \nu_n) > 0$ then

$$\lim_{n,m\to\infty} \prod_{k=n+1}^{m} H(\mu_k, \nu_k) = 1.$$

Hence, by (1.9), (ψ_n) is a Cauchy sequence in $L^2(\mathbb{R}^\infty, \mu)$ and so there is $\psi \in L^2(\mathbb{R}^\infty, \mu)$ such that $\lim_{n\to\infty} \|\psi_n - \psi\|_{L^2(\mathbb{R}^\infty, \mu)} = 0$.
Let prove now that $\nu \prec \mu$ and $\frac{d\nu}{d\mu}(x) = (\psi(x))^2$, $x \in \mathbb{R}^\infty$, i.e.

$$\nu(B) = \int_B (\psi(x))^2\, \mu(dx)$$

for any $B \in \mathcal{B}(\mathbb{R}^\infty)$. To this purpose it follows from Hölder's inequality and (1.9) that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^\infty} |\psi_m(x)^2 - \psi_n(x)^2| \, \mu(dx)\right)^2 \\
\leq\; & \int_{\mathbb{R}^\infty} |\psi_m(x) + \psi_n(x)|^2 \, \mu(dx) \int_{\mathbb{R}^\infty} |\psi_m(x) - \psi_n(x)|^2 \, \mu(dx) \\
\leq\; & 4 \int_{\mathbb{R}^\infty} |\psi_m(x) - \psi_n(x)|^2 \, \mu(dx) \\
=\; & 8 \left(1 - \prod_{k=n+1}^{m} H(\mu_k, \nu_k)\right)
\end{aligned}$$

for $n < m$. Thus,

$$\lim_{n\to\infty} \|\psi_n^2 - \psi^2\|_{L^1(\mathbb{R}^\infty,\mu)} = 0.$$

Finally let $B \in \mathcal{B}(\mathbb{R}^\infty)$ and set $\chi_n(x) := \chi_B(P_n x)$, $x \in \mathbb{R}^\infty$, where $\chi_B(\cdot)$ denotes the characteristic function of the measurable set B and $P_n x := (x_1, \ldots, x_n, 0, \ldots)$. So we have

$$\begin{aligned}
\int_{\mathbb{R}^\infty} \chi_n(x) \, \nu(dx) &= \int_{\mathbb{R}^n} \chi_B(x_1, \ldots, x_n, 0, \ldots) \, \nu_1(dx_1) \ldots \nu_n(dx_n) \\
&= \int_{\mathbb{R}^n} \chi_n(x) \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) \prod_{k=1}^{n} \mu_k(dx_k) \\
&= \int_{\mathbb{R}^\infty} \chi_n(x) \psi_n(x)^2 \, \mu(dx).
\end{aligned}$$

Since $\psi_n^2 \to \psi^2$ in $L^1(\mathbb{R}^\infty, \mu)$ and by letting $n \to \infty$ we obtain

$$\nu(B) = \int_{\mathbb{R}^\infty} \psi(x)^2 \, \mu(dx).$$

In a similar way one can see that $\mu \prec \nu$. So we obtain $\mu \sim \nu$. Finally, since $\mu \sim \nu$, we have

$$\begin{aligned}
H(\mu, \nu) &= \int_{\mathbb{R}^\infty} \psi(x) \, \mu(dx) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^\infty} \psi_n(x) \, \mu(dx) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \, \mu_k(dx_k) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} H(\mu_k, \nu_k).
\end{aligned}$$

So we obtain (1.8).
(ii) If $\prod_{k=1}^{\infty} H(\mu_k,\nu_k) = 0$ then for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\prod_{k=1}^{n} H(\mu_k,\nu_k) < \varepsilon$. Let $B_n \in \mathcal{B}(\mathbb{R}^n)$ with

$$B_n := \{(x_1,\dots,x_n) \in \mathbb{R}^n : \psi_n(x_1,\dots,x_n,0,\dots)^2 = \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) > 1\}.$$

Then,

$$\begin{aligned}
\left(\prod_{k=1}^{n} \mu_k\right)(B_n) &= \int_{B_n} \left(\prod_{k=1}^{n} \mu_k\right)(dx) \\
&< \int_{B_n} \psi_n(x_1,\dots,x_n,0,\dots) \left(\prod_{k=1}^{n} \mu_k\right)(dx) \\
&= \int_{B_n} \prod_{k=1}^{n} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \left(\prod_{k=1}^{n} \mu_k\right)(dx) \\
&\leq \prod_{k=1}^{n} H(\mu_k,\nu_k) < \varepsilon.
\end{aligned}$$

By the same computation we obtain

$$\left(\prod_{k=1}^{n} \nu_k\right)(\mathbb{R}^n \setminus B_n) \leq \prod_{k=1}^{n} H(\mu_k,\nu_k) < \varepsilon.$$

Therefore, if we set $B := B_n \times \prod_{k=n+1}^{\infty} \mathbb{R}$, then

$$\mu(B) < \varepsilon \text{ and } \nu(\mathbb{R}^\infty \setminus B) < \varepsilon.$$

This proves that $\mu \perp \nu$. Suppose now that $\mu \perp \nu$. Then there exists $B \in \mathcal{B}(\mathbb{R}^\infty)$ such that $\mu(B) = 0$ and $\nu(\mathbb{R}^\infty \setminus B) = 0$. So by Hölder's inequality, it follows that

$$\begin{aligned}
H(\mu,\nu) &= \int_B \sqrt{\frac{d\mu}{d\gamma}(x)\frac{d\nu}{d\gamma}(x)}\,\gamma(dx) + \int_{\mathbb{R}^\infty \setminus B} \sqrt{\frac{d\mu}{d\gamma}(x)\frac{d\nu}{d\gamma}(x)}\,\gamma(dx) \\
&\leq \left(\int_B \frac{d\mu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac{1}{2}} \left(\int_B \frac{d\nu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac{1}{2}} + \\
&\quad \left(\int_{\mathbb{R}^\infty \setminus B} \frac{d\mu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^\infty \setminus B} \frac{d\nu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac{1}{2}} \\
&= \mu(B)^{\frac{1}{2}}\nu(B)^{\frac{1}{2}} + \mu(\mathbb{R}^\infty \setminus B)^{\frac{1}{2}}\nu(\mathbb{R}^\infty \setminus B)^{\frac{1}{2}} = 0.
\end{aligned}$$

Therefore, (1.8) holds. This end the proof of the theorem. □

Let prove now the **Cameron-Martin formula**. We note here that the measure space $(H, \mathcal{B}(H))$ can be identified with $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$.

Corollary 1.3.5 *Let $B \in \mathcal{L}_1^+(H)$ such that $\ker B = \{0\}$ and $\mu := \mathcal{N}(0, B)$ and $\nu := \mathcal{N}(m, B)$ be two Gaussian measures on $(H, \mathcal{B}(H))$. Then the following assertions hold.*

(i) *The Gaussian measures μ and ν are equivalent if and only if $m \in B^{\frac{1}{2}}(H)$. Moreover the Radon-Nikodym derivative is given by*

$$\frac{d\nu}{d\mu}(x) = \exp\left(-\frac{1}{2}|B^{-\frac{1}{2}}m|^2 + < B^{-\frac{1}{2}}x, B^{-\frac{1}{2}}m >\right).$$

(ii) *The measures μ and ν are singular if and only if $m \notin B^{\frac{1}{2}}(H)$.*

Proof: We will apply Theorem 1.3.4 to the Gaussian measures μ and ν. To this purpose let compute the associated Hellinger integral using (1.8). It follows from Proposition 1.2.10 that

$$\begin{aligned} H(\mu_k, \nu_k) &= \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}\, \mu_k(dx_k) \\ &= e^{-\frac{m_k^2}{4\lambda_k}} \int_{\mathbb{R}} e^{\frac{m_k x_k}{2\lambda_k}}\, \mathcal{N}(0, \lambda_k)(dx_k) \\ &= e^{-\frac{m_k^2}{8\lambda_k}}. \end{aligned}$$

So by (1.8) we obtain

$$H(\mu, \nu) = \prod_{k=1}^{\infty} e^{-\frac{m_k^2}{8\lambda_k}}.$$

This implies that

$$\begin{aligned} H(\mu, \nu) > 0 \quad &\Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{m_k^2}{\lambda_k} < \infty \\ &\Longleftrightarrow \quad m \in B^{\frac{1}{2}}(H). \end{aligned}$$

Moreover, in this case, it follows from Theorem 1.3.4 that

$$\begin{aligned} \frac{d\nu}{d\mu}(x) &= \prod_{k=1}^{\infty} \frac{d\nu_k}{d\mu_k}(x) \\ &= \prod_{k=1}^{\infty} e^{-\frac{m_k^2}{2\lambda_k}} e^{\frac{x_k m_k}{\lambda_k}} \\ &= \exp\left(-\frac{1}{2}|B^{-\frac{1}{2}}m|^2 + \langle B^{-\frac{1}{2}}x, B^{-\frac{1}{2}}m\rangle\right), \end{aligned}$$

where $x = \sum_{k=1}^{\infty} x_k e_k$ with $x_k := \langle x, e_k\rangle$ for an orthonormal basis (e_n) of H such that $Be_n = \lambda_n e_n$ for $n \in \mathbb{N}$. Here we used Proposition 1.3.3.
Finally it is clear that the measures μ and ν are singular if and only if $m \notin B^{\frac{1}{2}}(H)$. □

Exercise 1.3.6 (The Feldman-Hajek theorem)
Let consider two linear operators $B_1, B_2 \in \mathcal{L}_1^+(H)$ *with* $\ker B_1 = \ker B_2 = \{0\}$ *and an orthonormal basis* (e_n) *of* H *such that* $B_1 e_n = \lambda_n e_n$, $n \in \mathbb{N}$, *where* $\lambda_n > 0$ *for all* $n \in \mathbb{N}$. *On* $(H, \mathcal{B}(H))$ *we consider the Gaussian measures* $\mu_1 := \mathcal{N}(0, B_1)$ *and* $\mu_2 := \mathcal{N}(0, B_2)$.

1. **The commutative case:** *Suppose that* $B_1 B_2 = B_2 B_1$. *By using Theorem 1.3.4 show that*

   a. *if* $\sum_{n=1}^\infty \frac{(\lambda_n - \alpha_n)^2}{(\lambda_n + \alpha_n)^2} < \infty$, *then* $\mu_1 \sim \mu_2$. *In this case*

   $$\frac{d\mu_2}{d\mu_1}(x) = \prod_{n=1}^{\infty} exp\left(-\frac{(\lambda_n - \alpha_n)}{2\lambda_n \alpha_n}\langle x, e_n\rangle^2\right),$$

   b. *if* $\sum_{n=1}^\infty \frac{(\lambda_n - \alpha_n)^2}{(\lambda_n + \alpha_n)^2} = \infty$, *then* $\mu_1 \perp \mu_2$.

   Here $\alpha_n > 0$, $n \in \mathbb{N}$, *are such that* $B_2 e_n = \alpha_n e_n$, $n \in \mathbb{N}$.

2. **The General case:**

   (a) *Assume that there is* $S \in \mathcal{L}_2^+(H)$ *such that*

   $$B_2 = B_1^{\frac{1}{2}}(Id - S)B_1^{\frac{1}{2}}.$$

   Show that $\mu_1 \sim \mu_2$.

   (b) *Assume that* $S \in \mathcal{L}_1^+(H)$ *and* $\|S\| < 1$. *Show that*

   $$\frac{d\mu_2}{d\mu_1}(x) = [\det(I - S)]^{-\frac{1}{2}} \exp(-\frac{1}{2}\langle S(I-S)^{-1}B_1^{\frac{1}{2}}x, B_1^{\frac{1}{2}}x\rangle), x \in H.$$

   Here $\mathcal{L}_2^+(H)$ *is the set of positive Hilbert-Schmidt bounded linear operators on* H. *That is,* $B \in \mathcal{L}_2^+(H)$ *if and only if* $B \in \mathcal{L}(H)$, B *positive and* $\sum_{n=1}^\infty |Be_n|^2 < \infty$.

CHAPTER 2

HEAT EQUATIONS IN HILBERT SPACES

In this chapter, H is a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H.
For $\varphi \in C_b(H)$, the space of continuous and bounded functions $\varphi : H \to \mathbb{R}$, we say that φ is differentiable in the direction $e_k,\ k \in \mathbb{N}$, if the limit

$$D_k\varphi(x) := \lim_{h\to 0} \frac{1}{h} \left(\varphi(x + he_k) - \varphi(x)\right), \qquad x \in H$$

exists in $C_b(H)$. The operator D_k will be considered as the linear operator in $C_b(H)$ defined by

$$D(D_k) := \left\{\varphi \in C_b(H) : \lim_{h\to 0} \frac{1}{h} \left(\varphi(\cdot + he_k) - \varphi(\cdot)\right) \text{ exists in } C_b(H)\right\}$$

and

$$D_k\varphi(x) = \lim_{h\to 0} \frac{1}{h} \left(\varphi(x + he_k) - \varphi(x)\right), \quad \varphi \in D(D_k),\ x \in H,\ h \in \mathbb{R}.$$

We start by showing that D_k is a closed operator on $C_b(H)$, for every $k \in \mathbb{N}$.
In fact, let $(\varphi_n)_{n\in\mathbb{N}} \subseteq D(D_k)$, and $\varphi, \psi\ \in C_b(H)$ such that

$$\varphi_n \longrightarrow \varphi \quad \text{and} \quad D_k\varphi_n \longrightarrow \psi \text{ in } C_b(H).$$

We consider $\phi_n, \phi \in C(C[-1, 1], C_b(H))$ defined by

$$\phi(h)(x) := \varphi(x + he_k) \quad \text{and} \quad \phi_n(h)(x) := \varphi_n(x + he_k),$$

$$x \in H,\ h \in [-1, 1] \text{ and } n \in \mathbb{N}.$$

Then ϕ_n is differentiable, as a function of the variable h, and

$$\frac{d}{dh}\phi_n(h)(x) = D_k\varphi_n(x + he_k).$$

So we have

$$\phi_n(h) - \phi_n(0) = \int_0^h \frac{d\phi_n}{dh}(s)\,ds$$

and by the assumption we obtain

$$\phi(h) - \phi(0) = \int_0^h \psi(\cdot + se_k)\,ds,$$

which implies that $\varphi \in D(D_k)$ and $D_k\varphi = \psi$.
In a similar way we can define partial derivatives of any order.
Now, we fix a sequence $(\lambda_n)_{n\in\mathbb{N}},\ \lambda_n > 0$ for $n \in \mathbb{N}$. In this chapter we are interested to solve the heat equation

$$(HE) \quad \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n D_n^2 u(t,x), & t > 0,\ x \in H, \\ u(0,x) = \varphi(x), & x \in H,\ \varphi \in C_b(H) \end{cases}$$

and to study the regularity of the solution u of (HE) in the case $\dim H = \infty$. For this purpose, let consider its finite dimensional approximation

$$(HE)_n \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{k=1}^{n}\lambda_k D_k^2 u(t,x), & t > 0,\ x \in H, \\ u(0,x) = \varphi(x), & x \in H,\ \varphi \in C_b(H). \end{cases}$$

It is easy to see that, for all $\varphi \in C_b(H)$, $(HE)_n$ has a unique classical solution given by

$$\begin{cases} u_n(t,x) = (2\pi t)^{-\frac{n}{2}}(\lambda_1 \ldots \lambda_n)^{-\frac{1}{2}}\int_{\mathbb{R}^n} e^{-\sum_{k=1}^n \frac{\xi_k^2}{2t\lambda_k}}\varphi(x - \sum_{k=1}^n \xi_k e_k)\,d\xi, \\ \hfill \text{if } t > 0 \\ u_n(0,x) = \varphi(x), \quad x \in H. \end{cases}$$

If we denote by

$$x_k := \langle x, e_k \rangle,\ x \in H$$

and

$$B_n := \begin{pmatrix} \lambda_1\, 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

then

$$u_n(t,x) = \int_{\mathbb{R}^n}\varphi\left(y + \sum_{k=n+1}^{\infty} x_k e_k\right)\mathcal{N}(x, tB_n)(dy),\ \ x \in H,\ t > 0.$$

In the sequel we denote by

$$P_t^{(n)}\varphi(x) := u_n(t,x)$$

for $t \geq 0,\ x \in H,\ n \in \mathbb{N}$, and $\varphi \in C_b(H)$. By an easy computation one has, for all $n \in \mathbb{N}$, $(P_t^{(n)})_{t\geq 0}$ is a semigroup on $C_b(H)$. Moreover, on $C_b(H)$, $(P_t^{(n)})$ is not strongly continuous at 0. In order to have strong continuity at 0 we have to work, for example, in $BUC(H)$, the space of all bounded and uniformly continuous functions from H into $\mathbb{R}$. Now, it is well-known that $(P_t^{(n)})$ is an analytic semigroup on $BUC(H)$ and

$$\|P_t^{(n)}\varphi\|_\infty \leq \|\varphi\|_\infty$$

for $\varphi \in BUC(H),\ t \geq 0$, and $n \in \mathbb{N}$.
Now, one asks under which conditions the limit

$$\lim_{n\to\infty} u_n(t,x) \quad \text{exists in } BUC(H)$$

for all $\varphi \in BUC(H)$?
A necessary condition for the existence of the above limit is

$$\sum_{n=1}^{\infty} \lambda_n < \infty.$$

In fact, let $\varphi(x) :=\exp\left(-\frac{1}{2}\|x\|^2\right)$. By applying Proposition 1.2.8 with $\alpha = -1,\ m = x$, and $B = tB_n$ one has

$$u_n(t,x) = \prod_{k=1}^{n} (1+\lambda_k t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{k=1}^{n} \frac{x_k^2}{1+\lambda_k t} - \frac{1}{2}\sum_{k=n+1}^{\infty} x_k^2 \right).$$

If $\lim_{n\to\infty} u_n(t,x)$ exists, then $\prod_{k=1}^{\infty}(1+t\lambda_k)^{-\frac{1}{2}}$ exists for $t > 0$. Hence,

$$\log \prod_{k=1}^{\infty}(1+t\lambda_k) = \sum_{k=1}^{\infty} \log(1+t\lambda_k),\ t > 0$$

exists. In particular, $\lim_{k\to\infty} \lambda_k = 0$. Set $M := \sup_n \lambda_n$. Then we have

$$mt\lambda_k \leq \log(1+t\lambda_k) \leq t\lambda_k,\ t > 0,\ k \in \mathbb{N},$$

where $m := \inf\{\frac{1}{\alpha}\log(1+\alpha),\ 0 < \alpha \leq M\}$. Therefore,

$$\sum_{k=1}^{\infty} \lambda_k < \infty$$

and

$$\lim_{n\to\infty} u_n(t,x) = u(t,x) = \prod_{k=1}^{\infty}(1+\lambda_k t)^{-\frac{1}{2}} e^{-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x_k^2}{1+t\lambda_k}}, \quad t > 0,\ x \in H.$$

If $\sum_{k=1}^{\infty} \lambda_k = \infty$, then

$$\lim_{n\to\infty} u_n(t,x) = \begin{cases} 0 & \text{if } x = 0,\ t \neq 0 \\ 1 & \text{if } x = 0,\ t = 0. \end{cases}$$

Hence, u_n does not converge to a continuous function.
Now, in the sequel we assume that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Set $Bx := \sum_{k=1}^{\infty} \lambda_k x_k,\ x \in H$. Then $B \in \mathcal{L}_1^+(H)$, $\ker B = \{0\}$, and Equation (HE) can be written as follows:

$$(HE) \quad \begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\mathrm{Tr}[BD^2 u(t,x)], & t > 0,\ x \in H, \\ u(0,x) = \varphi(x), & x \in H, \end{cases}$$

where $\varphi \in BUC(H)$.

Many results of this chapter can be found in the monographs [12] and [13].

2.1 Construction of the heat semigroup

In this section we are concerned with the construction of the solution of Equation (HE). To this purpose we suppose without loss of generality that $\lambda_k > 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \lambda_k < \infty$. The semigroup $(P_t^{(n)})$ can be written as

$$P_t^{(n)}\varphi = \prod_{k=1}^{n} T_k(t)\varphi, \quad t \geq 0,\ \varphi \in BUC(H),$$

where

$$T_k(t)\varphi(x) := \begin{cases} (2\pi t\lambda_k)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2t\lambda_k}} \varphi(x - se_k)\, ds & \text{if } t > 0 \\ \varphi(x), & \text{if } t = 0 \end{cases}$$

for $x \in H$ and $\varphi \in BUC(H)$. Note that $T_k(\cdot)$ is a C_0-semigroup of contractions on $BUC(H)$ for $k \in \mathbb{N}$. Before proving the strong convergence of P_t^n, $t \geq 0$, on $BUC(H)$, we recall some definitions and fix some notations.

We denote by $BUC^1(H)$ the subspace of $BUC(H)$ of all functions $\varphi : H \to \mathbb{R}$ which are Fréchet differentiable on H and the Fréchet derivative $D\varphi : H \to H$ is uniformly continuous and bounded. For $\varphi \in BUC^1(H)$ we set

$$\|\varphi\|_1 := \|\varphi\|_\infty + \sup_{x\in H} \|D\varphi(x)\|.$$

In the sequel we need the subspace $BUC^{1,1}(H)$ of $BUC^1(H)$ consisting of all functions $\varphi \in BUC^1(H)$ such that $D\varphi : H \to H$ is Lipschitz continuous and, for $\varphi \in BUC^{1,1}(H)$, we set

$$\|\varphi\|_{1,1} := \|\varphi\|_1 + \sup_{x,y\in H, x\neq y} \frac{\|D\varphi(x) - D\varphi(y)\|}{\|x - y\|}.$$

Theorem 2.1.1 *For all $\varphi \in BUC(H)$, the limit*

$$P_t\varphi := \lim_{n\to\infty} P_t^n\varphi$$

exists in $BUC(H)$, uniformly in t on bounded subsets of $\mathbb{R}^+$. Moreover (P_t) is a C_0-semigroup on $BUC(H)$ and

$$\|P_t\varphi\|_\infty \le \|\varphi\|_\infty$$

for $t \ge 0$ and $\varphi \in BUC(H)$.

Proof: Let compute first

$$\begin{aligned} P_t^n\varphi - P_t^{n-1}\varphi &= \prod_{k=1}^{n} T_k(t)\varphi - \prod_{k=1}^{n-1} T_k(t)\varphi \\ &= \prod_{k=1}^{n-1} T_k(t)(T_n(t)\varphi - \varphi), \end{aligned}$$

and hence,

$$\|P_t^n\varphi - P_t^{n-1}\varphi\|_\infty \le \|T_n(t)\varphi - \varphi\|_\infty, \quad t \ge 0,\ \varphi \in BUC(H),\ n \in \mathbb{N}.$$

So, for $\varphi \in BUC^{1,1}(H)$, we have

$$\begin{aligned} (T_n(t)\varphi - \varphi)(x) &= (2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \left(\varphi(x - se_n) - \varphi(x)\right) ds \\ &= (2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 -\frac{\partial}{\partial\gamma}\varphi\left(x - s(1-\gamma)e_n\right) d\gamma\, ds \\ &= -(2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 < D\varphi\left(x - s(1-\gamma)e_n\right), \\ &\quad se_n > d\gamma\, ds. \end{aligned}$$

Since,

$$\int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} < D\varphi(x), se_n > \, ds =< D\varphi(x), e_n > \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} s\, ds = 0,$$

it follows that

$$T_n(t)\varphi(x) - \varphi(x) = -(2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 < D\varphi(x - s(1-\gamma)e_n) - D\varphi(x),$$

$se_n > d\gamma\, ds.$

Thus,

$$|T_n(t)\varphi(x) - \varphi(x)| \le (2\pi\lambda_n t)^{\frac{1}{2}} \|\varphi\|_{1,1} \int_{\mathbb{R}} s^2 e^{-\frac{s^2}{2\lambda_n t}} ds = \lambda_n t \|\varphi\|_{1,1}.$$

Hence,

$$\|T_n(t)\varphi - \varphi\|_\infty \le \lambda_n t \|\varphi\|_{1,1}$$

for $t \ge 0,\ \varphi \in BUC^{1,1}(H)$, and $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&\|P_t^{n+p}\varphi - P_t^n\varphi\|_\infty \\
\le\ & \|\prod_{k=1}^{n+p} T_k(t)\varphi - \prod_{k=1}^{n+p-1} T_k(t)\varphi\|_\infty + \cdots + \|\prod_{k=1}^{n+1} T_k(t)\varphi - \prod_{k=1}^{n+1} T_k(t)\varphi\|_\infty \\
\le\ & \|T_{n+p}(t)\varphi - \varphi\|_\infty + \cdots + \|T_{n+1}(t)\varphi - \varphi\|_\infty \\
\le\ & t\|\varphi\|_{1,1} \sum_{k=n+1}^{n+p} \lambda_k, \quad n, p \in \mathbb{N}.
\end{aligned}$$

Since $\sum_{n=1}^\infty \lambda_n < \infty$, it follows that $(P_t^n\varphi)_n$ is a Cauchy sequence in $BUC(H)$, uniformly for t in bounded subsets of $\mathbb{R}_+$. Thus, the limit exists in $BUC(H)$ for all $\varphi \in BUC^{1,1}(H)$. Since $BUC^{1,1}(H)$ is dense in $BUC(H)$ (see [28] or [23]) and $\|P_t^n\| \le 1$ for all $n \in \mathbb{N}$ and $t \ge 0$, the limit exists for all $\varphi \in BUC(H)$ and will be denoted by

$$P_t\varphi := \lim_{n\to\infty} P_t^n\varphi, \qquad t \ge 0,\ \varphi \in BUC(H).$$

The family $(P_t)_{t\ge0}$ satisfies $P_{t+s}\varphi = P_tP_s\varphi,\ P_0\varphi = \varphi$ for all $t, s \ge 0$. This follows from the estimates $\|P_t^n\| \le 1$ and the fact that (P_t^n) is a semigroup on $BUC(H)$. The strong continuity of $(P_t)_{t\ge0}$ follows from the uniform convergence of P_t^n on bounded subsets of $\mathbb{R}_+$, and the strong continuity of $(P_t^n)_{t\ge0}$ for every $n \in \mathbb{N}$. □

Remark 2.1.2 *An other proof of Theorem 2.1.1, using the Mittag-Leffler theorem, can be found in [2]. In this work the authors find conditions implying the convergence of the infinite product of commuting C_0-semigroups.*

Let show now that the semigroup $(P_t)_{t\ge0}$ is given by a Gaussian measure.

Theorem 2.1.3 *If we denote by $\mu := \mathcal{N}(x, tB)$ the Gaussian measure with means $x \in H$ and covariance operator tB, then*

$$(P_t\varphi)(x) = \int_H \varphi(y)\mathcal{N}(x, tB)(dy)$$

for $\varphi \in BUC(H)$, and $t > 0$, where $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, \ldots)$.

Proof: For $n \in \mathbb{N}$, $\varphi \in BUC(H)$, $t > 0$, and $x \in B^{\frac{1}{2}}(H)$, it follows from the Cameron-Martin formula (see Corollary 1.3.5) that

$$\begin{aligned}
&\int_H \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right) \mathcal{N}(x, tB)(dy) \\
= &\int_{\mathbb{R}^n} \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right) \cdot \\
&\qquad \exp\left(-\frac{1}{2t}|B^{-\frac{1}{2}}x|^2 + \frac{1}{t}\langle B_n^{-\frac{1}{2}}y, B_n^{-\frac{1}{2}}x\rangle\right) \mathcal{N}(0, tB_n)(dy) \\
= &\int_{\mathbb{R}^n} \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right) \\
&\qquad \exp\left(-\frac{1}{2t}\left(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2\right)\right) \mathcal{N}(x, tB_n)(dy) \\
= &\exp\left(-\frac{1}{2t}(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2)\right) \left(P_t^n\varphi\right)(x).
\end{aligned}$$

So it follows from Theorem 2.1.1 that

$$\lim_{n\to\infty} \left(P_t^n\varphi\right)(x)\exp\left(-\frac{1}{2}(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2)\right) = \left(P_t\varphi\right)(x).$$

So by the dominated convergence theorem and Lemma 1.2.7 we obtain

$$\begin{aligned}
\left(P_t\varphi\right)(x) &= \int_H \varphi(y)\mathcal{N}(x, tB)(dy) \\
&= \int_H \varphi(y + x)\mathcal{N}(0, tB)(dy), \quad x \in B^{\frac{1}{2}}(H).
\end{aligned}$$

Since $\overline{B^{\frac{1}{2}}(H)} = H$ (see Remark 1.3.2), it follows that

$$(P_t\varphi)(x) = \int_H \varphi(y + x)\mathcal{N}(0, tB)(dy), \quad x \in H,$$

and the theorem follows now from Lemma 1.2.7. □

2.2 REGULARITY OF THE HEAT SEMIGROUP

Let prove first the differentiability of $P_t\varphi$ in any direction e_k, $k \in \mathbb{N}$, for $t > 0$ and $\varphi \in BUC(H)$.

Proposition 2.2.1 *Let $\varphi \in BUC(H)$ and $t > 0$. Then $P_t\varphi \in D(D_k)$ for all $k \in \mathbb{N}$ and*

$$D_k P_t\varphi(x) = \frac{1}{\lambda_k t}\int_H y_k\varphi(x + y)\mathcal{N}(0, tB)(dy), \qquad x \in H.$$

Proof: By the Cameron-Martin formula (see Corollary 1.3.5) we know that

$$P_t\varphi(x) = \int_H \varphi(y)\exp\left(-\frac{1}{2t}|B^{-\frac{1}{2}}x|^2 + \frac{1}{t} < B^{-\frac{1}{2}}y, B^{-\frac{1}{2}}x >\right)\mathcal{N}(0,tB)(dy)$$

for $t > 0,\ x \in H$ and $\varphi \in BUC(H)$.
It is now easy to see that $P_t\varphi$ is differentiable in the direction e_k and by Lemma 1.2.7 we obtain

$$\begin{aligned} D_kP_t\varphi(x) &= \frac{1}{t\lambda_k}\int_H (y_k - x_k)\varphi(y)\mathcal{N}(x,tB)(dy) \\ &= \frac{1}{t\lambda_k}\int_H y_k\varphi(x+y)\mathcal{N}(0,tB)(dy). \end{aligned}$$

□

By applying the Cameron-Martin formula to the derivatives $D_kP_t\varphi$ obtained in Proposition 2.2.1 one obtains by similar arguments the following result.

Proposition 2.2.2 *For $\varphi \in BUC(H)$ and $t > 0$ we have $P_t\varphi \in D(D_lD_k)$ for all $l,k \in \mathbb{N}$, and*

$$D_lD_kP_t\varphi(x) = \frac{1}{\lambda_l\lambda_k t^2}\int_H y_ly_k\varphi(x+y)\mathcal{N}(0,tB)(dy) - \frac{\delta_{l,k}}{\lambda_l t}P_t\varphi(x), \quad x \in H,$$

where $\delta_{l,k} := \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \neq k. \end{cases}$

Now, we are interested in global regularity properties of the semigroup (P_t) on $BUC(H)$. To this purpose we define two subspaces $BUC^1_B(H)$ and $BUC^2_B(H)$ of $BUC(H)$.

Definition 2.2.3 *We said that a function $\varphi \in BUC(H)$ is in $BUC^1_B(H)$ if*

(i) $\varphi \in \bigcap_{k=1}^\infty D(D_k)$;

(ii) $\sup_{x\in H}\sum_{k=1}^\infty \lambda_k|D_k\varphi(x)|^2 < \infty$;

(iii) *the mapping $D_B\varphi : H \to H;\ x \mapsto \sum_{k=1}^\infty \sqrt{\lambda_k}D_k\varphi(x)e_k$ is uniformly continuous.*

It is clear that $BUC^1(H) \subseteq BUC^1_B(H)$ and $D_B\varphi(x) = B^{\frac{1}{2}}D\varphi(x)$ for $x \in H$, and $\varphi \in BUC^1(H)$.

Definition 2.2.4 *A function $\varphi \in BUC(H)$ is in $BUC^2_B(H)$ if*

(i) $\varphi \in \bigcap_{l,k=1}^\infty D(D_lD_k)$;

(ii) $\sup_{x\in H}\sum_{l=1}^\infty \left(\sum_{k=1}^\infty \sqrt{\lambda_l\lambda_k}D_lD_k\varphi(x)y_k\right)^2 \le C^2|y|^2$ *for all $y \in H$ and some constant $C > 0$;*

(iii) *the mapping* $D_B^2\varphi$ *defined by* $D_B^2\varphi(x) : H \to \mathcal{L}(H);\, x \mapsto D_B^2\varphi(x)$, *where*

$$\langle D_B^2\varphi(x)y, z\rangle := \sum_{l,k=1}^{\infty} \sqrt{\lambda_l\lambda_k} D_l D_k \varphi(x) y_l z_k, \quad y, z \in H,$$

is uniformly continuous.

We propose now to show some auxiliary results.

Lemma 2.2.5 *The linear operator*

$$D_B : BUC_B^1(H) \to BUC(H, H)$$

is closed.

Proof: Let $(\varphi_n) \subset BUC_B^1(H)$, $\varphi \in BUC(H)$, and $F \in BUC(H,H)$ are such that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_\infty = 0, \text{ and } \lim_{n\to\infty} \|D_B\varphi - F\|_{BUC(H,H)} = 0.$$

For any $k \in \mathbb{N}$, we have

$$\begin{aligned} &\lim_{n\to\infty} \sup_{x\in H} |\langle D_B\varphi_n(x) - F(x), e_k\rangle| = \\ &= \lim_{n\to\infty} \sup_{x\in H} \left| \sqrt{\lambda_k} D_k\varphi_n(x) - \langle F(x), e_k\rangle \right| = 0. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \sup_{x\in H} \left| D_k\varphi(x) - \frac{1}{\sqrt{\lambda_k}} \langle F(x), e_k\rangle \right| = 0.$$

Since D_k is closed in $BUC(H)$, it follows that $\varphi \in D(D_k)$ and

$$D_k\varphi(x) = \frac{1}{\sqrt{\lambda_k}} \langle F(x), e_k\rangle, \quad k \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k |D_k\varphi(x)|^2 &= \sum_{k=1}^{\infty} |\langle F(x), e_k\rangle|^2 \\ &= |F(x)|^2 \le \|F\|_\infty^2. \end{aligned}$$

Moreover,

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k\varphi(x) e_k = \sum_{k=1}^{\infty} \langle F(x), e_k\rangle e_k = F(x)$$

is uniformly continuous. Therefore, $\varphi \in BUC_B^1(H)$ and $D_B\varphi = F$. □

Lemma 2.2.6 *For $\varphi \in \bigcap_{l,k=1}^{\infty} D(D_l D_k)$ and $x \in H$, we define $D^2_{B_n}\varphi(x)$ by*

$$\langle D^2_{B_n}\varphi(x)y, z\rangle = \sum_{l,k=1}^{n} \sqrt{\lambda_l \lambda_k} D_l D_k \varphi(x) y_l z_k, \quad y, z \in H.$$

Assume that

(i) *there is a constant $c > 0$ such that*

$$\left|\langle D^2_{B_n}\varphi(x)y, z\rangle\right| \le c|y||z|, \quad \forall x, y, z \in H,\ n \in \mathbb{N};$$

(ii) *for all $y, z \in H$, the limit*

$$\lim_{n\to\infty} \langle D^2_{B_n}\varphi(x)y, z\rangle \text{ exists uniformly in } x \in H.$$

Then, $\varphi \in BUC^2_B(H)$ and

$$\lim_{n\to\infty} \sup_{x\in H} \left|\langle D^2_{B_n}\varphi(x)y, z\rangle - \langle D^2_B\varphi(x)y, z\rangle\right| = 0, \quad y, z \in H.$$

Proof: From the assumptions we have

(i) $\varphi \in \bigcap_{l,k=1}^{\infty} D(D_l D_k)$;

(ii) $\sup_{x\in H} \left|\sum_{l=1}^{n} \left(\sum_{k=1}^{n} \sqrt{\lambda_l\lambda_k} D_l D_k \varphi(x) y_k\right) z_l\right| \le c|y||z|$ for all $n \in \mathbb{N}$ and $y, z \in H$. Thus,

$$\sup_{x\in H} \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \sqrt{\lambda_l\lambda_k} D_l D_k \varphi(x) y_k\right)^2 \le c^2|y|^2, \quad \forall n \in \mathbb{N}.$$

(iii) Since the limit $\lim_{n\to\infty}\langle D^2_{B_n}\varphi(x)y, z\rangle$ exists uniformly in $x \in H$, for all $y, z \in H$, it follows that the mapping

$$D^2_B\varphi : H \to \mathcal{L}(H);\ x \mapsto D^2_B\varphi(x)$$

is uniformly continuous.

Thus, $\varphi \in BUC^2_B(H)$. The last assertion follows easily from the definition of $D^2_{B_n}\varphi$. □

We are now able to show global regularity results for the heat semigroup (P_t).

Theorem 2.2.7 *Let $\varphi \in BUC(H)$ and $t > 0$. Then $P_t\varphi \in BUC^1_B(H)$ and*

$$\langle D_B P_t \varphi(x), z\rangle = \frac{1}{t}\int_H \langle z, B^{-\frac{1}{2}}y\rangle \varphi(x+y)\mathcal{N}(0, tB)(dy), \quad x, z \in H.$$

Moreover,

$$\|D_B P_t\varphi(x)\| \le \frac{1}{\sqrt{t}}\|\varphi\|_\infty, \quad \forall x \in H.$$

Proof: By Proposition 2.2.1 we have, $P_t\varphi \in D(D_k)$ for all $k \in \mathbb{N}$, and

$$\sum_{k=1}^{n} \sqrt{\lambda_k} D_k P_t\varphi(x) z_k = \sum_{k=1}^{n} \frac{1}{t\sqrt{\lambda_k}} \int_H y_k z_k \varphi(x+y) \mathcal{N}(0,tB)(dy).$$

So by the Hölder inequality we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{n} \sqrt{\lambda_k} D_k P_t\varphi(x) z_k\right|^2 &\le \frac{\|\varphi\|_\infty^2}{t^2} \int_H \left(\sum_{k=1}^{n} \frac{y_k z_k}{\sqrt{\lambda_k}}\right)^2 \mathcal{N}(0,tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{l,k=1}^{n} \frac{z_l z_k}{\sqrt{\lambda_l \lambda_k}} \int_H y_l y_k \mathcal{N}(0,tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{k=1}^{n} \frac{z_k^2}{\lambda_k} \int_H y_k^2 \mathcal{N}(0,tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{k=1}^{n} \frac{z_k^2}{\lambda_k} \int_{\mathbb{R}} y_k^2 \mathcal{N}(0,t\lambda_k)(dy_k) \\
&= \frac{\|\varphi\|_\infty^2}{t} \sum_{k=1}^{n} z_k^2.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{n} \lambda_k |D_k P_t\varphi(x)|^2 \le \frac{\|\varphi\|_\infty^2}{t}, \quad \forall n \in \mathbb{N}.$$

It remains to prove that the mapping

$$D_B P_t\varphi : x \mapsto \sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k P_t\varphi(x) e_k$$

is uniformly continuous. First, we note that, by the last estimate, the series

$$D_B P_t\varphi(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k P_t\varphi(x) e_k$$

converges and we have

$$\langle D_B P_t\varphi(x), z\rangle = \frac{1}{t} \int_H \langle z, B^{-\frac{1}{2}} y\rangle \varphi(x+y) \mathcal{N}(0,tB)(dy), \quad z \in H.$$

Now, we introduce the uniform continuity modulus of $\varphi \in BUC(H)$,

$$\omega_\varphi(t) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in H,\ |x-y| \le t\}, \quad t \ge 0.$$

Since φ is uniformly continuous, it is easy to see that ω_φ is continuous in $[0,\infty)$. Let $x, y \in H$. By Hölder's inequality and Proposition 1.3.1, we obtain

$$
\begin{aligned}
& |\langle D_B P_t\varphi(x) - D_B P_t\varphi(y), z\rangle|^2 \\
= & \left|\frac{1}{t}\int_H \langle z, B^{-\frac{1}{2}}\alpha\rangle(\varphi(x+\alpha) - \varphi(y+\alpha))\mathcal{N}(0,tB)(d\alpha)\right|^2 \\
\leq & \frac{\omega_\varphi(|x-y|)^2}{t^2}\int_H |\langle z, B^{-\frac{1}{2}}\alpha\rangle|^2\mathcal{N}(0,tB)(d\alpha) \\
= & \frac{\omega_\varphi(|x-y|)^2}{t}|z|^2.
\end{aligned}
$$

Hence,

$$\|D_B P_t\varphi(x) - D_B P_t\varphi(y)\| \leq \frac{1}{\sqrt{t}}\omega_\varphi(|x-y|).$$

Then, $P_t\varphi \in BUC^1_B(H)$ for all $\varphi \in BUC(H)$ and $t > 0$. Moreover, by the same computation as above, we obtain

$$\|D_B P_t\varphi(x)\| \leq \frac{1}{\sqrt{t}}\|\varphi\|_\infty$$

for all $\varphi \in BUC(H)$, $t > 0$, and $x \in H$. □

More global regularity is given by the following theorem.

Theorem 2.2.8 *For $\varphi \in BUC(H)$ and $t > 0$, we have $P_t\varphi \in BUC^2_B(H)$ and*

$$
\begin{aligned}
\langle D^2_B P_t\varphi(x)z_1, z_2\rangle = & \frac{1}{t^2}\int_H \langle z_1, B^{-\frac{1}{2}}y\rangle\langle z_2, B^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy) \\
& -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x)
\end{aligned}
$$

for $z_1, z_2, x \in H$. If in addition $\varphi \in BUC^1_B(H)$, then

$$\langle D^2_B P_t\varphi(x)z_1, z_2\rangle = \frac{1}{t}\int_H \langle D_B\varphi(x+y), z_2\rangle\langle z_1, B^{-\frac{1}{2}}y\rangle\mathcal{N}(0,tB)(dy)$$

for $x, z_1, z_2 \in H$. Moreover, for all $x \in H$,

$$
\begin{aligned}
\|D^2_B P_t\varphi(x)\|_{\mathcal{L}(H)} & \leq \frac{\sqrt{2}}{t}\|\varphi\|_\infty \quad \text{for } \varphi \in BUC(H), \qquad (2.1) \\
\|D^2_B P_t\varphi(x)\|_{\mathcal{L}(H)} & \leq \frac{1}{\sqrt{t}}\|D_B\varphi\|_{BUC(H,H)} \quad \text{for } \varphi \in BUC^1_B(H).
\end{aligned}
$$

Proof: From Proposition 2.2.2 it follows that

$$
\begin{aligned}
\langle D^2_{B_n} P_t\varphi(x)z_1, z_2\rangle = & \frac{1}{t^2}\int_H \langle z_1, B_n^{-\frac{1}{2}}y\rangle\langle z_2, B_n^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy) \\
& -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x), \quad z_1, z_2, x \in H.
\end{aligned}
$$

It is easy to see that all the assumptions of Lemma 2.2.6 are satisfied. Thus, $P_t\varphi \in BUC^2_B(H)$ and

$$\begin{aligned}\langle D^2_B P_t\varphi(x)z_1, z_2\rangle &= \frac{1}{t^2}\int_H \langle z_1, B^{-\frac{1}{2}}y\rangle\langle z_2, B^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy)\\ &\quad -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x), \quad z_1, z_2, x \in H.\end{aligned}$$

Hence, by Hölder's inequality and Theorem 2.1.3, we obtain

$$\begin{aligned}&|\langle D^2_B P_t\varphi(x)z, z\rangle|^2 =\\ &= \left|\frac{1}{t^2}\int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^2\varphi(x+y)\mathcal{N}(0,tB)(dy) - \frac{1}{t}|z|^2 P_t\varphi(x)\right|^2\\ &= \frac{1}{t^4}\left|\int_H \left(|\langle z, B^{-\frac{1}{2}}y\rangle|^2 - t|z|^2\right)\varphi(x+y)\mathcal{N}(0,tB)(dy)\right|^2\\ &\le \frac{\|\varphi\|^2_\infty}{t^4}\int_H \left(|\langle z, B^{-\frac{1}{2}}y\rangle|^2 - t|z|^2\right)^2\mathcal{N}(0,tB)(dy).\end{aligned}$$

Since

$$\begin{aligned}\int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^4\mathcal{N}(0,tB)(dy) &= 3t^2|z|^4 \text{ and}\\ \int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^2\mathcal{N}(0,tB)(dy) &= t|z|^2 \quad \text{(see Proposition 1.3.1)},\end{aligned}$$

it follows that

$$|\langle D^2_B P_t\varphi(x)z, z\rangle|^2 \le \frac{2}{t^2}|z|^4\|\varphi\|^2_\infty$$

for all $x, z \in H$. Consequently,

$$\|D^2_B P_t\varphi(x)\|_{\mathcal{L}(H)} \le \frac{\sqrt{2}}{t}\|\varphi\|_\infty, \quad \forall x \in H.$$

The second equality can be obtained similarly, by using Theorem 2.2.7 and the last estimate is a consequence of Proposition 1.3.1. □

We propose now to prove an additional regularity result, which will be needed to solve (HE).
We start by the following auxiliary result, where the proof can be founded in [15, Lemma XI.9.14 (a), p. 1098].

Lemma 2.2.9 *Let $B \in \mathcal{L}(H)$ and suppose that there is a constant $c > 0$ such that, for all finite rank linear operator N in $\mathcal{L}(H)$, $|\mathrm{Tr}(NB)| \le c\|N\|$. Then B is a trace class operator on H and*

$$\mathrm{Tr}\ B \le c\,.$$

The following result was proved first by L. Gross [19] by using probabilistic methods.

Theorem 2.2.10 *For $\varphi \in BUC^1(H)$ and $t > 0$, we have $D_B^2 P_t\varphi(x)$ is a trace class operator on H for all $x \in H$, and*

$$\text{Tr } (D_B^2 P_t\varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), y > \mathcal{N}(0, tB)(dy) \ , \qquad x \in H.$$

Moreover, $\text{Tr}D_B^2 P_t\varphi(\cdot) \in BUC(H)$ *and*

$$|\text{Tr}D_B^2 P_t\varphi(x)| \leq \frac{1}{\sqrt{t}}\|\varphi\|_1(\text{Tr}B)^{\frac{1}{2}}.$$

Proof: Since $\varphi \in BUC^1(H)$, it follows that, for $z_1 \in H$,

$$\begin{aligned} < DP_t\varphi(x), B^{\frac{1}{2}}z_1 > &= \int_H < D\varphi(x+y), B^{\frac{1}{2}}z_1 > \mathcal{N}(0, tB)(dy) \\ &= P_t\psi(x), \end{aligned}$$

where $\psi(x) :=< D\varphi(x), B^{\frac{1}{2}}z_1 >,\ x \in H$. From Theorem 2.2.7 we have

$$\begin{aligned} < D_B P_t\psi(x), z_2 > &= \frac{1}{t}\int_H < z_2, B^{-\frac{1}{2}}y > \psi(x+y)\mathcal{N}(0, tB)(dy) \\ &= \frac{1}{t}\int_H < z_2, B^{-\frac{1}{2}}y >< D\varphi(x+y), B^{\frac{1}{2}}z_1 \\ &> \mathcal{N}(0, tB)(dy) \end{aligned} \tag{2.2}$$

for $z_2 \in H$. On the other hand, by an easy computation, one can see,

$$< D_B P_t\psi(x), z_2 >=< D_B^2 P_t\varphi(x)z_1, z_2 > .$$

Hence,

$$\begin{aligned} &< D_B^2 P_t\varphi(x)z_1, z_2 > = \\ &= \tfrac{1}{t}\textstyle\int_H < D\varphi(x+y), B^{\frac{1}{2}}z_1 >< z_2, B^{-\frac{1}{2}}y > \mathcal{N}(0, tB)(dy). \end{aligned}$$

Now, take $N \in \mathcal{L}(H)$ a finite rank operator. We obtain

$$\begin{aligned} &< ND_B^2 P_t\varphi(x)z_1, z_2 > = \\ &= \tfrac{1}{t}\textstyle\int_H < D\varphi(x+y), B^{\frac{1}{2}}z_1 >< N^*z_2, B^{-\frac{1}{2}}y > \mathcal{N}(0, tB)(dy). \end{aligned}$$

Hence,

$$\text{Tr}(ND_B^2 P_t\varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), B^{\frac{1}{2}}NB^{-\frac{1}{2}}y > \mathcal{N}(0, tB)(dy),$$

and by Hölder's inequality, we obtain

$$\begin{aligned} |\mathrm{Tr}(ND_B^2 P_t\varphi(x))|^2 &\leq \frac{\|\varphi\|_1^2}{t^2}\int_H |B^{\frac{1}{2}}NB^{-\frac{1}{2}}y|^2\mathcal{N}(0,tB)(dy) \\ &= \frac{\|\varphi\|_1^2}{t^2} t\ \mathrm{Tr}(B^{\frac{1}{2}}NN^*B^{\frac{1}{2}}) \quad \text{(see Example 1.2.9.(b))} \\ &= \frac{\|\varphi\|_1^2}{t}\mathrm{Tr}(NN^*B). \end{aligned}$$

Thus,

$$|\mathrm{Tr}(ND_B^2 P_t\varphi(x))| \leq \frac{1}{\sqrt{t}}\|\varphi\|_1\|N\|(\mathrm{Tr}B)^{\frac{1}{2}},\ x \in H.$$

So, by Lemma 2.2.9, $\mathrm{Tr}(D_B^2 P_t\varphi(x)) < \infty$ for all $x \in H$. Moreover,

$$\mathrm{Tr}(D_B^2 P_t\varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), y > \mathcal{N}(0,tB)(dy), \quad x \in H,$$

and

$$|\mathrm{Tr}(D_B^2 P_t\varphi(x))| \leq \frac{1}{\sqrt{t}}\|\varphi\|_1(\mathrm{Tr}B)^{\frac{1}{2}}\ , \quad x \in H.$$

The uniform continuity of $\mathrm{Tr}(D_B^2 P_t\varphi(\cdot))$ follows from the fact that $\varphi \in BUC^1(H)$. □

2.3 Solutions of (HE) and characterization of the generator of (P_t)

We denote by $(G, D(G))$ the generator of (P_t) on $BUC(H)$.
First, we propose to compare G with the following operator
$D(G_0) :=$

$$\left\{\varphi \in BUC_B^2(H),\ D_B^2\varphi(x) \in \mathcal{L}_1(H),\ \forall\, x \in H \text{ and } \mathrm{Tr}(D_B^2\varphi(\cdot)) \in BUC(H)\right\},$$

$$G_0\varphi = \frac{1}{2}\mathrm{Tr}(D_B^2\varphi),$$

where $\mathcal{L}_1(H)$ denotes the set of $S \in \mathcal{L}(H)$ with $\mathrm{Tr}\, S < \infty$.

Proposition 2.3.1 *The following hold:*

(a) $\overline{D(G_0)} = BUC(H)$;

(b) $\overline{G_0} = G$.

Proof: (a) Let $\varphi \in BUC(H)$. Since $BUC^1(H)$ is dense in $BUC(H)$, it follows that, for any $\varepsilon > 0$ there is $\varphi_\varepsilon \in BUC^1(H)$ such that $\|\varphi - \varphi_\varepsilon\|_\infty < \frac{\varepsilon}{2}$. On the other hand, from the strong continuity of (P_t) we have, for any $\varepsilon > 0$ there exists $\delta > 0$ with

$$0 < t < \delta \Longrightarrow \|\varphi_\varepsilon - P_t\varphi_\varepsilon\|_\infty < \frac{\varepsilon}{2}.$$

Thus, for $0 < t < \delta$,

$$\|\varphi - P_t\varphi_\varepsilon\| < \varepsilon.$$

Now, (a) follows from Theorem 2.2.10.
(b) Let $\varphi \in D(G_0)$ and take $g(t) := P_t\varphi$ and $g_n(t) : P_t^n\varphi$.
It follows from Theorem 2.1.1 that

$$g_n \longrightarrow g \quad \text{in } C\left([0,1]; BUC(H)\right).$$

Moreover,

$$\frac{dg_n}{dt}(t) = \frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2 g_n(t) = \frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2 P_t^n\varphi = P_t^n\left(\frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2\varphi\right).$$

Hence,

$$\frac{dg_n}{dt}(t) \longrightarrow P_t(G_0\varphi) \quad \text{in } C\left([0,1], BUC(H)\right).$$

Consequently, $\dfrac{dg}{dt}(t) = P_t(G_0\varphi)$ and by taking $t = 0$ we have $\varphi \in D(G)$ and $G\varphi = G_0\varphi$, i.e., $G_0 \subseteq G$. In particular G_0 is closable. Now, take $\varphi \in D(G)$, $\lambda > 0$ and set $\psi := \lambda\varphi - G\varphi$. We know that there is $(\psi_n)_{n\in\mathbb{N}} \subseteq BUC^1(H)$ such that $\psi_n \to \psi$ in $BUC(H)$. Since (P_t) is a semigroup of contractions on $BUC(H)$, we can define $\varphi_n := R(\lambda, G)\psi_n$. It is clear that $\varphi_n \to \varphi$ in $BUC(H)$. Since $\varphi_n = \displaystyle\int_0^\infty e^{-\lambda t} P_t\psi_n dt$, it follows from Theorem 2.2.10 that

$$\varphi_n \in D(G_0) \quad \text{and} \quad \|G_0\varphi_n\|_\infty \le \left(\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{t}}dt\right)(\mathrm{Tr}B)^{\frac{1}{2}}\|\psi_n\|_1.$$

Moreover, since

$$G_0\varphi_n = G\varphi_n = \lambda R(\lambda, G)\psi_n - \psi_n,$$

it follows that

$$\lim_{n\to\infty} G_0\varphi_n = \lambda R(\lambda, G)\psi - \psi = GR(\lambda, G)\psi = G\varphi.$$

This proves that $\overline{G_0} = G$. □

We solve now the heat equation. Let $\varphi \in BUC^1(H)$ and set

$$u(t,x) = P_t\varphi(x), \quad t \ge 0, \; x \in H.$$

From Theorem 2.2.10 we know that $P_t\varphi \in D(G_0)$ for $t > 0$. Since $G_0 \subseteq G$ we obtain

$$\frac{d}{dt}P_t\varphi = GP_t\varphi = G_0P_t\varphi,\ t > 0.$$

Thus, we have the following result.

Theorem 2.3.2 *Let $\varphi \in BUC^1(H)$. Then the function*

$$u(t,x) = P_t\varphi(x), \quad t > 0,$$

is a classical solution of (HE) with $u(0,x) = \varphi(x),\ x \in H$.

An other characterization of the generator $(G, D(G))$ of the heat semigroup (P_t) on $BUC(H)$, which will play an important role in Section 2.4, is given by the following proposition.

Proposition 2.3.3 *The set*

$$\begin{aligned} D_0(G) \quad := \quad & \{\varphi \in BUC^{1,1}(H) : D_kD_l\varphi \in BUC(H), \\ & \textit{for all } k,l \in \mathbb{N},\ \sup_{k,l\in\mathbb{N}} \|D_kD_l\varphi\|_\infty < \infty\} \end{aligned}$$

is a P_t-invariant core for G. Moreover,

$$G\varphi = \sum_{k=1}^{\infty} \lambda_k D_k^2\varphi \quad \textit{for } \varphi \in D_0(G).$$

Proof: Let show first that, for $\varphi \in BUC^{1,1}(H)$,

$$\sup_{l,k\in\mathbb{N}} \|D_lD_kP_t\varphi\|_\infty \le \|\varphi\|_{1,1}, \quad t > 0. \tag{2.3}$$

Let $\varphi \in BUC^{1,1}(H)$ and $k \in \mathbb{N}$. Since D_k is closed and $D_kP_t^n\varphi = P_t^nD_k\varphi$ for $t \ge 0$ and $n \in \mathbb{N}$, it follows from Theorem 2.1.1 that

$$D_kP_t\varphi = P_tD_k\varphi$$

for all $t \ge 0$. So by Proposition 2.2.1 we have

$$D_kP_t\varphi \in D(D_l) \quad \text{for all } t > 0, \text{ and } l \in \mathbb{N}.$$

Thus, by Theorem 2.1.3, we deduce that

$$\begin{aligned} &|D_lD_kP_t\varphi(x)| = \\ &\quad = \quad |D_lP_tD_k\varphi(x)| \\ &\quad = \quad \left|\lim_{h\to 0}\frac{1}{h}(P_tD_k\varphi(x+he_l) - P_tD_k\varphi(x)\right| \\ &\quad = \quad \left|\lim_{h\to 0}\int_H \frac{1}{h}(D_k\varphi(x+y+he_l) - D_k\varphi(x+y)\mathcal{N}(0,tB)(dy)\right| \\ &\quad \le \quad \|\varphi\|_{1,1} \end{aligned}$$

for all $l, k \in \mathbb{N}$, and $x \in H$. This proves (2.3). So we obtain

$$P_t D_0(G) \subseteq D_0(G), \quad \forall t \geq 0.$$

From Proposition A.2.5, it suffices now to prove that $D_0(G)$ is dense in $BUC(H)$. This can be seen by using (2.3) and exactly the same proof as in Proposition 2.3.1.(a). □

We end this section by the following remark.

Remark 2.3.4 *If we compare the result of Theorem 2.2.8 and Theorem A.2.7 then the following question arise:*

Is the semigroup (P_t) analytic or at least differentiable on $BUC(H)$?

The answer is negative (see [27]) and will be given in the following section (see Corollary 2.4.2).

2.4 The spectrum of the infinite dimensional Laplacian

Let H be a separable, infinite dimensional, real Hilbert space and let (e_k) be an orthonormal basis. We shall regard $BUC(\mathbb{R}^n)$ as a subspace of $BUC(H)$ via the isometric embedding

$$J_n : BUC(\mathbb{R}^n) \to BUC(H), \quad (J_n\varphi)(x) := \varphi(x_1, \ldots, x_n),$$

for $\varphi \in BUC(\mathbb{R}^n)$, $x \in H$, and $x_k := \langle x, e_k \rangle$. Let $\lambda_k > 0$ with $\sum_{k=1}^{\infty} \lambda_k < \infty$ be given. We know from Theorem 2.1.1 that the infinite dimensional heat equation (HE) on $BUC(H)$ is solved by the C_0-semigroup of contractions

$$P_t\varphi = \lim_{n\to\infty} P_t^n \varphi, \quad \varphi \in BUC(H),$$

where the above limit exists in $BUC(H)$ uniformly in t on bounded subsets of $[0, \infty)$. We recall that for $\varphi \in BUC(H)$, $x \in H$ and $t > 0$,

$$P_t^n \varphi(x) := (2\pi t)^{-\frac{n}{2}} (\lambda_1 \cdots \lambda_n)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\sum_{k=1}^{n} \frac{y_k^2}{2t\lambda_k}} \varphi\Big(x - \sum_{k=1}^{n} y_k e_k\Big) dy. \tag{2.4}$$

Let compute the spectrum of the generator $(G, D(G)$ of the semigroup (P_t) on $BUC(H)$.

Theorem 2.4.1 *The spectrum of G is the left half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \leq 0\}$ and $\sigma(P_t) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Moreover, every $\lambda \in \sigma(G)$ is an approximate eigenvalue.*

Proof: Note that the restriction of P_t to $BUC(\mathbb{R}^n)$ coincides with the semigroup generated by $G_n := \sum_{k=1}^n \lambda_k\, D_k^2$. In particular, G_n is the part of G in $BUC(\mathbb{R}^n)$ and, hence, $R(\lambda, G_n) = R(\lambda, G)_{|BUC(\mathbb{R}^n)}$ for $\lambda \in \rho(G) \cap \rho(G_n)$. Therefore, for these values of λ, the sequence $\|R(\lambda, G_n)\|$ is bounded.
Let $V : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ be the isometry defined by

$$(V\varphi)(x) := \varphi(\sqrt{\frac{\lambda_1}{2}}x_1, \ldots, \sqrt{\frac{\lambda_n}{2}}x_n), \qquad \varphi \in BUC(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

A simple change of variables in (2.4) shows that $e^{tG_n} = V^{-1}e^{t\Delta_n}V$ for $t \geq 0, n \in \mathbb{N}$, where Δ_n denotes the Laplacian on $\mathbb{R}^n$. This implies that

$$R(\lambda, G_n) = V^{-1}R(\lambda, \Delta_n)V \quad \text{ for } \lambda \in \Sigma_\pi := \{0 \neq \lambda \in \mathbb{C} : |\arg\ \lambda| < \pi\},$$

so that $\|R(\lambda, G_n)\| = \|R(\lambda, \Delta_n)\|$ for $\lambda \in \Sigma_\pi$ and $n \in \mathbb{N}$.

Fix $\lambda \in \Sigma_\pi$ with $Re\ \lambda < 0$. For $n \in \mathbb{N}$, the function $g_{\lambda,n}(x) := e^{\frac{\lambda}{2n}|x|^2}$, $x \in \mathbb{R}^n$, belongs to $BUC(\mathbb{R}^n)$ and $\|g_{\lambda,n}\|_\infty = 1$. Setting

$$f_{\lambda,n}(x) := (\lambda - \Delta_n)g_{\lambda,n}(x) - \frac{\lambda^2}{n^2}|x|^2 e^{\frac{\lambda}{2n}|x|^2}, \quad x \in \mathbb{R}^n,$$

we compute

$$\|f_{\lambda,n}\|_\infty = \frac{2|\lambda|^2}{ne|Re\ \lambda|}.$$

So we derive

$$\|R(\lambda, G_n)\| = \|R(\lambda, \Delta_n)\| \geq \frac{\|R(\lambda, \Delta_n)f_{\lambda,n}\|_\infty}{\|f_{\lambda,n}\|_\infty} = \frac{ne|Re\ \lambda|}{2|\lambda|^2}.$$

Since the sequence $\|R(\lambda, G_n)\|$ is unbounded, λ must belong to the spectrum of G. From standard spectral theory of C_0-semigroups, cf. [16, Chap. IV], now follows the first and second assertion.

To prove the last assertion, we observe that $i\mathbb{R}$ is contained in the approximate point spectrum of G. Let $\lambda = -a^2 + ib$ for $a > 0$ and $b \in \mathbb{R}$. The first part of the proof applies to the operator $\tilde{G}$ on $BUC(H)$ corresponding to the sequence $(\lambda_2, \lambda_3, \cdots)$. Thus there exist $g_n \in D_0(\tilde{G})$ such that $\|g_n\|_\infty = 1$ and $\|\tilde{G}g_n - ibg_n\|_\infty \to 0$ as $n \to \infty$. We now define

$$f_n(x) : \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, g_n(x_2, x_3, \cdots), \quad x \in H.$$

Clearly, $f_n \in D_0(G)$, $\|f_n\|_\infty = 1$, and

$$Gf_n(x) = \sum_{k=1}^{\infty} \lambda_k\, D_k^2\, f_n(x) = -a^2 f_n(x) + \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, (\tilde{G}g_n)(x_2, x_3, \cdots),$$

$x \in H$.

As a result, λ is an approximate eigenvalue of G. □

As a consequence of Theorem A.2.10 and (11) we immediately obtain the following result from [14], see also [18], [29] and [2].

Corollary 2.4.2 *The semigroup (P_t) is not eventually norm continuous an hence not eventually differentiable on $BUC(H)$.*

CHAPTER 3

THE ORNSTEIN-UHLENBECK SEMIGROUP

In this chapter we are concerned with the Ornstein-Uhlenbeck semigroup, first on $C_b(H)$, and finally on L^p–spaces with invariant measure. The Ornstein-Uhlenbeck semigroup is related to the solution of the following linear stochastic differential equation

$$(SDE)\ \begin{cases} dX(t,x) = AX(t,x)dt + Q^{\frac{1}{2}}dW(t), \quad t \geq 0 \\ X(0,x) = x \in H, \end{cases}$$

where $Q \in \mathcal{L}(H)$ is selfadjoint and nonnegative and A generates a C_0–semigroup $(e^{tA})_{t\geq 0}$ on H. The process W is a standard cylindrical Wiener process on H. Under appropriate assumptions (see [12]) the solution to (SDE) is a Gaussian and Markov process in H, called the Ornstein-Uhlenbeck process. The associated Ornstein-Uhlenbeck semigroup on $B_b(H)$, the space of bounded and Borel functions from H into $\mathbb{R}$, is given by

$$R_t\varphi(x) := \mathbb{E}\left(\varphi(X(t,x))\right), \quad t \geq 0,\, x \in H,\, \varphi \in B_b(H).$$

This is the semigroup solution of the associated Kolmogorov equation

$$(KE)\ \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\mathrm{Tr}(QD^2u(t,x) + \langle x, A^*Du(t,x)\rangle, \quad t > 0,\, x \in H, \\ u(0,x) = \varphi(x), \quad x \in H. \end{cases}$$

The basic assumption in this chapter is

$$\text{(H1)} \qquad Q_t := \int_0^t e^{sA}Qe^{sA^*}\, ds \in \mathcal{L}_1^+(H), \quad t > 0.$$

Under (H1) and by the change of variables

$$v(t, e^{tA}x) := u(t,x), \quad t \geq 0,\, x \in H,$$

one can see (cf. [8], [4]) that v is the unique solution of the parabolic equation

$$(PE)\ \begin{cases} \frac{\partial}{\partial t}v(t,x) = \frac{1}{2}\mathrm{Tr}\left(e^{tA}Qe^{tA^*}D^2v(t,x)\right), & t>0,\, x\in H,\\ v(0,x)=\varphi(x), \quad x\in H, \end{cases}$$

and is given by

$$v(t,x) = \int_H \varphi(x+y)\mathcal{N}(0,Q_t)(dy), \quad x\in H,\, t\geq 0,$$

where $\varphi \in BUC^2(H)$. Therefore, if we suppose (H1) then the Ornstein-Uhlenbeck semigroup is given by

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x+y)\mathcal{N}(0,Q_t)(dy), \quad x\in H,\, t\geq 0,$$

for $\varphi \in B_b(H)$. Now, by Lemma 1.2.7, we have, for $\varphi\in B_b(H)$,

$$R_t\varphi(x) = \int_H \varphi(y)\mathcal{N}(e^{tA}x,Q_t)(dy), \quad x\in H,\, t\geq 0.$$

3.1 The Ornstein-Uhlenbeck semigroup on $C_b(H)$

The aim of this section is to study the global regularity of the Ornstein-Uhlenbeck semigroup $(R_t)_{t\geq 0}$ on $C_b(H)$. Existence and uniqueness of a classical solution for (KE) will be also considered.

In this section we assume the *controllability condition* (see [31])

$$\text{(H2)} \qquad e^{tA}(H) \subseteq Q_t^{\frac{1}{2}}(H) \quad \text{for all } t>0.$$

If we suppose in addition that $(e^{tA})_{t\geq 0}$ is exponentially stable, that is, there are constants $M\geq 1$ and $\omega>0$ such that $\|e^{tA}\|\leq Me^{-t\omega}$ for all $t\geq 0$, then it follows from the strong continuity of the semigroup $(e^{tA})_{t\geq 0}$ and Exercise 3.3.22 that, for any $t>0$, the subspace $Q_t^{\frac{1}{2}}(H)$ is dense in H and so, by Remark 1.3.2,

$$\ker Q_t = \{0\} \quad \text{for all } t>0.$$

This will be needed for the application of the Cameron-Martin formula. Regularity properties of the semigroup $(R_t)_{t\geq 0}$ are given by the following result.

Theorem 3.1.1 *Suppose that (H1) and (H2) are satisfied and* $\ker Q_t = \{0\}$ *for all* $t>0$. *Then, for any* $\varphi\in B_b(H)$ *and* $t>0$, *we have* $R_t\varphi\in BUC^\infty(H)$

and in particular, for $x, y, z \in H$,

$$
\begin{aligned}
\langle DR_t\varphi(x), y\rangle &= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle \varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh), \\
\langle D^2 R_t\varphi(x)y, z\rangle &= \int_H \left[\langle \Lambda_t y, Q_t^{-\frac{1}{2}} v\rangle\langle \Lambda_t z, Q_t^{-\frac{1}{2}} v\rangle - \langle \Lambda_t y, \Lambda_t z\rangle\right] \cdot \\
&\qquad \varphi(e^{tA}x + v)\mathcal{N}(0, Q_t)(dv),
\end{aligned}
$$

where $\Lambda_t := Q_t^{-\frac{1}{2}} e^{tA}$, $t > 0$. *Moreover,*

$$
\begin{aligned}
|DR_t\varphi(x)| &\leq \|\Lambda_t\|\|\varphi\|_\infty, \\
\|D^2 R_t\varphi(x)\| &\leq \sqrt{2}\|\Lambda_t\|^2\|\varphi\|_\infty.
\end{aligned}
$$

Furthermore, if for any $t > 0$, $R_t B_b(H) \subset C_b(H)$, then (H2) holds.

Proof: Let $t > 0$, $\varphi \in B_b(H)$ and $x \in H$. Since, by (H2), $e^{tA}x \in Q_t^{\frac{1}{2}}(H)$, it follows from the Cameron-Martin formula (see Corollary 1.3.5) that $\mathcal{N}(e^{tA}x, Q_t) \sim \mathcal{N}(0, Q_t)$ and

$$
\frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0, Q_t)}(y) = \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right).
$$

Thus,

$$
R_t\varphi(x) = \int_H \varphi(y) \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right)\mathcal{N}(0, Q_t)(dy).
$$

Therefore, by a change of variables (see Lemma 1.2.7), we obtain

$$
\begin{aligned}
\langle DR_t\varphi(x), y\rangle &= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}}(h - e^{tA}x)\rangle \varphi(h)\mathcal{N}(e^{tA}x, Q_t)(dh) \\
&= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle \varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh).
\end{aligned}
$$

So by Proposition 1.3.1 we have

$$
\begin{aligned}
|\langle DR_t\varphi(x), y\rangle|^2 &\leq \|\varphi\|_\infty \int_H |\langle \Lambda_t y, Q_t^{\frac{1}{2}} h\rangle|^2 \mathcal{N}(0, Q_t)(dh) \\
&= \|\varphi\|_\infty |\Lambda_t y|^2
\end{aligned}
$$

for all $y \in H$. Similarly one obtains the second derivative of $R_t\varphi$ and the estimate follows by a simple computation. Let now prove the last assertion. Suppose that for any $\varphi \in B_b(H)$, the function $R_t\varphi(\cdot)$ is continuous and there is $x_0 \in H$ such that $e^{tA}x_0 \notin Q_t^{\frac{1}{2}}(H)$. It follows from the Cameron-Martin formula (Corollary 1.3.5) that, for any $n \in \mathbb{N}$, $\mathcal{N}(\frac{1}{n}e^{tA}x_0, Q_t) \perp \mathcal{N}(0, Q_t)$. This means that , for any $n \in \mathbb{N}$, there is $\Gamma_n \in \mathcal{B}(H)$ with

$$
\mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma_n) = 0 \text{ and } \mathcal{N}(0, Q_t)(\Gamma_n) = 1.
$$

If we set $\Gamma := \cap_{n\in\mathbb{N}}\Gamma_n$, then

$$\mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma) = 0 \text{ and } \mathcal{N}(0, Q_t)(\Gamma) = 1.$$

Now, we consider the characteristic function $\varphi := \chi_\Gamma$. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} R_t\varphi\left(\frac{x_0}{n}\right) &= \mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma) = 0 \text{ and} \\ R_t\varphi(0) &= \mathcal{N}(0, Q_t)(\Gamma) = 1. \end{aligned}$$

Hence, the function $R_t\varphi(\cdot)$ is not continuous at zero. This end the proof of the theorem. □

We show now that the Ornstein-Uhlenbeck semigroup $(R_t)_{t\geq 0}$ solves the Kolmogorov equation (KE) in the following sense.
We say that a function $u(t,x)$, $t \geq 0, x \in H$, is a *classical solution* of (KE) if

(a) $u : [0,\infty) \times H \to \mathbb{R}$ is continuous and $u(0,\cdot) = \varphi$,

(b) $u(t,\cdot) \in BUC^2(H)$ for all $t > 0$, and $QD^2u(t,x)$ is a trace class operator on H for all $x \in H$ and $t > 0$,

(c) $Du(t,x) \in D(A^*)$ for all $x \in H$ and $t > 0$,

(d) for any $x \in H$, $u(\cdot, x)$ is continuously differentiable on $(0,\infty)$ and fulfills (KE)

Under appropriate conditions we show now the existence and the uniqueness of a classical solution for (KE) (cf. [13, Theorem 6.2.4]).

Theorem 3.1.2 *Suppose(H1), (H2) and* $\ker Q_t = \{0\}$ *for all* $t > 0$. *If* $\Lambda_t A$ *has a continuous extension* $\overline{\Lambda_t A}$ *on* H *and* $\Lambda_t Q^{\frac{1}{2}}$ *is a Hilbert-Schmidt operator on* H *for every* $t > 0$*, then (KE) has a unique classical solution.*

Proof: For $\varphi \in B_b(H)$ we know, from Theorem 3.1.1, that, for any $t > 0$, $R_t\varphi \in BUC^\infty(H)$ and

$$\langle DR_t\varphi(x), Ay\rangle = \int_H \langle \Lambda_t Ay, Q_t^{-\frac{1}{2}}h\rangle \varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh)$$

for $y \in D(A)$, $t > 0$ and $x \in H$. So by Proposition 1.3.1, we obtain

$$|\langle DR_t\varphi(x), Ay\rangle| \leq \|\varphi\|_\infty \|\overline{\Lambda_t A}\||y|, \quad \forall y \in D(A),$$

for $t > 0$ and $x \in H$. Hence, $DR_t\varphi(x) \in D(A^*)$ for all $x \in H$ and $t > 0$.
Again from Theorem 3.1.1 we deduce that

$$\begin{aligned} &\langle D^2R_t\varphi(x)Q^{\frac{1}{2}}e_j, Q^{\frac{1}{2}}e_j\rangle = \\ = \ &\int_H \left(\langle \Lambda_t Q^{\frac{1}{2}}e_j, Q_t^{-\frac{1}{2}}y\rangle^2 - |\Lambda_t Q^{\frac{1}{2}}e_j|^2\right) \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy) \end{aligned}$$

for $x \in H$, $t > 0$ and $j \in \mathbb{N}$. It follows from Proposition 1.3.1 that

$$\left|\langle D^2 R_t \varphi(x) Q^{\frac{1}{2}} e_j, Q^{\frac{1}{2}} e_j \rangle\right| \leq 2|\Lambda_t Q^{\frac{1}{2}} e_j|^2 \|\varphi\|_\infty$$

for $x \in H$ and $t > 0$. This implies that $QD^2 R_t \varphi(x)$ is a trace class operator on H for all $x \in H$ and $t > 0$.
For any $x \in H$, the function $t \mapsto R_t \varphi(x)$ fulfills (KE) follows from a straightforward computation and is left to the reader. The uniqueness follows from the fact that Equation (PE) has a unique solution for an initial data $\varphi \in BUC^2(H)$. □

If the semigroup $(e^{tA})_{t \geq 0}$ is exponentially stable then the assumption "$\Lambda_t Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H" is automatically satisfied as the following corollary shows.

Corollary 3.1.3 *Assume (H1) and (H2). If $\Lambda_t A$ has a continuous extension $\overline{\Lambda_t A}$ on H for every $t > 0$ and $(e^{tA})_{t \geq 0}$ is exponentially stable then (KE) has a unique classical solution.*

Proof: It suffices to prove that the assumptions of Theorem 3.1.2 are satisfied. Since

$$\Lambda_t = Q_t^{-\frac{1}{2}} e^{tA} = (Q_t^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}})(Q_\infty^{-\frac{1}{2}} e^{\frac{t}{2}A}) e^{\frac{t}{2}A}, \quad t > 0,$$

it follows from Exercise 3.3.22 that Λ_t is a trace class operator and hence $\Lambda_t Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H for every $t > 0$. □

3.2 Sobolev spaces with respect to Gaussian measures on H

In this section we propose to define and study the Sobolev spaces $W^{1,2}(H, \mu)$, $W_B^{1,2}(H, \mu)$ and $W^{2,2}(H, \mu)$, where $\mu := \mathcal{N}(0, B)$ and $B \in \mathcal{L}_1^+(H)$. Without loss of generality we suppose that $\ker B = \{0\}$ and consider an orthonormal system (e_k) and positive numbers λ_k with $Be_k = \lambda_k e_k$ for $k \in \mathbb{N}$.
Define the subspaces $\mathcal{E}(H)$ and $\mathcal{E}_A(H)$ of $BUC(H)$ by

$$\begin{aligned} \mathcal{E}(H) &:= \operatorname{Span}\{e^{i\langle x, h\rangle}; h \in H\} \\ \mathcal{E}_A(H) &:= \operatorname{Span}\{e^{i\langle x, h\rangle}; h \in D(A^*)\}. \end{aligned}$$

In the sequel the following lemma will play a crucial role.

Lemma 3.2.1 *For any $\varphi \in BUC(H)$, there is a sequence $(\varphi_{n,k})_{n,k \in \mathbb{N}} \subset \mathcal{E}(H)$ with*

(a) $\lim_{k\to\infty}\lim_{n\to\infty}\varphi_{n,k}(x)=\varphi(x), \quad \forall x\in H,$

(b) $\|\varphi_{n,k}\|_\infty \le \|\varphi\|_\infty, \quad \forall n,k\in\mathbb{N}.$

Thus, $\mathcal{E}(H)$ (resp. $\mathcal{E}_A(H)$) is dense in $L^2(H,\mu)$.

Proof: Since $D(A^*)$ is dense in H and $BUC(H)$ is dense in $L^2(H,\mu)$, and by the dominated convergence theorem, it suffices to show the existence of such a sequence.
To this purpose we assume first that $\dim H := d < \infty$ and consider the function φ_n satisfying

(i) φ_n is periodic with period n in all coordinate x_k, $k=1,\dots,d$,

(ii) $\varphi_n(x)=\varphi(x), \quad \forall x\in[-n-\frac{1}{2},n-\frac{1}{2}]^d$,

(iii) $\|\varphi_n\|_\infty\le\|\varphi\|_\infty$.

Hence,

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x), \quad \forall x\in H.$$

On the other hand, any function φ_n, $n\in\mathbb{N}$, can be approximate, by using Fourier series, by functions in $\mathcal{E}(H)$. This proves the lemma for finite dimensional Hilbert spaces.
In the general case, let $\varphi\in BUC(H)$. Take

$$\psi_k(x):=\varphi(x_1,x_2,\dots,x_k,0,\dots), \quad x\in H,\, k\in\mathbb{N}.$$

Then it follows from the first step that there is $(\varphi_{n,k})_{n,k\in\mathbb{N}}\subset\mathcal{E}(H)$ with

$$\begin{aligned}\lim_{n\to\infty}\varphi_{n,k}(x) &= \psi_k(x), \quad \forall x\in H,\\ \|\varphi_{n,k}\|_\infty &\le \|\psi_k\|_\infty\le\|\varphi\|_\infty.\end{aligned}$$

Therefore, for any $x\in H$,

$$\lim_{k\to\infty}\lim_{n\to\infty}\varphi_{n,k}(x)=\varphi(x), \quad \forall x\in H.$$

□

For any $k\in\mathbb{N}$ we define the partial derivative in the direction e_k by

$$D_k\varphi(x):=\lim_{t\to 0}\frac{1}{t}(\varphi(x+te_k)-\varphi(x)), \quad x\in H$$

for $\varphi\in\mathcal{E}_A(H)$ (or $\varphi\in\mathcal{E}(H)$). We note that for $\varphi(x):=e^{i\langle x,h\rangle}$, we have $D_k\varphi(x)=ihe^{i\langle x,h\rangle}$ for $x,h\in H$.
The following proposition gives an integration by part formula.

Proposition 3.2.2 *For $\varphi,\tilde\varphi\in\mathcal{E}(H)$ and $k\in\mathbb{N}$ the following holds*

$$\int_H D_k\varphi(x)\tilde\varphi(x)\mu(dx)=-\int_H\varphi(x)D_h\tilde\varphi(x)\mu(dx)+\frac{1}{\lambda_k}\int_H x_k\varphi(x)\tilde\varphi(x)\mu(dx).$$

Proof: For $\varphi, \tilde{\varphi} \in \mathcal{E}(H)$ we have

$$\begin{aligned}
\int_H D_k\varphi(x)\tilde{\varphi}(x)\mu(dx) &= \int_H ih_k e^{i\langle x,h\rangle} e^{i\langle x,\tilde{h}\rangle}\mu(dx) \\
&= ih_k \int_H e^{i\langle x,h+\tilde{h}\rangle}\mu(dx) \\
&= ih_k e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle} \quad \text{and} \\
\int_H \varphi(x)D_k\tilde{\varphi}(x)\mu(dx) &= i\tilde{h}_k e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\frac{1}{\lambda_k}\int_H x_k\varphi(x)\tilde{\varphi}(x)\mu(dx) = \\
&= \frac{1}{\lambda_k}\int_H x_k e^{i\langle x,h+\tilde{h}\rangle}\mu(dx) \\
&= \frac{1}{i\lambda_k}\frac{d}{dt}\left(\int_H e^{it\langle x,e_k\rangle}e^{i\langle x,h+\tilde{h}\rangle}\mu(dx)\right)_{|t=0} \\
&= \frac{1}{i\lambda_k}\frac{d}{dt}\left(\int_H e^{i\langle x,te_k+h+\tilde{h}\rangle}\mu(dx)\right)_{|t=0} \\
&= \frac{1}{i\lambda_k}\frac{d}{dt}\left[\exp\left(-\frac{1}{2}\langle B(te_k+h+\tilde{h}),te_k+h+\tilde{h}\rangle\right)\right]_{|t=0} \\
&= \frac{1}{i\lambda_k}\left[-\lambda_k(h_k+\tilde{h}_k)e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}\right] \\
&= i(h_k+\tilde{h}_k)e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}.
\end{aligned}$$

This proves the integration by part formula. □

The following proposition permits us to define the first Sobolev space with respect to the Gaussian measure μ.

Proposition 3.2.3 *For any $k \in \mathbb{N}$, the operator D_k with domain $\mathcal{E}(H)$ is closable on $L^2(H,\mu)$.*

Proof: Let $(\varphi_n) \subset \mathcal{E}(H)$ be such that $\lim_{n\to\infty}\varphi_n = 0$ and $\lim_{n\to\infty} D_k\varphi_n = \psi$ in $L^2(H,\mu)$. By Proposition 3.2.2 we have

$$\int_H D_k\varphi_n(x)\varphi(x)\mu(dx) + \int_H \varphi_n(x)D_k\varphi(x)\mu(dx) = \frac{1}{\lambda_k}\int_H x_k\varphi_n(x)\varphi(x)\mu(dx).$$

By Hölder's inequality, one can estimate the right hand side of the above equation and obtains

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_H x_k\varphi_n(x)\varphi(x)\mu(dx)\right|^2 \le \\
&\le \lim_{n\to\infty}\left(\int_H \varphi_n(x)^2\mu(dx)\cdot\int_H x_k^2\varphi(x)^2\mu(dx)\right) = 0
\end{aligned}$$

for $\varphi \in \mathcal{E}(H)$. Hence,

$$\int_H \psi(x)\varphi(x)\mu(dx) = 0, \quad \forall \varphi \in \mathcal{E}(H).$$

Since $\mathcal{E}(H)$ is dense in $L^2(H,\mu)$, it follows that $\psi \equiv 0$. □

In the sequel we use the notation $D_k := \overline{D_k}$ for $k \in \mathbb{N}$.

Definition 3.2.4 *The first order Sobolev space $W^{1,2}(H,\mu)$ is defined by*

$$W^{1,2}(H,\mu) := \{\varphi \in L^2(H,\mu) : \varphi \in D(D_k),\ \forall k \in \mathbb{N},\ \textit{and}\ \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2 \mu(dx) < \infty\}.$$

For $\varphi \in W^{1,2}(H,\mu)$, we denote by

$$D\varphi(x) := \sum_{k=1}^{\infty} D_k\varphi(x) e_k, \quad x \in H,$$

the gradient of φ at x, which exists as a $L^2(H,\mu)$–function and hence for almost every $x \in H$. It is clear that $W^{1,2}(H,\mu)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{W^{1,2}(H,\mu)} := \langle \varphi, \psi \rangle_{L^2(H,\mu)} + \int_H \langle D\varphi(x), D\psi(x)\rangle \mu(dx), \quad \varphi, \psi \in W^{1,2}(H,\mu),$$

is a Hilbert space.

Now, we show that Proposition 3.2.2 remains valid in $W^{1,2}(H,\mu)$. To this purpose we need the following lemma.

Lemma 3.2.5 *If $\varphi \in W^{1,2}(H,\mu)$, then, for any $k \in \mathbb{N}$, $x_k\varphi \in L^2(H,\mu)$.*

Proof: It is easy to see that Proposition 3.2.2 holds for all $\varphi \in W^{1,2}(H,\mu)$ and $\tilde{\varphi} \in \mathcal{E}(H)$. So if we apply Proposition 3.2.2 with $\varphi = x_k g$ and $\tilde{\varphi} = g$ for $k \in \mathbb{N}$ and $g \in \mathcal{E}(H)$, then

$$\begin{aligned}\int_H x_k^2 g(x)^2 \mu(dx) = \\ &= \lambda_k \int_H (g(x) + x_k D_k g(x)) g(x)\mu(dx) + \lambda_k \int_H x_k g(x) D_k g(x) \mu(dx) \\ &= \lambda_k \int_H g(x)^2 \mu(dx) + 2\lambda_k \int_H x_k g(x) D_k g(x)\mu(dx).\end{aligned}$$

So by Young's inequality we obtain

$$\begin{aligned}\int_H x_k^2 g(x)^2 \mu(dx) \leq \\ &\leq \lambda_k \int_H g(x)^2 \mu(dx) + \frac{1}{2}\int_H x_k^2 g(x)^2 \mu(dx) + 2\lambda_k^2 \int_H D_k g(x)^2 \mu(dx).\end{aligned}$$

Thus,

$$\int_H x_k^2 g(x)^2\mu(dx) \le 2\lambda_k \int_H g(x)^2\mu(dx) + 4\lambda_k^2 \int_H D_k g(x)^2\mu(dx).$$

This end the proof of the lemma. □

From the above lemma we obtain the following corollaries.

Corollary 3.2.6 *If $\varphi \in W^{1,2}(H,\mu)$, then $|x|\varphi \in L^2(H,\mu)$ and the following holds*

$$\int_H |x|^2\varphi(x)^2\mu(dx) \le 2\mathrm{Tr}B \int_H \varphi(x)^2\mu(dx) + 4\|B\|^2 \int_H |D\varphi(x)|^2\mu(dx).$$

Corollary 3.2.7 *For $\varphi,\psi \in W^{1,2}(H,\mu)$ the following holds*

$$\int_H D_k\varphi(x)\psi(x)\mu(dx) + \int_H \varphi(x)D_k\psi(x)\mu(dx) = \frac{1}{\lambda_k}\int_H x_k\varphi(x)\psi(x)\mu(dx).$$

By the same proof as for the first derivative one can see that, for any $h,k \in \mathbb{N}$ the operator $D_hD_k : \mathcal{E}(H) \to L^2(H,\mu)$ is closable on $L^2(H,\mu)$ and as before we use the notation $D_hD_k := \overline{D_hD_k}$.

Definition 3.2.8 *The second order Sobolev space $W^{2,2}(H,\mu)$ is defined by*

$$W^{2,2}(H,\mu) :=$$
$$\{\varphi \in L^2(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_hD_k) \ \textit{and} \ \sum_{h,k=1}^{\infty}\int_H |D_hD_k\varphi(x)|^2\mu(dx) < \infty\}.$$

If $\varphi \in W^{2,2}(H,\mu)$, then, for a.e. $x \in H$ one can define a Hilbert-Schmidt operator $D^2\varphi(x)$ (since $\sum_{h,k\in\mathbb{N}} |D_hD_k\varphi(x)|^2 < \infty$ for a.e. $x \in H$) by

$$\langle D^2\varphi(x)y, z\rangle := \sum_{h,k=1}^{\infty} D_hD_k\varphi(x)y_hz_k, \quad y,z \in H, \text{ a.e. } x \in H.$$

It is easy to see that $W^{2,2}(H,\mu)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{2,2}(H,\mu)} := \langle\varphi,\psi\rangle_{W^{1,2}(H,\mu)} + \sum_{h,k=1}^{\infty}\int_H \langle D_hD_k\varphi(x), D_hD_k\psi(x)\rangle\mu(dx)$$

is a Hilbert space.

In a similar way one can obtain the following useful result.

Proposition 3.2.9 *If $\varphi \in W^{2,2}(H,\mu)$, then $|x|\varphi \in W^{1,2}(H,\mu)$, $|x|^2\varphi \in L^2(H,\mu)$ and the following estimates hold*

$$\begin{aligned}\int_H |x|^2|D\varphi(x)|^2\mu(dx) &\le 2\int_H \varphi(x)^2\mu(dx) + 4\mathrm{Tr}B\int_H |D\varphi(x)|^2\mu(x) + \\ &\quad 8\|B\|^2\int_H \mathrm{Tr}(D^2\varphi(x))^2\mu(dx),\\ \int_H |x|^4\varphi(x)^2\mu(dx) &\le c\left(\int_H \varphi(x)^2\mu(dx) + \int_H |D\varphi(x)|^2\mu(dx) + \right.\\ &\quad \left.\int_H \mathrm{Tr}(D^2\varphi(x))^2\mu(dx)\right).\end{aligned}$$

For the characterization of the generator of the Ornstein-Uhlenbeck semigroup on $L^2(H,\mu)$ we need the notion of Malliavin derivatives.

We consider the operator $D_B : \mathcal{E}(H) \to L^2(H,\mu;H)$ defined by

$$D_B\varphi := B^{\frac{1}{2}}D\varphi \quad \text{for } \varphi \in \mathcal{E}(H).$$

Here $L^2(H,\mu;H)$ denotes the space of all strongly measurable functions $\Phi : H \to H$ satisfying $\int_H |\Phi(x)|^2\mu(dx) < \infty$.

Proposition 3.2.10 *The operator D_B with domain $\mathcal{E}(H)$ is closable in $L^2(H,\mu;H)$.*

Proof: Let $(\varphi_n) \subset \mathcal{E}(H)$ and $F \in L^2(H,\mu;H)$ are such that $\lim_{n\to\infty}\varphi_n = 0$ in $L^2(H,\mu)$ and $\lim_{n\to\infty} D_B\varphi_n = F$ in $L^2(H,\mu;H)$. This means that

$$\begin{aligned}&\lim_{n\to\infty}\int_H |D_B\varphi_n(x) - F(x)|^2\mu(dx) = \\ &= \lim_{n\to\infty}\int_H \sum_{k=1}^{\infty} |\sqrt{\lambda_k}D_k\varphi_n(x) - F_k(x)|^2\mu(dx) = 0.\end{aligned}$$

Since we have supposed that $\ker B = \{0\}$, it follows that, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} D_k\varphi_n = \frac{1}{\sqrt{\lambda_k}}F_k \quad \text{in } L^2(H,\mu).$$

So by Proposition 3.2.3 we have, for any $k \in \mathbb{N}$, $F_k \equiv 0$, which proves the claim. □

As before we use the notation $D_B := \overline{D_B}$ and this will be called the *Malliavin derivative*. In a similar way we define the following spaces

$$\begin{aligned}W_B^{1,2}(H,\mu) &:= \{\varphi \in L^2(H,\mu) : D_B\varphi \in L^2(H,\mu;H)\},\\ W_B^{2,2}(H,\mu) &:= \{\varphi \in L^2(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_hD_k) \text{ and}\\ &\qquad \sum_{h,k=1}^{\infty}\int_H \lambda_h\lambda_k|D_hD_k\varphi(x)|^2\mu(dx) < \infty\}.\end{aligned}$$

3.3 The Ornstein-Uhlenbeck semigroup on L^p-spaces with invariant measure

The aim of this section is to study the Ornstein-Uhlenbeck semigroup on L^p–spaces with respect to an invariant measure.
Under appropriate assumptions we prove the existence and uniqueness of an invariant measure μ for the Ornstein-Uhlenbeck semigroup (R_t). This allows us to extend (R_t) to a C_0–semigroup on $L^p(H,\mu)$, $1 \le p < \infty$. We find sufficient conditions for the existence and uniqueness of a classical solution for (KE) on $L^p(H,\mu)$, $1 < p < \infty$ and finally we characterize the domain of the generator of the symmetric Ornstein-Uhlenbeck semigroup on $L^2(H,\mu)$.

In order to have an invariant measure for the Ornstein-Uhlenbeck semigroup we suppose in this section the following assumptions

(H3) $A : D(A) \to H$ generates a C_0 – semigoup $(e^{tA})_{t\ge 0}$ satisfying $\|e^{tA}\| \le Me^{-\omega t}$ for some constants $M \ge 1$, $\omega > 0$.

(H4) $Q \in \mathcal{L}(H)$ is a symmetric and positive operator and

$$Q_t := \int_0^t e^{sA} Q e^{sA^*}\, ds \in \mathcal{L}_1^+(H), \quad t \ge 0.$$

If we set $Q_\infty x := \int_0^\infty e^{sA} Q e^{sA^*}\, ds$, $x \in H$, then

$$Q_\infty x = \sum_{n=0}^{\infty} \int_n^{n+1} e^{sA} Q e^{sA^*}\, ds = \sum_{n=0}^{\infty} e^{nA} Q_1 e^{nA^*} x, \quad x \in H.$$

Hence,

$$\mathrm{Tr} Q_\infty \le M^2 \mathrm{Tr} Q_1 \sum_{n=0}^{\infty} e^{-2\omega n} < \infty,$$

which implies that $Q_\infty \in \mathcal{L}_1^+(H)$.
The following result shows the existence and uniqueness of invariant measure for the Ornstein-Uhlenbeck semigroup.

Proposition 3.3.1 *Assume that (H3) and (H4) hold. Then the Gaussian measure $\mu := \mathcal{N}(0, Q_\infty)$ is the unique invariant measure for the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge 0}$. This means that, for all $\varphi \in BUC(H)$,*

$$\int_H R_t \varphi(x) \mu(dx) = \int_H \varphi(x) \mu(dx).$$

Moreover, for all $\varphi \in BUC(H)$ and $x \in H$,

$$\lim_{t\to\infty} R_t \varphi(x) = \int_H \varphi(x) \mu(dx).$$

Proof: It follows from Lemma 3.2.1 that it suffices to show the proposition for $\varphi \in \mathcal{E}_A(H)$. For $\varphi_h(x) := e^{i\langle h,x\rangle}$, $x, h \in H$, we have

$$\begin{aligned}
\int_H R_t\varphi_h(x)\mu(dx) &= \int_H \int_H e^{i\langle h, e^{tA}x+y\rangle} \mathcal{N}(0,Q_t)(dy)\mu(dx) \\
&= \int_H e^{i\langle e^{tA}x,h\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \mu(dx) \\
&= e^{-\frac{1}{2}\langle Q_t h,h\rangle - \frac{1}{2}\langle Q_\infty e^{tA^*}h, e^{tA^*}h\rangle} \\
&= e^{-\frac{1}{2}\langle (Q_t + e^{tA}Q_\infty e^{tA^*})h,h\rangle} \\
&= \int_H \varphi_h(x)\mu(dx),
\end{aligned}$$

where the last equality follows from the equation

$$Q_t + e^{tA}Q_\infty e^{tA^*} = Q_\infty, \quad t \geq 0. \tag{3.1}$$

On the other hand, we obtain

$$\begin{aligned}
\lim_{t\to\infty} R_t\varphi_h(x) &= \lim_{t\to\infty} e^{i\langle e^{tA}h,x\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \\
&= e^{-\frac{1}{2}\langle Q_\infty h,h\rangle} \\
&= \int_H \varphi_h(x)\mu(dx).
\end{aligned}$$

For the uniqueness, we suppose that there is an invariant measure ν for (R_t). In particular ν satisfies

$$\int_H R_t\varphi_h(x)\nu(dx) = \int_H \varphi_h(x)\nu(dx)$$

for $\varphi_h(x) := e^{i\langle h,x\rangle}$, $x, h \in H$. This implies that

$$e^{-\frac{1}{2}\langle Q_t h,h\rangle}\widehat{\nu}(e^{tA^*}h) = \widehat{\nu}(h).$$

So by letting $t \to \infty$ we obtain

$$\widehat{\nu}(h) = e^{-\frac{1}{2}\langle Q_\infty h,h\rangle} = \widehat{\mu}(h)$$

and the uniqueness follows now from the characterization of Gaussian measures (see Theorem 1.2.5). □

Now, one can extend the semigroup $(R_t)_{t\geq 0}$ to a C_0–semigroup on $L^p(H,\mu)$,
$1 \leq p < \infty$.

Theorem 3.3.2 *Assume that (H3) and (H4) are satisfied. Then, for all $t \geq 0$, R_t can be extended to a bounded linear operator on $L^p(H,\mu)$ and $(R_t)_{t\geq 0}$ defines a C_0–semigroup of contractions on $L^p(H,\mu)$ for $1 \leq p < \infty$.*

Proof: Let $t \geq 0$ and $\varphi \in BUC(H)$. By Hölder's inequality we have

$$|R_t\varphi(x)|^p \leq (R_t|\varphi|^p)(x), \quad x \in H.$$

Hence,

$$\begin{aligned}\int_H |R_t\varphi(x)|^p \mu(dx) &\leq \int_H R_t|\varphi|^p(x)\mu(dx) \\ &= \int_H |\varphi(x)|^p \mu(dx).\end{aligned}$$

So, the first assertion follows from the density of $BUC(H)$ in $L^p(H,\mu)$ for $1 \leq p < \infty$ and we have

$$\|R_t\varphi\|_{L^p(H,\mu)} \leq \|\varphi\|_{L^p(H,\mu)}, \quad t \geq 0, \, \varphi \in L^p(H,\mu).$$

Finally, the strong continuity follows from the dominated convergence theorem. □

As in Section 3.1 we show that $u(t,x) := (R_t\varphi)(x)$, $t \geq 0, x \in H$, and $\varphi \in L^p(H,\mu)$ is the unique classical solution of (KE), which means that

(a) u is continuous on $[0,\infty) \times H$, $u(t,\cdot) \in C^2(H)$ for all $t > 0$,

(b) $QD^2u(t,x)$ is a trace class operator on H and $Du(t,x) \in D(A^*)$ for every $t > 0$ and $x \in H$,

(c) A^*Du and $\mathrm{Tr}(QD^2u)$ are two continuous functions on $(0,\infty) \times H$ and u satisfies (KE) for all $t > 0$ and $x \in D(A)$.

This result can be found in [6, Theorem 5].
To this purpose we need the following lemmas (see [6, Proposition 2] and [5, Proposition 1] or [13, Theorem 10.3.5]).

Lemma 3.3.3 *Suppose (H2), (H3) and (H4). Then the following hold.*

(i) *The family* $S_0(t) := Q_\infty^{-\frac{1}{2}} e^{tA} Q_\infty^{\frac{1}{2}}$, $t \geq 0$, *defines a* C_0*–semigroup of contractions on* H.

(ii) *The operators* $S_0(t)S_0^*(t)$, $t > 0$, *satisfy*

$$\|S_0(t)S_0^*(t)\| < 1 \text{ and}$$
$$\Lambda_t\Lambda_t^*(Q_\infty^{-\frac{1}{2}}e^{tA})^*(I - S_0(t)S_0^*(t))^{-1}(Q_\infty^{-\frac{1}{2}}e^{tA}).$$

(iii) *For* $0 < t_0 < t_1$, *the function* $[t_0,t_1] \ni t \mapsto \Lambda_t \in \mathcal{L}(H)$ *is bounded.*

Lemma 3.3.4 *Assume (H2), (H3) and (H4) and let* $\varphi \in L^p(H,\mu)$, $1 < p < \infty$*. Then, for any* $t > 0$*,* $(R_t\varphi)(\cdot) \in C^\infty(H)$ *and*

$$|D^n R_t\varphi(x)| \le c(t,n,p,\varphi) < \infty$$

uniformly on bounded subsets of H *for* $n = 0, 1, \dots$ *and some constant* $c(t,n,p,\varphi) > 0$.

Proof of Lemma 3.3.3: (i) It follows from (H2) and Exercise 3.3.22 that $S_0(t)$, $t \ge 0$, are bounded linear operators on H and

$$S_0^*(t) = \overline{Q_\infty^{\frac{1}{2}} e^{tA^*} Q_\infty^{-\frac{1}{2}}}, \quad t \ge 0,$$

which can be defined on H, since $\ker Q_\infty = \{0\}$ and hence, $\overline{Q_\infty^{\frac{1}{2}}(H)} = H$ by Remark 1.3.2. Now, from (3.1), we obtain

$$0 \le \langle Q_t x, x\rangle = \langle (I - S_0(t)S_0^*(t))Q_\infty^{\frac{1}{2}}x, Q_\infty^{\frac{1}{2}}x\rangle, \quad t \ge 0, x \in H.$$

Hence, $\|S_0^*(t)Q_\infty^{\frac{1}{2}}x\| \le \|Q_\infty^{\frac{1}{2}}x\|$, $t \ge 0$, $x \in H$. Since $\overline{Q_\infty^{\frac{1}{2}}(H)} = H$, we deduce that

$$\|S_0(t)\| \le 1, \quad t \ge 0. \tag{3.2}$$

The semigroup property can be easily verified. It suffices now to show that $S_0(\cdot)$ is weakly continuous at zero. Let $x, y \in H$. Then,

$$\lim_{t\to 0^+} \langle S_0(t)x, Q_\infty^{\frac{1}{2}}y\rangle = \langle x, Q_\infty^{\frac{1}{2}}y\rangle,$$

and the weak continuity follows from (3.2) and the density of $Q_\infty^{\frac{1}{2}}(H)$ in H.
(ii) From (3.1) and Exercise 3.3.22 it follows that

$$I - S_0(t)S_0^*(t) = (Q_\infty^{-\frac{1}{2}}Q_t^{\frac{1}{2}})(\overline{Q_t^{\frac{1}{2}}Q_\infty^{-\frac{1}{2}}}), \quad t > 0.$$

By Exercise 3.3.22 we have that $Q_\infty^{-\frac{1}{2}}Q_t^{\frac{1}{2}}$ has a bounded inverse and so does $I - S_0(t)S_0^*(t)$ for $t > 0$. Since $I - S_0(t)S_0^*(t)$ is selfadjoint and positive, we deduce that

$$\|S_0(t)S_0^*(t)\| < 1 \quad \text{for all } t > 0.$$

On the other hand, by Exercise 3.3.22, we have

$$\begin{aligned}
\Lambda_t^*\Lambda_t &= (Q_t^{-\frac{1}{2}}e^{tA})^*(Q_t^{-\frac{1}{2}}e^{tA}) \\
&= (Q_\infty^{-\frac{1}{2}}e^{tA})^*(Q_t^{-\frac{1}{2}}Q_\infty^{\frac{1}{2}})^*(Q_t^{-\frac{1}{2}}Q_\infty^{\frac{1}{2}})(Q_\infty^{-\frac{1}{2}}e^{tA}) \\
&= (Q_\infty^{-\frac{1}{2}}e^{tA})^*(I - S_0(t)S_0^*(t))^{-1}(Q_\infty^{-\frac{1}{2}}e^{tA})
\end{aligned}$$

for every $t > 0$.
(iii) Take $a > 0$ such that

$$\|S_0(t_0)S_0^*(t_0)\| < a < 1.$$

Then,

$$\begin{aligned} \|S_0(t)S_0^*(t)\| &= \|S_0(t-t_0)S_0(t_0)S_0^*(t_0)S_0^*(t-t_0)\| \\ &\leq \|S_0(t_0)S_0^*(t_0)\| < a \end{aligned}$$

for $t \in [t_0, t_1]$. Now, (iii) follows from the identity

$$Q_\infty^{-\frac{1}{2}} e^{tA} = (Q_\infty^{-\frac{1}{2}} e^{t_0 A}) e^{(t-t_0)A}$$

for $t \in [t_0, t_1]$. □

Proof of Lemma 3.3.4: We fix $t > 0$ and $\varphi \in L^p(H, \mu)$. Suppose without loss of generality that

$$\int_H |\varphi(e^{tA}x + y)|^p \mathcal{N}(0, Q_t)(dy) < \infty \quad \text{for } x = 0. \tag{3.3}$$

Let consider a sequence $(\varphi_n) \subset B_b(H)$ with $|\varphi_n(x)| \leq |\varphi(x)|$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for μ–a.a. x and hence, by Exercise 3.3.20, for $\mathcal{N}(0, Q_t)$–a.a. x. So, by (3.3), φ_n converges also to φ in $L^p(H, \mathcal{N}(0, Q_t))$. On the other hand, we know from Theorem 3.1.1 that $R_t\varphi_n \in BUC^\infty(H)$. So, by the Cameron-Martin formula and Hölder's inequality, we obtain

$$\begin{aligned} & |R_t\varphi(x) - R_t\varphi_n(x)| \\ \leq & \int_H |\varphi(e^{tA}x + y) - \varphi_n(e^{tA}x + y)| \mathcal{N}(0, Q_t)(dy) \\ = & \int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right) |\varphi(y) - \varphi_n(y)| \mathcal{N}(0, Q_t)(dy) \\ \leq & \left(\int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right)^q \mathcal{N}(0, Q_t)(dy)\right)^{\frac{1}{q}} \\ & \left(\int_H |\varphi(y) - \varphi_n(y)|^p \mathcal{N}(0, Q_t)(dy)\right)^{\frac{1}{p}} \end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Thus, it follows from Proposition 1.3.3 that

$$\sup_{\|x\| \leq K} |R_t\varphi(x) - R_t\varphi_n(x)| \leq \sup_{\|x\| \leq K} \exp\left(\frac{q-1}{2}|\Lambda_t x|^2\right) \|\varphi - \varphi_n\|_{L^p(H, \mathcal{N}(0, Q_t))}$$

for $t > 0$ and any constant $K > 0$. This implies that $R_t\varphi \in C(H)$.
On the other hand, from Exercise 3.3.21 and the Cameron-Martin formula,

we have

$$
\begin{aligned}
& |\langle DR_t\varphi_n(x) - DR_t\varphi_m(x), y\rangle| \\
\leq\ & \int_H |\langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle (\varphi_n(e^{tA}x+h) - \varphi_m(e^{tA}x+h))| \mathcal{N}(0,Q_t)(dh) \\
\leq\ & \left(\int_H |\langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle|^{r'} \mathcal{N}(0,Q_t)(dh) \right)^{\frac{1}{r'}} \\
& \left(\int_H |\varphi_n(e^{tA}x+h) - \varphi_m(e^{tA}x+h)|^r \mathcal{N}(0,Q_t)(dh) \right)^{\frac{1}{r}} \\
=\ & c_r|\Lambda_t y| \Big(\int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} h\rangle \right) \\
& |\varphi_n(h) - \varphi_m(h)|^r \mathcal{N}(0,Q_t)(dh) \Big)^{\frac{1}{r}} \\
\leq\ & c_r|\Lambda_t y| \left(\int_H \exp\left(-\frac{b}{2}|\Lambda_t x|^2 + b\langle \Lambda_t x, Q_t^{-\frac{1}{2}} h\rangle \right) \mathcal{N}(0,Q_t)(dh) \right)^{\frac{1}{rb}} \\
& \left(\int_H |\varphi_n(h) - \varphi_m(h)|^p \mathcal{N}(0,Q_t)(dh) \right)^{\frac{1}{p}},
\end{aligned}
$$

where $\frac{1}{r} + \frac{1}{r'} = 1$, $r > 1$, and $\frac{1}{b} + \frac{r}{p} = 1$. So, by Proposition 1.3.3, it follows that

$$
|DR_t\varphi_n(x) - DR_t\varphi_m(x)| \leq c(t,p) \exp\left(\frac{b-1}{2r} |\Lambda_t x|^2 \right) \|\varphi_n - \varphi_m\|_{L^p(H,\mathcal{N}(0,Q_t))}
$$

for $x \in H$. Thus, $DR_t\varphi_n$ converges uniformly on bounded subsets of H to a continuous function. Using Theorem 3.1.1 and by the same argument one can show the result for arbitrary n. □

The following result shows the existence and uniqueness of the classical solution for (KE), for any $\varphi \in L^p(H,\mu)$, $1 < p < \infty$.

Theorem 3.3.5 *Let (H2), (H3) and (H4) hold. If the operator $\Lambda_t A$ has a continuous extension $\overline{\Lambda_t A}$ on H then the function $(t,x) \mapsto (R_t\varphi)(x)$ is the unique classical solution for (KE) for any $\varphi \in L^p(H,\mu)$, $1 < p < \infty$.*

Proof: As in Theorem 3.1.2 we prove first that, for every $\varphi \in L^p(H,\mu)$, and $x \in H$,

$$
DR_t\varphi(x) \in D(A^*) \quad \text{for all } t > 0.
$$

Let $t > 0$ and $\varphi \in L^p(H,\mu)$ be fixed. We know from Theorem 3.1.1 and Lemma 3.3.4 that, for $y \in D(A)$,

$$
\langle DR_t\varphi(x), Ay\rangle = \int_H \langle \Lambda_t Ay, Q_t^{-\frac{1}{2}} h\rangle \varphi(e^{tA}x+h) \mathcal{N}(0,Q_t)(dh).
$$

Thus, by Hölder's inequality and Exercise 3.3.21, we obtain

$$\begin{aligned}|\langle DR_t\varphi(x), Ay\rangle| &\leq \left(\int_H |\langle \Lambda_t Ay, Q_t^{-\frac{1}{2}}h\rangle|^{r'}\mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r'}} \\ &\quad \left(\int_H |\varphi(e^{tA}x+h)|^r\mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r}} \\ &\leq c_r|\Lambda_t Ay|\,(R_t|\varphi|^r(x))^{\frac{1}{r}} \qquad (3.4)\\ &\leq c_r\|\overline{\Lambda_t A}\||y|\,(R_t|\varphi|^r(x))^{\frac{1}{r}}\end{aligned}$$

for $x \in H$, $\frac{1}{r'}+\frac{1}{r}=1$, $1<r<p$, and all $y \in D(A)$. Since $|\varphi|^r \in L^{\frac{p}{r}}(H,\mu)$, it follows from Lemma 3.3.4 that

$$c(r,\varphi,x) := c_r\,(R_t|\varphi|^r(x))^{\frac{1}{r}} < \infty.$$

Hence, $DR_t\varphi(x) \in D(A^*)$ for $t>0$ and $x \in H$.
On the other hand, by Theorem 3.1.1 and Lemma 3.3.4, we have $D^2R_t\varphi(x)$ exists for all $x \in H$ and

$$\langle D^2R_t\varphi(x)e_j, e_j\rangle = \int_H \left[|\langle \Lambda_t e_j, Q_t^{-\frac{1}{2}}y\rangle|^2 - |\Lambda_t e_j|^2\right]\varphi(e^{tA}x+y)\mathcal{N}(0,Q_t)(dy).$$

Take $1<r<p$. Then, it follows from Hölder's inequality and Exercise 3.3.21 that

$$\begin{aligned}|\langle D^2R_t\varphi(x)e_j, e_j\rangle| &\leq \left(\int_H \left[|\langle \Lambda_t e_j, Q_t^{-\frac{1}{2}}y\rangle|^2 - |\Lambda_t e_j|^2\right]^{r'}\mathcal{N}(0,Q_t)(dy)\right)^{\frac{1}{r'}} \\ &\quad \left(\int_H |\varphi(e^{tA}x+y)|^r\mathcal{N}(0,Q_t)(dy)\right)^{\frac{1}{r}} \\ &\leq c_r|\Lambda_t e_j|^2\,(R_t|\varphi|^r(x))^{\frac{1}{r}} \qquad (3.5)\end{aligned}$$

for $x \in H$, and $\frac{1}{r'}+\frac{1}{r}=1$, $1<r<p$. By the same argument as above and Corollary 3.1.3 we have $c(r,\varphi,x) := c_r\,(R_t|\varphi|^r(x))^{\frac{1}{r}} < \infty$ and

$$\sum_{j=1}^{\infty}|\langle D^2R_t\varphi(x)e_j,e_j\rangle| \leq c(r,\varphi,x)\sum_{j=1}^{\infty}|\Lambda_j e_j|^2 < \infty.$$

This shows that $D^2R_t\varphi(x)$ is a trace class operator on H for $x \in H$, $t>0$ and $\varphi \in L^p(H,\mu)$. From Corollary 3.1.3 we know that (KE) has a unique classical solution $u(t,x) := R_t\varphi(x)$ for $\varphi \in B_b(H)$. Now, for $\varphi \in L^p(H,\mu)$, there is a sequence $(\varphi_n) \subset B_b(H)$ with $|\varphi_n(x)| \leq |\varphi(x)|$ and $\lim_{n\to\infty}\varphi_n(x) =$

$\varphi(x)$ for μ–a.a. $x \in H$. It follows from Exercise 3.3.23 that

$$\begin{aligned}
&|R_t\varphi_n(x) - R_t\varphi(x)| \leq \\
&\leq \left(\int_H k(t,x,y)^q \mu(dy)\right)^{\frac{1}{q}} \|\varphi_n - \varphi\|_{L^p(H,\mu)} \\
&= \det(I - S_0(t)S_0^*(t))^{\frac{1-q}{2q}} \det(I + (q-1)S_0(t)S_0^*(t))^{-\frac{1}{2q}} \\
&\quad \exp\left(\frac{q-1}{2}\langle (I + (q-1)S_0(t)S_0^*(t))^{-1} Q_\infty^{-\frac{1}{2}} e^{tA}x, Q_\infty^{-\frac{1}{2}} e^{tA}x\rangle\right)
\end{aligned}$$

for $t > 0$, $x \in H$ and $\frac{1}{q} + \frac{1}{p} = 1$. So, by Lemma 3.3.3(iii), $R_t\varphi_n(x) \to R_t\varphi(x)$ uniformly in $(t,x) \in [t_0, t_1] \times \{x \in H : |x| \leq K\}$ for $0 < t_0 < t_1$ and any constant $K > 0$. Again by Exercise 3.3.23, we obtain

$$\begin{aligned}
&R_t|\varphi|^r(x) \leq \\
&\leq \left(\int_H k(t,x,y)^{\frac{p}{r}} \mu(dy)\right)^{\frac{r}{p}} \|\varphi\|^r_{L^p(H,\mu)} \\
&= \det(I - S_0(t)S_0^*(t))^{\frac{r-p}{2p}} \det(I + (\frac{p}{r} - 1)S_0(t)S_0^*(t))^{-\frac{r}{2p}} \\
&\quad \exp\left(\frac{p-r}{2r}\langle (I + (\frac{p}{r} - 1)S_0(t)S_0^*(t))^{-1} Q_\infty^{-\frac{1}{2}} e^{tA}x, Q_\infty^{-\frac{1}{2}} e^{tA}x\rangle\right)
\end{aligned}$$

for $t > 0$, $x \in H$ and $1 < r < p$. So, by Lemma 3.3.3(iii), (3.4) and (3.5), it follows that $\frac{\partial}{\partial t} R_t\varphi_n(x)$ converges uniformly in $(t,x) \in [t_0, t_1] \times \{x \in H : |x| \leq K\}$. Hence the function $(t,x) \mapsto R_t\varphi(x)$ is a classical solution for (KE). The uniqueness follows from Theorem 3.1.2. □

We propose now to characterize symmetric Ornstein-Uhlenbeck semigroups on $L^2(H,\mu)$. To this purpose we need the following lemma.

Lemma 3.3.6 *Assume that (H3) and (H4) hold. Then the operator Q_∞ is the only positive and symmetric solution of the following Lyapunov equation*

$$\langle Q_\infty x, A^*y\rangle + \langle Q_\infty A^*x, y\rangle = -\langle Qx, y\rangle, \quad x, y \in D(A^*). \tag{3.6}$$

Proof: For $x, y \in D(A^*)$, by using integration by part, we have

$$\begin{aligned}
\langle Q_\infty x, A^*y\rangle &= \int_0^\infty \langle e^{sA}Qe^{sA^*}x, A^*y\rangle\, ds \\
&= \int_0^\infty \langle Qe^{sA^*}x, \frac{d}{ds}e^{sA^*}y\rangle\, ds \\
&= -\langle Qx, y\rangle - \langle Q_\infty A^*x, y\rangle.
\end{aligned}$$

Suppose now that there is a positive and symmetric opertor $R \in \mathcal{L}(H)$ solution of the Lyapunov equation (3.6). Then we obtain

$$\frac{d}{dt}\langle Re^{tA^*}x, e^{tA^*}x\rangle = -\langle Qe^{tA^*}x, e^{tA^*}x\rangle, \quad x \in D(A^*).$$

So by integrating between 0 and t we obtain

$$\langle Re^{tA^*}x, e^{tA^*}x\rangle - \langle Rx, x\rangle = -\langle Q_t x, x\rangle, \quad x \in D(A^*).$$

Now, by letting $t \to \infty$ we get

$$\langle Rx, x\rangle = \langle Q_\infty x, x\rangle \quad \text{for all } x \in D(A^*).$$

This implies that $R = Q_\infty$. □

Symmetric Ornstein-Uhlenbeck semigroups on $L^2(H, \mu)$ are characterized by the following result.

Proposition 3.3.7 *Suppose (H3) and (H4) hold. Then the following assertion are equivalent*

(i) $(R_t)_{t\geq 0}$ *is symmetric in* $L^2(H, \mu)$.

(ii) $Q_\infty e^{tA^*} = e^{tA}Q_\infty$ *for all* $t \geq 0$.

(iii) $Qe^{tA^*} = e^{tA}Q$ *for all* $t \geq 0$.

If $(R_t)_{t\geq 0}$ *is symmetric then* $Q_\infty = -\frac{1}{2}A^{-1}Q$.

Proof: For $\varphi(x) := e^{i\langle x,h\rangle}$ and $\tilde{\varphi}(x) := e^{i\langle x,\tilde{h}\rangle}$, $x, h \in H$, we have

$$\begin{aligned} R_t\varphi(x) &= e^{i\langle e^{tA}x,h\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \quad \text{and} \\ R_t\tilde{\varphi}(x) &= e^{i\langle e^{tA}x,\tilde{h}\rangle - \frac{1}{2}\langle Q_t \tilde{h},\tilde{h}\rangle}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx) &= e^{-\frac{1}{2}\langle Q_t h,h\rangle}\int_H e^{i\langle x,\tilde{h}+e^{tA^*}h\rangle}\mu(dx) \\ &= e^{-\frac{1}{2}\langle Q_t h,h\rangle}e^{i\langle Q_\infty(\tilde{h}+e^{tA^*}h),\tilde{h}+e^{tA^*}h\rangle} \\ &= e^{-\frac{1}{2}\langle (Q_t+e^{tA}Q_\infty e^{tA^*})h,h\rangle}e^{-\frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle}e^{-\langle Q_\infty e^{tA^*}h,\tilde{h}\rangle}. \end{aligned}$$

So by (3.1) we obtain

$$\int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx)e^{-\frac{1}{2}\langle Q_\infty h,h\rangle - \frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle - \langle Q_\infty e^{tA^*}h,\tilde{h}\rangle}.$$

By the same computation we have

$$\int_H R_t\tilde{\varphi}(x)\varphi(x)\mu(dx)e^{-\frac{1}{2}\langle Q_\infty h,h\rangle - \frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle - \langle Q_\infty e^{tA^*}\tilde{h},h\rangle}.$$

Therefore,

$$\begin{aligned} \int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx) &= \int_H R_t\tilde{\varphi}(x)\varphi(x)\mu(dx) \text{ if and only if} \\ e^{-\langle Q_\infty e^{tA^*}h,\tilde{h}\rangle} &= e^{-\langle Q_\infty e^{tA^*}\tilde{h},h\rangle} \text{ if and only if} \\ Q_\infty e^{tA^*} &= e^{tA}Q_\infty. \end{aligned}$$

Hence the equivalence (i) $\Leftrightarrow$ (ii) follows from the density of $\mathcal{E}_A(H)$ in $L^2(H,\mu)$ (see Lemma 3.2.1).
The implication (iii) $\Rightarrow$ (ii) is trivial. It remains to prove (ii) $\Rightarrow$ (iii). To this purpose we consider $x \in D(A^*)$. It follows from (ii) that $Q_\infty x \in D(A)$ and

$$Q_\infty A^* x = A Q_\infty x.$$

So by Lemma 3.3.6 it follows that $2AQ_\infty = -Q$ and hence

$$Q_\infty = -\frac{1}{2} A^{-1} Q,$$

which proves the last assertion of the theorem. Again by Lemma 3.3.6 we have

$$\begin{aligned}\langle Q e^{tA^*} x, y\rangle &= -\langle Q_\infty e^{tA^*} x, A^* y\rangle - \langle Q_\infty A^* e^{tA^*} x, y\rangle \\ &= -\langle Q_\infty x, A^* e^{tA^*} y\rangle - \langle Q_\infty A^* x, e^{tA^*} y\rangle.\end{aligned}$$

On the other hand, it follows from Lemma 3.3.6 that

$$\begin{aligned}\langle e^{tA} Q x, y\rangle &= \langle Qx, e^{tA^*} y\rangle \\ &= -\langle Q_\infty x, A^* e^{tA^*} y\rangle - \langle Q_\infty A^* x, e^{tA^*} y\rangle.\end{aligned}$$

This implies that

$$\langle Q e^{tA^*} x, y\rangle = \langle e^{tA} Q x, y\rangle, \quad x, y \in D(A^*),\ t \ge 0,$$

which is equivalent to $Qe^{tA^*} = e^{tA}Q$ for all $t \ge 0$. □

In the particular case where A is selfadjoint we have the following result.

Corollary 3.3.8 *If the following assumptions are satisfied*

1. $A : D(A) \to H$ *is selfadjoint and there is* $\omega > 0$ *such that* $\langle Ax, x\rangle \le -\omega |x|^2$ *for all* $x \in D(A)$,
2. $Qe^{tA} = e^{tA}Q$ *for all* $t \ge 0$,
3. $QA^{-1} \in \mathcal{L}(H)$ *is a trace class operator,*

then $(R_t)_{t\ge 0}$ *is symmetric on* $L^2(H,\mu)$.

Proof: In this particular case we have

$$Q_t = Q \int_0^t e^{2sA}\, ds = \frac{1}{2} QA^{-1}(e^{2tA} - I), \quad t \ge 0.$$

From the third assumption we have $\mathrm{Tr} Q < \infty$ and the second assumption is exactly the third assertion in Proposition 3.3.7. This end the proof of the corollary. □

In the special case $Q = I$ we obtain

Corollary 3.3.9 *Assume that $A : D(A) \to H$ is selfadjoint, there is $\omega > 0$ such that $\langle Ax, x\rangle \le -\omega|x|^2$ for all $x \in D(A)$, A^{-1} is a trace class operator and $Q = I$. Then $(R_t)_{t\ge 0}$ is symmetric on $L^2(H,\mu)$.*

We propose now to describe the generator L_p of the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge0}$ on $L^p(H,\mu)$ $1 \le p < \infty$.
We set

$$L_0\varphi(x) := \frac{1}{2}\mathrm{Tr}(QD^2\varphi(x)) + \langle x, A^*D\varphi(x)\rangle, \quad x \in H,\ \varphi \in \mathcal{E}_A(H).$$

Proposition 3.3.10 *If the assumptions (H3) and (H4) are satisfied, then $\mathcal{E}_A(H)$ is a core for L_p.*

Proof: For $\varphi(x) := e^{i\langle h,x\rangle}$, $h \in D(A^*)$, $x \in H$, we have

$$\begin{aligned} R_t\varphi(x) &= \int_H e^{i\langle h, e^{tA}x+y\rangle}\mathcal{N}(0,Q_t)(dy) \\ &= e^{i\langle e^{tA^*}h,x\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \in \mathcal{E}_A(H). \end{aligned}$$

Hence,

$$R_t\mathcal{E}_A(H) \subseteq \mathcal{E}_A(H), \quad \forall t \ge 0.$$

On the other hand we know that

$$\begin{aligned} \lim_{t\to0^+}\frac{1}{t}(R_t\varphi-\varphi)(x) &= e^{i\langle h,x\rangle}\left(i\langle A^*h,x\rangle - \frac{1}{2}\langle Qh,h\rangle\right) \\ &= L_0\varphi(x), \quad x \in H. \end{aligned}$$

So by the dominated convergence theorem we obtain

$$\lim_{t\to0^+}\left\|\frac{1}{t}(R_t\varphi-\varphi) - L_0\varphi\right\|_{L^p(H,\mu)} = 0.$$

Thus, $\mathcal{E}_A(H) \subset D(L_p)$ and the assertion follows from the density of $\mathcal{E}_A(H)$ in $L^p(H,\mu)$ (see Lemma 3.2.1) and Proposition A.2.5. □

In the remaining part of this section we propose to describe exactly the domain $D(L_2)$ of the generator of the symmetric Ornstein-Uhlenbeck semigroup on $L^2(H,\mu)$. To this purpose we need some auxiliary results. The following result was proved independently in [3] and [17].

Proposition 3.3.11 *Assume (H3) and (H4). Then the following hold*

$$\begin{aligned} \int_H L_0\varphi(x)\tilde\varphi(x)\mu(dx) &= \int_H \langle Q_\infty D\tilde\varphi(x), A^*D\varphi(x)\rangle\mu(dx) \\ \int_H L_0\varphi(x)\varphi(x)\mu(dx) &= -\frac{1}{2}\int_H \langle Q^{\frac12}D\varphi(x), Q^{\frac12}D\varphi(x)\rangle\mu(dx) \end{aligned}$$

for φ, $\tilde\varphi \in \mathcal{E}_A(H)$.

Proof: For $\varphi(x) := e^{i\langle h,x\rangle}$, $\tilde{\varphi}(x) := e^{i\langle \tilde{h},x\rangle}$, $h, \tilde{h} \in D(A^*)$, $x \in H$, we have

$$\begin{aligned}
&\int_H L_0\varphi(x)\tilde{\varphi}(x)\mu(dx) \\
= &\int_H e^{i\langle h,x\rangle}\left(i\langle A^*h, x\rangle - \frac{1}{2}\langle Qh, h\rangle\right) e^{i\langle \tilde{h},x\rangle}\mu(dx) \\
= &\; i\int_H \langle A^*h, x\rangle e^{i\langle h+\tilde{h},x\rangle}\mu(dx) - \frac{1}{2}\langle Qh, h\rangle e^{-\frac{1}{2}\langle Q_\infty(h+\tilde{h}),h+\tilde{h}\rangle} \\
= &\; \frac{d}{dt}\left(\int_H e^{i\langle tA^*h+h+\tilde{h},x\rangle}\mu(dx)\right)_{|t=0} - \frac{1}{2}\langle Qh, h\rangle e^{-\frac{1}{2}\langle Q_\infty(h+\tilde{h}),h+\tilde{h}\rangle} \\
= &\; -\left(\langle Q_\infty A^*h, h+\tilde{h}\rangle + \frac{1}{2}\langle Qh, h\rangle\right) e^{-\frac{1}{2}\langle Q_\infty(h+\tilde{h}),h+\tilde{h}\rangle}.
\end{aligned}$$

Hence, it follows from Proposition 3.3.6 that

$$\begin{aligned}
\int_H \langle Q_\infty D\tilde{\varphi}(x), A^*D\varphi(x)\rangle\mu(dx) &= -\langle A^*h, Q_\infty\tilde{h}\rangle e^{-\frac{1}{2}\langle Q_\infty(h+\tilde{h}),h+\tilde{h}\rangle} \\
&= \int_H L_0\varphi(x)\tilde{\varphi}(x)\mu(dx).
\end{aligned}$$

In particular, and again by Proposition 3.3.6, we obtain

$$\begin{aligned}
\int_H L_0\varphi(x)\varphi(x)\mu(dx) &= \int_H \langle Q_\infty D\varphi(x), A^*D\varphi(x)\rangle\mu(dx) \\
&= -\frac{1}{2}\int_H \langle QD\varphi(x), D\varphi(x)\rangle\mu(dx).
\end{aligned}$$

This end the proof of the proposition. □

Remark 3.3.12 *If the Ornstein Uhlenbeck semigroup is symmetric, then it follows from Proposition 3.3.7 that*

$$\int_H L_0\varphi(x)\tilde{\varphi}(x)\mu(dx) = -\frac{1}{2}\int_H \langle QD\varphi(x), D\tilde{\varphi}(x)\rangle\mu(dx) \tag{3.7}$$

for φ, $\tilde{\varphi} \in \mathcal{E}_A(H)$.

For the proof of the next proposition we need the following lemma.

Lemma 3.3.13 *Assume that* $\ker Q = \{0\}$ *and* $Q_\infty^{\frac{1}{2}}(H) \subset Q^{\frac{1}{2}}(H)$. *Then the operator*

$$D_Q : \mathcal{E}_A(H) \subset L^2(H,\mu) \to L^2(H,\mu;H);\ \varphi \mapsto Q^{\frac{1}{2}}D\varphi$$

is closable.

Proof: From the closed graph theorem we have $K := \overline{Q^{-\frac{1}{2}}Q_\infty^{\frac{1}{2}}}$ is a bounded linear operator on H. Its adjoint is given by $K^* = Q_\infty^{\frac{1}{2}}Q^{-\frac{1}{2}}$.
Let $(\varphi_n) \subset \mathcal{E}_A(H)$ and $F \in L^2(H,\mu;H)$ with $\lim_{n\to\infty} \|\varphi_n\|_{L^2(H,\mu)} = 0$ and $\lim_{n\to\infty} \|D_Q\varphi_n - F\|_{L^2(H,\mu;H)} = 0$. Hence,

$$Q_\infty^{\frac{1}{2}} D\varphi_n = K^* Q^{\frac{1}{2}} D\varphi_n \to K^* F$$

in $L^2(H,\mu;H)$ as $n \to \infty$. Now, it follows from Proposition 3.2.10 that $K^*F \equiv 0$ and therefore $F \equiv 0$. This can be obtain by considering the orthonormal basis of eigenfunctions e_n, $n \in \mathbb{N}$, of Q_∞ and the fact that $\ker Q_\infty = \{0\}$. □

As in Section 2 we define The spaces

$W_Q^{1,2}(H,\mu) := D(\overline{D_Q})$ and
$W_Q^{2,2}(H,\mu) :=$

$$:= \quad \{\varphi \in W_Q^{1,2}(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_h D_k), \int_H \mathrm{Tr}(QD^2\varphi(x))^2 \mu(dx) < \infty\}.$$

In the following result we obtain that $D((-L_2)^{\frac{1}{2}}) = W_Q^{1,2}(H,\mu)$ for symmetric Ornstein-Uhlenbeck semigroups on $L^2(H,\mu)$.

Proposition 3.3.14 *Suppose (H3), (H4),* $\ker Q = \{0\}$*, and* $Q_\infty^{\frac{1}{2}}(H) \subseteq Q^{\frac{1}{2}}(H)$. *Then,*

$$D(L_2) \subset W_Q^{1,2}(H,\mu).$$

Moreover, for any $\varphi \in D(L_2)$*,*

$$\int_H L_2\varphi(x)\varphi(x)\mu(dx) - \frac{1}{2}\int_H \langle QD\varphi(x), D\varphi(x)\rangle \mu(dx).$$

In the case where $(R_t)_{t\geq 0}$ *is symmetric, one has*

$$D((-L_2)^{\frac{1}{2}}) = W_Q^{1,2}(H,\mu).$$

Proof: Let $\varphi \in D(L_2)$. It follows from Proposition 3.3.10 that there is $(\varphi_n) \subset \mathcal{E}_A(H)$ with

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^2(H,\mu)} = 0 \text{ and } \lim_{n\to\infty} \|L_0\varphi_n - L_2\varphi\|_{L^2(H,\mu)} = 0.$$

By Proposition 3.3.11, we have

$$\begin{aligned} &\int_H \langle Q^{\frac{1}{2}} D(\varphi_n - \varphi_m)(x), Q^{\frac{1}{2}} D(\varphi_n - \varphi_m)(x)\rangle \mu(dx) \\ = \quad & -2\int_H L_0(\varphi_n - \varphi_m)(x)(\varphi_n - \varphi_m)(x)\mu(dx). \end{aligned}$$

Now, one can apply Lemma 3.3.13 and hence $\varphi \in W_Q^{1,2}(H,\mu)$ and

$$\int_H L_2\varphi(x)\varphi(x)\mu(dx) - \frac{1}{2}\int_H \langle QD\varphi(x), D\varphi(x)\rangle\mu(dx).$$

On the other hand the last assertion follows from

$$\int_H |(-L_2)^{\frac{1}{2}}\varphi(x)|^2\mu(dx) = \int_H |Q^{\frac{1}{2}}D\varphi(x)|^2\mu(dx).$$

□

Remark 3.3.15 *The bilinear form*

$$a(\varphi,\tilde{\varphi}) := \int_H \langle Q_\infty D\tilde{\varphi}(x), A^* D\varphi(x)\rangle\mu(dx), \quad \varphi,\, \tilde{\varphi} \in \mathcal{E}_A(H)$$

is not always continuous on $W_Q^{1,2}(H,\mu)\times W_Q^{1,2}(H,\mu)$ and therefore not in general a Dirichlet form. The continuity of the bilinear form a can be proved under some additional conditions (see [3] or [17]). In [9] it is proved that a is a Dirichlet form provided that $Q = I$, which implies that $AQ_\infty \in \mathcal{L}(H)$.

Suppose now that the assumptions of Corollary 3.3.9 are satisfied. Then $Q_\infty = -\frac{1}{2}A^{-1}$. Let consider an orthonormal system $(e_n) \subset H$ and $(\alpha_n) \subset (0,\infty)$ such that

$$Ae_n = -\alpha_n e_n, \quad n \in \mathbb{N}.$$

The following proposition is the main tool used for the characterization of the domain of L_2.

Proposition 3.3.16 *Suppose that the assumptions of Corollary 3.3.9 are satisfied. Then,*

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}}D\varphi(x)|^2\mu(dx) = 2\int_H (L_2\varphi(x))^2\mu(dx)$$

for $\varphi \in \mathcal{E}_A(H)$.

Proof: For $\varphi \in \mathcal{E}_A(H)$ we have $D_j(L_2\varphi) = L_2 D_j\varphi - \alpha_j D_j\varphi$. Hence, by Proposition 3.3.14,

$$\begin{aligned}
&\int_H D_j\varphi(x)D_j(L_2\varphi)(x)\mu(dx)\\
&= \int_H D_j\varphi(x)L_2(D_j\varphi)(x)\mu(dx) - \alpha_j\int_H |D_j\varphi(x)|^2\mu(dx)\\
&= -\frac{1}{2}\int_H \langle DD_j\varphi(x), DD_j\varphi(x)\rangle\mu(dx) - \alpha_j\int_H |D_j\varphi(x)|^2\mu(dx).
\end{aligned}$$

Now, if we take the sum over $j \in \mathbb{N}$, we obtain

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}} D\varphi(x)|^2\mu(dx)$$

$$= -\int_H \langle D\varphi(x), D(L_2\varphi)(x)\rangle\mu(dx).$$

Since $L_2\varphi \in W^{1,2}(H,\mu)$, it follows from Remark 3.3.12 that

$$\int_H \langle D\varphi(x), D(L_2\varphi)(x)\rangle\mu(dx) = -2\int_H |L_2\varphi(x)|^2\mu(dx).$$

Thus,

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}} D\varphi(x)|^2\mu(dx) = 2\int_H |L_2\varphi(x)|^2\mu(dx).$$

□

For the characterization of the domain of L_2 we need the following space

$$W^{1,2}_{(-A)}(H,\mu) := \{\varphi \in W^{1,2}(H,\mu) : \int_H |(-A)^{\frac{1}{2}} D\varphi(x)|^2\mu(dx) = \sum_{k\in\mathbb{N}}\int_H \alpha_k |D_k\varphi(x)|^2\mu(dx) < \infty\}.$$

Endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}_{(-A)}(H,\mu)} := \varphi,\psi\rangle_{L^2(H,\mu)} + \int_H \langle(-A)^{\frac{1}{2}} D\varphi(x), (-A)^{\frac{1}{2}} D\psi(x)\rangle\mu(dx),$$

$W^{1,2}_{(-A)}(H,\mu)$ is Hilbert space.

Theorem 3.3.17 *Assume that the assumptions of Corollary 3.3.9 hold. Then,*

$$D(L_2) = W^{2,2}(H,\mu) \cap W^{1,2}_{(-A)}(H,\mu).$$

Proof: Let $\varphi \in D(L_2)$. By Proposition 3.3.10 there is $(\varphi_n) \subset \mathcal{E}_A(H)$ with $\varphi_n \to \varphi$ and $L_2\varphi_n \to L_2\varphi$ in $L^2(H,\mu)$. For $n,m \in \mathbb{N}$, it follows from Proposition 3.3.16 that

$$2\int_H |L_2(\varphi_n - \varphi_m)(x)|^2\mu(dx) = \frac{1}{2}\int_H \mathrm{Tr}\left((D^2(\varphi_n-\varphi_m)(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}} D(\varphi_n-\varphi_m)(x)|^2\mu(dx).$$

Therefore (φ_n) is a Cauchy sequence in both spaces $W^{2,2}(H,\mu)$ and $W^{1,2}_{(-A)}(H,\mu)$. This implies that

$$D(L_2) \subseteq W^{2,2}(H,\mu) \cap W^{1,2}_{(-A)}(H,\mu).$$

Now, if $\varphi \in W^{2,2}(H,\mu) \cap W^{1,2}_{(-A)}(H,\mu)$ then one can find a sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ such that φ_n converges to φ in both spaces $W^{2,2}(H,\mu)$ and $W^{1,2}_{(-A)}(H,\mu)$. The other inclusion follows now from Proposition 3.3.16. □

In the more general assumptions given in Corollary 3.3.8 one has to prove the formula

$$\frac{1}{2}\int_H \mathrm{Tr}\left((QD^2\varphi(x))^2\right)\mu(dx) + \int_H \langle(-AQ)D\varphi(x), D\varphi(x)\rangle\mu(dx) = \\ = 2\int_H (L_2\varphi(x))^2\mu(dx). \quad (3.8)$$

The proof of (3.8) is similar to that of Proposition 3.3.16. As in the proof of Theorem 3.3.17, (3.8) implies the following general result.

Theorem 3.3.18 *Suppose that the assumptions of Corollary 3.3.8 hold. Then,*

$$D(L_2) = \{\varphi \in W^{2,2}_Q(H,\mu) : \int_H \langle(-AQ)D\varphi(x), D\varphi(x)\rangle\mu(dx) < \infty\}.$$

Remark 3.3.19 *Theorem 3.3.17 and 3.3.18 are due to Da Prato [10]. In the finite dimensional case Lunardi [24] proved first that $D(L_2) = W^{2,2}(\mathbb{R}^N,\mu)$, by making heavy use of interpolation theory. A simpler proof of the same result can be found in [11]. Recently, this result was extended to $p \in (1,\infty)$ (see [25] or [26]).*

Exercise 3.3.20 *Assume (H1) and (H2). Prove that $\mathcal{N}(0,Q_t)$ is $\mathcal{N}(0,Q_\infty)$–absolutely continuous.*

Exercise 3.3.21 *Let $1 < p < \infty$, and $B \in \mathcal{L}^+_1(H)$ with $\ker B = \{0\}$. Show that*

$$\int_H |\langle h, B^{-\frac{1}{2}}y\rangle|^p \mathcal{N}(0,B)(dy) = |h|^p \int_{\mathbb{R}} |y|^p \mathcal{N}(0,1)(dy).$$

This generalizes the case $p = 2$ proved in Proposition 1.3.1.

Exercise 3.3.22 *Assume (H1) and (H2). Show that*

(i) $Q_t^{\frac{1}{2}}(H) = Q_\infty^{\frac{1}{2}}(H)$.

(ii) *For any $t > 0$, $S_0(t) := Q_\infty^{-\frac{1}{2}} e^{tA} Q_\infty^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H.*

(iii) *Deduce that e^{tA} is a trace class operator on H for every $t > 0$.*

Exercise 3.3.23 *Assume (H2), (H3) and (H4).*

(a) *Show that*

$$Q_t = Q_\infty^{\frac{1}{2}}(I - S_0(t)S_0^*(t))Q_\infty^{\frac{1}{2}}, \quad t \geq 0.$$

(b) *By using the Cameron-Martin formula and the Feldman-Hajek theorem (see Exercise 1.3.6) show that*

$$R_t\varphi(x) = \int_H k(t,x,y)\varphi(y)\mu(dy), \quad \mu - \text{a.a.}\, x \in H,$$

with

$$\begin{aligned} k(t,x,y) :=& \\ &\exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle (I - S_0(t)S_0^*(t))^{-1}S_0(t)Q_\infty^{-\frac{1}{2}}x, Q_\infty^{-\frac{1}{2}}y\rangle\right)\cdot \\ &\cdot\det(I - S_0(t)S_0^*(t))^{-\frac{1}{2}}\cdot \\ &\cdot\exp\left(-\frac{1}{2}\langle S_0(t)S_0^*(t)(I - S_0(t)S_0^*(t))^{-1}Q_\infty^{-\frac{1}{2}}y, Q_\infty^{-\frac{1}{2}}y\rangle\right) \end{aligned}$$

for $t > 0$, and $x, y \in H$.

(c) *Show that, for any $1 < q < \infty$,*

$$\begin{aligned} \int_H & k(t,x,y)^q\mu(dy) = \\ & \det(I - S_0(t)S_0^*(t))^{\frac{1-q}{2}}\det(I + (q-1)S_0(t)S_0^*(t))^{-\frac{1}{2}} \\ & \exp\left(\frac{q(q-1)}{2}\langle (I + (q-1)S_0(t)S_0^*(t))^{-1}Q_\infty^{-\frac{1}{2}}e^{tA}x, Q_\infty^{-\frac{1}{2}}e^{tA}x\rangle\right) \end{aligned}$$

for $t > 0$ and $x \in H$, (see [6, Lemma 3]).

Exercise 3.3.24 *Suppose (H2), (H3) and (H4). Use the formula*

$$\langle DR_t\varphi(x), y\rangle = \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}}h\rangle\varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh),$$

which, by Lemma 3.3.4, remains valid for $t > 0$ and $\varphi \in L^p(H,\mu)$ to prove that

$$R_t L^p(H,\mu) \subset W^{1,p}(H,\mu)$$

for $t > 0$ and $1 \leq p < \infty$. Deduce from [7] that the Ornstein-Uhlenbeck semigroup (R_t) is immediately compact in $L^p(H,\mu)$.

APPENDIX

A.1 THE CLASSICAL BOCHNER THEOREM

In this section we recall the classical theorem of Bochner and for the sake of completeness we will give the proof.
First of all we say that a family Λ of probability measures on $(E, \mathcal{B}(E))$ is **tight** if for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset E$ such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon \quad \text{for all } \mu \in \Lambda.$$

Here E is a separable Banach space and $\mathcal{B}(E)$ its Borel σ-field.
A sequence of measures (μ_p) on $(E, \mathcal{B}(E))$ is said to be **weakly convergent** to a measure μ if for every $\varphi \in C_b(E)$

$$\lim_{p\to\infty} \int_E \varphi(x)\mu_p(dx) = \int_E \varphi(x)\mu(dx).$$

A family Λ of measures on $(E, \mathcal{B}(E))$ is said **relatively compact** if for an arbitrary sequence $(\mu_p) \subset \Lambda$ contains a weakly convergent subsequence (μ_{p_k}) to a measure μ on $(E, \mathcal{B}(E))$.
The following result is due to Prokhorov (cf. [12, Theorem 2.3]).

Theorem A.1.1 *A set Λ of probability measures on $(E, \mathcal{B}(E))$ is tight if and only if is relatively compact.*

For the proof of the Bochner theorem we need the following lemma.

Lemma A.1.2 *Assume that (μ_p) is a sequence of probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$. If $\varphi_p(z) := \widehat{\mu_p}(z)$ converges to $\varphi(z)$ for all $z \in \mathbb{R}^N$ and if this convergence is uniform in $\{z \in \mathbb{R}^N : |z| \leq a\}$ for a small number a, then $\{\mu_p : p \in \mathbb{N}\}$ is tight.*

Proof: Since φ_p is continuous and (φ_p) converges uniformly in a neighborhood of 0, it follows that φ is continuous at 0 and $\varphi(0) = 1$. Hence, for any $\varepsilon > 0$ there is $\delta \in (0, a)$ such that

$$|\varphi(z) - 1| < \varepsilon \quad \text{for all } |z| < \delta.$$

It follows now from the uniform convergence of (φ_p) to φ in $\{z \in \mathbb{R}^N : |z| < \delta\}$ that there exists $M = M(\varepsilon)$ independent of z such that

$$|\varphi_p(z) - 1| < \frac{\varepsilon}{2}, \quad \forall p \geq M, \forall |z| < \delta.$$

So, by Fubini's theorem we have

$$\begin{aligned} 1 - \frac{\varepsilon}{2} &< \frac{1}{(2\delta)^N} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \Re\varphi_p(z)\, dz \\ &= \frac{1}{(2\delta)^N} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \int_{\mathbb{R}^N} \cos\langle z, x\rangle \mu_p(dx)\, dz \\ &= \int_{\mathbb{R}^N} \left(\frac{\sin \delta x_1}{\delta x_1}\right) \cdots \left(\frac{\sin \delta x_N}{\delta x_N}\right) \mu_p(dx), \end{aligned}$$

where the last equality can be seen by induction.
Since $\left|\frac{\sin \delta x_j}{\delta x_j}\right|$ is dominated by 1 on $[-R, R]$ and by $\frac{1}{\delta R}$ elsewhere, we obtain

$$\int_{\mathbb{R}^N} \left(\frac{\sin \delta x_1}{\delta x_1}\right) \cdots \left(\frac{\sin \delta x_N}{\delta x_N}\right) \mu_p(dx) \leq \mu_p([-R, R]^N) + \left(\frac{1}{\delta R}\right)^N.$$

Take now $R := \frac{1}{\delta}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{N}}$, it follows that

$$1 - \varepsilon < \mu_p([-R, R]^N) \quad \text{for all } p \geq M.$$

This gives the proof of the lemma. □

We are now ready to show the classical Bochner theorem. The arguments are taking from the proof in one dimensional case (see [20, Theorem 2.6.6]).

Theorem A.1.3 *A functional $\varphi : \mathbb{R}^N \to \mathbb{C}$ is the Fourier transform of a probability measure on $\mathbb{R}^N$ if and only if φ is a continuous positive definite functional satisfying $\varphi(0) = 1$.*

Proof: It suffices to prove the sufficiency. Assume that $\varphi : \mathbb{R}^N \to \mathbb{C}$ is a continuous positive definite functional with $\varphi(0) = 1$. Then, by Lemma 1.1.3, φ is uniformly continuous and bounded. Take now $g : \mathbb{R}^N \to \mathbb{C}$ integrable, bounded and uniformly continuous. If we set $\dot{y} := (y_2, \ldots, y_N) \in \mathbb{R}^{N-1}$ then we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(\xi-\eta)g(\xi)\overline{g(\eta)}\,d\xi d\eta \\
= &\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\xi_1-\eta_1,\dot{\xi}-\dot{\eta})g(\xi_1,\dot{\xi})\overline{g(\eta_1,\dot{\eta})}\,d\xi_1 d\eta_1\right) d\dot{\xi}d\dot{\eta} \\
= &\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}} \lim_{p\to\infty}\sum_{l,k=-p^2}^{p^2} \varphi\left(\frac{l}{p}-\frac{k}{p},\dot{\xi}-\dot{\eta}\right) g\left(\frac{l}{p},\dot{\xi}\right)\overline{g\left(\frac{k}{p},\dot{\eta}\right)}\left(\frac{1}{p}\right)^2 d\dot{\xi}d\dot{\eta} \\
\geq &\ 0. \qquad (9)
\end{aligned}$$

Put $g(\xi) := \mathcal{N}(0, \frac{p}{4}Id_{\mathbb{R}^N})(\xi)e^{-i\langle x,\xi\rangle}$, $\xi, x \in \mathbb{R}^N$. Since

$$\begin{aligned}
\mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) &= \mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(-\xi) \text{ and} \\
\mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) * \mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) &= \mathcal{N}(0,\frac{p}{2}Id_{\mathbb{R}^N})(\xi)
\end{aligned}$$

it follows that, for $x \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} g(\xi+\eta)\overline{g(\eta)}\,d\eta = e^{-i\langle x,\xi\rangle}\frac{1}{(p\pi)^{\frac{N}{2}}}e^{-\frac{|\xi|^2}{p}},$$

where $\mathcal{N}(0, \frac{p}{4}Id_{\mathbb{R}^N})(\xi) : \frac{1}{(\pi(p/2))^{\frac{N}{2}}}e^{-2\frac{|\xi|^2}{p}}$ for $\xi \in \mathbb{R}^N$ and $Id_{\mathbb{R}^N}$ denotes the identity operator in $\mathbb{R}^N$. So by (9) we obtain

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(\xi-\eta)g(\xi)\overline{g(\eta)}\,d\xi d\eta \\
&= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} g(\xi+\eta)\overline{g(\eta)}\,d\eta\right)\varphi(\xi)\,d\xi \\
&= \frac{1}{(p\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle x,\xi\rangle}\,d\xi.
\end{aligned}$$

Thus,

$$f_p(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle x,\xi\rangle}\,d\xi \geq 0$$

for $x \in \mathbb{R}^N$. Define the measure $\mu_p(dx) := f_p(x)dx$. We propose to show now that μ_p is a probability measure on $\mathbb{R}^N$. First, by applying Fubini's theorem, observe that

$$\begin{aligned}
&\mu_p([-a_1,a_1]\times\ldots\times[-a_N,a_N]) \\
= &\frac{1}{(2\pi)^N}\int_{-a_1}^{a_1}\cdots\int_{-a_N}^{a_N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\xi_1x_1}\ldots e^{-i\xi_Nx_N}\,d\xi\,dx_1\ldots dx_N \\
= &\frac{1}{\pi^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{\sin a_1\xi_1}{\xi_1}\right)\ldots\left(\frac{\sin a_N\xi_N}{\xi_N}\right)d\xi.
\end{aligned}$$

On the other hand, for $m \in \mathbb{N}$, we have

$$\frac{1}{m^N}\int_0^m \dots \int_0^m \mu_p([-a_1,a_1]\times\dots\times[-a_N,a_N])\,da_1\dots da_N = \int_0^1 \dots \int_0^1 \mu_p([-a_1 m,a_1 m]\times\dots\times[-a_N m,a_N m])\,da_1\dots da_N.$$

Since $\mu_p([-a_1 m,a_1 m]\times\dots\times[-a_N m,a_N m]) \uparrow \mu_p(\mathbb{R}^N)$ as $m\to\infty$, it follows from the monotone convergence theorem that

$$\begin{aligned}
&\mu_p(\mathbb{R}^N)\\
= &\lim_{m\to\infty}\frac{1}{m^N}\int_0^m\dots\int_0^m \mu_p([-a_1,a_1]\times\dots\times[-a_N,a_N])\,da\\
= &\lim_{m\to\infty}\frac{1}{(\pi m)^N}\int_0^m\dots\int_0^m\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{\sin a_1\xi_1}{\xi_1}\right)\dots\left(\frac{\sin a_N\xi_N}{\xi_N}\right)d\xi\,da\\
= &\lim_{m\to\infty}\frac{1}{\pi^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{1-\cos m\xi_1}{m\xi_1^2}\right)\dots\left(\frac{1-\cos m\xi_N}{m\xi_N^2}\right)d\xi\\
= &\lim_{m\to\infty}\frac{1}{\pi^N}\int_{\mathbb{R}^N}\varphi(\frac{\xi_1}{m},\dots,\frac{\xi_N}{m})e^{-\frac{|\xi|^2}{mp}}\left(\frac{1-\cos\xi_1}{\xi_1^2}\right)\dots\left(\frac{1-\cos\xi_N}{\xi_N^2}\right)d\xi,
\end{aligned}$$

where $a := (a_1,\dots,a_N)$. Since $\varphi(0)=1$ and

$$\frac{1-\cos\xi_j}{\xi_j^2}\geq 0,\ \int_{\mathbb{R}}\frac{1-\cos\xi_j}{\xi_j^2}=\pi,\quad \forall j=1,\dots,N, \tag{10}$$

it follows from the dominated convergence theorem that

$$\mu_p(\mathbb{R}^N)=\varphi(0)=1.$$

Let compute now the Fourier transform of μ_p. For $a_j\geq 0$ and $m\in\mathbb{N}$, observe that

$$\begin{aligned}
&\left|\frac{1}{(2\pi)^N}\int_{-a_N m}^{a_N m}\dots\int_{-a_1 m}^{a_1 m} e^{i\langle z,x\rangle}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle\xi,x\rangle}\,d\xi dx\right|\\
\leq\ &\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle\xi,x\rangle}\,d\xi dx\\
=\ &\mu_p(\mathbb{R}^N)=1.
\end{aligned}$$

So it follows from the dominated convergence theorem that

$$\begin{aligned}
&\widehat{\mu_p}(z)\\
&= \lim_{m\to\infty} \frac{1}{(2\pi)^N} \int_0^1 \cdots \int_0^1 \left[\int_{-a_N m}^{a_N m} \cdots \int_{-a_1 m}^{a_1 m} \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} e^{-i\langle \xi - z, x\rangle}\, d\xi dx \right] da\\
&= \lim_{m\to\infty} \frac{1}{\pi^N} \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} \left(\frac{\sin a_1 m(\xi_1 - z_1)}{\xi_1 - z_1} \right) \cdots\\
&\qquad \left(\frac{\sin a_N m(\xi_N - z_N)}{\xi_N - z_N} \right) d\xi da\\
&= \lim_{m\to\infty} \frac{1}{\pi^N} \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} \left(\frac{1-\cos m(\xi_1 - z_1)}{m(\xi_1 - z_1)^2} \right) \cdots \left(\frac{1-\cos m(\xi_N - z_N)}{m(\xi_N - z_N)^2} \right) d\xi\\
&= \lim_{m\to\infty} \frac{1}{\pi^N} \int_{\mathbb{R}^N} \varphi(z + \frac{\xi}{m}) e^{-\frac{|z+\frac{\xi}{m}|^2}{p}} \left(\frac{1-\cos \xi_1}{\xi_1^2} \right) \cdots \left(\frac{1-\cos \xi_N}{\xi_N^2} \right) d\xi.
\end{aligned}$$

So, again by the dominated convergence theorem and (10), we obtain

$$\widehat{\mu_p}(z) = \varphi(z) e^{-\frac{|z|^2}{p}}, \quad z \in \mathbb{R}^N.$$

Finally,

$$\lim_{p\to\infty} \widehat{\mu_p}(z) = \varphi(z)$$

uniformly in $|z| \leq 1$. The theorem follows now from Lemma A.1.2 and Theorem A.1.1. □

A.2 C_0-SEMIGROUPS

In this section we give a general discussion of the abstract Cauchy problem for unbounded linear operators on a Banach space and its relation to the theory of C_0-semigroups. For more details we refer to the recent books of Engel-Nagel [16] and Arendt-Batty-Hieber-Neubrander [1]. A particular attention will be dedicated to the class of eventually norm continuous C_0-semigroups.

We consider the abstract Cauchy problem

$$(ACP) \qquad \begin{cases} \frac{du}{dt}(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where A is a possibly unbounded linear operator with domain $D(A)$ on a Banach space X and $x \in X$. A *classical solution* of (ACP) is a function $u \in C^1(\mathbb{R}_+, X)$ such that $u(t) \in D(A)$ for all $t \geq 0$ and u satisfies (ACP).

Now we introduce C_0-semigroups.

Definition A.2.1 *A family $T(\cdot) := (T(t))_{t\geq 0}$ of bounded linear operators on X is called a C_0-semigroup if*

(i) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0, \quad \forall x \in X$,

(ii) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$ *and* $T(0) = Id$.

The generator of $T(\cdot)$ is the linear operator A defined by

$$\begin{aligned} D(A) &= \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists } \}, \\ Ax &= \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A). \end{aligned}$$

One can prove that the generator is always a closed and densely defined operator. The domain $D(A)$ satisfies

$$T(t)D(A) \subseteq D(A) \text{ and } AT(t)x = T(t)Ax, \quad \forall t \geq 0.$$

Moreover, for $x \in D(A)$,

$$\frac{d}{dt}T(t)x = AT(t)x, \quad t \geq 0.$$

This shows that for $x \in D(A)$ the problem (ACP) has a classical solution $u(\cdot) := T(\cdot)x$. We say that (ACP) is *well-posed* if for each initial value $x \in D(A)$ there is a unique classical solution $u(\cdot, x)$ satisfying

> for any sequence $(x_n) \subset D(A)$ with $\lim_{n\to\infty} \|x_n - x\| = 0$ for $x \in D(A)$, the corresponding classical solutions $u(\cdot, x_n)$ converges to $u(\cdot, x)$ uniformly on compact subsets of $\mathbb{R}_+$.

The following theorem shows that wellposedness is equivalent to generation of C_0-semigroups.

Theorem A.2.2 *Let A be a linear operator with domain $D(A)$ on a Banach space X. Then the following assertion are equivalent:*

(a) *A is the generator of a C_0-semigroup on X.*

(b) *The abstract Cauchy problem (ACP) associated with A is well-posed.*

On the other hand, for a C_0-semigroup $T(\cdot)$, one has

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0,$$

for some constants $\omega \in \mathbb{R}$ and $M \geq 1$. If we denote by

$$\omega_0(A) := \inf\{\omega \in \mathbb{R} : \text{ there is } M_\omega \geq 1 \text{ with } \|T(t)\| \leq M_\omega e^{\omega t}, \forall t \geq 0\}$$

the growth bound of the C_0-semigroup $T(\cdot)$ with generator A, then $(\omega_0(A), \infty) \subset \rho(A)$, the resolvent set of A, and the resolvent $R(\lambda, A)$ of A is given by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x\, dt, \quad x \in X, \lambda > \omega_0(A).$$

In the following proposition we collect some properties of C_0-semigroups and their generators.

Proposition A.2.3 *Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. If $(A, D(A))$ denotes its generator then the following assertions hold:*

(i) $\int_0^t T(s)x\,ds \in D(A)$ *and* $A\int_0^t T(s)x\,ds = T(t)x - x$ *for all* $x \in X$ *and* $t \geq 0$.

(ii) $A\int_0^t T(s)x\,ds = \int_0^t T(s)Ax\,ds = T(t)x - x$ *for all* $x \in D(A)$ *and* $t \geq 0$.

(iii) $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ *for all* $x \in X$.

(iv) $R(\lambda, A)T(t) = T(t)R(\lambda, A)$ *for all* $\lambda \in \rho(A)$ *and* $t \geq 0$.

In many applications it is difficult to identify the domain of the generator of a C_0-semigroup. It is often the case that one can find a "large" subspace of $D(A)$ as defined now.

Definition A.2.4 *A subspace D of $D(A)$, the domain of a linear operator A on a Banach space X is called a core for A if D is dense in $D(A)$ for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

A useful criterion for subspaces to be a core for the generator of a C_0-semigroup is given by the following proposition.

Proposition A.2.5 *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and D be a subspace of $D(A)$. If D is dense in X and invariant under $(T(t))_{t\geq 0}$, then D is a core for A.*

We propose now to introduce different classes of semigroups. In the sequel we denote the sector in $\mathbb{C}$ of angle δ by

$$\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \delta\} \setminus \{0\}.$$

Definition A.2.6 *A family $(T(z))_{z\in\Sigma_\theta\cup\{0\}} \subset \mathcal{L}(X)$ on a Banach space X is called an analytic semigroup (of angle $\theta \in (0, \frac{\pi}{2}]$) if*

(a1) $T(0) = Id$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\theta$.

(a2) *The map $z \mapsto T(z)$ is analytic in Σ_θ.*

(a3) $\lim_{\Sigma_{\theta'}\ni z\to 0} T(z)x = x$ *for all* $x \in X$ *and* $0 < \theta' < \theta$.

If, in addition

(a4) $\|T(z)\|$ *is bounded in* $\Sigma_{\theta'}$ *for every* $0 < \theta' < \theta$,

we call $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ a bounded analytic semigroup.

The following theorem gives useful characterization of generators of bounded analytic semigroups.

Theorem A.2.7 *Let $(A, D(A)$ be an operator on a Banach space X. Then the following assertions are equivalent:*

(i) *A generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ on X.*

(ii) *A generates a bounded C_0-semigroup $T(\cdot)$ on X with $\mathrm{rg}(T(t)) \subset D(A)$ for all $t > 0$, and*

$$\|AT(t)\| \leq \frac{M}{t}$$

for some positive constant M.

(iii) *There is $\delta \in (0, \frac{\pi}{2})$ such that $e^{\pm i\delta}A$ generate bounded C_0-semigroups on X.*

(iv) *$\Sigma_{\theta+\frac{\pi}{2}} \subset \rho(A)$ and for each $\varepsilon \in (0, \theta)$ there is $M_\varepsilon \geq 1$ such that*

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|} \quad \text{for all } 0 \neq \lambda \in \overline{\Sigma}_{\theta+\frac{\pi}{2}-\varepsilon}.$$

From (ii) above we see that if $T(\cdot)$ is an analytic semigroup, then the maps $0 < t \mapsto T(t)x$ are differentiable for every $x \in X$. This motivate the following definition.

Definition A.2.8 *A C_0-semigroup $T(\cdot)$ on a Banach space X is called eventually (resp. immediately) differentiable if there is $t_0 \geq 0$ such that the maps $(t_0, \infty) \ni t \mapsto T(t)x$ (resp. $(0, \infty) \ni t \mapsto T(t)x$) are differentiable for every $x \in X$.*

A characterization of differentiable semigroups in terms of the spectrum and the growth of the resolvent can be proved (cf. [16, Theorem II.4.14]).

Finally we recall the class of eventually norm continuous C_0-semigroups.

Definition A.2.9 *A C_0-semigroup $T(\cdot)$ on a Banach space X is called eventually (resp. immediately) norm continuous if there is $t_0 \geq 0$ such that the mapping $(t_0, \infty) \ni t \mapsto T(t) \in \mathcal{L}(X)$ (resp. $(0, \infty) \ni t \mapsto T(t) \in \mathcal{L}(X)$) is norm continuous.*

It is an easy exercise to see that the following implications between the three classes of semigroups hold:

$$\begin{aligned}&\text{analytic} \Longrightarrow \text{immediately differentiable} \Longrightarrow \text{immediately norm continuous},\\ &\text{analytic} \Longrightarrow \text{eventually differentiable} \Longrightarrow \text{eventually norm continuous.}\end{aligned} \quad (11)$$

On Hilbert spaces eventually norm C_0-semigroups are completely characterized (cf. [16, Theorem II.4.20]). But in general Banach spaces such a characterization remain open. However a necessary condition can be obtained as the following theorem shows.

Theorem A.2.10 *If $(A, D(A))$ is the generator of an eventually norm continuous C_0-semigroup $T(\cdot)$ on a Banach space X, then, for every $a \in \mathbb{R}$, the set*

$$\{\lambda \in \sigma(A) : \Re\lambda \geq a\}$$

is bounded.

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