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# Espansioni Binomiali Non-Commutative e Relazioni di Serie Inverse

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**Abstract**

Il metodo del calcolo combinatorio è applicato per dimostrare alcune formule di espansioni relative a due variabili non commutative che ci portano a stabilire una generale relazione reciproca di serie inverse con due sequenze di doppio indice.

## Non-Commutative Binomial Expansions and Inverse Series Relations

***Abstract**: Combinatorial computation method is applied to demonstrate several expansion formulas related to two non commutative variables which lead us to establish a general pair of inverse series relations involving two double-indexed sequences.*

## 1 Introduction

Following the convention in the literature, we denote the $q$-shifted factorial by

$$(x;q)_0 \equiv 1 \quad \text{and} \quad (x;q)_n = \prod_{k=1}^{n}(1-xq^{k-1}),\ n=1,2,\cdots \tag{1}$$

and the Gaussian binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{2}$$

Extending his previous work [3], Benaoum [4] introduced the $(p,q)$-deformed quantum plane and the related $(p,q)$-analysis, and established successfully the Newton $(p,q)$-binomial formulae.

**Theorem 1 (Benaoum [4, Eq 7]).** *For two variables $x$ and $y$ satisfying the non-commutative relation*

$$xy = qyx + py^2 \quad \textit{with} \quad \gamma = 1+p-q$$

*there holds the following binomial expansion*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_k \Big(\frac{\gamma}{1-q}\Big)^k y^k x^{n-k}. \tag{3}$$

This result reduces, for $p = 0$, to the Gaussian binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k x^{n-k} \qquad \text{with} \quad xy = qyx \tag{4}$$

and for $q = 1$, to the Newton $p$-binomial formula [3, Eq 5]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \langle p \rangle_k y^k x^{n-k} \quad \text{with} \quad xy = yx + py^2 \tag{5}$$

where $\langle c \rangle_k$ is defined by

$$\langle c \rangle_0 = 1 \quad \text{and} \quad \langle c \rangle_n = \prod_{k=0}^{n-1} \big(1 + kc\big) \quad \text{for} \quad n = 1, 2, \cdots. \tag{6}$$

Recently, Zhang and Wang [5] investigated properties of the $(p, q)$-binomial coefficients and obtained the $(p, q)$-analogue of the classical multinomial theorem.

By means of combinatorial computation, this paper will demonstrate several expansion formulas which simplify explicitly the earlier results obtained in [5]. They lead us to establish a general pair of inverse series relations which make us in turn understand deeply dual expansions.

## 2 Binomial and Monomial Expansion Formulas

For the partitions into parts $\leq n$ with the number of parts $\leq m$, their generating function is given (cf. [2, p 117]) by the $q$-binomial coefficient $\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]$. Then the generating function of the partitions into exactly $m$-parts with each part $\leq n$ is given by

$$\begin{bmatrix} m+n \\ n \end{bmatrix} - \begin{bmatrix} m+n-1 \\ n \end{bmatrix} = q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix}$$

which brings us to the following:

**Lemma 2.** *For two natural numbers $n, k$ and a complex indeterminate $q$, the multisum defined by*

$$P_{k,n}(q) = \sum_{\substack{m_1+m_2+\cdots+m_n=k \\ m_\iota \geq 0,\ 1 \leq \iota \leq n}} q^{\sum_{\iota=1}^{n} \iota m_\iota} \tag{7}$$

*possesses the closed forms*

$$P_{k,n}(q) = q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \tag{8}$$

*and*

$$P_{k,n}(q^{-1}) = q^{-nk} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}. \tag{9}$$

*Proof.* The first result follows from the fact that $P_{k,n}(q)$ is the explicit form of the generating function for the partitions into parts $\leq n$ with exactly $k$-parts. For the $q$-binomial coefficient, noting that

$$\begin{bmatrix} n \\ k \end{bmatrix}\Big|_{q\to 1/q} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}$$

we have

$$P_{k,n}(q^{-1}) = \left\{ q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \right\}\Big|_{q\to 1/q} = q^{-nk} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

which proves the second formula in the lemma. □

Now it is ready for us to establish consequently the following expansion formulae which is the common extension of relations displayed in [3, Eq 4] and [4, Eq 6].

**Theorem 3.** *With two non-commutative variables $x$ and $y$ satisfying $xy = qyx + py^2$, we have the monomial expansion*

$$x^m y^n = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q^n;q)_k}{(1-q)^k} p^k q^{n(m-k)}\, y^{n+k} x^{m-k}. \tag{10}$$

*Proof.* Recall the formulae obtained by Zhang and Wang [5, Prop 1]

$$x^m y^n = \sum_{k=0}^{m} \frac{(q^{1+m-k};q)_k}{(1-q)^k} p^k y^{n+k} x^{m-k} \sum_{\substack{r_1+r_2+\cdots+r_n=k \\ r_\iota \geq 0,\ 1\leq \iota \leq n}} q^{mn-\sum_{\iota=1}^{n} \iota r_\iota}$$

where the inner sum may be expressed in terms of $q^{mn}P_{k,n}(q^{-1})$. In view of Lemma 2, the expansion admits the simplification stated in the theorem. □

Denoting by $\delta_{k,n}$ the Kronecker delta, which is equal one for $k = n$ and to zero for $k \neq n$ and recalling the terminating version of the Euler formulae for the $q$-finite differences [1, p 66]

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k \tag{11}$$

we have the explicit orthogonal relation

$$\begin{aligned}\delta_{0,\ell} &= (q^{1-\ell};q)_\ell = \frac{(q^n;q)_\ell}{(1-q)^\ell} \begin{bmatrix} m \\ \ell \end{bmatrix} \left(pq^{-n}\right)^\ell (q^{1-\ell};q)_\ell \\ &= \frac{(q^n;q)_\ell}{(1-q)^\ell} \begin{bmatrix} m \\ \ell \end{bmatrix} \left(pq^{-n}\right)^\ell \sum_{k=0}^{\ell} (-1)^k \begin{bmatrix} \ell \\ k \end{bmatrix} q^{\binom{1+k}{2}-k\ell}.\end{aligned}$$

Then the following trivial identity

$$y^n x^m = \sum_{\ell=0}^{m} y^{n+\ell} x^{m-\ell} \delta_{0,\ell}$$

may be expanded as a double sum

$$\begin{aligned} y^n x^m &= \sum_{\ell=0}^{m} y^{n+\ell} x^{m-\ell} \begin{bmatrix} m \\ \ell \end{bmatrix} \frac{(q^n;q)_\ell}{(1-q)^\ell} \left(pq^{-n}\right)^\ell \sum_{k=0}^{\ell} (-1)^k \begin{bmatrix} \ell \\ k \end{bmatrix} q^{\binom{k+1}{2}-k\ell} \\ &= \sum_{k=0}^{m} (-1)^k \sum_{\ell=k}^{m} y^{n+\ell} x^{m-\ell} \begin{bmatrix} m \\ \ell \end{bmatrix} \begin{bmatrix} \ell \\ k \end{bmatrix} \frac{(q^n;q)_\ell}{(1-q)^\ell} p^\ell q^{\binom{k+1}{2}-(n+k)\ell}. \end{aligned}$$

Noticing the binomial symmetry

$$\begin{bmatrix} m \\ \ell \end{bmatrix} \begin{bmatrix} \ell \\ k \end{bmatrix} = \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m-k \\ \ell-k \end{bmatrix} \tag{12}$$

and then performing replacement $\ell = k + i$ for the inner sum with respect to summation index $\ell$, we can manipulate the double sum as follows

$$\begin{aligned} y^n x^m &= \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k+1}{2}} \sum_{\ell=k}^{m} y^{n+\ell} x^{m-\ell} \begin{bmatrix} m-k \\ \ell-k \end{bmatrix} \frac{(q^n;q)_\ell}{(1-q)^\ell} p^\ell q^{-(n+k)\ell} \\ &= \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q^n;q)_k}{(1-q)^k} p^k q^{\binom{k+1}{2}-m(n+k)} \\ &\times \sum_{i=0}^{m-k} y^{n+k+i} x^{m-k-i} \begin{bmatrix} m-k \\ i \end{bmatrix} \frac{(q^{n+k};q)_i}{(1-q)^i} p^i q^{(n+k)(m-k-i)}. \end{aligned}$$

The last sum with respect to $i$ reduces to $x^{m-k}y^{n+k}$ in view of Equation (10), which enables us to simplify Proposition 3 in [5] to the following:

**Theorem 4.** *With two non-commutative variables $x$ and $y$ satisfying $xy = qyx + py^2$, we have the expansion*

$$y^n x^m = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q^n;q)_k}{(1-q)^k} p^k q^{\binom{k+1}{2}-m(n+k)} x^{m-k} y^{n+k}. \tag{13}$$

**Remark**: In [5], the coefficient $C(n,k)$ has been treated in Proposition 2 and 3 at length. Our theorem presented here clearly gives it in the following explicit form

$$C(n,k) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{1+k}{2}-n(k+1)}. \tag{14}$$

Now substituting (13) into the $(p,q)$-binomial relation (3) and then applying the binomial symmetry (12), we may reformulate the double sum as follows:

$$\begin{aligned}(x+y)^n \; &= \; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_k \Big(\frac{\gamma}{1-q}\Big)^k \\ \times \quad &\sum_{\ell=k}^{n} (-1)^{k+\ell} \begin{bmatrix} n-k \\ \ell-k \end{bmatrix} \frac{(q^k;q)_{\ell-k}}{(1-q)^{\ell-k}} p^{\ell-k} q^{\binom{1+\ell-k}{2}-(n-k)\ell} x^{n-\ell} y^{\ell} \\ = \quad &\sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix} x^{n-\ell} y^{\ell} \sum_{k=0}^{\ell} (-1)^{k+\ell} \begin{bmatrix} \ell \\ k \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_k \Big(\frac{\gamma}{1-q}\Big)^k \\ &\qquad\qquad \times \frac{(q^k;q)_{\ell-k}}{(1-q)^{\ell-k}} p^{\ell-k} q^{\binom{1+\ell-k}{2}-(n-k)\ell}.\end{aligned}$$

Denote by $S$ the last sum with respect to $k$. Changing the summation index $k$ by $\ell-k$ and then performing the following substitutions

$$\begin{aligned}\begin{bmatrix} \ell \\ k \end{bmatrix} \quad &= \quad (-1)^k \frac{(q^{-\ell};q)_k}{(q;q)_k} q^{k\ell-\binom{k}{2}} \\ (q^{\ell-k};q)_k \quad &= \quad (-1)^k (q^{1-\ell};q)_k q^{k\ell-\binom{1+k}{2}} \\ \Big(\frac{p}{\gamma};q\Big)_{\ell-k} \quad &= \quad (-1)^k \frac{(p/\gamma;q)_\ell (\gamma/p)^k}{(q^{1-\ell}\gamma/p;q)_k} q^{\binom{1+k}{2}-k\ell}\end{aligned}$$

we can rewrite $S$ as follows

$$\begin{aligned}S \quad &= \quad \sum_{k=0}^{\ell} (-1)^k \begin{bmatrix} \ell \\ k \end{bmatrix} \frac{(q^{\ell-k};q)_k}{(1-q)^k} \Big(\frac{p}{\gamma};q\Big)_{\ell-k} \Big(\frac{\gamma}{1-q}\Big)^{\ell-k} p^k q^{\binom{1+k}{2}-(n+k-\ell)\ell} \\ &= \quad \Big(\frac{p}{\gamma};q\Big)_\ell \Big(\frac{\gamma}{1-q}\Big)^\ell q^{\ell(\ell-n)} \sum_{k=0}^{\ell} \frac{(q^{-\ell};q)_k (q^{1-\ell};q)_k}{(q;q)_k (q^{1-\ell}\gamma/p;q)_k} q^k\end{aligned}$$

where the last sum may be evaluated as

$$\frac{(\gamma/p;q)_\ell}{(q^{1-\ell}\gamma/p;q)_\ell} q^{\ell(1-\ell)} \;=\; \Big\{-\frac{p}{\gamma}\Big\}^\ell \frac{(\gamma/p;q)_\ell}{(p/\gamma;q)_\ell} q^{-\binom{\ell}{2}}$$

thanks to the reversal of the terminating $q$-Gauss summation theorem (cf. Bailey [1, p 68])

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k (a;q)_k}{(q;q)_k (c;q)_k} q^k = a^n \frac{(c/a;q)_n}{(c;q)_n}. \tag{15}$$

Therefore we have

$$S = (-1)^\ell \Big(\frac{\gamma}{p};q\Big)_\ell \Big(\frac{p}{1-q}\Big)^\ell q^{\binom{1+\ell}{2}-n\ell}$$

which leads us to the dual form of the Newton $(p,q)$-binomial formula.

**Theorem 5.** *With two variables $x$ and $y$ satisfying the non-commutative relation $xy = qyx + py^2$, we have*

$$(x+y)^n = \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \Big(\frac{\gamma}{p}; q\Big)_\ell \Big(\frac{p}{1-q}\Big)^\ell q^{\binom{1+\ell}{2}-n\ell} x^{n-\ell} y^\ell. \tag{16}$$

For $p = 0$ and $q = 1$, this equation reduces, respectively, to the equivalent form of the Gaussian binomial formula

$$(x+y)^n = \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix} q^{\ell(\ell-n)} x^{n-\ell} y^\ell \quad \text{with} \quad xy = qyx \tag{17}$$

and to the dual form of the Newton $p$-binomial formula (5)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \langle -p \rangle_k x^{n-k} y^k \quad \text{with} \quad xy = yx + py^2. \tag{18}$$

Recalling the $(p,q)$-binomial coefficient defined in [4, Eq 8]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,p)} = \begin{bmatrix} n \\ k \end{bmatrix} \Big(\frac{p}{\gamma}; q\Big)_k \Big(\frac{\gamma}{1-q}\Big)^k$$

then the $(p,q)$-Vandermonde convolution identity obtained in [5, Thm 7] may be restated as

$$\begin{aligned} \begin{bmatrix} n+m \\ \ell \end{bmatrix}_{(q,p)} &= \sum_{i+j+k=\ell} \begin{bmatrix} m \\ i \end{bmatrix}_{(q,p)} \begin{bmatrix} n \\ j \end{bmatrix}_{(q,p)} \frac{(q^{m-i-k+1}; q)_k}{(1-q)^k} \\ &\times \sum_{r_1+r_2+\cdots+r_j=k} p^k q^{(m-i)j - \sum_{\iota=1}^{j} \iota r_\iota}. \end{aligned}$$

Applying Lemma 2 to the multisum displayed in the last line, we find that it equals

$$p^k q^{(m-i)j} P_{k,j}(q^{-1}) = p^k q^{(m-i-k)j} \begin{bmatrix} j+k-1 \\ k \end{bmatrix}$$

which reduces the above mentioned $(p,q)$-convolution to the following:

**Proposition 6 (Zhang and Wang [5, Thm 7]).**

$$\begin{aligned} \begin{bmatrix} m+n \\ \ell \end{bmatrix} \Big(\frac{p}{\gamma}; q\Big)_\ell &= \sum_{i+j+k=\ell} \begin{bmatrix} m \\ i+k \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} i+k \\ k \end{bmatrix} (q^j; q)_k \\ &\times \Big(\frac{p}{\gamma}; q\Big)_i \Big(\frac{p}{\gamma}; q\Big)_j \Big(\frac{p}{\gamma}\Big)^k q^{j(m-i-k)}. \end{aligned}$$

This is a composition of the $q$-Gauss summation formulae, which can be directly verified. In fact, denote by $\mathcal{L}(zw)$ and $\mathcal{R}(zw)$ the left and right member of the above displayed equation respectively. The triple sum on the right hand side has essentially two free summation variables, assuming $i$ and $j$. Then replacing $i+k$ by $\jmath$ and noting that

$$\begin{aligned}
\begin{bmatrix} i+k \\ k \end{bmatrix} &= \begin{bmatrix} \jmath \\ i \end{bmatrix} = (-1)^i \frac{(q^{-\jmath};q)_i}{(q;q)_i} q^{i\jmath-\binom{i}{2}} \\
(q^j;q)_k &= (q^j;q)_{\jmath-i} = (-1)^i \frac{(q^j;q)_\jmath}{(q^{1-\ell};q)_i} q^{\binom{1+i}{2}-i\ell}
\end{aligned}$$

we may express $\mathcal{R}(zw)$ as follows

$$\begin{aligned}
\mathcal{R}(zw) &= \sum_{\jmath+j=\ell} \begin{bmatrix} m \\ \jmath \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_j \Big(\frac{p}{\gamma}\Big)^\jmath (q^j;q)_\jmath q^{j(m-\jmath)} \\
&\times \sum_{i=0}^{\jmath} \frac{(q^{-\jmath};q)_i (p/\gamma;q)_i}{(q;q)_i (q^{1-\ell};q)_i} \Big(\frac{\gamma}{p}\Big)^i q^{i(1+\jmath-\ell)}
\end{aligned}$$

where the last sum may be evaluated as $(q^{1-\ell}\gamma/p;q)_\jmath/(q^{1-\ell};q)_\jmath$ in view of the terminating $q$-Gauss summation formulae (cf. Bailey [1, p 68])

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k (a;q)_k}{(q;q)_k (c;q)_k} \Big(\frac{c}{a} q^n\Big)^k = \frac{(c/a;q)_n}{(c;q)_n} \tag{19}$$

whose reversal has been anticipated in Equation (15).

By means of transformations

$$\begin{aligned}
\begin{bmatrix} m \\ \jmath \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} &= \begin{bmatrix} m \\ \ell \end{bmatrix} \frac{(q^{-n};q)_j (q^{-\ell};q)_j}{(q;q)_j (q^{1+m-\ell};q)_j} q^{j(1+n+\ell-j)} \\
(q^{1-\ell}\gamma/p;q)_{\ell-j} &= \Big(-\frac{\gamma}{p}\Big)^{\ell-j} \frac{(p/\gamma;q)_\ell}{(p/\gamma;q)_j} q^{\binom{j}{2}-\binom{\ell}{2}} \\
\frac{(q^j;q)_\jmath}{(q^{1-\ell};q)_\jmath} &= \frac{(q^j;q)_{\ell-j}}{(q^{1-\ell};q)_{\ell-j}} = (-1)^{\ell+j} q^{\binom{\ell}{2}-\binom{j}{2}}
\end{aligned}$$

we now can write $\mathcal{R}(zw)$ in the following simplified form

$$\begin{aligned}
\mathcal{R}(zw) &= \sum_{\jmath+j=\ell} \begin{bmatrix} m \\ \jmath \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(q^{1-\ell}\gamma/p;q)_\jmath}{(q^{1-\ell};q)_\jmath} \Big(\frac{p}{\gamma};q\Big)_j \Big(\frac{p}{\gamma}\Big)^\jmath (q^j;q)_\jmath q^{j(m-\jmath)} \\
&= \begin{bmatrix} m \\ \ell \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_\ell \sum_{j=0}^{\ell} \frac{(q^{-n};q)_j (q^{-\ell};q)_j}{(q;q)_j (q^{1+m-\ell};q)_j} q^{j(1+m+n)}
\end{aligned}$$

where the last sum equals

$$\frac{(q^{1+m+n-\ell};q)_\ell}{(q^{1+m-\ell};q)_\ell} = \begin{bmatrix} m+n \\ \ell \end{bmatrix} \Big/ \begin{bmatrix} m \\ \ell \end{bmatrix}$$

thanks again to the terminating $q$-Gauss summation theorem (19).

Therefore we have arrived at the point:

$$\mathcal{R}(zw) = \begin{bmatrix} m+n \\ \ell \end{bmatrix} \Big(\frac{p}{\gamma};q\Big)_\ell = \mathcal{L}(zw)$$

which completes the proof as desired.

# 3 Double-indexed inverse series relations

Suggested by the monomial expansions, we now establish the following $q$-analogue of the classical $\delta$-inversions.

**Theorem 7.** *The system of equations*

$$f(m,n) = \sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} g(m-k,n+k),\ m,n=0,1,2,\cdots \tag{20a}$$

*is equivalent to the system*

$$g(m,n) = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} f(m-k,n+k),\ m,n=0,1,2,\cdots. \tag{20b}$$

*Proof.* It is sufficient to show that one system implies another. Supposing that the second relation is valid and substituting it into the first, we need to confirm that the resulting double sum simplifies to $f(m,n)$. This can be accomplished as follows

$$\begin{aligned} & \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \sum_{i=0}^{m-k} q^{\binom{i}{2}} \begin{bmatrix} m-k \\ i \end{bmatrix} f(m-k-i,n+k+i) \\ = & \sum_{\ell=0}^{m} (-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} f(m-\ell,n+\ell) \sum_{i=0}^{\ell} (-1)^i \begin{bmatrix} \ell \\ i \end{bmatrix} q^{\binom{i}{2}} \\ = & \sum_{\ell=0}^{m} (-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} f(m-\ell,n+\ell)(1;q)_\ell \equiv f(m,n) \end{aligned}$$

where $\ell = i+k$ and the binomial symmetry (12) have been recalled from the first to the middle line; and the Euler $q$-finite differences (11) from the middle to the last line. □

Specifying two sequences appeared in the inverse relations by

$$\begin{aligned} f(m,n) &= \frac{(q;q)_{n-1}}{(1-q)^n} p^n q^{-\binom{n}{2}-mn} x^m y^n \\ g(m,n) &= \frac{(q;q)_{n-1}}{(1-q)^n} p^n q^{-\binom{n}{2}} y^n x^m \end{aligned}$$

we recover the dual relations displayed in Theorem 3 and 4:

$$\begin{cases} x^m y^n = \displaystyle\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q^n;q)_k}{(1-q)^k} p^k q^{n(m-k)}\, y^{n+k} x^{m-k} \\ y^n x^m = \displaystyle\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q^n;q)_k}{(1-q)^k} (-p)^k q^{\binom{k+1}{2}-m(n+k)} x^{m-k} y^{n+k}. \end{cases}$$

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