CHAPTER E

Basic Hypergeometric Series

This chapter introduces the basic hypergeometric series. Its convergence condition will be determined. The fundamental transformations and summation formulae will be covered briefly.

E1. Introduction and notation

E1.1. Definition. Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq q^{-n}$ with $n \in \mathbb{N}_0$ for all $j = 1, 2, \cdots, s$. Then the basic hypergeometric series with variable z is defined by

$$ {}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \, z^n \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r}. $$

Remark If there is a numerator parameter $a_i = q^{-k}$ with $k \in \mathbb{N}_0$, then the q-hypergeometric series is terminating, which is in fact a polynomial of z. When the series is nonterminating, we assume that $|q| < 1$ for convenience.

E1.2. Convergence condition. For the q-hypergeometric series just defined, the convergence conditions are as follows:

(A) If $s > r$, the series is convergent for all $z \in \mathbb{C}$;
(B) If $s < r$, the series is convergent only when $z = 0$;
(C) If $s = r$, the series is convergent for $|z| < 1$.

PROOF. Denote by T_n the summand of q-hypergeometric series

$$ T_n := \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \, z^n. $$

To determine the convergence conditions, we consider the term-ratio:

$$\frac{T_{n+1}}{T_n} = z\frac{(1-q^na_0)(1-q^na_1)\cdots(1-q^na_r)}{(1-q^{n+1})(1-q^nb_1)\cdots(1-q^na_s)}(-q^n)^{s-r}.$$

On account of $|q| < 1$, we have $|q^n| \to 0$ as $n \to +\infty$. Hence we get the following limit:

$$\lim_{n\to+\infty}\left|\frac{T_{n+1}}{T_n}\right| = \begin{cases} 0, & r < s \\ +\infty, & r > s \text{ and } z \neq 0 \\ |z|, & r = s. \end{cases}$$

According to the D'Alembert ratio test, the convergence conditions stated in the Theorem follow immediately. □

E1.3. Classification. For the basic hypergeometric series, suppose $r = s$, the very important case. If the product of denominator parameters is equal to the base q times the product of numerator parameters, i.e.,

$$qa_0a_1\cdots a_r = b_1b_2\cdots b_r$$

then the ${}_{1+r}\phi_r$-series is called *balanced* or *Saalschützian.*

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same product:

$$qa_0 = a_1b_1 = \cdots = a_rb_r$$

then we say that the ${}_{1+r}\phi_r$-series is *well-poised.* In particular, it is said to be *very-well-poised* if we have $a_1 = -a_2 = q\sqrt{a_0}$ in addition. These pairs of parameters appear in the basic hypergeometric sum as a linear fraction

$$\frac{1-a_0q^{2k}}{1-a_0} = \frac{(q\sqrt{a_0};q)_k}{(\sqrt{a_0};q)_k} \times \frac{(-q\sqrt{a_0};q)_k}{(-\sqrt{a_0};q)_k}.$$

E1.4. Examples. In terms of q-series, we can reformulate the Euler and Gauss summation formulae as follows:

$$(z;\,q)_\infty = {}_1\phi_1\left[\begin{matrix}-\\-\end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty}\frac{(-z)^k}{(q;\,q)_k}q^{\binom{k}{2}}$$

$$\frac{1}{(z;\,q)_\infty} = {}_1\phi_0\left[\begin{matrix}0\\-\end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty}\frac{z^k}{(q;\,q)_k}.$$

They will be used to demonstrate the q-binomial theorem.

E1.5. Ordinary hypergeometric series. In comparison with the basic hypergeometric series, we present here briefly the ordinary hypergeometric series, its convergence condition and classification. The details can be found in the book by Bailey (1935).

Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N}_0$ for $j = 1, 2, \cdots, s$. Then the ordinary hypergeometric series with variable z is defined by

$$ {}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{n!\,(b_1)_n \cdots (b_s)_n}\, z^n $$

where the (rising) shifted factorial is defined by

$$ (c)_0 = 1 \quad \textit{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \textit{for} \quad n = 1, 2, \cdots. $$

Classification Similar to basic hypergeometric series, we consider the case $r = s$ for ordinary hypergeometric series. If the sum of denominator parameters is equal to one plus the sum of numerator parameters, i.e.,

$$ 1 + a_0 + a_1 + \cdots + a_r = b_1 + b_2 + \cdots + b_r $$

then the ${}_{1+r}F_r$-series is called *balanced* or *Saalschützian.*

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same sum:

$$ 1 + a_0 = a_1 + b_1 = \cdots = a_r + b_r $$

then we say that the ${}_{1+r}F_r$-series is *well-poised.* In particular, it is said to be *very-well-poised* if we have $a_1 = 1 + a_0/2$ in addition. The last pair of parameters appear in the (ordinary) hypergeometric sum as a linear fraction

$$ \frac{a_0 + 2k}{a_0} \quad = \quad \frac{(1 + a_0/2)_k}{(a_0/2)_k}. $$

Convergence condition for the (ordinary) hypergeometric series is determined as follows:

- if $r < s$, the ${}_{1+r}F_s$-series converges for all $z \in \mathbb{C}$;
- if $r > s$, the ${}_{1+r}F_s$-series diverges for all $z \in \mathbb{C}$ except for $z = 0$;
- if $r = s$, the ${}_{1+r}F_r$-series converges for $|z| < 1$, and when

$$ \begin{aligned} z &= +1 \quad \text{if} \quad \Re(B - A) > 0 \\ z &= -1 \quad \text{if} \quad \Re(B - A) > -1 \end{aligned} $$

where A and B are defined respectively by

$$A = \sum_{i=0}^{r} a_i \quad \text{and} \quad B = \sum_{j=1}^{r} b_j.$$

Remark Noting that the limit relation between ordinary and q-shifted factorials

$$\lim_{q\to 1} \frac{(q^c;\, q)_k}{(1-q)^k} \quad = \quad (c)_k$$

we can consider the (ordinary) hypergeometric series as the limit of the basic hypergeometric series:

$${}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, z\right] = \lim_{q\to 1} {}_{1+r}\phi_s\left[\begin{matrix} q^{a_0}, q^{a_1}, \cdots, q^{a_r} \\ q^{b_1}, \cdots, q^{b_s} \end{matrix} \,\middle|\, q;\, \frac{(-1)^{r-s}\, z}{(1-q)^{r-s}}\right].$$

This explains why there exist generally the q-counterparts for the (ordinary) hypergeometric series identities.

E2. The q-Gauss summation formula

This section will prove the q-binomial theorem, the q-Gauss summation formula as well as the q-Chu-Vandermonde convolution.

E2.1. The q-binomial theorem. In terms of hypergeometric series, the classical binomial theorem reads as follows:

$${}_1F_0\left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} z^n = \frac{1}{(1-z)^c}, \quad (|z| < 1).$$

Its q-analog is given by the following q-binomial theorem:

$${}_1\phi_0\left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, q;\, z\right] = \frac{(cz;\, q)_\infty}{(z;\, q)_\infty} = \sum_{n=0}^{\infty} \frac{(c;\, q)_n}{(q;\, q)_n} z^n, \quad (|z| < 1). \tag{E2.1}$$

For $c = 0$ this identity reduces to Gauss summation formula. Replacing z by z/c and then letting $c \to \infty$, we recover from it the Euler formula.

PROOF. In fact, expanding the numerator and the denominator respectively according to Euler and Gauss summation formulae, we have

$$\begin{aligned}\frac{(cz;\ q)_\infty}{(z;\ q)_\infty} &= \sum_{i=0}^{\infty}(-1)^i\frac{c^i z^i}{(q;\ q)_i}q^{\binom{i}{2}}\sum_{j=0}^{\infty}\frac{z^j}{(q;\ q)_j}\\ &= \sum_{n=0}^{\infty}\frac{z^n}{(q;\ q)_n}\sum_{i=0}^{n}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}n\\ i\end{bmatrix}c^i = \sum_{n=0}^{\infty}\frac{(c;\ q)_n}{(q;\ q)_n}z^n\end{aligned}$$

where the last line follows from the finite q-differences. □

E2.2. The q-Gauss summation formula. The q-binomial theorem can be generalized to the following theorem.

For three complex numbers a, b and c with $|c/ab| < 1$, there holds

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\;\middle|\; q;\ c/ab\right] = \sum_{n=0}^{\infty}\frac{(a;\ q)_n(b;\ q)_n}{(q;\ q)_n(c;\ q)_n}\big(c/ab\big)^n = \frac{(c/a;\ q)_\infty(c/b;\ q)_\infty}{(c;\ q)_\infty(c/ab;\ q)_\infty}.$$

PROOF. We can manipulate, by means of the q-binomial theorem (E2.1), the infinite series as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\Big(\frac{c}{ab}\Big)^n &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(q^nc;q)_\infty}{(q;q)_n(q^nb;q)_\infty}\Big(\frac{c}{ab}\Big)^n\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{c}{ab}\Big)^n\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}\big(q^nb\big)^k\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{q^kc}{ab}\Big)^n\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\frac{(q^kc/b;q)_\infty}{(q^kc/ab;q)_\infty}\\ &= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/ab;q)_k}{(q;q)_k}b^k\\ &= \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\end{aligned}$$

which establishes the q-Gauss summation formula. □

E2.3. The q-analog of Chu-Vandermonde convolution. The terminating case of the q-Gauss summation formula can be reformulated as the

q-analogues of the Chu-Vandermonde convolution:

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\ q^n\, c/b\right] = \frac{(c/b;\ q)_n}{(c;\ q)_n} \tag{E2.2a} $$

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\ q\right] = \frac{(c/b;\ q)_n}{(c;\ q)_n} b^n \tag{E2.2b} $$

$$ \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \begin{bmatrix} x+y \\ n \end{bmatrix}. \tag{E2.2c} $$

Proof. The first formula is the case $a = q^{-n}$ of the q-Gauss theorem, which can be reformulated to other two identities.

By definition of q-hypergeometric series, rewrite (E2.2a) explictly as

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} \,\middle|\, q; q^n c/b\right] = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k (c; q)_k} (q^n c/b)^k = \frac{(c/b;\ q)_n}{(c;\ q)_n}. $$

Considering that

$$ (x; q)_{n-k} = (-1)^k x^{-k} q^{\binom{k+1}{2}-nk} \frac{(x; q)_n}{(q^{1-n}/x; q)_k} $$

we can manipulate the reversed series as follows:

$$ \begin{aligned} \frac{(c/b;\ q)_n}{(c;\ q)_n} &= \sum_{k=0}^{n} \frac{(q^{-n}; q)_{n-k} (b; q)_{n-k}}{(q; q)_{n-k} (c; q)_{n-k}} (q^n c/b)^{n-k} \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{1-n}/c; q)_k}{(q; q)_k (q^{1-n}/b; q)_k} q^k \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\ q\right] \end{aligned} $$

which is equivalent to

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\ q\right] = (-1)^n q^{-\binom{n}{2}} \Big(\frac{b}{c}\Big)^n \frac{(c/b; q)_n}{(b; q)_n} $$

in view of

$$ (q^{-n}; q)_n = (-1)^n q^{-\binom{n+1}{2}} (q; q)_n. $$

Performing the parameter replacements

$$ \begin{aligned} B &\to q^{1-n}/c \\ C &\to q^{1-n}/b \end{aligned} $$

and then applying the relation

$$ (q^{1-n}/C; q)_n = (-1)^n q^{-\binom{n}{2}} C^{-n} (C; q)_n $$

we can restate the last formula as:

$$ {}_2\phi_1 \left[\begin{matrix} q^{-n}, & B \\ & C \end{matrix} \,\middle|\, q;\ q \right] = B^n \frac{(C/B;q)_n}{(C;q)_n} $$

which is the second formula (E2.2b).

Writing the q-binomial coefficients in terms of q-shifted factorials

$$ \begin{aligned} \begin{bmatrix} x \\ k \end{bmatrix} &= \frac{(q^{x-k+1};q)_k}{(q;q)_k} &&= (-1)^k q^{xk-\binom{k}{2}} \frac{(q^{-x};q)_k}{(q;q)_k} \\ \begin{bmatrix} y \\ n-k \end{bmatrix} &= \frac{(q^{y-n+k+1};q)_{n-k}}{(q;q)_{n-k}} &&= (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_n} \frac{(q^{y-n+1};q)_n}{(q^{y-n+1};q)_k} \end{aligned} $$

we can express the q-binomial sum in terms of q-series:

$$ \begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k}{(q;q)_k (q^{y-n+1};q)_k} q^k \\ &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx}\ {}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{-x} \\ & q^{y-n+1} \end{matrix} \,\middle|\, q; q \right]. \end{aligned} $$

Evaluate the last q-series by (E2.2b):

$$ {}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{-x} \\ & q^{y-n+1} \end{matrix} \,\middle|\, q;\ q \right] = q^{-nx} \frac{(q^{x+y-n+1};q)_n}{(q^{y-n+1};q)_n} $$

we find consequently the following q-binomial identity

$$ \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \frac{(q^{x+y-n+1};q)_n}{(q;q)_n} = \begin{bmatrix} x+y \\ n \end{bmatrix} $$

which is, in fact, the convolution formula (E2.2c). □

E3. Transformations of Heine and Jackson

E3.1. Jackson's ${}_2\phi_2$-series transformation.

$$ {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z \right] = \frac{(az;q)_\infty}{(z;q)_\infty} \times {}_2\phi_2 \left[\begin{matrix} a, & c/b \\ c, & az \end{matrix} \,\middle|\, q;\ bz \right]. $$

Proof. According to the q-Chu-Vandermonde formula, we have

$$ \frac{(b;q)_n}{(c;q)_n} = {}_2\phi_1 \left[\begin{matrix} q^{-n}, & c/b \\ & c \end{matrix} \,\middle|\, q;\ q^n b \right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (c/b;q)_k}{(q;q)_k (c;q)_k} (bq^n)^k. $$

Then the q-hypergeometric series in question can be expressed as a double sum:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\frac{(b;q)_n}{(c;q)_n} z^n \\
&= \sum_{n=0}^{\infty} z^n \frac{(a;q)_n}{(q;q)_n} \times \sum_{k=0}^{n} \frac{(q^{-n};q)_k (c/b;q)_k}{(q;q)_k (c;q)_k} (bq^n)^k \\
&= \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k (c;q)_k} b^k \times \sum_{n=k}^{\infty} \frac{(a;q)_n (q^{-n};q)_k}{(q;q)_n} \left(zq^k\right)^n.
\end{aligned}
$$

For the last sum with respect to n, changing by $j := n-k$ on the summation index and then applying transformations

$$
\begin{aligned}
(a;q)_{j+k} &= (a;q)_k (aq^k;q)_j \\
\frac{(q^{-j-k};q)_k}{(q;q)_{j+k}} &= \frac{(-1)^k q^{-k(j+k)+\binom{k}{2}}}{(q;q)_j}
\end{aligned}
$$

we can evaluate it, by means of (E2.1) with $c \to aq^k$, as follows:

$$
\begin{aligned}
\sum_{n=k}^{\infty} \frac{(a;q)_n (q^{-n};q)_k}{(q;q)_n} \left(zq^k\right)^n &= (-z)^k q^{\binom{k}{2}} (a;q)_k \times \sum_{j=0}^{\infty} \frac{(aq^k;q)_j}{(q;q)_j} z^j \\
&= (-z)^k q^{\binom{k}{2}} (a;q)_k \times {}_1\phi_0\left[\begin{matrix} aq^k \\ - \end{matrix} \,\middle|\, q;\ z\right] \\
&= (-z)^k q^{\binom{k}{2}} (a;q)_k \times \frac{(q^k az;q)_k}{(z;q)_k} \\
&= (-z)^k q^{\binom{k}{2}} \frac{(a;q)_k}{(az;q)_k} \frac{(az;q)_\infty}{(z;q)_\infty}.
\end{aligned}
$$

We have therefore established

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] &= \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{(a;q)_k (c/b;q)_k}{(q;q)_k (az;q)_k (c;q)_k} (bz)^k \\
&= \frac{(az;q)_\infty}{(z;q)_\infty} \, {}_2\phi_2\left[\begin{matrix} a, & c/b \\ c, & az \end{matrix} \,\middle|\, q;\ bz\right]
\end{aligned}
$$

which is Jackson's transformation. □

E3.2. Heine's q-Euler transformations.

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\Big|\, q;\ z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}\,\Big|\, q;\ b\right] \tag{E3.1a}
$$

$$
= \frac{[c/b, bz; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\Big|\, q; c/b\right] \tag{E3.1b}
$$

$$
= \frac{(abz/c; q)_\infty}{(z; q)_\infty} \times {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\Big|\, q;\ abz/c\right]. \tag{E3.1c}
$$

Proof. Substituting the q-factorial fraction

$$
\frac{(b; q)_n}{(c; q)_n} = \frac{(b; q)_\infty}{(c; q)_\infty} \times \frac{(q^n c; q)_\infty}{(q^n b; q)_\infty}
$$

into the q-hypergeometric series

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\Big|\, q;\ z\right] = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n
$$

and then applying the q-binomial theorem (E2.1):

$$
\frac{(q^n c; q)_\infty}{(q^n b; q)_\infty} = {}_1\phi_0\left[\begin{matrix} c/b \\ - \end{matrix}\,\Big|\, q;\ q^n b\right] = \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} q^{nk} b^k
$$

we can manipulate the q-series as follows:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\Big|\, q;\ z\right] &= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} q^{nk} b^k \\
&= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} b^k \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (zq^k)^n.
\end{aligned}
$$

Again by means of (E2.1), evaluating the last sum with respect to n as

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (zq^k)^n = \frac{(q^k az; q)_\infty}{(q^k z; q)_\infty}
$$

and then simplifying the series with

$$
\begin{aligned}
(q^k az; q)_\infty &= \frac{(az; q)_\infty}{(az; q)_k} \\
(q^k z; q)_\infty &= \frac{(z; q)_\infty}{(z; q)_k}
\end{aligned}
$$

we derive the following expression

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k(z;q)_k}{(q;q)_k(az;q)_k}b^k\\
&= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;\ b\right]
\end{aligned}$$

which is the first transformation (E3.1a).

Applying the transformation just established to the series on the right hand side, we have

$${}_2\phi_1\left[\begin{matrix}z, & c/b\\ & az\end{matrix}\,\middle|\, q;\ b\right] = \frac{(c/b;q)_\infty(bz;q)_\infty}{(az;q)_\infty(b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;\ c/b\right]$$

whose combination with the first one result in

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;\ b\right]\\
&= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;c/b\right].
\end{aligned}$$

This is the second transformation (E3.1b).

Applying again the first transformation, we get

$${}_2\phi_1\left[\begin{matrix}b, abz/c\\ bz\end{matrix}\,\middle|\, q;c/b\right] = \frac{(abz/c;q)_\infty(c;q)_\infty}{(bz;q)_\infty(c/b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a, c/b\\ c\end{matrix}\,\middle|\, q;abz/c\right].$$

This leads us to the following

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;\ c/b\right]\\
&= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a, c/b\\ c\end{matrix}\,\middle|\, q;\ abz/c\right]
\end{aligned}$$

which is exactly the third transformation (E3.1c).

The last transformation can also be derived by means of the Jackson transformation stated in E3.1. In fact, interchanging a and b in the Jackson formula, we have

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] = \frac{(bz;q)_\infty}{(z;q)_\infty}\times{}_2\phi_2\left[\begin{matrix}b, & c/a\\ c, & bz\end{matrix}\,\middle|\, q;\ az\right].$$

While the q-series on the right hand side of (E3.1c) can be transformed, by means of the Jackson identity, into the following

$${}_2\phi_1\left[\begin{matrix}c/a, & c/b\\ & c\end{matrix}\,\middle|\, q;\ abz/c\right] = \frac{(bz;q)_\infty}{(abz/c;q)_\infty}\times{}_2\phi_2\left[\begin{matrix}c/a, & b\\ c, & bz\end{matrix}\,\middle|\, q;\ az\right].$$

Equating both expressions, we have

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\Big|\, q;\, z\right] = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \times {}_2\phi_1\left[\begin{matrix}c/a, & c/b\\ & c\end{matrix}\,\Big|\, q;\, abz/c\right].$$

The proof of (E3.1c) is therefore completed again. □

Remark The Heine's transformations (E3.1a-E3.1b-E3.1c) may be considered as q-analogues of the Pfaff-Euler Transformations for the (ordinary) hypergeometric series:

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\Big|\, z\right] &= (1-z)^{-a}\quad {}_2F_1\left[\begin{matrix}a, & c-b\\ & c\end{matrix}\,\Big|\, \frac{z}{z-1}\right]\\ &= (1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a, & c-b\\ & c\end{matrix}\,\Big|\, z\right].\end{aligned}$$

E3.3. The Bailey-Daum summation formula.

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & qa/b\end{matrix}\,\Big|\, q;\, -q/b\right] = (-q;q)_\infty\frac{\left[qa, q^2a/b^2; q^2\right]_\infty}{\left[qa/b, -q/b; q\right]_\infty}, \qquad (|q/b|<1).$$

Proof. Applying the Heine transformation (E3.1a)

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\Big|\, q;\, z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\Big|\, q;\; b\right]$$

we can proceed as follows:

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}b,\ a\\ qa/b\end{matrix}\,\Big|\, q; -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/b, & -q/b\\ & -q\end{matrix}\,\Big|\, q;\, a\right]\\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n}a^n.\end{aligned}$$

Simplifying the last sum with relations

$$\begin{aligned}(q^2/b^2;q^2)_n &= (q/b;q)_n(-q/b;q)_n\\ (q^2;q^2)_n &= (q;q)_n(-q;q)_n\end{aligned}$$

and then evaluating it by means of the q-binomial theorem (E2.1), we have

$$\sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n}a^n = \sum_{n=0}^{\infty}\frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n}a^n = \frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty}$$

which results consequently in the following

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a, & b\\ & qa/b\end{matrix}\,\Big|\, q;\, -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\times\frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty}\\ &= \frac{(-q;q)_\infty(aq;q^2)_\infty(q^2a/b^2;q^2)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\end{aligned}$$

thanks to the shifted factorial relation

$$(a;q)_\infty \;=\; (aq;q^2)_\infty(a;q^2)_\infty.$$

This proves the Bailey-Daum summation theorem. $\square$

E3.4. Infinite series transformation. Armed with the q-series transformation formulae, we apply again inverse series relations (D1.3a-D1.3b) to establish another infinite series transformation, which will be used in turn to prove two infinite series identities of Rogers-Ramanujan type.

Recalling the inverse series relations (D1.3a-D1.3b), if we take the g-sequence

$$g(n) \;=\; \frac{(\lambda;q)_n}{(q\lambda;q^2)_n}q^{\binom{n}{2}} \quad \text{with} \quad n=0,1,2,\cdots$$

then the dual sequence will be determined by

$$f(n) \;=\; \begin{cases} 0, & n-\text{odd} \\ (-1)^m\,\left[q,\lambda;q^2\right]_m q^{m^2-m}, & n=2m. \end{cases}$$

We have accordingly from (D1.4b) the infinite series transformation:

$$\sum_{n=0}^{\infty}\frac{\lambda^n q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q\lambda;\,q^2)_n} \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{1-q^{4k}\lambda}{(\lambda;q)_\infty}\frac{(\lambda;\,q^2)_k}{(q^2;\,q^2)_k}q^{5k^2-k}\lambda^{2k}. \tag{E3.2}$$

Proof. Substituting $g(k)$ into (D1.3a) and then rewriting the q-Gauss binomial coefficient, we have

$$\begin{aligned} f(n) \;&=\; \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}(q^k\lambda;\,q)_n\,g(k) \\ &=\; \sum_{k=0}^{n}\frac{(q^{-n};\,q)_k}{(q;\,q)_k}q^{\binom{n}{2}+\binom{k+1}{2}}\frac{(\lambda;\,q)_{n+k}}{(q\lambda;\,q^2)_k}. \end{aligned}$$

By means of factorization

$$(q\lambda;\,q^2)_k \;=\; (\sqrt{q\lambda};\,q)_k\times(-\sqrt{q\lambda};\,q)_k$$

we can express $f(n)$ in terms of a terminating q-hypergeometric series

$$\begin{aligned} f(n) \;&=\; q^{\binom{n}{2}}(\lambda;\,q)_n\,\times\,\sum_{k=0}^{n}\begin{bmatrix}q^{-n}, & q^n\lambda \\ q, & \pm\sqrt{q\lambda}\end{bmatrix}q\,\Bigg]_k q^{\binom{k+1}{2}} \\ &=\; q^{\binom{n}{2}}(\lambda;\,q)_n\,\times\,{}_2\phi_2\left[\begin{matrix}q^{-n}, & q^n\lambda \\ \sqrt{q\lambda}, & -\sqrt{q\lambda}\end{matrix}\,\Big|\,q;\,-q\right]. \end{aligned}$$

Rewriting Jackson's transformation formula stated in E3.1

$$ {}_2\phi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix} \,\Big|\, q;\ \frac{bd}{ac}\right] \;=\; \frac{(d/a;q)_\infty}{(d;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} a, & b/c \\ & b \end{matrix} \,\Big|\, q;\ d/a\right] $$

we can further reformulate $f(n)$ as follows:

$$ f(n) \;=\; q^{\binom{n}{2}} \frac{(\lambda;\ q)_n}{(-\sqrt{q\lambda};\ q)_n} \times {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{-n}\sqrt{q/\lambda} \\ & \sqrt{q\lambda} \end{matrix} \,\Big|\, q;\ -q^n\sqrt{q\lambda}\right]. $$

Evaluating the last series by means of the Bailey-Daum formula stated in E3.3:

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & qa/b \end{matrix} \,\Big|\, q;\ -q/b\right] \;=\; (-q;q)_\infty \frac{\left[qa, q^2a/b^2; q^2\right]_\infty}{\left[qa/b, -q/b; q\right]_\infty}, \quad (|q/b| < 1) $$

we find that

$$ f(n) \;=\; q^{\binom{n}{2}} (\lambda;\ q)_n \left[\begin{matrix} q^{1-n}, & q^{1+n}\lambda \\ q, & q\lambda \end{matrix} \,\Big|\, q^2\right]_\infty . $$

If n is odd, we have $f(n) = 0$ for $(q^{1-n};\ q^2)_\infty = 0$. Suppose $n = 2m$ instead, we have the following reduction

$$ \begin{aligned} f(n) \;&=\; q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \left[\begin{matrix} q^{1-2m}, & q^{1+2m}\lambda \\ q, & q\lambda \end{matrix} \,\Big|\, q^2\right]_\infty \\ &=\; q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \frac{(q^{1-2m};\ q^2)_m}{(q\lambda;\ q^2)_m} \\ &=\; (-1)^m q^{m^2-m} \left[q,\ \lambda;\ q^2\right]_m . \end{aligned} $$

Substituting $g(n)$ and $f(k)$ into (D1.4b), we establish (E3.2). $\square$

E3.5. Two further identities of Rogers-Ramanujan type. Specifying with $\lambda \to 1$ and $\lambda \to q^2$ in (E3.2), we derive the following identities of Rogers-Ramanujan type:

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\ q)_n\ (q;\ q^2)_n} \;&=\; \frac{\left[q^{10}, q^4, q^6; q^{10}\right]_\infty}{(q;\ q)_\infty} \\ \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2+3n}{2}}}{(q;\ q)_n (q;\ q^2)_{n+1}} \;&=\; \frac{\left[q^{10}, q^2, q^8; q^{10}\right]_\infty}{(q;\ q)_\infty} . \end{aligned} $$

Proof. Putting $\lambda \to 1$ in (E3.2) and then separating the first term from the right hand side, we derive

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q;\,q^2)_n} &= \frac{1}{(q;q)_\infty}\left\{1+\sum_{k=1}^{\infty}(-1)^k\left\{1+q^{2k}\right\}q^{5k^2-k}\right\}\\ &= \frac{1}{(q;q)_\infty}\left\{1+\sum_{k=1}^{\infty}(-1)^k q^{5k^2-k}+\sum_{k=1}^{\infty}(-1)^k q^{5k^2+k}\right\}.\end{aligned}$$

Performing replacement $k \to -k$ in the last sum and then applying the Jacobi triple product identity, we reduce the sum inside $\{\cdots\}$ as

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{10\binom{k}{2}+4k} = \left[q^{10},\, q^4,\, q^6;\, q^{10}\right]_\infty$$

which leads us to the first identity:

$$\sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q;\,q^2)_n} = \frac{\left[q^{10},\, q^4,\, q^6;\, q^{10}\right]_\infty}{(q;q)_\infty}.$$

When $\lambda \to q^2$, we can similarly write (E3.2) as

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{\frac{3n^2+3n}{2}}}{(q;\,q)_n(q;\,q^2)_{n+1}} &= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\left\{1-q^{4k+2}\right\}q^{5k^2+3k}\\ &= \frac{1}{(q;q)_\infty}\left\{\sum_{k=0}^{\infty}(-1)^k q^{5k^2+3k}+\sum_{k=0}^{\infty}(-1)^{k+1} q^{5k^2+7k+2}\right\}.\end{aligned}$$

Replacing k by $-k-1$ in the second sum and then applying the Jacobi triple product identity, we find that the sum inside $\{\cdots\}$ equals

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{10\binom{k}{2}+8k} = \left[q^{10},\, q^2,\, q^8;\, q^{10}\right]_\infty$$

which results in the second identity:

$$\sum_{n=0}^{\infty} \frac{q^{\frac{3n^2+3n}{2}}}{(q;\,q)_n(q;\,q^2)_{n+1}} = \frac{\left[q^{10},\, q^2,\, q^8;\, q^{10}\right]_\infty}{(q;q)_\infty}.$$

This completes proofs of two infinite series identities of Rogers-Ramanujan type. □

E4. The q-Pfaff-Saalschütz summation theorem

The formula under the title reads as the following

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \,\Big|\, q;\ q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix} \,\Big|\, q;\ q\right]_n. \tag{E4.1} $$

E4.1. Proof. Recall the q-Euler transformation (E3.1c):

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \,\Big|\, q;\ abz/c\right] = \frac{(z;q)_\infty}{(abz/c;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\Big|\, q;\ z\right] $$

which can be reformulated through the q-binomial theorem (E2.1), as a product of two basic hypergeometric series:

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \,\Big|\, q;\ abz/c\right] = {}_1\phi_0\left[\begin{matrix} c/ab, \\ - \end{matrix} \,\Big|\, q;\ abz/c\right] \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\Big|\, q;\ z\right]. $$

Extracting the coefficient of z^n from both members, we have

$$ \frac{(c/a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(ab/c)^n = \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}\frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}}(ab/c)^{n-k} $$

which can be restated equivalently as

$$ \begin{aligned} \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} &= \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}\frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k}q^k \\ &= {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \,\Big|\, q;\ q\right] \end{aligned} $$

in view of shifted factorial fraction

$$ \begin{aligned} \frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}} &= \frac{(c/ab;q)_n}{(q;q)_n}\frac{(q^{n-k+1};q)_k}{(q^{n-k}c/ab;q)_k} \\ &= \frac{(c/ab;q)_n}{(q;q)_n}\frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k}(qab/c)^k. \end{aligned} $$

This completes the proof of the q-Saalschütz formula. □

E4.2. The formula (E4.1) can also be proved by means of series rearrangement.

Recalling the q-Chu-Vandermonde formula (E2.2a), we have

$$ \frac{(a;q)_k}{(c;q)_k} = {}_2\phi_1\left[\begin{matrix} q^{-k}, & c/a \\ & c \end{matrix} \,\Big|\, q;\ q^k a\right] = \sum_{j=0}^{k} \frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j}q^{kj}a^j. $$

Then the q-hypergeometric series in (E4.1) can be written as a double sum:

$$\begin{aligned}
\text{LHS(E4.1)} &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k(q^{1-n}ab/c;q)_k} q^k \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(b;q)_k}{(q;q)_k(q^{1-n}ab/c;q)_k} q^k \sum_{j=0}^{k} \frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j} q^{kj}a^j \\
&= \sum_{j=0}^{n} \frac{(c/a;q)_j}{(q;q)_j(c;q)_j} a^j \sum_{k=j}^{n} \frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k} q^{k(j+1)}
\end{aligned}$$

where we have changed the summation order.

Denote by $\Omega(j)$ the last sum with respect to k. Changing the summation index with $i := k-j$ and then applying relations

$$\begin{aligned}
(x;q)_{i+j} &= (x;q)_i(q^ix;q)_j = (x;q)_j(q^jx;q)_i \\
(q^{-i-j};q)_j &= (-1)^j q^{-j(i+j)+\binom{j}{2}}(q^{i+1};q)_j
\end{aligned}$$

we can reduce $\Omega(j)$ as follows:

$$\begin{aligned}
\Omega(j) &= \sum_{k=j}^{n} \frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k} q^{k(j+1)} \\
&= \sum_{i=0}^{n-j} \frac{(q^{-n};q)_{i+j}(b;q)_{i+j}(q^{-i-j};q)_j}{(q;q)_{i+j}(q^{1-n}ab/c;q)_{i+j}} q^{(i+j)(j+1)} \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} \sum_{i=0}^{n-j} \frac{(q^{-n+j};q)_i(q^jb;q)_i}{(q;q)_i(q^{1-n+j}ab/c;q)_i} q^i \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} {}_2\phi_1\left[\begin{matrix} q^{-n+j},\ q^jb \\ q^{1-n+j}ab/c \end{matrix} \,\middle|\, q;\ q\right].
\end{aligned}$$

Applying now the q-Chu-Vandermonde formula (E2.2b), we can evaluate the q-series on the right hand side as

$${}_2\phi_1\left[\begin{matrix} q^{-n+j}, & bq^j \\ & q^{1-n+j}ab/c \end{matrix} \,\middle|\, q;\ q\right] = \frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}} \left(bq^j\right)^{n-j}$$

which results consequently in

$$\begin{aligned}
\Omega(j) &= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} \times \frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}} \left(bq^j\right)^{n-j} \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-j}a/c;q)_j} \times \frac{(q^{1-n}a/c;q)_n}{(q^{1-n}ab/c;q)_n} \left(bq^j\right)^{n-j} \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n} \frac{(q^{-n};q)_j(b;q)_j}{(c/a;q)_j} \left(q^nc/ab\right)^j.
\end{aligned}$$

Substituting the last expression of $\Omega(j)$ into the ${}_3\phi_2$-series and then applying the q-Chu-Vandermonde formula (E2.2a), we get the following evaluation

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \,\middle|\, q;\, q\right] &= \sum_{j=0}^{n} a^j \frac{(c/a;q)_j}{(q;q)_j(c;q)_j}\Omega(j) \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\,{}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} \,\middle|\, q; q^n c/b\right] \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\frac{(c/b;q)_n}{(c;q)_n}
\end{aligned}$$

which is equivalent to the q-Pfaff-Saalschütz formula (E4.1).

E5. The terminating q-Dougall-Dixon formula

It is, in fact, a very-well-poised terminating series identity

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a \end{matrix} \,\middle|\, q; \frac{q^{1+n}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \,\middle|\, q\right]_n. \tag{E5.1}$$

Proof. Based on the Carlitz inversions presented in (D1.1), we can derive the identity directly as the dual relation of the q-Pfaff-Saalschütz formula (E4.1).

Recalling the q-Pfaff-Saalschütz theorem (E4.1)

$${}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \,\middle|\, q;\, q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix} \,\middle|\, q;\, q\right]_n$$

we can restate it under parameter replacements as

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix} \,\middle|\, q;\, q\right] &= \frac{(q^{1-n}/b;q)_n(c;q)_n}{(qa/b;q)_n(q^{-n}c/a;q)_n} \\
&= \frac{(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n.
\end{aligned}$$

In order to apply the Carlitz inversions, we reformulate the q-series

$${}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix} \,\middle|\, q;\, q\right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^n a;q)_k(qa/bc;q)_k}{(q;q)_k(qa/b;q)_k(qa/c;q)_k}q^k$$

in terms of the q-binomial sum

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(q^k a;q)_n \frac{(a;q)_k(qa/bc;q)_k}{(qa/b;q)_k(qa/c;q)_k} \\
&= q^{\binom{n}{2}}\frac{(a;q)_n(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n
\end{aligned}$$

where we have used the following transformations:

$$\begin{aligned}\frac{(q^{-n};q)_k}{(q;q)_k} &= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} q^{-\binom{n}{2}-k} \\ (q^n a;q)_k &= \frac{(a;q)_{n+k}}{(a;q)_n} = \frac{(a;q)_k (q^k a;q)_n}{(a;q)_n}.\end{aligned}$$

Specifying the ϕ-polynomials with $a_k = 1$ and $b_k = -q^k a$ in the Carlitz inversions (D1.1a-D1.1b), which implies

$$\phi(x;n) := (ax;q)_n = \prod_{i=0}^{n-1}(1-axq^i)$$

and then choosing two sequences

$$\begin{aligned}f(n) &:= q^{\binom{n}{2}} \frac{(a;q)_n (b;q)_n (c;q)_n}{(qa/b;q)_n (qa/c;q)_n} \left(\frac{qa}{bc}\right)^n \\ g(k) &:= \frac{(a;q)_k (qa/bc;q)_k}{(qa/b;q)_k (qa/c;q)_k}\end{aligned}$$

we write down directly the dual relation

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}a}{(q^n a;q)_{k+1}} q^{\binom{k}{2}} \frac{(a;q)_k (b;q)_k (c;q)_k}{(qa/b;q)_k (qa/c;q)_k} \left(\frac{qa}{bc}\right)^k \\ &= \frac{(a;q)_n (qa/bc;q)_n}{(qa/b;q)_n (qa/c;q)_n}.\end{aligned}$$

Feeding back the q-binomial coefficient to factorial fraction

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{(q^{n-k+1};q)_k}{(q;q)_k} = (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k} \\ \frac{1-q^{2k}a}{1-a} &= \frac{(q\sqrt{a};q)_k}{(\sqrt{a};q)_k} \frac{(-q\sqrt{a};q)_k}{(-\sqrt{a};q)_k}\end{aligned}$$

we reformulate the dual relation in terms of q-series

$$\frac{(qa;q)_n (qa/bc;q)_n}{(qa/b;q)_n (qa/c;q)_n} = {}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a \end{matrix} \,\middle|\, q; \frac{q^{1+n}a}{bc} \right]$$

which is the terminating q-Dixon formula (E5.1). □

E6. The Sears balanced transformations

Replacing the base q with its inverse $1/q$ and then observing that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \times q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}$$

we can restate the Carlitz inversions in an equivalent form

$$\begin{cases} f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \phi(q^{-k};n)\, g(k), & n = 0,1,2,\cdots \quad \text{(E6.1a)} \\ g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \dfrac{a_k + q^{-k}b_k}{\phi(q^{-n};k+1)}\, f(k), & n = 0,1,2,\cdots \quad \text{(E6.1b)} \end{cases}$$

which will be used in this section to prove the Sears transformations on balanced basic hypergeometric series.

E6.1. The Sears balanced transformations.

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q \right] && \text{(E6.2a)} \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q \right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} && \text{(E6.2b)} \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q \right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n}. && \text{(E6.2c)} \end{aligned}$$

E6.2. Proof of (E6.2a-E6.2c). The second transformation formula is a consequence of the first. In fact, applying the symmetric property to (E6.2b) and then transform it by the first transformation (E6.2a-E6.2b), we have

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q \right] \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/c, & a, & b/e \\ & bd/ce, & b, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q \right] \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/c, & bd/ace, & d/c \\ & bd/ce, & bd/ac, & q^{1-n}/c \end{matrix} \,\middle|\, q;\ q \right] \times \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n}. \end{aligned}$$

Substituting this result into (E6.2b), we find the transformation

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q \right] \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q \right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q \right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n} \\ = \ & {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q \right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n} \end{aligned}$$

which is the second formula (E6.2a-E6.2c).

E6.3. Proof of (E6.2a-E6.2b). Let the ϕ-polynomials be defined by

$$\phi(x;n) \;=\; (acex/bd;q)_n \quad\rightleftharpoons\quad a_k = 1 \text{ and } b_k = -q^k ace/bd.$$

Then the corresponding inversions (E6.1a-E6.1b) become the following:

$$f(n) \;=\; \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} (q^{-k}ace/bd;q)_n\, g(k) \tag{E6.3a}$$

$$g(n) \;=\; \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}}\, f(k). \tag{E6.3b}$$

By means of two q-shifted factorial relations

$$\frac{(q^{-n};q)_k}{(q;q)_k} \;=\; (-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}-\binom{n}{2}-k}$$

$$(q^{1-n}ace/bd;q)_k \;=\; \frac{(q^{-n}ace/bd;q)_{k+1}}{1-q^{-n}ace/bd}$$

we can rewrite the ${}_4\phi_3$-series displayed in (E6.2a) as a q-binomial sum

$${}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e\\ & b, & d, & q^{1-n}ace/bd\end{matrix}\;\Big|\; q;\; q\right] \frac{1-bd/ace}{1-q^n bd/ace} q^{\binom{n+1}{2}}$$
$$= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\Big|\; q\right]_k .$$

Then the first transformation of Sears (E6.2a-E6.2b) can be stated equivalently as

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\Big|\; q\right]_k \tag{E6.4a}$$

$$= {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e\\ b,\ bd/ce,\ q^{1-n}a/d\end{matrix}\;\Big|\; q;q\right] \left[\begin{matrix} d/a, bd/ce\\ d, qbd/ace\end{matrix}\;\Big|\; q\right]_n q^{\binom{n+1}{2}}. \tag{E6.4b}$$

This expression matches perfectly with the relation (E6.3b), where two sequences have been specified by

$$f(k) := \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\Big|\; q\right]_k \tag{E6.5a}$$

$$g(n) := \left[\begin{matrix} d/a, bd/ce\\ d, qbd/ace\end{matrix}\;\Big|\; q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e\\ b,\ bd/ce,\ q^{1-n}a/d\end{matrix}\;\Big|\; q;q\right] q^{\binom{n+1}{2}}. \tag{E6.5b}$$

Therefore in order to demonstrate the first transformation (E6.4a-E6.4b) of Sears, it suffices to prove the following dual relation, which corresponds to the relation (E6.3a):

$$\left[\begin{matrix} a, c, e \\ b,\ d \end{matrix} \,\middle|\, q\right]_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}(q^{-k}ace/bd;q)_n \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix} \,\middle|\, q\right]_k q^{\binom{k+1}{2}} \tag{E6.6a}$$

$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-k}a/d \end{matrix} \,\middle|\, q;\ q\right]. \tag{E6.6b}$$

Let Ξ stand for the double sum on the right. We should therefore verify that Ξ reduces to the factorial fraction on the left.

Recalling the definition of q-hypergeometric series

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q;\ q\right] \\ = &\sum_{i=0}^{k}\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ q, & b, & bd/ce, & q^{1-k}a/d \end{matrix} \,\middle|\, q\right]_i q^i \end{aligned}$$

and the relation of q-binomial coefficient in terms of factorial fraction

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \frac{(q^{-n};q)_k}{(q;q)_k} q^{nk-\binom{k}{2}}$$

we can rearrange the double sum as follows:

$$\begin{aligned} \Xi &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}(q^{-k}ace/bd;q)_n \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix} \,\middle|\, q\right]_k q^{\binom{k+1}{2}} \\ &\times \sum_{i=0}^{k}\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ q, & b, & bd/ce, & q^{1-k}a/d \end{matrix} \,\middle|\, q\right]_i q^i \\ &= \sum_{i=0}^{n}\left[\begin{matrix} a, b/c, b/e \\ q, b, bd/ce \end{matrix} \,\middle|\, q\right]_i q^i \sum_{k=i}^{n}\left[\begin{matrix} q^{-n}, d/a, bd/ce \\ q, d, qbd/ace \end{matrix} \,\middle|\, q\right]_k \\ &\times\ (q^{-k}ace/bd;q)_n \frac{(q^{-k};q)_i}{(q^{1-k}a/d;q)_i} q^{k(n+1)}. \end{aligned}$$

For the inner sum, performing the replacement $j := k - i$ on summation index and then applying relations

$$\begin{aligned} \frac{(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i} &= \left(\frac{d}{qa}\right)^i \frac{(q^{1+j};q)_i}{(q^j d/a;q)_i} = \left(\frac{d}{qa}\right)^i \frac{(q;q)_{i+j}}{(d/a;q)_{i+j}} \frac{(d/a;q)_j}{(q;q)_j} \\ (q^{-i-j}ace/bd;q)_n &= \frac{(q^{-i-j}ace/bd;q)_{i+j}}{(q^{n-i-j}ace/bd;q)_{i+j}}(ace/bd;q)_n \\ &= \frac{(qbd/ace;q)_{i+j}}{(q^{1-n}bd/ace;q)_i} \frac{(ace/bd;q)_n}{(q^{1+i-n}bd/ace;q)_j} q^{-n(i+j)} \end{aligned}$$

we can reduce it to the following

$$\begin{aligned}
&\sum_{k=i}^{n}\left[\begin{matrix} q^{-n}, & d/a, & bd/ce \\ q, & d, & qbd/ace \end{matrix}\,\middle|\, q\right]_k \frac{(q^{-k}ace/bd;q)_n(q^{-k};q)_i}{(q^{1-k}a/d;q)_i}q^{k(n+1)}\\
=&\sum_{j=0}^{n-i}\left[\begin{matrix} q^{-n}, & d/a, & bd/ce \\ q, & d, & qbd/ace \end{matrix}\,\middle|\, q\right]_{i+j} \frac{(q^{-i-j}ace/bd;q)_n(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i}q^{(i+j)(n+1)}\\
=&(ace/bd;q)_n\left[\begin{matrix} q^{-n},\ bd/ce \\ d,\ q^{1-n}bd/ace \end{matrix}\,\middle|\, q\right]_i\left(\frac{d}{a}\right)^i\sum_{j=0}^{n-i}\left[\begin{matrix} q^{i-n},\ d/a,\ q^ibd/ce \\ q,\ q^id,\ q^{1+i-n}bd/ace \end{matrix}\,\middle|\, q\right]_j q^j.
\end{aligned}$$

The last sum with respect to j can be evaluated by means of the q-Saalschütz formula as follows:

$${}_3\phi_2\left[\begin{matrix} q^{i-n}, & d/a, & q^ibd/ce \\ & q^id, & q^{1+i-n}bd/ace \end{matrix}\,\middle|\, q;\ q\right] = \left[\begin{matrix} q^ia, & ce/b \\ q^id, & ace/bd \end{matrix}\,\middle|\, q\right]_{n-i}.$$

Substituting this result into the double sum expression of Ξ and then applying transformation

$$\frac{(ce/b;q)_{n-i}}{(ace/bd;q)_{n-i}} = \frac{(ce/b;q)_n}{(ace/bd;q)_n}\frac{(q^{1-n}bd/ace;q)_i}{(q^{1-n}b/ce;q)_i}\left(\frac{a}{d}\right)^i$$

we reduce the double sum to a single ${}_3\phi_2$-series:

$$\begin{aligned}
\Xi &= \sum_{i=0}^{n}\left[\begin{matrix} a,\ b/c,\ b/e \\ q,\ b,\ bd/ce \end{matrix}\,\middle|\, q\right]_i\left(\frac{qd}{a}\right)^i\left[\begin{matrix} q^{-n},\ bd/ce \\ d,\ q^{1-n}bd/ace \end{matrix}\,\middle|\, q\right]_i\\
&\times\ (ace/bd;q)_n\left[\begin{matrix} q^ia, & ce/b \\ q^id, & ace/bd \end{matrix}\,\middle|\, q\right]_{n-i}\\
&= (ce/b;q)_n\frac{(a;q)_n}{(d;q)_n}\sum_{i=0}^{n}q^i\left[\begin{matrix} b/c,\ b/e,\ q^{-n} \\ q,\ b,\ q^{1-n}b/ce \end{matrix}\,\middle|\, q\right]_i.
\end{aligned}$$

Evaluating the last sum with respect to i through the q-Saalschütz formula

$${}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c, & b/e \\ & b, & q^{1-n}b/ce \end{matrix}\,\middle|\, q;q\right] = \left[\begin{matrix} c, & e \\ b, & ce/b \end{matrix}\,\middle|\, q\right]_n$$

which is equivalent to

$$\Xi = \left[\begin{matrix} a,\ c,\ e \\ b,\ d \end{matrix}\,\middle|\, q\right]_n.$$

This completes the proof of (E6.2a-E6.2b). □

E6.4. The q-Kummer-Thomae-Whipple's formulae. As the limiting cases $n \to \infty$ of Sears' transformations, we have the non-terminating

q-Kummer-Thomae-Whipple's formulae:

$$\begin{aligned}{}_3\phi_2\left[\begin{matrix}a,\, c,\, e\\ b,\, d\end{matrix}\,\middle|\, q;\, \frac{bd}{ace}\right] &= {}_3\phi_2\left[\begin{matrix}a, & b/c, & b/e\\ & b, & bd/ce\end{matrix}\,\middle|\, q;\, \frac{d}{a}\right]\times\frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty}\\ &= {}_3\phi_2\left[\begin{matrix}b/c,\, d/c,\, bd/ace\\ bd/ac,\, bd/ce\end{matrix}\,\middle|\, q;\, c\right]\times\frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty}.\end{aligned}$$

Other transformations on terminating series derived from Sears' transformation may be displayed as follows:

$$\begin{aligned}{}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q; q\right] &\overset{a\rightleftharpoons e}{\underset{e\to 0}{=\!=}} {}_3\phi_2\left[\begin{matrix}q^{-n}, & b/a,\, b/c\\ & b,\, bd/ac\end{matrix}\,\middle|\, q; q\right]\frac{(bd/ac; q)_n}{(d; q)_n}\left(\frac{qc}{b}\right)^n\\ {}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q; q\right] &\underset{e\to 0}{=\!=} {}_3\phi_2\left[\begin{matrix}q^{-n},\, a, & b/c\\ & b,\, q^{1-n}a/d\end{matrix}\,\middle|\, q;\, \frac{qc}{d}\right]\frac{(d/a; q)_n}{(d; q)_n}a^n\\ {}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q; q\right] &\underset{e\to 0}{=\!=} {}_3\phi_2\left[\begin{matrix}q^{-n}, & b/c, d/c\\ & q^{1-n}/c, bd/ac\end{matrix}\,\middle|\, q;\, \frac{q}{a}\right]\frac{[c, bd/ac; q]_n}{[b, d; q]_n}a^n.\end{aligned}$$

$$\begin{aligned}{}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\overset{a\rightleftharpoons e}{\underset{e\to\infty}{=\!=}} {}_3\phi_2\left[\begin{matrix}q^{-n}, & b/a,\, b/c\\ & b,\, bd/ac\end{matrix}\,\middle|\, q; q^n d\right]\frac{(bd/ac; q)_n}{(d; q)_n}\\ {}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to\infty}{=\!=} {}_3\phi_2\left[\begin{matrix}q^{-n}, & a, & b/c\\ & b, & q^{1-n}a/d\end{matrix}\,\middle|\, q; q\right]\frac{(d/a; q)_n}{(d; q)_n}\\ {}_3\phi_2\left[\begin{matrix}q^{-n},\, a,\, c\\ b,\, d\end{matrix}\,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to\infty}{=\!=} {}_3\phi_2\left[\begin{matrix}q^{-n}, & b/c,\, d/c\\ & q^{1-n}/c,\, bd/ac\end{matrix}\,\middle|\, q; q\right]\frac{[c, bd/ac; q]_n}{[b, d; q]_n}.\end{aligned}$$

E7. Watson's q-Whipple transformation

E7.1. The Watson transformation. One of the most important basic hypergeometric transformations reads as

$$ {}_8\phi_7\left[\begin{matrix}a,\, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix}\,\middle|\, q;\, \frac{q^{2+n}a^2}{bcde}\right] \tag{E7.1a}$$

$$= \left[\begin{matrix}qa, qa/bc\\ qa/b, qa/c\end{matrix}\,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix}q^{-n}, & b, & c, & qa/de\\ & qa/d, & qa/e, & q^{-n}bc/a\end{matrix}\,\middle|\, q; q\right]. \tag{E7.1b}$$

Proof. In view of the definition of q-hypergeometric series, we can write (E7.1a) explicitly as

$$\begin{aligned}\mathrm{Eq(E7.1a)} &= \sum_{k=0}^{n}\left[\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-n}\\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ aq^{n+1}\end{matrix}\,\Big|\, q\right]_k\left(\frac{q^{1+n}a}{bc}\right)^k\\ &\times\left[\begin{matrix}d,\ e\\ qa/d,\ qa/e\end{matrix}\,\Big|\, q\right]_k\left(\frac{qa}{de}\right)^k.\end{aligned}$$

Recalling the q-Paff-Saalschütz theorem, we have

$$\begin{aligned}\left[\begin{matrix}d,\ e\\ qa/d,\ qa/e\end{matrix}\,\Big|\, q\right]_k\left(\frac{qa}{de}\right)^k &= {}_3\phi_2\left[\begin{matrix}q^{-k},\ q^ka,\ qa/de\\ qa/d,\ qa/e\end{matrix}\,\Big|\, q\,;\,q\right]\\ &= \sum_{i=0}^{k}\left[\begin{matrix}q^{-k},\ q^ka,\ qa/de\\ q,\ qa/d,\ qa/e\end{matrix}\,\Big|\, q\right]_i q^i.\end{aligned}$$

Therefore substituting this result into Eq(E7.1a) and changing the order of the double sum, we obtain

$$\begin{aligned}\mathrm{Eq(E7.1a)} &= \sum_{i=0}^{n}\left[\begin{matrix}qa/de\\ q,\ qa/d,\ qa/e\end{matrix}\,\Big|\, q\right]_i q^i\\ &\times \sum_{k=i}^{n}\frac{1-q^{2k}a}{1-a}\left[\begin{matrix}b,\ c,\ q^{-n}\\ qa/b,\ qa/c,\ q^{n+1}a\end{matrix}\,\Big|\, q\right]_k\\ &\times \frac{(a;\,q)_{k+i}(q^{-k};\,q)_i}{(q;\,q)_k}\left(\frac{q^{1+n}a}{bc}\right)^k.\end{aligned}$$

Indicate with Ω the inner sum with respect to k. Putting $k-i=j$ and observing that

$$\frac{(q^{-i-j};q)_i}{(q;q)_{i+j}} = \frac{(-1)^i q^{-\binom{i+1}{2}-ij}}{(q;q)_j}$$

we have

$$\begin{aligned}\Omega &= (-1)^i\, q^{-\binom{i+1}{2}}\,(a;q)_{2i}\,\frac{1-q^{2i}a}{1-a}\left[\begin{matrix}b,\ c,\ q^{-n}\\ qa/b,\ qa/c,\ q^{n+1}a\end{matrix}\,\Big|\, q\right]_i\left(\frac{q^{1+n}\,a}{b\,c}\right)^i\\ &\times\sum_{j=0}^{n-i}\frac{1-q^{2i+2j}a}{1-q^{2i}a}\left[\begin{matrix}q^{2i}a,\ q^ib,\ q^ic,\ q^{-n+i}\\ q,\ q^{1+i}a/b,\ q^{1+i}a/c,\ q^{1+n+i}a\end{matrix}\,\Big|\, q\right]_j\left(\frac{q^{1+n-i}a}{bc}\right)^j\\ &= (-1)^i\, q^{-\binom{i+1}{2}}\,(qa;q)_{2i}\left[\begin{matrix}b,\ c,\ q^{-n}\\ qa/b,\ qa/c,\ q^{n+1}a\end{matrix}\,\Big|\, q\right]_i\left(\frac{q^{1+n}\,a}{b\,c}\right)^i\\ &\times\ {}_6\phi_5\left[\begin{matrix}q^{2i}a,\ q^{1+i}\sqrt{a},\ -q^{1+i}\sqrt{a},\ q^ib,\ q^ic,\ q^{-n+i}\\ q^i\sqrt{a},\ -q^i\sqrt{a},\ q^{1+i}a/b,\ q^{1+i}a/c,\ q^{1+n+i}a\end{matrix}\,\Big|\, q;\,\frac{q^{1+n-i}a}{bc}\right].\end{aligned}$$

Evaluating the last series by the terminating q-Dougall-Dixon formula (E5.1), we obtain

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} q^{2i}a,\ q^{1+i}\sqrt{a},\ -q^{1+i}\sqrt{a}, & q^i b, & q^i c, & q^{-n+i} \\ q^i\sqrt{a}, \quad -q^i\sqrt{a}, & q^{i+1}a/b, & q^{i+1}a/c, & q^{1+n+i}a \end{matrix} \,\middle|\, q;\ \frac{q^{1+n-i}a}{bc}\right] \\
&= \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix} \,\middle|\, q\right]_{n-i}
\end{aligned}$$

which implies the following:

$$\begin{aligned}
\mathrm{Eq(E7.1a)} &= \sum_{i=0}^{n}\left[\begin{matrix} qa/de \\ q,\ qa/d,\ qa/e \end{matrix} \,\middle|\, q\right]_i q^i \times \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix} \,\middle|\, q\right]_{n-i} \\
&\times \quad (-1)^i q^{-\binom{i+1}{2}} (qa;q)_{2i} \left[\begin{matrix} b, & c, & q^{-n} \\ qa/b, & qa/c, & q^{n+1}a \end{matrix} \,\middle|\, q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i.
\end{aligned}$$

Noting that for the shifted factorials, there hold relations:

$$\begin{aligned}
(qa/bc;q)_{n-i} &= (-1)^i q^{\binom{i}{2}-ni}\left(\frac{bc}{a}\right)^i \frac{(qa/bc;q)_n}{(q^{-n}bc/a;q)_i} \\
(qa;q)_{2i}\frac{(q^{1+2i}a;q)_{n-i}}{(q^{1+n}a;q)_i} &= (qa;q)_n.
\end{aligned}$$

Consequently, we have the following expression

$$\begin{aligned}
\mathrm{Eq(E7.1a)} &= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \,\middle|\, q\right]_n \sum_{i=0}^{n}\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ q, & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \,\middle|\, q\right]_i q^i \\
&= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \,\middle|\, q;q\right]
\end{aligned}$$

which is exactly (E7.1b). □

E7.2. Rogers-Ramanujan identities. In view of $|q| < 1$ and

$$x \to \infty \qquad \Longrightarrow \qquad (x;\ q)_k \sim (-1)^k q^{\binom{k}{2}} x^k$$

the limiting case $b, c, d, e,\ n \to \infty$ of the Watson transformation reads as:

$$\sum_{m=0}^{\infty}\frac{q^{m^2}a^m}{(q;\ q)_m} = \frac{1}{(qa;\ q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,\frac{1-q^{2k}a}{1-a}\,\frac{(a;\ q)_k}{(q;\ q)_k}q^{5\binom{k}{2}+2k}a^{2k}. \tag{E7.2}$$

This transformation can provide us an alternative demonstration of the well-known Rogers-Ramanujan identities (D3.2a) and (D3.2b):

$$\begin{aligned}
\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\ q)_m} &= \frac{1}{(q;\ q^5)_\infty\,(q^4;\ q^5)_\infty} \\
\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\ q)_m} &= \frac{1}{(q^2;\ q^5)_\infty\,(q^3;\ q^5)_\infty}.
\end{aligned}$$

In fact, observe first that

$$\frac{1-q^{2k}a}{1-a}\frac{(a;\,q)_k}{(q;\,q)_k} = \frac{1-q^{2k}a}{1-q^ka}\frac{(qa;\,q)_k}{(q;\,q)_k} \xlongequal{a\to1} \begin{cases}1, & k=0\\ 1+q^k, & k>0.\end{cases}$$

Then letting $a\to1$, we can restate the transformation (E7.2) as

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k(1+q^k)q^{5\binom{k}{2}+2k}\Big\}$$

$$= \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+3k}\Big\}$$

$$= \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-2k}\Big\}.$$

Applying the Jacobi-triple product identity, we therefore establish the first Rogers-Ramanujan identity:

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+2k} = \frac{[q^5,q^2,q^3;q^5]}{(q;\,q)_\infty}.$$

Letting $a\to q$ instead, we can write the transformation (E7.2) as

$$\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=0}^{\infty}(-1)^k(1-q^{1+2k})q^{5\binom{k}{2}+4k}$$

$$= \frac{1}{(q;\,q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=0}^{\infty}(-1)^{k+1}q^{5\binom{k}{2}+6k+1}\Big\}$$

$$= \frac{1}{(q;\,q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-4k}\Big\}$$

where the last line is justified by $k\to k-1$ in the second sum. It leads us to the second Rogers-Ramanujan identity

$$\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+4k} = \frac{[q^5,q^4,q;q^5]}{(q;\,q)_\infty}$$

thanks again to the Jacobi-triple product identity. □

E7.3. Jackson's q-Dougall-Dixon formula.

$${}_8\phi_7\left[\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ \sqrt{a},\ -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix}\,\Big|\,q;q\right] \qquad \text{(E7.3a)}$$

$$= \left[\begin{matrix}qa,qa/bc,qa/bd,qa/cd\\ qa/b,qa/c,qa/d,qa/bcd\end{matrix}\,\Big|\,q\right]_n, \text{ where } \ q^{n+1}a^2=bcde. \qquad \text{(E7.3b)}$$

PROOF. When $q^{1+n}a^2 = bcde$ or equivalently $qa/de = q^{-n}bc/a$, the ${}_4\phi_3$-series in the Watson transformation reduces to a balanced ${}_3\phi_2$-series. Therefore, we have in this case the simplified form:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix}\,\middle|\, q; q\right] \\
&= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix}\,\middle|\, q\right]_n {}_3\phi_2\left[\begin{matrix} q^{-n}, & b, & c \\ & qa/d, & qa/e \end{matrix}\,\middle|\, q\,;\, q\right].
\end{aligned}$$

Evaluating the balanced series by the q-Pfaff-Saalschütz theorem, we have

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix}\,\middle|\, q;\, q\right] \\
&= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix}\,\middle|\, q\right]_n \times \left[\begin{matrix} qa/bd, & qa/cd \\ qa/d, & qa/bcd \end{matrix}\,\middle|\, q\right]_n
\end{aligned}$$

which is essentially the same as Jackson's q-Dougall-Dixon formula. $\square$

E7.4. The non-terminating ${}_6\phi_5$-summation formula.

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix}\,\middle|\, q;\, \frac{qa}{bcd}\right] && \text{(E7.4a)} \\
&= \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd \end{matrix}\,\middle|\, q\right]_\infty, \qquad \left|\frac{qa}{bcd}\right| < 1. && \text{(E7.4b)}
\end{aligned}$$

PROOF. Substituting $e = q^{1+n}a^2/bcd$ in the Jackson's q-Dougall-Dixon formula explicitly, we have

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & q^{1+n}a^2/bcd, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & q^{-n}bcd/a, & q^{n+1}a \end{matrix}\,\middle|\, q;\, q\right] \\
&= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix}\,\middle|\, q\right]_n.
\end{aligned}$$

For $n \to \infty$, recalling the limit relations

$$\frac{(q^{1+n}a^2/bcd;\, q)_k}{(q^{1+n}a;\, q)_k} \sim 1 \quad \text{and} \quad \frac{(q^{-n};\, q)_k}{(q^{-n}bcd/a;\, q)_k} \sim \Big(\frac{a}{bcd}\Big)^k$$

and then applying the Tannery limiting theorem, we get the non-terminating q-Dougall-Dixon formula (E7.4a-E7.4b). $\square$

We remark that when $d = q^{-n}$, the formula (E7.4a-E7.4b) reduces to the terminating q-Dougall-Dixon summation identity (E5.1).