CHAPTER D

The Carlitz Inversions and
Rogers-Ramanujan Identities

According to the Jacobi triple product identity, we have

\[
[q^4, \pm q, \pm q^3; q^4] = \sum_{k=-\infty}^{+\infty} (\mp 1)^k q^{2k^2+k}.
\]

The sum of both triple products can be evaluated as a single triple product:

\[
[q^4, -q, -q^3; q^4] + [q^4, q, q^3; q^4] = 2 \sum_{n=\infty}^{-\infty} q^{8n^2+2n} = 2 \sum_{n=\infty}^{-\infty} q^{16n^2+10n} = 2[q^{16}, -q^6, -q^{10}; q^{16}]_{\infty}.
\]

We can similarly treat their difference as follows:

\[
[q^4, -q, -q^3; q^4] - [q^4, q, q^3; q^4] = 2 \sum_{n=\infty}^{-\infty} q^{8n^2-6n+1} = 2 \sum_{n=\infty}^{-\infty} q^{16n^2+2n+1} = 2q[q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}.
\]

Dividing both equations by \( (q^4; q^4)_{\infty} \) and noting the fact that the odd natural numbers are congruent to 1 or to 3 modulo 4, we get two \( q \)-difference equations:

\[
(-q; q^2)^{\infty} + (q; q^2)^{\infty} = \frac{2}{(q^4; q^4)_{\infty}} \sum_{n} q^{8n^2+2n} \tag{D0.1a}
\]

\[
= 2\frac{[q^{16}, -q^6, -q^{10}; q^{16}]}{(q^4; q^4)_{\infty}} \tag{D0.1b}
\]

\[
(-q; q^2)^{\infty} - (q; q^2)^{\infty} = \frac{2q}{(q^4; q^4)_{\infty}} \sum_{n} q^{8n^2-6n} \tag{D0.2a}
\]

\[
= 2q\frac{[q^{16}, -q^2, -q^{14}; q^{16}]}{(q^4; q^4)_{\infty}} \tag{D0.2b}
\]
Further, if we specify with $x \mapsto \pm q^{1/2}$ in Euler’s $q$-difference formula
\[ (x; q)_\infty = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(q; q)_m} q^{\frac{m}{2}} \]
then we find that
\[ (\pm q^{1/2}; q)_\infty = \sum_{n=0}^{\infty} \frac{(\pm 1)^n q^{n^2/2}}{(q; q)_n} \]
whose linear combinations lead us to two summation formulae as follows:
\[ (-q^{1/2}; q)_\infty + (q^{1/2}; q)_\infty = 2 \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}} \]
\[ (-q^{1/2}; q)_\infty - (q^{1/2}; q)_\infty = 2q^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n(2n+1)}}{(q; q)_{2n+1}}. \]

Replacing the base $q$ by $q^{1/2}$ in (D0.1a-D0.1b) and (D0.2a-D0.2b), we can reformulate the left hand sides of both equations just displayed respectively as follows:
\[ (-q^{1/2}; q)_\infty + (q^{1/2}; q)_\infty = 2 \frac{[q^8, -q^5, -q^6; q^8]}{(q^2; q^2)_\infty} \]
\[ (-q^{1/2}; q)_\infty - (q^{1/2}; q)_\infty = 2q^{1/2} \frac{[q^8, -q, -q^7; q^8]}{(q^2; q^2)_\infty}. \]

Combining (D0.3a) and (D0.3b) respectively with (D0.4a) and (D0.4b), we establish two infinite series identities:
\[ \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}} = \left[ q^8, -q^5, -q^6; q^8 \right] \] (D0.5a)
\[ \sum_{n=0}^{\infty} \frac{q^{2n(2n+1)}}{(q; q)_{2n+1}} = \left[ q^8, -q, -q^7; q^8 \right] \] (D0.5b)

They are only very simple examples of classical partition identities of Roger-Ramanujan’s type. By means of inverse series relations, we establish a finite series transformation, which leads us to an elementary derivation to the celebrated Rogers-Ramanujan identities and their finite forms.
D1. Combinatorial inversions and series transformations

D1.1. The Carlitz inversions. Let \( \{a_i\} \) and \( \{b_j\} \) be two complex sequences such that the polynomials defined by

\[
\phi(x; 0) = 1 \quad \text{and} \quad \phi(x; n) = \prod_{k=0}^{n-1} (a_k + xb_k), \quad \text{for } n = 1, 2, \ldots
\]

differ from zero for \( x = q^n \) with \( n \) being non-negative integers. Then we have the following inverse series relations due to Carlitz (1973)

\[
\begin{align*}
F(n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^k \phi(q^k; n) G(k), \quad n = 0, 1, 2, \cdots \quad (D1.1a) \\
G(n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + q^k b_k}{\phi(q^n; k+1)} F(k), \quad n = 0, 1, 2, \cdots \quad (D1.1b)
\end{align*}
\]

which may be considered as \( q \)-analogue of Gould-Hsu Inversions (1973).

**Proof.** To prove the bilateral implications \((D1.1a) \iff (D1.1b)\), it is sufficient to verify one implication because one system of equations with \( F(n) \) in terms of \( G(k) \) can be considered as the (unique) solution of another system with \( G(n) \) in terms of \( F(k) \), and vice versa.

\( \iff \) We first reproduce the original proof due to Carlitz. Suppose that the relations of \( G(n) \) in terms of \( F(k) \) are valid. We have to verify the relations of \( F(n) \) in terms of \( G(k) \).

Substituting the relations of \( G(n) \) in terms of \( F(k) \) into the right hand sides of those of \( F(n) \) in terms of \( G(k) \) and observing that

\[
\binom{n}{k} \times \binom{k}{i} = \binom{n}{i} \times \binom{n-i}{k-i}
\]
we get the double sum
\[
\sum_{k=0}^{n} (-1)^k q^{(n-k)} \binom{n}{k} \phi(q^k; n) G(k)
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(n-k)} \phi(q^k; n) \sum_{i=0}^{k} \binom{k}{i} \frac{\alpha_i + q^i b_i}{\phi(q^k; i + 1)} F(i)
\]
\[
= \sum_{i=0}^{n} (a_i + q^i b_i) \binom{n}{i} F(i) \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{\phi(q^k; n)}{\phi(q^{n-i}; k + 1)} q^{(n-k)}
\]
\[
= \sum_{i=0}^{n} (a_i + q^i b_i) \binom{n}{i} F(i) \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell} \frac{\phi(q^{n-i}; \ell)}{\phi(q^{n-i}; i + 1)} q^{n-\ell-i} .
\]
Let \(S(i, n)\) stand for the inner sum with respect to \(\ell\):
\[
S(i, n) := \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell} q^{(n-i-\ell)} \frac{\phi(q^{n-i}; \ell)}{\phi(q^{n-i}; i + 1)} .
\]
It is trivial to see that
\[
S(n, n) = \frac{\phi(q^n; n)}{\phi(q^n; n + 1)} = \frac{1}{a_n + q^n b_n}
\]
which implies that the double sum reduces to \(F(n)\) when \(i = n\).

In order to prove that the double sum is equal to \(F(n)\), it suffices for us to verify that \(S(i, n) = 0\) for \(0 \leq i < n\).

Noting that \(\frac{\phi(q^{n-i}; n)}{\phi(q^{n-i}; i + 1)}\) is a polynomial of degree \(n - i - 1\) in \(q^\ell\), we can write it formally as
\[
\frac{\phi(q^{n-i}; n)}{\phi(q^{n-i}; i + 1)} = \sum_{j=0}^{n-i-1} C_j q^{(n-i-j-1)}
\]
where \(\{C_j\}\) are constants independent of \(\ell\). Therefore the sum \(S(i, n)\) can be reformulated accordingly as follows:
\[
S(i, n) = \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell} q^{(n-i-\ell)} \sum_{j=0}^{n-i-1} C_j q^{(n-i-j-1)}
\]
\[
= \sum_{j=0}^{n-i-1} C_j q^{(n-i-j)} \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell} q^{-\ell j}
\]
where we have applied the binomial relation
\[
\binom{n-i-\ell}{2} = \binom{n-i}{2} + \binom{\ell}{2} - \ell(n-i-1).
\]
Evaluating the sum with respect to $\ell$ by Euler’s $q$-difference formula (B5.3)
\[
\sum_{\ell=0}^{n-i} (-1)^{\ell} \binom{n-i}{\ell} q^{(j)\ell} = (q^{-j}; q)_{n-i}
\]
which vanishes for $0 \leq j < n - i$.

This completes the proof of the Carlitz inversions stated in D1.1. □

An alternative proof is worth to be included. Assuming that (D1.1a) is true for all $n \in \mathbb{N}_0$, we should verify the truth of (D1.1b).

In fact, substituting the first relation into the second, we reduce the question to the confirmation of the following orthogonal relation:
\[
\sum_{k=i}^{n} (-1)^{k+i} \left\{ a_k + q^k b_k \right\} \binom{n-i}{k-i} \phi(q^i; k) k^{-i} q^{(k-i)} = \begin{cases} 1, & i = n \\ 0, & i \neq n. \end{cases} \quad (D1.2)
\]

It is obvious that the relation is valid for $i = n$. We therefore need to verify it only when $i < n$. For that purpose, we introduce the sequence
\[
\tau_k := \binom{n-i-1}{k-i-1} \left( \phi(q^i; k) k^{-i} q^{(k-i)} \phi(q^n; k+1) \right).
\]

Then it is not hard to check that the summand in (D1.2) can be expressed as follows:
\[
\tau_k + \tau_{k+1} = \left\{ a_k + q^k b_k \right\} \binom{n-i}{k-i} \phi(q^i; k) k^{-i} q^{(k-i)}.
\]

Separating the two extreme terms indexed with $k = i$ and $k = n$ from the sum displayed in (D1.2)
\[
\tau_{i+1} = \frac{\phi(q^i; i+1)}{\phi(q^n; i+1)} \\
\tau_n = \frac{\phi(q^n; n)}{\phi(q^n; n)} q^{(n-i)}
\]

and then appealing for the telescoping method, we find that
\[
\text{LHS}(D1.2) = \tau_{i+1} + (-1)^{n+i} \tau_n + \sum_{i < k < n} (-1)^{k+i} \left\{ \tau_k + \tau_{k+1} \right\}
\]
\[
= \left\{ \tau_{i+1} + (-1)^{n+i} \tau_n \right\} - \left\{ \tau_{i+1} + (-1)^{n+i} \tau_n \right\} = 0.
\]

This completes the proof of (D1.2).
D1.2. Series transformation. For the polynomials \( \phi(x; n) = (\lambda x; q)_n \) specified with \( a_k = 1 \) and \( b_k = -q^k \lambda \), the inverse series relations displayed in D1.1 become the following:

\[
f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(n-k)} (q^k \lambda; q)_n g(k), \quad n = 0, 1, 2, \ldots \quad (D1.3a)
\]

\[
g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - q^{2k} \lambda}{(q^n \lambda; q)_{k+1}} f(k), \quad n = 0, 1, 2, \ldots \quad (D1.3b)
\]

By means of the finite version of Kummer’s theorem and rearrangement of double sums, we may establish finite and infinite series transformations

\[
\sum_{n=0}^{m} \frac{\lambda^n q^{n^2}}{(\lambda; q)_n} g(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1 - q^{2k} \lambda}{(q^m \lambda; q)_{k+1}} \lambda^k q^{k^2} f(k) \quad (D1.4a)
\]

\[
\sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q; q)_n (\lambda; q)_n} g(n) = \sum_{k=0}^{\infty} (-1)^k \frac{1 - q^{2k} \lambda}{(\lambda; q)_{\infty}} \lambda^k q^{k^2} f(k). \quad (D1.4b)
\]

**Proof.** By means of (D1.3b), we can express the left member of (D1.4a) as the following double sum

\[
\text{LHS}(D1.4a) = \sum_{n=0}^{m} \left[ \frac{\lambda^n q^{n^2}}{(\lambda; q)_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - q^{2k} \lambda}{(q^n \lambda; q)_{k+1}} f(k) \right]
\]

\[
= \sum_{k=0}^{m} (-1)^k (1 - q^{2k} \lambda) \binom{m}{k} f(k) \sum_{n=k}^{m} \frac{\lambda^n q^{n^2}}{(\lambda; q)_{n+k+1}}
\]

\[
= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1 - q^{2k} \lambda}{(\lambda; q)_{2k+1}} \lambda^k q^{k^2} f(k) \sum_{j=0}^{m-k} \frac{\lambda^j q^{j(2k)}}{(q^{2k+1} \lambda; q)_j}
\]

where we have applied relations on shifted factorials

\[
(\lambda; q)_{n+k+1} = (\lambda; q)_n (q^n \lambda; q)_k + 1 = (\lambda; q)_{2k+1} (q^{2k+1} \lambda; q)_{n-k} \quad (D1.5)
\]

and the substitution \( j := n - k \) on summation indices.

In view of the finite version of Kummer’s theorem stated in Corollary C1.2

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{x^k q^{k^2}}{(q x; q)_n} = \frac{1}{(q x; q)_n}
\]

we can evaluate the inner sum as the following closed form:

\[
\sum_{j=0}^{m-k} \binom{m-k}{j} \frac{\lambda^j q^{j(2k)}}{(q^{2k+1} \lambda; q)_j} = \frac{1}{(q^{2k+1} \lambda; q)_{m-k}}.
\]
Recalling (D1.5), we derive finally the following

$$\text{LHS(D1.4a)} = \sum_{k=0}^{m} (-1)^k \left[ \frac{m}{k} \right] \frac{1 - q^{2k} \lambda}{(\lambda; q)_{2k+1}} \frac{\lambda^k q^{2k}}{(q^{2k+1} \lambda; q)_{m-k}} f(k)$$

$$= \sum_{k=0}^{m} (-1)^k \left\{ -q^{2k} \lambda \right\} \left[ \frac{m}{k} \right] \frac{\lambda^k q^{2k}}{(\lambda; q)_{m+k+1}} f(k)$$

which is the first identity (D1.4a).

The second identity (D1.4b) follows from the limit $m \to \infty$ of (D1.4a). □

**D2. Finite $q$-differences and further transformation**

On account of the inverse series relations

$$\frac{q(n)}{x^n} = \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{(n-k)} x^k \quad (D2.1a)$$

$$q^{(n-k)} x^n = \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{(n-k)}(x; q)_k \quad (D2.1b)$$

we may determine, as an example of (D1.3a-D1.3b), two sequences as follows:

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}}(\lambda; q)_n \quad \Rightarrow \quad g(n) = (\lambda; q)_n.$$

They may be used to reformulate the finite series transformation (D1.4a) explicitly

$$\sum_{n=0}^{m} \left[ \frac{m}{n} \right] \lambda^n q^{n^2} = \sum_{k=0}^{m} (-1)^k \left[ \frac{m}{k} \right] \frac{1 - q^{2k} \lambda}{(q^k \lambda; q)_{m+1}} \lambda^{2k} q^{2k^2+\binom{k}{2}}. \quad (D2.2)$$

**Proof.** The first relation (D2.1a) is a restatement of Euler’s $q$-finite difference formula (B5.3). Specifying the Carlitz inversions stated in D1.1 with

$$\phi(x; n) = 1, \quad f(n) = x^n q^{(n)}(x; q)_n, \quad g(n) = (x; q)_n$$

we get the second relation (D2.1b) which is dual to the first one.

In order to verify that two sequences

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}}(\lambda; q)_n \quad \Rightarrow \quad g(n) = (\lambda; q)_n$$
satisfy (D1.3a-D1.3b), it is sufficient to show that
\[ \lambda^n q^{n^2 + \binom{n}{2}}(\lambda; q)_n = \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{\binom{n-k}{2}}(\lambda; q)k(q^k \lambda; q)_n \]
\[ = \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{\binom{n-k}{2}}(\lambda; q)_{n+k} \]
in view of the inverse series relations specified with \( \phi(x; n) = (\lambda x; q)_n \).

Applying (D2.1b) with \( x = q^n \lambda \), we confirm the last summation identity:
\[ \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{\binom{n-k}{2}}(\lambda; q)_{n+k} = (\lambda; q)_n \sum_{k=0}^{n} (-1)^k \left[ \frac{n}{k} \right] q^{\binom{n-k}{2}}(q^n \lambda; q)_k \]
\[ = (\lambda; q)_n q^{n^2 + \binom{n}{2}} \lambda^n. \]

The transformation (D2.2) follows from (D1.4a) with the \{f(k), g(n)\} sequences just displayed explicitly.

\[ \square \]

**D3. Rogers-Ramanujan identities and their finite forms**

**D3.1. Proposition.** With the specifications \( \lambda \mapsto 1 \) and \( \lambda \mapsto q \) in (D2.2), the finite forms of Rogers-Ramanujan identities can be derived as follows:

\[ \sum_{n=0}^{m} \left[ \frac{m}{n} \right] q^{n^2 + \binom{n}{2}} = \frac{(q; q)_m}{(q; q)_{2m}} \sum_{k=-m}^{m} (-1)^k \left[ \frac{2m}{m+k} \right] q^{\binom{n}{2} + 2k^2} \]  
(D3.1a)

\[ \sum_{n=0}^{m} \left[ \frac{m}{n} \right] q^{n^2 + n} = \frac{(q; q)_m}{(q; q)_{2m+1}} \sum_{k=-m}^{m+1} (-1)^k \left[ \frac{2m+1}{m+k} \right] q^{\binom{n}{2} + 2k^2 - k}. \]  
(D3.1b)

**Proof.** Separating the first term from (D2.2), we have
\[ \sum_{n=0}^{m} \left[ \frac{m}{n} \right] \lambda^n q^{n^2} = \frac{1 - \lambda}{(\lambda; q)_{m+1}} + \sum_{k=1}^{m} (-1)^k \left[ \frac{m}{k} \right] \frac{1 - q^{2k} \lambda}{(q^k \lambda; q)_{m+1}} \lambda^{2k} q^{2k^2 + \binom{k}{2}}. \]
we can further reformulate the sum as

$$
\sum_{n=0}^{m} \binom{m}{n} q^{n^2} = \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{1 - q^{2k}}{(q^k; q)_{m+1}} q^{2k^2 + \binom{k}{2}}
$$

$$
= \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{1 + q^k}{(q^{k+1}; q)_m} q^{2k^2 + \binom{k}{2}}.
$$

Its limiting case $\lambda \to 1$ may be manipulated as follows:

$$
\sum_{n=0}^{m} \binom{m}{n} q^{n^2} = \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{1 + q^k}{(q^{k+1}; q)_m} q^{2k^2 + \binom{k}{2}}
$$

$$
= \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{1 + q^k}{(q^{k+1}; q)_m} q^{2k^2 + \binom{k}{2}}.
$$

In view of the definition of $q$-Gauss binomial coefficient and the relation

$$
(q; q)_{m+k} = (q; q)_k q^{k+1}; q)_m
$$

we can further reformulate the sum as

$$
\sum_{n=0}^{m} \binom{m}{n} q^{n^2} = \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{(q; q)_m}{(q; q)_{m-k}} \frac{1 + q^k}{(q; q)_{m+k}} q^{2k^2 + \binom{k}{2}}
$$

$$
= \frac{1}{(q; q)_m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{2m}{m + k} q^{2k^2 + \binom{k}{2}} + \frac{(q; q)_m}{(q; q)_{2m}} \sum_{k=1}^{m} (-1)^k \left[ \binom{2m}{m + k} q^{2k^2 + \binom{k+1}{2}} \right].
$$

Performing the replacement $k \to -k$ in the last sum and noting that

$$
\binom{2m}{m - k} = \binom{2m}{m + k}
$$

we can combine the last three expressions as a single one:

$$
\sum_{n=0}^{m} \binom{m}{n} q^{n^2} = \sum_{k=-m}^{m} (-1)^k \binom{2m}{m + k} q^{2k^2 + \binom{k}{2}}
$$

which is the finite form of the first Rogers-Ramanujan identity (D3.1a).

Similarly, specifying (D2.2) with $\lambda \to q$, we have

$$
\sum_{n=0}^{m} \binom{m}{n} q^{n+n^2} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1 - q^{2k+1}}{(q^{k+1}; q)_{m+1}} q^{2k^2 + \binom{k}{2} + 2k}
$$

$$
= \frac{(q; q)_m}{(q; q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m + 1}{m - k} (1 - q^{2k+1}) q^{2k^2 + \binom{k}{2} + 2k}
$$

$$
= \frac{(q; q)_m}{(q; q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m + 1}{m - k} q^{2k^2 + \binom{k}{2} + 2k}
$$

$$
- \frac{(q; q)_m}{(q; q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m + 1}{m - k} q^{2k^2 + \binom{k}{2} + 4k + 1}.
$$
Replacing the summation index $k$ by $-1 - k$ in the second sum and then combining the result with the first one, we get the following simplified transformation

$$\sum_{n=0}^{m} \binom{m}{n} q^{n + n^2} = \frac{(q; q)_m}{(q; q)_{2m+1}} \sum_{k=-m-1}^{m} (-1)^k \left[ \frac{2m+1}{m-k} \right] q^{2k^2 + \binom{k}{2} + 2k}$$

which is equivalent to the second finite form (D3.1b) of Rogers-Ramanujan identities under parameter replacement $k \rightarrow -k$. □

**D3.2. Theorem.** Their limiting cases give rise, with the help of the Jacobi-triple product identity, to the celebrated Rogers-Ramanujan identities:

$$\frac{1}{(q; q)_\infty (q^4; q^3)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^{4+5k})(1 - q^{4+5k})} \quad \text{(D3.2a)}$$

$$\frac{1}{(q^2; q^3)_\infty (q^3; q^3)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2 + n}}{(q; q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^{2+5k})(1 - q^{3+5k})} \quad \text{(D3.2b)}$$

**Proof.** Letting $m \rightarrow \infty$, we can state (D3.1a) as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^\binom{k}{2} + 2k^2$$

$$= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^5 \binom{k}{2} + 2k.$$ 

The sum on the right hand side can be evaluated, by means of Jacobi triple product identity, as

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^5 \binom{k}{2} + 2k = [q^5, q^2, q^3, q^5]_\infty.$$ 

Therefore the first identity (D3.2a) follows consequently:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{[q^5, q^2, q^3, q^5]}{(q; q)_\infty} = \frac{1}{[q, q^4, q^3]_\infty}.$$ 

If we let $m \rightarrow \infty$ in (D3.1b), we find that

$$\sum_{n=0}^{\infty} \frac{q^{n^2 + n}}{(q; q)_n} = \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^\binom{k}{2} + 2k^2 - k$$

$$= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^5 \binom{k}{2} + k.$$
The sum on the right hand side reads as

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{5(2k)+k} = [q^5, q, q^4, q^5]_\infty$$

in view of Jacobi triple product identity.

Hence we have established the following

$$\sum_{n=0}^\infty q^{n^2+n} (q; q)_n = \frac{[q^5, q, q^4; q^5]_\infty}{(q; q)_\infty} = \frac{1}{[q^2, q^3; q^5]_\infty}$$

which is the second identity (D3.2b). □

Up to now, about ten proofs have been provided for this beautiful pair of identities. The most recent ones are, respectively, due to Baxter (1982) based on the statistical mechanics and Lepowsky-Milne (1978) through the character formula on infinite dimensional Lie algebra (Kac-Moody algebra [45, 1985]).