CHAPTER A

Partitions and Algebraic Structures

In this chapter, we introduce partitions of natural numbers and the Ferrers diagrams. The algebraic structures of partitions such as addition, multiplication and ordering will be studied.

A1. Partitions and representations

A partition is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots)$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots$$

and containing only finitely many non-zero terms.

The non-zero λ_k in λ are called the parts of λ. The number of parts of λ is the length of λ, denoted by $\ell(\lambda)$; and the sum of parts is the weight of λ, denoted by $|\lambda|$:

$$|\lambda| = \sum_{k \geq 1} \lambda_k = \lambda_1 + \lambda_2 + \cdots .$$

If $n = |\lambda|$ we say that λ is a partition of n, denoted by $n \dashv \lambda$.

The set of all partitions of n is denoted by $\mathcal{P}_n$. In particular, $\mathcal{P}_0$ consists of a single element, the unique partition of zero, which we denote by 0.

Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part:

$$\lambda = (1^{m_1} 2^{m_2} \cdots k^{m_k} \cdots)$$

means that exactly m_k copies of the parts of λ are equal to k. The number

$$m_k = m_k(\lambda) = Card\{i : k = \lambda_i\}$$

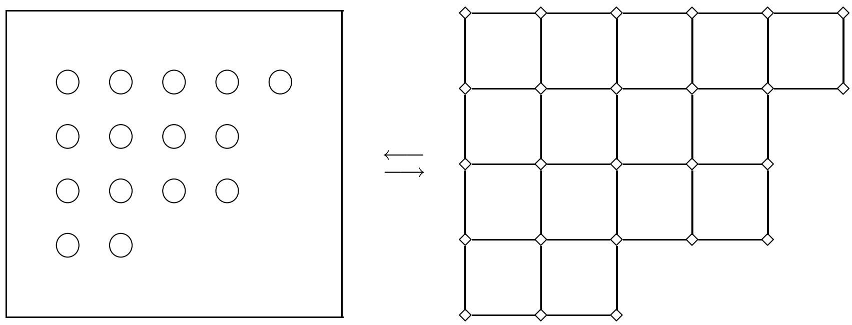
is called the multiplicity of k in λ.

A2. Ferrers diagrams of partitions

The diagram of a partition λ may be formally defined as the set of points (or unit squares)

$$\lambda \quad = \quad \Big\{(i,j) \mid 1 \leq j \leq \lambda_i, \; 1 \leq i \leq \ell(\lambda) \Big\}$$

drawn with the convention as matrices. For example, the diagram of the partition $\lambda = (5442)$ is shown as follows:



We shall usually denote the diagram of a partition λ by the same symbol. The conjugate of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ, i.e., the diagram obtained by reflection in the main diagonal. For example, the conjugate of (5442) is (44331). Hence λ'_k is the number of the nodes in the k-th column of λ, or equivalently

$$\lambda'_k = Card\big\{i : \lambda_i \geq k\big\}.$$

In particular, $\lambda'_1 = \ell(\lambda)$ and $\lambda_1 = \ell(\lambda')$. Obviously, we also have $\lambda'' = \lambda$ and $m_k = \lambda'_k - \lambda'_{k+1}$. Therefore we can dually express the Ferrers diagram of λ as

$$\lambda \;=\; \Big\{(i,j) \mid 1 \leq i \leq \lambda'_j, \; 1 \leq j \leq \ell(\lambda')\Big\}.$$

A2.1. Euler's theorem. The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n.

Proof. Let S be the set of partitions of n into distinct odd parts and T the set of self-conjugate partitions of n, the mapping

$$\begin{aligned} f \quad : \quad & S \to T \\ & \lambda \mapsto \mu \end{aligned}$$

defined by

$$\mu_i = \mu_i' := \frac{\lambda_i - 1}{2} + i \quad \text{where for all} \quad i = 1, 2, \cdots, \ell(\lambda).$$

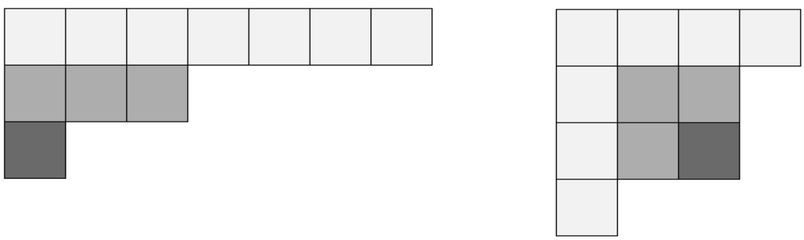
Obviously, μ is a selfconjugate partition with diagonal length equal to $\ell(\lambda)$ and the weight equal to $|\lambda|$, which can be justified as follows:

$$|\mu| + \ell^2(\lambda) = \sum_{i=1}^{\ell(\lambda)} (\mu_i + \mu_i') = |\lambda| + \sum_{i=1}^{\ell(\lambda)} (2i-1) = |\lambda| + \ell^2(\lambda).$$

From the Ferrers diagrams, we see that f is a bijection between S and T. Therefore they have the same cardinality $|S| = |T|$, which completes the proof. $\square$

For example, the image of partition $\lambda = (731)$ under f reads as $\mu = (4331)$. This can be illustrated as follows:

$$\lambda = (731) \quad \mapsto \quad \mu = (4331).$$



A2.2. Theorem on permutations. Let λ be a partition with $m \geq \lambda_1$ and $n \geq \lambda_1'$. Then the $m+n$ numbers

$$\lambda_i + n - i \quad (1 \leq i \leq n) \quad \text{and} \quad n - 1 + j - \lambda_j' \quad (1 \leq j \leq m)$$

are a permutation of $\{0, 1, 2, \cdots, m+n-1\}$.

Proof. Define three subsets of non-negative integers:

$$\begin{aligned} \mathcal{U} &: = \{\lambda_i + n - i \mid 1 \leq i \leq n\} \\ \mathcal{V} &: = \{n - 1 + j - \lambda_j' \mid 1 \leq j \leq m\} \\ \mathcal{W} &: = \{k \mid 0 \leq k \leq m+n-1\}. \end{aligned}$$

In order to prove the theorem, it suffices to show the following

(A) $\mathcal{U} \subseteq \mathcal{W}$ and $\mathcal{V} \subseteq \mathcal{W}$;
(B) The elements of $\mathcal{U}$ are distinct;
(C) The elements of $\mathcal{V}$ are distinct;
(D) $\mathcal{U} \cap \mathcal{V} = \emptyset$.

It is clearly true (A). Suppose that there exist i and j with $1 \leq i < j \leq n$ such that

$$\lambda_i + n - i \;=\; \lambda_j + n - j.$$

Keeping in mind of the partition λ, we see that it is absurd for $\lambda_i \geq \lambda_j$ and $n - i > n - j$. This proves (B). We can prove (C) similarly in view of the conjugate partition λ'. There remains only (D) to be confirmed.

Observe that the Ferrers diagram of λ is contained in the Ferrers diagram of (m^n), which is an $n \times m$ rectangle. We can identify the partition λ with the points inside its Ferrers diagram. If the point with coordinate (i, j) is inside λ, then we have $\lambda_i \geq i$ and $j \leq \lambda'_j$, which are equivalent to the inequality

$$\lambda_i - i \geq 0 \geq j - \lambda'_j \quad\Rightarrow\quad \lambda_i + n - i > n - 1 + j - \lambda'_j.$$

This means that for (i, j) inside the Ferrers diagram λ with $1 \leq i \leq n$ and $1 \leq j \leq m$, the corresponding $\lambda_i + n - i$ and $n - 1 + j - \lambda'_j$ can not be the common element in $\mathcal{U} \cap \mathcal{V}$.

Instead if the point with coordinate (i, j) lies outside λ, then we have $\lambda_i < i$ and $j > \lambda'_j$, which are equivalent to another inequality

$$\lambda_i - i < 0 \leq j - \lambda'_j - 1 \quad\Rightarrow\quad \lambda_i + n - i < n - 1 + j - \lambda'_j.$$

This implies that for (i, j) outside the Ferrers diagram λ with $1 \leq i \leq n$ and $1 \leq j \leq m$, the corresponding $\lambda_i + n - i$ and $n - 1 + j - \lambda'_j$ can not be again the common element in $\mathcal{U} \cap \mathcal{V}$.

In any case, we have verified that $\mathcal{U}$ and $\mathcal{V}$ have no common elements, which confirms (D).

The proof of Theorem A2.2 is hence completed. $\square$

A2.3. The hooklength formula. Let λ be a partition. The hooklength of λ at $(i, j) \in \lambda$ is defined to be

$$h(i, j) = 1 + \lambda_i + \lambda'_j - i - j.$$

If the diagram of λ is contained in the diagram of (m^n), define

$$\nu_k = \lambda_k + n - k \quad (1 \leq k \leq n).$$

Then the theorem on $\{m+n\}$-permutations can be used to demonstrate the following hooklength formulae:

$$\prod_{(i,j)\in\lambda} (1-q^{h(i,j)}) = \frac{\prod_{i\geq 1}\prod_{j=1}^{\nu_i}(1-q^j)}{\prod_{i<j}(1-q^{\nu_i-\nu_j})} \rightleftharpoons \prod_{(i,j)\in\lambda} h(i,j) = \frac{\prod_{i\geq 1}\nu_i!}{\prod_{i<j}(\nu_i-\nu_j)}.$$

Proof. Interchanging λ and λ' in permutation Theorem A2.2 and then putting $m = \lambda_1$ and $\lambda_1' \leq n$, we see that $m + \lambda_j' - j$ $(1 \leq j \leq m)$ and $m-1+j-\lambda_j$ $(1 \leq j \leq n)$ constitute a permutation of $\{0,1,2,\cdots,m+n-1\}$. Therefore we have a disjoint union:

$$\left\{q^{\lambda_1+\lambda_j'-j}\right\}_{j=1}^{\lambda_1} \biguplus \left\{q^{\lambda_1-1+j-\lambda_j}\right\}_{j=1}^{n} = \left\{q^j\right\}_{j=0}^{\lambda_1+n-1}.$$

According to the definition of the hooklength of λ, the identity can be reformulated as follows:

$$\left\{q^{h(1,j)}\right\}_{j=1}^{\lambda_1} \biguplus \left\{q^{\nu_1-\nu_j}\right\}_{j=2}^{n} = \left\{q^j\right\}_{j=1}^{\nu_1}.$$

Writing down this identity for the partition $(\lambda_i, \lambda_{i+1}, \cdots)$:

$$\left\{q^{h(i,j)}\right\}_{j=1}^{\lambda_i} \biguplus \left\{q^{\nu_i-\nu_j}\right\}_{j=1+i}^{n} = \left\{q^j\right\}_{j=1}^{\nu_i}$$

and then summing them over $i = 1, 2, \cdots, \ell(\lambda)$, we obtain

$$\sum_{(i,j)\in\lambda} q^{h(i,j)} + \sum_{i<j} q^{\nu_i-\nu_j} = \sum_{i\geq 1}\sum_{j=1}^{\nu_i} q^j.$$

Instead of summation, the multiplication leads us consequently to the following:

$$\prod_{(i,j)\lambda} (1-q^{h(i,j)}) = \frac{\prod_{i\geq 1}\prod_{j=1}^{\nu_i}(1-q^j)}{\prod_{i<j}(1-q^{\nu_i-\nu_j})}.$$

In particular dividing both sides by $(1-q)^{|\lambda|}$ and then setting $q = 1$, we find that

$$\prod_{(i,j)\in\lambda} h(i,j) = \frac{\prod_{i\geq 1}\nu_i!}{\prod_{i<j}(\nu_i-\nu_j)}.$$

This completes the proof of the hooklength formula. □

A3. Addition on partitions

Let λ and μ be partitions. We define $\lambda+\mu$ to be the sum of the sequences λ and μ:

$$(\lambda+\mu)_k = \lambda_k + \mu_k.$$

Also we define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ, arranged in descending order.

A3.1. Proposition. The operations $+$ and $\cup$ are dual each other

$$(\lambda \cup \mu)' = \lambda' + \mu' \quad \rightleftharpoons \quad (\lambda+\mu)' = \lambda' \cup \mu'.$$

Proof. The diagram of $\lambda\cup\mu$ is obtained by taking the rows of the diagrams of λ and μ and reassembling them in decreasing order. Hence the length of the k-th column of $\lambda \cup \mu$ is the sum of lengths of the k-th columns of λ and of μ, i.e.

$$(\lambda \cup \mu)'_k \;=\; \big|\{i|\lambda_i \geq k\}\big| \;+\; \big|\{j|\mu_j \geq k\}\big| \;=\; \lambda'_k + \mu'_k.$$

The converse follows from duality. $\square$

A3.2. Examples. For two symmetric partitions given by $\lambda = (321)$ and $\mu = (21)$, we then have

$$\lambda+\mu = (531) \quad \text{and} \quad \lambda \cup \mu = (32211).$$

Similarly, we consider a non-symmetric example. If $\lambda = (331)$ and $\mu = (21)$, then it is easy to compute $\lambda' = (322)$ and $\mu' = (21)$. Therefore

$$\lambda+\mu = (541) \quad \text{and} \quad \lambda \cup \mu = (33211)$$

and

$$\begin{aligned}(\lambda \cup \mu)' &= (532) = \lambda' + \mu' \\ (\lambda+\mu)' &= (32221) = \lambda' \cup \mu'.\end{aligned}$$

A4. Multiplication on partitions

Next, we define $\lambda \diamond \mu$ to be the component-wise product of the sequences λ and μ:

$$(\lambda \diamond \mu)_k = \lambda_k \mu_k.$$

Also we define $\lambda \times \mu$ to be the partition whose parts are $\min(\lambda_i, \mu_j)$ for all (i, j) with $1 \le i \le \ell(\lambda)$ and $1 \le j \le \ell(\mu)$, arranged in descending order.

A4.1. Proposition. For the operations "$\diamond$" and "$\times$", we have the dual relation:

$$(\lambda \times \mu)' = \lambda' \diamond \mu' \quad \rightleftharpoons \quad (\lambda \diamond \mu)' = \lambda' \times \mu'.$$

Proof. By definition of $\lambda \times \mu$, we can write

$$\begin{aligned}(\lambda \times \mu)'_k &= \left|\left\{(i, j) : \lambda_i \ge k \text{ and } \mu_j \ge k \,\middle|\, 1 \le i \le \ell(\lambda) \text{ and } 1 \le j \le \ell(\mu)\right\}\right| \\ &= \left|\left\{i :\ \lambda_i \ge k \,\middle|\, 1 \le i \le \ell(\lambda)\right\}\right| \times \left|\left\{j :\ \mu_j \ge k \,\middle|\, 1 \le j \le \ell(\mu)\right\}\right|.\end{aligned}$$

It reads equivalently as

$$(\lambda \times \mu)'_k = \lambda'_k \cdot \mu'_k = (\lambda' \diamond \mu')_k \quad \Longrightarrow \quad (\lambda \times \mu)' = \lambda' \diamond \mu'.$$

Another relation is a consequence of the dual property. □

A4.2. Examples. Consider the same partitions in the examples illustrated in A3.2. For $\lambda = (321)$ and $\mu = (21)$, we have

$$\lambda \diamond \mu = (62) \quad \text{and} \quad \lambda \times \mu = (221111).$$

The non-symmetric example with $\lambda = (331)$ and $\mu = (21)$ yields

$$\lambda \diamond \mu = (63) \quad \text{and} \quad \lambda \times \mu = (221111).$$

Moreover $\lambda' = (322)$ and $\mu' = (21)$ and so we have

$$\begin{aligned}(\lambda \times \mu)' \;&=\; (62) \;=\; \lambda' \diamond \mu' \\ (\lambda \diamond \mu)' \;&=\; (222111) \;=\; \lambda' \times \mu'.\end{aligned}$$

A5. Dominance partial ordering

A5.1. Young's lattice. Let $\mathcal{P}$ be the set of partitions of all non-negative integers. Order $\mathcal{P}$ component-wise; that is,

$$(\lambda_1, \lambda_2, \cdots) \preceq (\mu_1, \mu_2, \cdots) \quad \rightleftharpoons \quad \lambda_k \le \mu_k,\ \forall\, k \ge 1.$$

Then $\mathcal{P}$ is a partially ordered set. For two partitions λ, μ, we have

$$\begin{aligned}\lambda \vee \mu = \sup(\lambda, \mu) \quad &\text{where} \quad (\lambda \vee \mu)_k = \max(\lambda_k, \mu_k) \\ \lambda \wedge \mu = \inf(\lambda, \mu) \quad &\text{where} \quad (\lambda \wedge \mu)_k = \min(\lambda_k, \mu_k).\end{aligned}$$

Therefore $\mathcal{P}$ is a lattice, known as Young's lattice.

A5.2. Total orderings. Let L_n denote the reverse lexicographic ordering on the set $\mathcal{P}_n$ of partitions of n: that is to say, L_n is the subset of $\mathcal{P}_n \times \mathcal{P}_n$ consisting of all (λ, μ) such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda_k - \mu_k$ is positive. L_n is a total ordering. Another total ordering on $\mathcal{P}_n$ is L'_n, the set of all (λ, μ) such that either $\lambda = \mu$ or else the first non-vanishing difference $\lambda^*_k - \mu^*_k$ is negative, where $\lambda^*_k = \lambda_{1+n-k}$.

For example, when $n = 5$, L_5 and L'_5 arrange $\mathcal{P}_5$ in the sequence

$$L_5 \;=\; L'_5 \;=\; (5),\ (14),\ (1^2 3),\ (12^2),\ (1^3 2),\ (1^5).$$

However the orderings L_n and L'_n are distinct as soon as $n > 5$. This can be exemplified from two partitions $\lambda = (31^3)$ and $\mu = (2^3)$ as well as their orderings $(\lambda, \mu) \in L_6$ and $(\mu, \lambda) \in L'_6$.

In general, for $\lambda, \mu \in \mathcal{P}_n$, there holds

$$(\lambda, \mu) \in L_n \quad \rightleftharpoons \quad (\mu', \lambda') \in L'_n.$$

Proof. Suppose that $(\lambda, \mu) \in L_n$ and $\lambda \neq \mu$. Then for some integer $k \geq 1$ we have $\lambda_k - \mu_k > 0$ and $\lambda_i = \mu_i$ for $1 \leq i < k$. If we put $\ell = \lambda_k$ and consider the diagrams of λ and μ, we see immediately that $\lambda'_i = \mu'_i$ for $\ell < i \leq n$, and that $\lambda'_\ell > \mu'_\ell$, so that $(\mu', \lambda') \in L'_n$. The converse can be proved analogously. □

A5.3. Dominance partial ordering. An ordering is more important than either L_n or L'_n is the natural (partial) ordering N_n on $\mathcal{P}_n$ (also called the dominance partial ordering), which is defined through the partial sums as follows:

$$(\lambda, \mu) \in N_n \quad \rightleftharpoons \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k,\ \forall\, k \geq 1.$$

However, N_n is not a total ordering as soon as $n > 5$. For example, (31^3) and (2^3) are incomparable to N_6 as their partial sums are (3456) and (2466) respectively. We shall write $\lambda \geq \mu$ in place of $(\lambda, \mu) \in N_n$.

A5.4. Proposition. Let $\lambda, \mu \in \mathcal{P}_n$. Then

$$\begin{aligned} &\text{(A)} \quad \lambda \geq \mu \quad \Rightarrow \quad (\lambda, \mu) \in L_n \cap L'_n \\ &\text{(B)} \quad \lambda \geq \mu \quad \rightleftharpoons \quad \mu' \geq \lambda'. \end{aligned}$$

Proof. We prove (A) and (B) separately.

(A) Suppose that $\lambda \geq \mu$. Then either $\lambda_1 > \mu_1$, in which case $(\lambda, \mu) \in L_n$, or else $\lambda_1 = \mu_1$. In this case either $\lambda_2 > \mu_2$, in which case again $(\lambda, \mu) \in L_n$, or else $\lambda_2 = \mu_2$. Continuing in this way, we see that $(\lambda, \mu) \in L_n$. Also, for each $i \geq 1$, we have

$$\begin{aligned}
\lambda_{i+1} + \lambda_{i+2} + \cdots &= n - (\lambda_1 + \cdots + \lambda_i) \\
&\leq n - (\mu_1 + \cdots + \mu_i) \\
&= \mu_{i+1} + \mu_{i+2} + \cdots .
\end{aligned}$$

Hence the same reasoning as before shows that $(\lambda, \mu) \in L'_n$.

(B) Clearly it is enough to prove one implication. Suppose that $\mu' \ngeq \lambda'$. Then for some $k \geq 1$, we have

$$\lambda'_1 + \cdots + \lambda'_i \leq \mu'_1 + \cdots + \mu'_i, \quad 1 \leq i < k$$

and

$$\lambda'_1 + \cdots + \lambda'_k > \mu'_1 + \cdots + \mu'_k$$

which implies that $\lambda'_k > \mu'_k$. Let $u = \lambda'_k$, $v = \mu'_k$. Now that λ and μ are partitions of the same number n, it follows that

$$\lambda'_{k+1} + \lambda'_{k+2} + \cdots < \mu'_{k+1} + \mu'_{k+2} + \cdots .$$

Recalling that $\lambda'_{k+1} + \lambda'_{k+2} + \cdots$ is equal to the number of nodes in the diagram of λ which lie to the right of the kth column, we have

$$\lambda'_{k+1} + \lambda'_{k+2} + \cdots = \sum_{i=1}^{u} (\lambda_i - k).$$

Likewise

$$\mu'_{k+1} + \mu'_{k+2} + \cdots = \sum_{i=1}^{v} (\mu_i - k).$$

Hence we have

$$\sum_{i=1}^{v} (\mu_i - k) > \sum_{i=1}^{u} (\lambda_i - k) \geq \sum_{i=1}^{v} (\lambda_i - k)$$

in which the right-hand inequality holds because $u > v$ and $\lambda_i \geq k$ for $1 \leq i \leq u$. So we have

$$\mu_1 + \cdots + \mu_v > \lambda_1 + \cdots + \lambda_v$$

and therefore $\lambda \ngeq \mu$, which contradicts to the condition $\lambda \geq \mu$. $\square$

A5.5. Theorem. The set $\mathcal{P}_n$ of partitions of n is a lattice with respect to the natural ordering, which is confirmed by the following important theorem. Each pair of partitions λ, μ of n has a greatest lower bound $\tau = \inf(\lambda, \mu)$, defined by

$$\tau: \quad \sum_{i=1}^{k} \tau_i = \min\Big(\sum_{i=1}^{k} \lambda_i, \sum_{i=1}^{k} \mu_i\Big) \quad \text{for each} \quad k \geq 1$$

and a least upper bound $\sigma = \sup(\lambda, \mu)$ defined by $\sigma' = \inf(\lambda', \mu')$.

Proof. Let $\nu \in \mathcal{P}_n$ with $\lambda \geq \nu$ and $\mu \geq \nu$. We see that for $k = 1, 2, \cdots, n$, there hold

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_k &\geq \nu_1 + \nu_2 + \cdots + \nu_k \\ \mu_1 + \mu_2 + \cdots + \mu_k &\geq \nu_1 + \nu_2 + \cdots + \nu_k \end{aligned}$$

which is equivalent to $\nu \leq \tau = \inf(\lambda, \mu)$ in accordance with the definition of inf.

Now suppose that $\nu \in \mathcal{P}_n$ with $\nu \geq \lambda$ and $\nu \geq \mu$. By means of Proposition A5.4, we have

$$\begin{aligned} \nu \geq \lambda \quad &\Rightarrow \quad \lambda' \geq \nu' \\ \nu \geq \mu \quad &\Rightarrow \quad \mu' \geq \nu' \end{aligned}$$

which read as

$$\nu' \leq \sigma' = \inf(\lambda', \mu') \quad \rightleftharpoons \quad \nu \geq \sigma = \sup(\lambda, \mu).$$

This complete the proof of the theorem. $\square$

The example with $\lambda = (1^3 3)$, $\mu = (2^3)$ and $\sigma = (321)$ shows that it is not always true that

$$\sigma: \quad \sum_{i=1}^{k} \sigma_i = \max\Big(\sum_{i=1}^{k} \lambda_i, \sum_{i=1}^{k} \mu_i\Big), \ \forall\, k \geq 1$$

even we would have desired it.

In fact, the partial sums of λ and μ read respectively as (3456) and (2466), whose minimum is given by (2456). Therefore we have $\inf(\lambda, \mu) = (1^2 2^2)$. Similarly, for the conjugate partitions $\lambda' = (1^2 4)$ and $\mu' = (3^2)$, the corresponding partial sums are given respectively by (456) and (366). Their minimum reads as (356) and hence $\inf(\lambda', \mu') = (321)$ which leads us to $\sup(\lambda, \mu) = (321)$. However the maximum between the partial sums of λ and μ is (346). It corresponds to the partial sums of the sequence (312), which is even not a partition.