UNIVERSITÀ DEGLI STUDI DI LECCE
DIPARTIMENTO DI MATEMATICA
"Ennio De Giorgi"

Wenchang Chu Leontina Di Claudio

Dipartimento di Matematica
Università di Lecce
Lecce - Italia

CLASSICAL PARTITION IDENTITIES AND BASIC HYPERGEOMETRIC SERIES

QUADERNO **6/2004**

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Classical Partition Identities
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Basic Hypergeometric Series

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Preface

Basic hypergeometric series, often shortened as q-series, has developed rapidly during the past two decades. Its increasing importance to theoretical physics, computer science and classical mathematics (algebra, analysis and combinatorics) has widely been understood and accepted. Nowadays q-series is fully in its flourishing period and there is indeed a necessity to have an introduction book on the topic.

This book originates from the teaching experience of the first author. In the spring of 2000, a series of lectures entitled *Classical Partitions and Rogers-Ramanujan Identities* was delivered by the first author at Lecce University (Italy). The same program was then replayed in the summer of 2001 at Dalian University of Technology (China). In the spring of 2002 and 2004, these lectures have been extended to a course for PhD students again at Lecce University under the cover-title *Teoria dei Numeri*. The second author is one among the participants of these lectures.

The main purpose of the book is to present a brief introduction to basic hypergeometric series and applications to partition enumeration and number theory. As a short account to the theory of partitions, the first part (Chapters **A-B-C**) covers the algebraic aspects (basic structures: partially ordered sets and lattices), combinatorial aspects (generating functions and Durfee rectangles), and analytic aspects (the Jacobi triple products and Rogers-Ramanujan identities). Further development toward basic hypergeometric series and bilateral counterparts is dealt with in the second part (Chapters **D-E-F**). Applications to the representations of natural numbers by square sums and Ramanujan's congruences on partition function are presented in the third and the last part (Chapters **G-H**).

Lecce, November 1st, 2004

CHU Wenchang
DI CLAUDIO Leontina

Contents

CHAPTER A

Partitions and Algebraic Structures

In this chapter, we introduce partitions of natural numbers and the Ferrers diagrams. The algebraic structures of partitions such as addition, multiplication and ordering will be studied.

A1. Partitions and representations

A partition is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots)$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots$$

and containing only finitely many non-zero terms.

The non-zero λ_k in λ are called the parts of λ. The number of parts of λ is the length of λ, denoted by $\ell(\lambda)$; and the sum of parts is the weight of λ, denoted by $|\lambda|$:

$$|\lambda| = \sum_{k \geq 1} \lambda_k = \lambda_1 + \lambda_2 + \cdots .$$

If $n = |\lambda|$ we say that λ is a partition of n, denoted by $n \dashv \lambda$.

The set of all partitions of n is denoted by $\mathcal{P}_n$. In particular, $\mathcal{P}_0$ consists of a single element, the unique partition of zero, which we denote by 0.

Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part:

$$\lambda = (1^{m_1} 2^{m_2} \cdots k^{m_k} \cdots)$$

means that exactly m_k copies of the parts of λ are equal to k. The number

$$m_k = m_k(\lambda) = Card\{i : k = \lambda_i\}$$

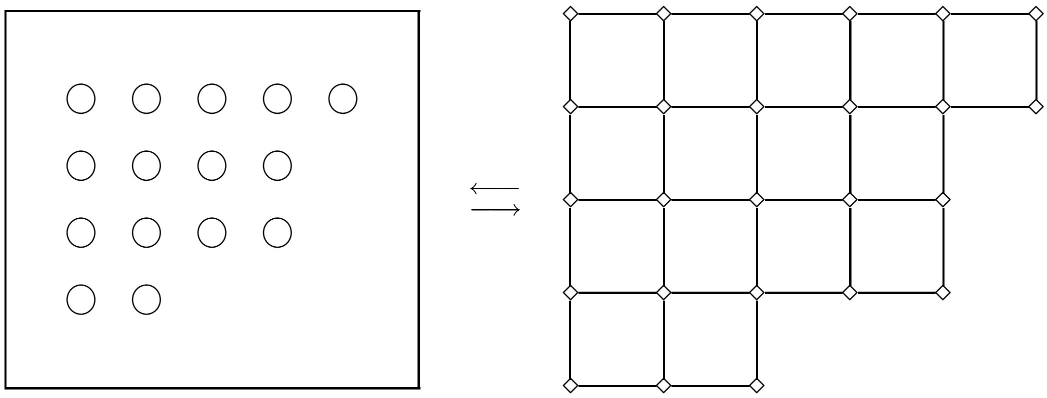
is called the multiplicity of k in λ.

A2. Ferrers diagrams of partitions

The diagram of a partition λ may be formally defined as the set of points (or unit squares)

$$\lambda \quad = \quad \Big\{(i,j) \mid 1 \leq j \leq \lambda_i,\ 1 \leq i \leq \ell(\lambda)\Big\}$$

drawn with the convention as matrices. For example, the diagram of the partition $\lambda = (5442)$ is shown as follows:



We shall usually denote the diagram of a partition λ by the same symbol. The conjugate of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ, i.e., the diagram obtained by reflection in the main diagonal. For example, the conjugate of (5442) is (44331). Hence λ'_k is the number of the nodes in the k-th column of λ, or equivalently

$$\lambda'_k = Card\big\{i : \lambda_i \geq k\big\}.$$

In particular, $\lambda'_1 = \ell(\lambda)$ and $\lambda_1 = \ell(\lambda')$. Obviously, we also have $\lambda'' = \lambda$ and $m_k = \lambda'_k - \lambda'_{k+1}$. Therefore we can dually express the Ferrers diagram of λ as

$$\lambda \ = \ \Big\{(i,j) \mid 1 \leq i \leq \lambda'_j,\ 1 \leq j \leq \ell(\lambda')\Big\}.$$

A2.1. Euler's theorem. The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n.

Proof. Let S be the set of partitions of n into distinct odd parts and T the set of self-conjugate partitions of n, the mapping

$$\begin{aligned} f \quad : \quad & S \to T \\ & \lambda \mapsto \mu \end{aligned}$$

defined by

$$\mu_i = \mu_i' := \frac{\lambda_i - 1}{2} + i \quad \text{where for all} \quad i = 1, 2, \cdots, \ell(\lambda).$$

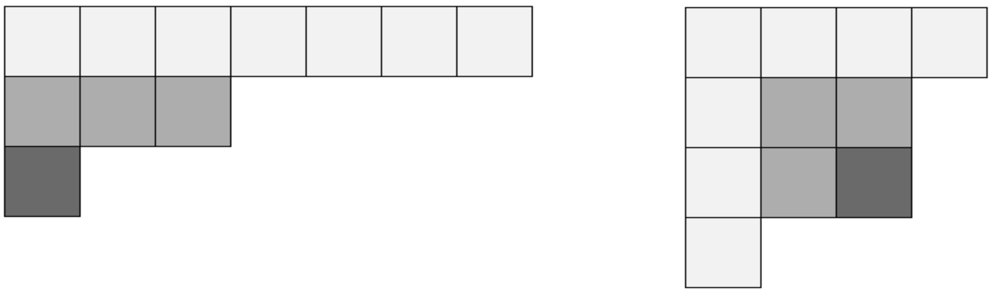
Obviously, μ is a selfconjugate partition with diagonal length equal to $\ell(\lambda)$ and the weight equal to $|\lambda|$, which can be justified as follows:

$$|\mu| + \ell^2(\lambda) = \sum_{i=1}^{\ell(\lambda)} (\mu_i + \mu_i') = |\lambda| + \sum_{i=1}^{\ell(\lambda)} (2i - 1) = |\lambda| + \ell^2(\lambda).$$

From the Ferrers diagrams, we see that f is a bijection between S and T. Therefore they have the same cardinality $|S| = |T|$, which completes the proof. □

For example, the image of partition $\lambda = (731)$ under f reads as $\mu = (4331)$. This can be illustrated as follows:

$$\lambda = (731) \quad \mapsto \quad \mu = (4331).$$



A2.2. Theorem on permutations. Let λ be a partition with $m \geq \lambda_1$ and $n \geq \lambda_1'$. Then the $m + n$ numbers

$$\lambda_i + n - i \quad (1 \leq i \leq n) \quad \text{and} \quad n - 1 + j - \lambda_j' \quad (1 \leq j \leq m)$$

are a permutation of $\{0, 1, 2, \cdots, m + n - 1\}$.

Proof. Define three subsets of non-negative integers:

$$\begin{aligned} \mathcal{U} \; : &= \; \{\lambda_i + n - i \,|\, 1 \leq i \leq n\} \\ \mathcal{V} \; : &= \; \{n - 1 + j - \lambda_j' \,|\, 1 \leq j \leq m\} \\ \mathcal{W} \; : &= \; \{k \,|\, 0 \leq k \leq m + n - 1\}. \end{aligned}$$

In order to prove the theorem, it suffices to show the following

(A) $\mathcal{U} \subseteq \mathcal{W}$ and $\mathcal{V} \subseteq \mathcal{W}$;
(B) The elements of $\mathcal{U}$ are distinct;
(C) The elements of $\mathcal{V}$ are distinct;
(D) $\mathcal{U} \cap \mathcal{V} = \emptyset$.

It is clearly true (A). Suppose that there exist i and j with $1 \le i < j \le n$ such that

$$\lambda_i + n - i \;=\; \lambda_j + n - j.$$

Keeping in mind of the partition λ, we see that it is absurd for $\lambda_i \ge \lambda_j$ and $n - i > n - j$. This proves (B). We can prove (C) similarly in view of the conjugate partition λ'. There remains only (D) to be confirmed.

Observe that the Ferrers diagram of λ is contained in the Ferrers diagram of (m^n), which is an $n \times m$ rectangle. We can identify the partition λ with the points inside its Ferrers diagram. If the point with coordinate (i, j) is inside λ, then we have $\lambda_i \ge i$ and $j \le \lambda'_j$, which are equivalent to the inequality

$$\lambda_i - i \ge 0 \ge j - \lambda'_j \quad \Rightarrow \quad \lambda_i + n - i > n - 1 + j - \lambda'_j.$$

This means that for (i, j) inside the Ferrers diagram λ with $1 \le i \le n$ and $1 \le j \le m$, the corresponding $\lambda_i + n - i$ and $n - 1 + j - \lambda'_j$ can not be the common element in $\mathcal{U} \cap \mathcal{V}$.

Instead if the point with coordinate (i, j) lies outside λ, then we have $\lambda_i < i$ and $j > \lambda'_j$, which are equivalent to another inequality

$$\lambda_i - i < 0 \le j - \lambda'_j - 1 \quad \Rightarrow \quad \lambda_i + n - i < n - 1 + j - \lambda'_j.$$

This implies that for (i, j) outside the Ferrers diagram λ with $1 \le i \le n$ and $1 \le j \le m$, the corresponding $\lambda_i + n - i$ and $n - 1 + j - \lambda'_j$ can not be again the common element in $\mathcal{U} \cap \mathcal{V}$.

In any case, we have verified that $\mathcal{U}$ and $\mathcal{V}$ have no common elements, which confirms (D).

The proof of Theorem A2.2 is hence completed. □

A2.3. The hooklength formula. Let λ be a partition. The hooklength of λ at $(i, j) \in \lambda$ is defined to be

$$h(i, j) = 1 + \lambda_i + \lambda'_j - i - j.$$

If the diagram of λ is contained in the diagram of (m^n), define

$$\nu_k = \lambda_k + n - k \quad (1 \leq k \leq n).$$

Then the theorem on $\{m+n\}$-permutations can be used to demonstrate the following hooklength formulae:

$$\prod_{(i,j)\in\lambda} (1-q^{h(i,j)}) = \frac{\prod_{i\geq 1}\prod_{j=1}^{\nu_i}(1-q^j)}{\prod_{i<j}(1-q^{\nu_i-\nu_j})} \rightleftharpoons \prod_{(i,j)\in\lambda} h(i,j) = \frac{\prod_{i\geq 1}\nu_i!}{\prod_{i<j}(\nu_i-\nu_j)}.$$

Proof. Interchanging λ and λ' in permutation Theorem A2.2 and then putting $m = \lambda_1$ and $\lambda'_1 \leq n$, we see that $m + \lambda'_j - j$ $(1 \leq j \leq m)$ and $m-1+j-\lambda_j$ $(1 \leq j \leq n)$ constitute a permutation of $\{0, 1, 2, \cdots, m+n-1\}$. Therefore we have a disjoint union:

$$\left\{q^{\lambda_1+\lambda'_j-j}\right\}_{j=1}^{\lambda_1} \biguplus \left\{q^{\lambda_1-1+j-\lambda_j}\right\}_{j=1}^{n} = \left\{q^j\right\}_{j=0}^{\lambda_1+n-1}.$$

According to the definition of the hooklength of λ, the identity can be reformulated as follows:

$$\left\{q^{h(1,j)}\right\}_{j=1}^{\lambda_1} \biguplus \left\{q^{\nu_1-\nu_j}\right\}_{j=2}^{n} = \left\{q^j\right\}_{j=1}^{\nu_1}.$$

Writing down this identity for the partition $(\lambda_i, \lambda_{i+1}, \cdots)$:

$$\left\{q^{h(i,j)}\right\}_{j=1}^{\lambda_i} \biguplus \left\{q^{\nu_i-\nu_j}\right\}_{j=1+i}^{n} = \left\{q^j\right\}_{j=1}^{\nu_i}$$

and then summing them over $i = 1, 2, \cdots, \ell(\lambda)$, we obtain

$$\sum_{(i,j)\in\lambda} q^{h(i,j)} + \sum_{i<j} q^{\nu_i-\nu_j} = \sum_{i\geq 1}\sum_{j=1}^{\nu_i} q^j.$$

Instead of summation, the multiplication leads us consequently to the following:

$$\prod_{(i,j)\lambda} (1-q^{h(i,j)}) = \frac{\prod_{i\geq 1}\prod_{j=1}^{\nu_i}(1-q^j)}{\prod_{i<j}(1-q^{\nu_i-\nu_j})}.$$

In particular dividing both sides by $(1-q)^{|\lambda|}$ and then setting $q = 1$, we find that

$$\prod_{(i,j)\in\lambda} h(i,j) = \frac{\prod_{i\geq 1}\nu_i!}{\prod_{i<j}(\nu_i-\nu_j)}.$$

This completes the proof of the hooklength formula. □

A3. Addition on partitions

Let λ and μ be partitions. We define $\lambda+\mu$ to be the sum of the sequences λ and μ:

$$(\lambda+\mu)_k = \lambda_k + \mu_k.$$

Also we define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ, arranged in descending order.

A3.1. Proposition. The operations $+$ and $\cup$ are dual each other

$$(\lambda \cup \mu)' = \lambda' + \mu' \quad \rightleftharpoons \quad (\lambda+\mu)' = \lambda' \cup \mu'.$$

Proof. The diagram of $\lambda\cup\mu$ is obtained by taking the rows of the diagrams of λ and μ and reassembling them in decreasing order. Hence the length of the k-th column of $\lambda\cup\mu$ is the sum of lengths of the k-th columns of λ and of μ, i.e.

$$(\lambda \cup \mu)'_k \;=\; \big|\{i|\lambda_i \geq k\}\big| \;+\; \big|\{j|\mu_j \geq k\}\big| \;=\; \lambda'_k + \mu'_k.$$

The converse follows from duality. $\square$

A3.2. Examples. For two symmetric partitions given by $\lambda = (321)$ and $\mu = (21)$, we then have

$$\lambda+\mu = (531) \quad \text{and} \quad \lambda\cup\mu = (32211).$$

Similarly, we consider a non-symmetric example. If $\lambda = (331)$ and $\mu = (21)$, then it is easy to compute $\lambda' = (322)$ and $\mu' = (21)$. Therefore

$$\lambda+\mu = (541) \quad \text{and} \quad \lambda\cup\mu = (33211)$$

and

$$\begin{aligned} (\lambda\cup\mu)' &= (532) = \lambda' + \mu' \\ (\lambda+\mu)' &= (32221) = \lambda' \cup \mu'. \end{aligned}$$

A4. Multiplication on partitions

Next, we define $\lambda \diamond \mu$ to be the component-wise product of the sequences λ and μ:

$$(\lambda \diamond \mu)_k = \lambda_k \mu_k.$$

Also we define $\lambda \times \mu$ to be the partition whose parts are $\min(\lambda_i, \mu_j)$ for all (i, j) with $1 \le i \le \ell(\lambda)$ and $1 \le j \le \ell(\mu)$, arranged in descending order.

A4.1. Proposition. For the operations "$\diamond$" and "$\times$", we have the dual relation:

$$(\lambda \times \mu)' = \lambda' \diamond \mu' \quad \rightleftharpoons \quad (\lambda \diamond \mu)' = \lambda' \times \mu'.$$

Proof. By definition of $\lambda \times \mu$, we can write

$$\begin{aligned}(\lambda \times \mu)'_k &= \left|\left\{(i, j) : \lambda_i \ge k \text{ and } \mu_j \ge k \;\middle|\; 1 \le i \le \ell(\lambda) \text{ and } 1 \le j \le \ell(\mu)\right\}\right| \\ &= \left|\left\{i :\ \lambda_i \ge k \;\middle|\; 1 \le i \le \ell(\lambda)\right\}\right| \times \left|\left\{j :\ \mu_j \ge k \;\middle|\; 1 \le j \le \ell(\mu)\right\}\right|.\end{aligned}$$

It reads equivalently as

$$(\lambda \times \mu)'_k = \lambda'_k \cdot \mu'_k = (\lambda' \diamond \mu')_k \quad \Longrightarrow \quad (\lambda \times \mu)' = \lambda' \diamond \mu'.$$

Another relation is a consequence of the dual property. □

A4.2. Examples. Consider the same partitions in the examples illustrated in A3.2. For $\lambda = (321)$ and $\mu = (21)$, we have

$$\lambda \diamond \mu = (62) \quad \text{and} \quad \lambda \times \mu = (221111).$$

The non-symmetric example with $\lambda = (331)$ and $\mu = (21)$ yields

$$\lambda \diamond \mu = (63) \quad \text{and} \quad \lambda \times \mu = (221111).$$

Moreover $\lambda' = (322)$ and $\mu' = (21)$ and so we have

$$\begin{aligned}(\lambda \times \mu)' &= (62) = \lambda' \diamond \mu' \\ (\lambda \diamond \mu)' &= (222111) = \lambda' \times \mu'.\end{aligned}$$

A5. Dominance partial ordering

A5.1. Young's lattice. Let $\mathcal{P}$ be the set of partitions of all non-negative integers. Order $\mathcal{P}$ component-wise; that is,

$$(\lambda_1, \lambda_2, \cdots) \preceq (\mu_1, \mu_2, \cdots) \quad \rightleftharpoons \quad \lambda_k \le \mu_k,\ \forall\, k \ge 1.$$

Then $\mathcal{P}$ is a partially ordered set. For two partitions λ, μ, we have

$$\begin{aligned}\lambda \vee \mu = \sup(\lambda, \mu) \quad &\text{where} \quad (\lambda \vee \mu)_k = \max(\lambda_k, \mu_k) \\ \lambda \wedge \mu = \inf(\lambda, \mu) \quad &\text{where} \quad (\lambda \wedge \mu)_k = \min(\lambda_k, \mu_k).\end{aligned}$$

Therefore $\mathcal{P}$ is a lattice, known as Young's lattice.

A5.2. Total orderings. Let L_n denote the reverse lexicographic ordering on the set $\mathcal{P}_n$ of partitions of n: that is to say, L_n is the subset of $\mathcal{P}_n \times \mathcal{P}_n$ consisting of all (λ, μ) such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda_k - \mu_k$ is positive. L_n is a total ordering. Another total ordering on $\mathcal{P}_n$ is L'_n, the set of all (λ, μ) such that either $\lambda = \mu$ or else the first non-vanishing difference $\lambda^*_k - \mu^*_k$ is negative, where $\lambda^*_k = \lambda_{1+n-k}$.

For example, when $n = 5$, L_5 and L'_5 arrange $\mathcal{P}_5$ in the sequence

$$L_5 \;=\; L'_5 \;=\; (5),\ (14),\ (1^2 3),\ (12^2),\ (1^3 2),\ (1^5).$$

However the orderings L_n and L'_n are distinct as soon as $n > 5$. This can be exemplified from two partitions $\lambda = (31^3)$ and $\mu = (2^3)$ as well as their orderings $(\lambda, \mu) \in L_6$ and $(\mu, \lambda) \in L'_6$.

In general, for $\lambda, \mu \in \mathcal{P}_n$, there holds

$$(\lambda, \mu) \in L_n \quad \rightleftharpoons \quad (\mu', \lambda') \in L'_n.$$

Proof. Suppose that $(\lambda, \mu) \in L_n$ and $\lambda \neq \mu$. Then for some integer $k \geq 1$ we have $\lambda_k - \mu_k > 0$ and $\lambda_i = \mu_i$ for $1 \leq i < k$. If we put $\ell = \lambda_k$ and consider the diagrams of λ and μ, we see immediately that $\lambda'_i = \mu'_i$ for $\ell < i \leq n$, and that $\lambda'_\ell > \mu'_\ell$, so that $(\mu', \lambda') \in L'_n$. The converse can be proved analogously. □

A5.3. Dominance partial ordering. An ordering is more important than either L_n or L'_n is the natural (partial) ordering N_n on $\mathcal{P}_n$ (also called the dominance partial ordering), which is defined through the partial sums as follows:

$$(\lambda, \mu) \in N_n \quad \rightleftharpoons \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k,\ \forall\, k \geq 1.$$

However, N_n is not a total ordering as soon as $n > 5$. For example, (31^3) and (2^3) are incomparable to N_6 as their partial sums are (3456) and (2466) respectively. We shall write $\lambda \geq \mu$ in place of $(\lambda, \mu) \in N_n$.

A5.4. Proposition. Let $\lambda, \mu \in \mathcal{P}_n$. Then

$$\begin{array}{llll} \text{(A)} & \lambda \geq \mu & \Rightarrow & (\lambda, \mu) \in L_n \cap L'_n \\ \text{(B)} & \lambda \geq \mu & \rightleftharpoons & \mu' \geq \lambda'. \end{array}$$

Proof. We prove (A) and (B) separately.

(A) Suppose that $\lambda \geq \mu$. Then either $\lambda_1 > \mu_1$, in which case $(\lambda, \mu) \in L_n$, or else $\lambda_1 = \mu_1$. In this case either $\lambda_2 > \mu_2$, in which case again $(\lambda, \mu) \in L_n$, or else $\lambda_2 = \mu_2$. Continuing in this way, we see that $(\lambda, \mu) \in L_n$. Also, for each $i \geq 1$, we have

$$\begin{aligned} \lambda_{i+1} + \lambda_{i+2} + \cdots &= n - (\lambda_1 + \cdots + \lambda_i) \\ &\leq n - (\mu_1 + \cdots + \mu_i) \\ &= \mu_{i+1} + \mu_{i+2} + \cdots . \end{aligned}$$

Hence the same reasoning as before shows that $(\lambda, \mu) \in L'_n$.

(B) Clearly it is enough to prove one implication. Suppose that $\mu' \not\geq \lambda'$. Then for some $k \geq 1$, we have

$$\lambda'_1 + \cdots + \lambda'_i \leq \mu'_1 + \cdots + \mu'_i, \quad 1 \leq i < k$$

and

$$\lambda'_1 + \cdots + \lambda'_k > \mu'_1 + \cdots + \mu'_k$$

which implies that $\lambda'_k > \mu'_k$. Let $u = \lambda'_k$, $v = \mu'_k$. Now that λ and μ are partitions of the same number n, it follows that

$$\lambda'_{k+1} + \lambda'_{k+2} + \cdots < \mu'_{k+1} + \mu'_{k+2} + \cdots .$$

Recalling that $\lambda'_{k+1} + \lambda'_{k+2} + \cdots$ is equal to the number of nodes in the diagram of λ which lie to the right of the kth column, we have

$$\lambda'_{k+1} + \lambda'_{k+2} + \cdots = \sum_{i=1}^{u} (\lambda_i - k).$$

Likewise

$$\mu'_{k+1} + \mu'_{k+2} + \cdots = \sum_{i=1}^{v} (\mu_i - k).$$

Hence we have

$$\sum_{i=1}^{v} (\mu_i - k) > \sum_{i=1}^{u} (\lambda_i - k) \geq \sum_{i=1}^{v} (\lambda_i - k)$$

in which the right-hand inequality holds because $u > v$ and $\lambda_i \geq k$ for $1 \leq i \leq u$. So we have

$$\mu_1 + \cdots + \mu_v > \lambda_1 + \cdots + \lambda_v$$

and therefore $\lambda \not\geq \mu$, which contradicts to the condition $\lambda \geq \mu$. $\square$

A5.5. Theorem. The set $\mathcal{P}_n$ of partitions of n is a lattice with respect to the natural ordering, which is confirmed by the following important theorem. Each pair of partitions λ, μ of n has a greatest lower bound $\tau = \inf(\lambda, \mu)$, defined by

$$\tau : \quad \sum_{i=1}^{k} \tau_i = \min\Big(\sum_{i=1}^{k} \lambda_i, \sum_{i=1}^{k} \mu_i\Big) \quad \text{for each} \quad k \geq 1$$

and a least upper bound $\sigma = \sup(\lambda, \mu)$ defined by $\sigma' = \inf(\lambda', \mu')$.

Proof. Let $\nu \in \mathcal{P}_n$ with $\lambda \geq \nu$ and $\mu \geq \nu$. We see that for $k = 1, 2, \cdots, n$, there hold

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_k \quad &\geq \quad \nu_1 + \nu_2 + \cdots + \nu_k \\ \mu_1 + \mu_2 + \cdots + \mu_k \quad &\geq \quad \nu_1 + \nu_2 + \cdots + \nu_k \end{aligned}$$

which is equivalent to $\nu \leq \tau = \inf(\lambda, \mu)$ in accordance with the definition of inf.

Now suppose that $\nu \in \mathcal{P}_n$ with $\nu \geq \lambda$ and $\nu \geq \mu$. By means of Proposition A5.4, we have

$$\begin{aligned} \nu \geq \lambda \quad &\Rightarrow \quad \lambda' \geq \nu' \\ \nu \geq \mu \quad &\Rightarrow \quad \mu' \geq \nu' \end{aligned}$$

which read as

$$\nu' \leq \sigma' = \inf(\lambda', \mu') \quad \rightleftharpoons \quad \nu \geq \sigma = \sup(\lambda, \mu).$$

This complete the proof of the theorem. $\square$

The example with $\lambda = (1^3 3)$, $\mu = (2^3)$ and $\sigma = (321)$ shows that it is not always true that

$$\sigma : \quad \sum_{i=1}^{k} \sigma_i = \max\Big(\sum_{i=1}^{k} \lambda_i, \sum_{i=1}^{k} \mu_i\Big), \; \forall\, k \geq 1$$

even we would have desired it.

In fact, the partial sums of λ and μ read respectively as (3456) and (2466), whose minimum is given by (2456). Therefore we have $\inf(\lambda, \mu) = (1^2 2^2)$. Similarly, for the conjugate partitions $\lambda' = (1^2 4)$ and $\mu' = (3^2)$, the corresponding partial sums are given respectively by (456) and (366). Their minimum reads as (356) and hence $\inf(\lambda', \mu') = (321)$ which leads us to $\sup(\lambda, \mu) = (321)$. However the maximum between the partial sums of λ and μ is (346). It corresponds to the partial sums of the sequence (312), which is even not a partition.

CHAPTER B

Generating Functions of Partitions

For a complex sequence $\{\alpha_n | n = 0, 1, 2, \cdots\}$, its generating function with a complex variable q is defined by

$$A(q) := \sum_{n=0}^{\infty} \alpha_n \, q^n \quad \rightleftharpoons \quad \alpha_n = [q^n] \, A(q).$$

When the sequence has finite non-zero terms, the generating function reduces to a polynomial. Otherwise, it becomes an infinite series. In that case, we suppose in general $|q| < 1$ from now on.

B1. Basic generating functions of partitions

Given three complex indeterminates x, q and n with $|q| < 1$, the shifted factorial is defined by

$$\begin{aligned} (x;\, q)_\infty &= \prod_{k=0}^{\infty} (1 - xq^k) \\ (x;\, q)_n &= \frac{(x;\, q)_\infty}{(q^n x;\, q)_\infty}. \end{aligned}$$

When n is a natural number in particular, it reduces to

$$(x;\, q)_0 = 1 \quad \text{and} \quad (x;\, q)_n = \prod_{k=0}^{n-1} (1 - q^k x) \quad \text{for} \quad n = 1, 2, \cdots.$$

We shall frequently use the following abbreviated notation for shifted factorial fraction:

$$\left[\begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \,\middle|\, q \right]_n = \frac{(a;q)_n (b;q)_n \cdots (c;q)_n}{(\alpha;q)_n (\beta;q)_n \cdots (\gamma;q)_n}.$$

B1.1. Partitions with parts in $\mathbb{S}$. We first investigate the generating functions of partitions with parts in $\mathbb{S}$, where the basic set $\mathbb{S} \subseteq \mathbb{N}$ with $\mathbb{N}$ being the set of natural numbers.

Let $\mathbb{S}$ be a set of natural numbers and $p(n|\mathbb{S})$ denote the number of partitions of n into elements of $\mathbb{S}$ (*or in other words, the parts of partitions belong to* $\mathbb{S}$). Then the univariate generating function is given by

$$\sum_{n=0}^{\infty} p(n|\mathbb{S})\, q^n \;=\; \prod_{k\in\mathbb{S}} \frac{1}{1-q^k}. \tag{B1.1a}$$

If we denote further by $p_\ell(n|\mathbb{S})$ the number of partitions with exactly ℓ-parts in $\mathbb{S}$, then the bivariate generating function is

$$\sum_{\ell,n\geq 0} p_\ell(n|\mathbb{S})\, x^\ell\, q^n \;=\; \prod_{k\in\mathbb{S}} \frac{1}{1-xq^k}. \tag{B1.1b}$$

Proof. For $|q| < 1$, we can expand the right member of the equation (B1.1a) according to the geometric series

$$\prod_{k\in\mathbb{S}} \frac{1}{1-q^k} \;=\; \prod_{k\in\mathbb{S}} \sum_{m_k=0}^{\infty} q^{km_k} \;=\; \sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k}.$$

Extracting the coefficient of q^n from both sides, we obtain

$$[q^n]\prod_{k\in\mathbb{S}} \frac{1}{1-q^k} \;=\; [q^n]\sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k} \;=\; \sum_{\substack{\sum_{k\in\mathbb{S}} km_k=n\\ m_k\geq 0:\, k\in\mathbb{S}}} 1.$$

The last sum is equal to the number of solutions of the Diophantine equation

$$\sum_{k\in\mathbb{S}} km_k = n$$

which enumerates the partitions $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ of n into parts in $\mathbb{S}$.

This completes the proof of (B1.1a). The bivariate generating function (B1.1b) can be verified similarly.

In fact, consider the formal power series expansion

$$\prod_{k\in\mathbb{S}} \frac{1}{1-xq^k} \;=\; \prod_{k\in\mathbb{S}} \sum_{m_k=0}^{\infty} x^{m_k} q^{km_k} \;=\; \sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}$$

in which the coefficient of $x^\ell q^n$ reads as

$$\begin{aligned}[x^\ell q^n]\prod_{k\in\mathbb{S}}\frac{1}{1-xq^k} &= [x^\ell q^n]\sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}\\ &= \sum_{\substack{\left.\begin{array}{l}\sum_{k\in\mathbb{S}} m_k=\ell\\ \sum_{k\in\mathbb{S}} km_k=n\end{array}\right\}\\ m_k\geq 0:\, k\in\mathbb{S}}} 1.\end{aligned}$$

The last sum enumerates the solutions of the system of Diophantine equations

$$\begin{cases}\displaystyle\sum_{k\in\mathbb{S}} m_k &= \ell\\ \displaystyle\sum_{k\in\mathbb{S}} km_k &= n\end{cases}$$

which are the number of partitions $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ of n with exactly ℓ-parts in $\mathbb{S}$. $\square$

B1.2. Partitions into distinct parts in $\mathbb{S}$. Next we study the generating functions of partitions into distinct parts in $\mathbb{S}$.

If we denote by $Q(n|\mathbb{S})$ and $Q_\ell(n|\mathbb{S})$ the corresponding partition numbers with distinct parts from $\mathbb{S}$, then their generating functions read respectively as

$$\sum_{n=0}^{\infty} Q(n|\mathbb{S})\,q^n = \prod_{k\in\mathbb{S}}\big(1+q^k\big) \tag{B1.2a}$$

$$\sum_{\ell,n\geq 0} Q_\ell(n|\mathbb{S})\,x^\ell q^n = \prod_{k\in\mathbb{S}}\big(1+xq^k\big). \tag{B1.2b}$$

PROOF. For the first identity, observing that

$$1+q^k = \sum_{m_k=0,1} q^{km_k}$$

we can reformulate the product on the right hand side as

$$\prod_{k\in\mathbb{S}}\big(1+q^k\big) = \prod_{k\in\mathbb{S}}\sum_{m_k=0,1} q^{km_k} = \sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k}.$$

Extracting the coefficient of q^n, we obtain

$$[q^n]\prod_{k\in\mathbb{S}}\big(1+q^k\big) = [q^n]\sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k} = \sum_{\substack{\sum_{k\in\mathbb{S}} km_k=n\\ m_k=0,1:\, k\in\mathbb{S}}} 1.$$

The last sum enumerates the solutions of Diophantine equation

$$\sum_{k\in\mathbb{S}} km_k = n \quad \text{with} \quad m_k = 0,1$$

which is equal to $Q(n|\mathbb{S})$, the number of partitions of n into distinct parts in $\mathbb{S}$.

Instead, we can proceed similarly for the second formula as follows:

$$\prod_{k\in\mathbb{S}}(1+xq^k) \;=\; \prod_{k\in\mathbb{S}}\sum_{m_k=0,1} x^{m_k}q^{km_k} \;=\; \sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}.$$

The coefficient of $x^\ell q^n$ leads us to the following

$$\begin{aligned}
[x^\ell q^n]\prod_{k\in\mathbb{S}}(1+xq^k) &= [x^\ell q^n]\sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}\\
&= \sum_{\substack{\left.\begin{array}{l}\sum_{k\in\mathbb{S}} m_k \;= \ell\\ \sum_{k\in\mathbb{S}} km_k = n\end{array}\right\}\\ m_k=0,1:\, k\in\mathbb{S}}} 1.
\end{aligned}$$

The last sum equals the number of solutions of the system of Diophantine equations

$$\left.\begin{array}{l}\displaystyle\sum_{k\in\mathbb{S}} m_k \;= \ell\\ \displaystyle\sum_{k\in\mathbb{S}} km_k = n\end{array}\right\} \quad \text{with} \quad m_k = 0,1$$

which correspond to the partitions $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ of n with exactly ℓ distinct parts in $\mathbb{S}$. $\square$

B1.3. Classical generating functions. When $\mathbb{S} = \mathbb{N}$, the set of natural numbers, the corresponding generating functions may be displayed, respectively, as

$$\frac{1}{(q;\,q)_\infty} = \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \sum_{n=0}^{\infty} p(n)\,q^n \tag{B1.3a}$$

$$\frac{1}{(qx;\,q)_\infty} = \prod_{m=1}^{\infty} \frac{1}{1-xq^m} = \sum_{\ell,n\geq 0} p_\ell(n)\,x^\ell q^n \tag{B1.3b}$$

$$(-q;\,q)_\infty = \prod_{m=1}^{\infty} (1+q^m) = \sum_{n=0}^{\infty} Q(n)\,q^n \tag{B1.3c}$$

$$(-qx;\,q)_\infty = \prod_{m=1}^{\infty} (1+xq^m) = \sum_{\ell,n\geq 0} Q_\ell(n)\,x^\ell q^n. \tag{B1.3d}$$

Manipulating the generating function of the partitions into odd numbers in the following manner

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} &= \prod_{k=1}^{\infty} \frac{1}{1-q^k} \times \prod_{k=1}^{\infty} (1-q^{2k}) \\ &= \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} = \prod_{k=1}^{\infty} (1+q^k)\end{aligned}$$

we see that it results in the generating function of the partitions into distinct parts. We have therefore proved the following theorem due to Euler. *The number of partitions of n into odd numbers equals to the number of partitions of n into distinct parts.*

B2. Classical partitions and the Gauss formula

B2.1. Proposition. Let $p_m(n)$ be the number of partitions into exactly m parts (*or dually, partitions with the largest part equal to m*). Its generating function reads as

$$\sum_{n=0}^{\infty} p_m(n)\,q^n = \frac{q^m}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B2.1}$$

Proof. For $\mathbb{S}=\mathbb{N}$, the generating function of $\{p_\ell(n|\mathbb{N})\}$ reads as

$$\begin{aligned}\sum_{\ell,n\geq 0} p_\ell(n)\,x^\ell q^n &= \sum_{\ell\geq 0} x^\ell \sum_{n\geq 0} p_\ell(n) q^n\\ &= \prod_{k=1}^{\infty}\frac{1}{1-xq^k} = \frac{1}{(qx;q)_\infty}.\end{aligned}$$

Extracting the coefficient of x^m, we get

$$\sum_{n=0}^{\infty} p_m(n)\,q^n = [x^m]\frac{1}{(qx;q)_\infty}.$$

For $|q|<1$, the function $1/(qx;q)_\infty$ is analytic at $x=0$. We can therefore expand it in MacLaurin series:

$$\frac{1}{(qx;q)_\infty} = \sum_{\ell=0}^{\infty} A_\ell(q)x^\ell \tag{B2.2}$$

where the coefficients $\{A_\ell(q)\}$ are independent of x to be determined. Performing the replacement $x\to x/q$, we can restate the expansion just displayed as

$$\frac{1}{(x;q)_\infty} = \sum_{\ell=0}^{\infty} A_\ell(q)x^\ell q^{-\ell}. \tag{B2.3}$$

It is evident that (B2.2) equals $(1-x)$ times (B2.3), which results in the functional equation

$$\sum_{\ell=0}^{\infty} A_\ell(q)x^\ell = (1-x)\sum_{\ell=0}^{\infty} A_\ell(q)x^\ell q^{-\ell}.$$

Extracting the coefficient of x^m from both expansions, we get

$$A_m(q) = A_m(q)q^{-m} - A_{m-1}(q)q^{1-m}$$

which is equivalent to the following recurrence relation

$$A_m(q) = \frac{q}{1-q^m}A_{m-1}(q) \quad \text{where} \quad m=1,2,\cdots.$$

Iterating this recursion for m-times, we find that

$$A_m(q) = \frac{q^m\,A_0(q)}{(1-q^m)(1-q^{m-1})\cdots(1-q)} = \frac{q^m}{(q;q)_m}A_0(q).$$

Noting that $A_0(q)=1$, we get finally

$$\sum_{n=0}^{\infty} p_m(n)\,q^n = [x^m]\frac{1}{(qx;q)_\infty} = \frac{q^m}{(q;q)_m}.$$

This completes the proof of Proposition B2.1. $\square$

A combinatorial proof. Let $p(n|\lambda_1 = m)$ be the number of partitions of n with the first part λ_1 equal to m. Then $p_m(n) = p(n|\lambda_1 = m)$ because the partitions enumerated by $p_m(n)$ are conjugate with those enumerated by $p(n|\lambda_1 = m)$. Therefore they have the same generating functions:

$$\sum_{n=0}^{\infty} p_m(n)\, q^n \;=\; \sum_{n=0}^{\infty} p(n|\lambda_1 = m)\, q^n.$$

All the partition of n enumerated by $p(n|\lambda_1 = m)$ have the first part $\lambda_1 = m$ in common and the remaining parts constitute the partitions of $n - m$ with each part $\leq m$. Therefore we have

$$\begin{aligned}\sum_{n=0}^{\infty} p(n|\lambda_1 = m)\, q^n \;&=\; \sum_{n=m}^{\infty} p(n-m|\lambda_1 \leq m)\, q^n \;=\; q^m \sum_{n=0}^{\infty} p(n|\lambda_1 \leq m)\, q^n \\ &=\; q^m \sum_{n=0}^{\infty} p(n|\{1,2,\cdots,m\})\, q^n \;=\; \frac{q^m}{(q;q)_m}\end{aligned}$$

where the first line is justified by replacement $n \to n + m$ on summation index, while the second is a consequence of (B1.1a).

This confirms again the generating function (B2.1). □

B2.2. Proposition. Let $p^m(n)$ be the number of partitions into $\leq m$ parts *(or dually, partitions into parts* $\leq m$*).* Then we have the generating function

$$\sum_{n=0}^{\infty} p^m(n)\, q^n \;=\; \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \tag{B2.4}$$

which yields a finite summation formula

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \;=\; 1 + \sum_{k=1}^{m} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Proof. Notice that $p^m(n)$, the number of partitions into $\leq m$ parts is equal to the number of partitions into parts $\leq m$ in view of conjugate partitions. We get immediately from (B1.1a) the generating function (B2.4).

The classification of the partitions of n into $\leq m$ parts with respect to the number k of parts yields

$$p^m(n) = p_0(n) + p_1(n) + p_2(n) + \cdots + p_m(n).$$

The corresponding generating function results in

$$\sum_{n=0}^{\infty} p^m(n)\, q^n \;=\; \sum_{k=0}^{m}\sum_{n=0}^{\infty} p_k(n)\, q^n \;=\; \sum_{k=0}^{m} \frac{q^k}{(q;q)_k}.$$

Recalling the first generating function expression (B2.4), we get the second formula from the last relation. □

B2.3. Gauss' classical partition identity.

$$\prod_{n=0}^{\infty} \frac{1}{1-xq^n} \;=\; 1+\sum_{m=1}^{\infty} \frac{x^m}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B2.5}$$

Proof. In fact, we have already established this identity from the demonstration of the last theorem, where it has been displayed explicitly in (B2.3).

Alternatively, classifying all the partitions with respect to the number of parts, we can manipulate the bivariate generating function

$$\begin{aligned}\frac{1}{(xq;q)_\infty} &= \sum_{\ell,n=0}^{\infty} p_\ell(n) x^\ell q^n \;=\; \sum_{\ell=0}^{\infty} x^\ell \sum_{n=0}^{\infty} p_\ell(n) q^n \\ &= \sum_{\ell=0}^{\infty} \frac{x^\ell q^\ell}{(1-q)(1-q^2)\cdots(1-q^\ell)}\end{aligned}$$

which is equivalent to Gauss' classical partition identity. □

B2.4. Theorem. Let $p_\ell(n|m)$ be the number of partitions of n with exactly ℓ-parts $\leq m$. Then we have its generating function

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|m)\, x^\ell q^n \;=\; \frac{1}{(1-qx)(1-q^2x)\cdots(1-q^mx)}.$$

The classification with respect to the maximum part k of partitions produces another identity

$$\frac{1}{(1-qx)(1-q^2x)\cdots(1-q^mx)} = 1+x\sum_{k=1}^{m} \frac{q^k}{(1-qx)(1-q^2x)\cdots(1-q^kx)}.$$

Proof. The first generating function follows from (B1.1b).

From the first generating function, we see that the bivariate generating function of partitions into parts $\leq k$ reads as

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|k)\, x^\ell q^n \;=\; \frac{1}{(qx;q)_k}.$$

Putting an extra part $\lambda_1 = k$ with enumerator xq^k over the partitions enumerated by the last generating function, we therefore derive the bivariate generating function of partitions into ℓ parts with the first one $\lambda_1 = k$ as follows:

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|\lambda_1 = k)\, x^\ell q^n \;=\; \frac{xq^k}{(qx;q)_k}.$$

Classifying the partitions of n into exactly ℓ parts with each parts $\leq m$ according to the first part $\lambda_1 = k$, we get the following expression

$$\begin{aligned}\sum_{\ell,n=0}^{\infty} p_\ell(n|m)\, x^\ell q^n &= \sum_{k=0}^{m}\sum_{\ell,n=0}^{\infty} p_\ell(n|\lambda_1 = k) x^\ell q^n \\ &= 1 + x\sum_{k=1}^{m} \frac{q^k}{(qx;q)_k}\end{aligned}$$

which is the second identity. □

B3. Partitions into distinct parts and the Euler formula

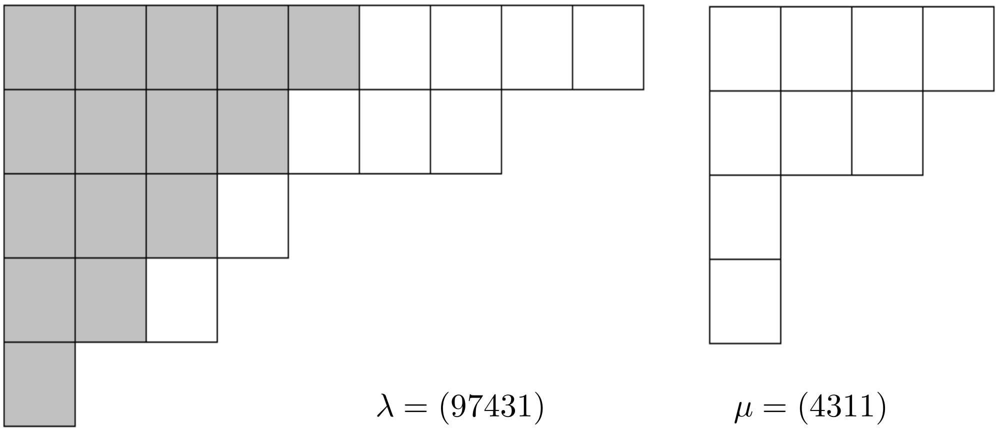
B3.1. Theorem. Let $Q_m(n)$ be the number of partitions into exactly m distinct parts. Its generating function reads as

$$\sum_{n=0}^{\infty} Q_m(n)\, q^n \;=\; \frac{q^{\binom{1+m}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B3.1}$$

Proof. Let $\lambda = (\lambda_1 > \lambda_2 \cdots > \lambda_m > 0)$ be a partition enumerated by $Q_m(n)$. Based on λ, define another partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0)$ by

$$\mu_k := \lambda_k - (m - k + 1) \quad \text{for} \quad k = 1, 2, \cdots, m. \tag{B3.2}$$

It is obvious that μ is a partition of $|\lambda| - \binom{1+m}{2}$ into $\leq m$ parts. As an example, the following figures show this correspondence between two partitions $\lambda = (97431)$ and $\mu = (4311)$.



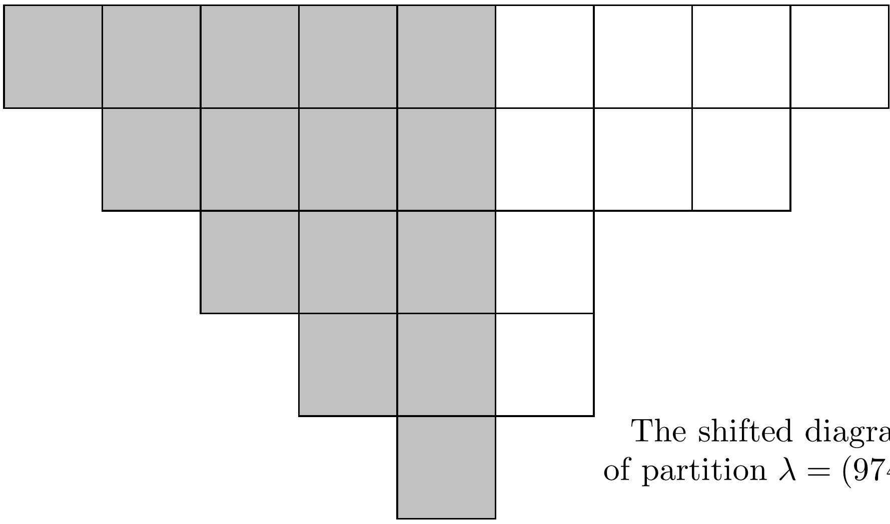
$\lambda = (97431)$ $\mu = (4311)$

It is not difficult to verify that the mapping (B3.2) is a bijection between the partitions of n with exactly m distinct parts and the partitions of $n - \binom{1+m}{2}$ with $\leq m$ parts. Therefore the generating function of $\{Q_m(n)\}_n$ is equal to that of $\{p^m\big(n - \binom{1+m}{2}\big)\}_n$, the number of partitions of $n - \binom{1+m}{2}$ with the number of parts $\leq m$:

$$\begin{aligned}\sum_{n=0}^{\infty} Q_m(n)\, q^n &= \sum_{n=0}^{\infty} p^m\big(n - \tbinom{1+m}{2}\big)\, q^n \\ &= q^{\binom{1+m}{2}} \sum_{n=0}^{\infty} p^m(n)\, q^n = \frac{q^{\binom{1+m}{2}}}{(q;q)_m}\end{aligned}$$

thanks for the generating function displayed in (B2.4). This completes the proof of Theorem B3.1. $\square$

Instead of the ordinary Ferrers diagram, we can draw a shifted diagram of λ as follows (see the figure). Under the first row of λ_1 squares, we put λ_2 squares lined up vertically from the second column. For the third row, we put λ_3 squares beginning from the third column. Continuing in this way, the last row of λ_m squares will be lined up vertically from the m-th column.



The shifted diagram of partition $\lambda = (97431)$

From the shifted diagram of λ, we see that all the partitions enumerated by $Q_m(n)$ have one common triangle on the left whose weight is $\binom{1+m}{2}$. The remaining parts right to the triangle are partitions of $n - \binom{m+1}{2}$ with $\leq m$ parts. This reduces the problem of computing the generating function to the case just explained.

B3.2. Classifying all the partitions with distinct parts according to the number of parts, we get Euler's classical partition identity

$$\prod_{n=0}^{\infty}(1 - xq^n) \;=\; 1 + \sum_{m=1}^{\infty} \frac{(-1)^m\, x^m q^{\binom{m}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)} \tag{B3.3}$$

which can also be verified through the correspondence between partitions into distinct odd parts and self-conjugate partitions.

Proof. Considering the bivariate generating function of $Q_m(n)$, we have

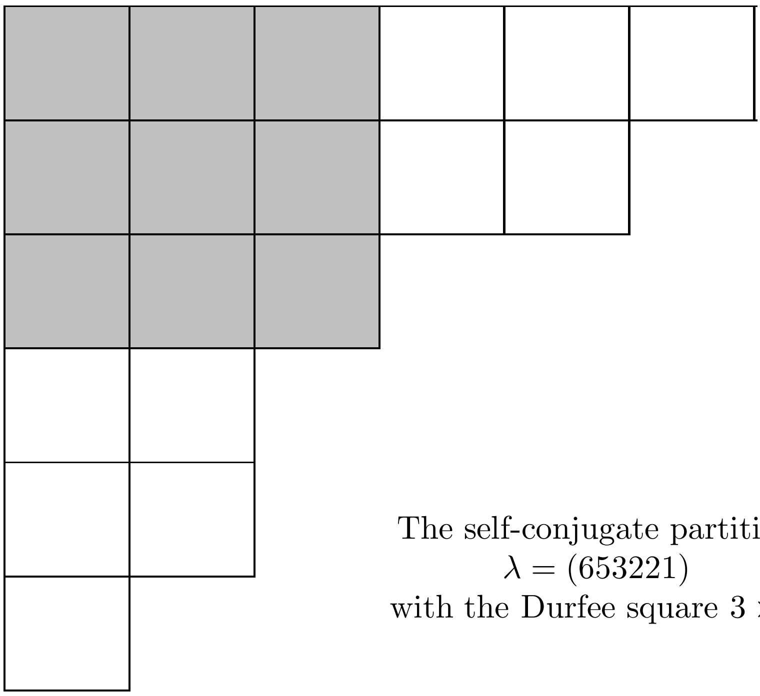
$$\prod_{k=1}^{\infty}(1 + xq^k) \;=\; \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} Q_m(n) q^n.$$

Recalling (B3.1) and then noting that $Q_0(n) = \delta_{0,n}$, we deduce that

$$\prod_{k=1}^{\infty}(1+xq^k) \;=\; 1+\sum_{m=1}^{\infty}\frac{x^m q^{\binom{1+m}{2}}}{(q;q)_m}$$

which becomes the Euler identity under parameter replacement $x \to -x/q$.

In view of Euler's Theorem A2.1, we have a bijection between the partitions into distinct odd parts and the self-conjugate partitions.



The self-conjugate partition
$\lambda = (653221)$
with the Durfee square 3×3

For a self-conjugate partition with the main diagonal length equal to m (which corresponds exactly to the length of partitions into distinct odd parts), it consists of three pieces: the first piece is the square of $m \times m$ on the top-left with bivariate enumerator $x^m q^{m^2}$, the second piece right to the square is a partition with $\leq m$ parts enumerated by $1/(q;q)_m$ and the third piece under the square is in effect the conjugate of the second one. Therefore the partitions right to the square and under the square $m \times m$ are altogether enumerated by $1/(q^2;q^2)_m$.

Classifying the self-conjugate partitions according to the main diagonal length m, multiplying both generating functions together and summing m

over $0 \le m \le \infty$, we find the following identity:

$$\prod_{n=0}^{\infty}(1 + xq^{1+2n}) \;=\; \sum_{m=0}^{\infty} \frac{x^m q^{m^2}}{(q^2;q^2)_m}$$

where the left hand side is the bivariate generating function of the partitions into odd distinct parts.

It is trivial to verify that under replacements

$$x \to -xq^{-1/2} \quad \text{and} \quad q \to q^{1/2}$$

the last formula is exactly the identity displayed in (B3.3). □

Unfortunately, there does not exist the closed form for the generating function of $Q^m(n)$, numbers of partitions into $\le m$ distinct parts.

B3.3. Dually, if we classify the partitions into distinct parts $\le m$ according to their maximum part. Then we can derive the following finite and infinite series identities

$$\prod_{j=1}^{m}(1 + q^j x) \;=\; 1 + x\sum_{k=1}^{m} q^k \prod_{i=1}^{k-1}(1 + q^i x) \tag{B3.4a}$$

$$\prod_{j=1}^{\infty}(1 + q^j x) \;=\; 1 + x\sum_{k=1}^{\infty} q^k \prod_{i=1}^{k-1}(1 + q^i x). \tag{B3.4b}$$

Proof. For the partitions into distinct parts with the maximum part equal to k, their bivariate generating function is given by

$$q^k x \prod_{i=1}^{k-1}(1 + q^i x) \quad \text{which reduces to 1 for } k = 0.$$

Classifying the partitions into distinct parts $\le m$ according to their maximum part k with $0 \le k \le m$, we get

$$(-qx;q)_m \;=\; 1 + x\sum_{k=1}^{m} q^k(-qx;q)_{k-1}.$$

The second identity follows from the first one with $m \to \infty$. □

B4. Partitions and the Gauss q-binomial coefficients

B4.1. Lemma. Let $p_\ell(n|m)$ and $p^\ell(n|m)$ be the numbers of partitions of n into ℓ and $\leq \ell$ parts, respectively, with each part $\leq m$. We have the generating functions:

$$\sum_{\ell,n\geq 0} p_\ell(n|m)\, x^\ell q^n = \frac{1}{(1-xq)(1-xq^2)\cdots(1-xq^m)} \tag{B4.1a}$$

$$\sum_{\ell,n\geq 0} p^\ell(n|m)\, x^\ell q^n = \frac{1}{(1-x)(1-xq)\cdots(1-xq^m)}. \tag{B4.1b}$$

The first identity (B4.1a) is a special case of the generating function shown in (B1.1b).

On account of the length of partitions, we have

$$p^\ell(n|m) = p_0(n|m) + p_1(n|m) + \cdots + p_\ell(n|m).$$

Manipulating the triple sum and then applying the geometric series, we can calculate the corresponding generating function as follows:

$$\begin{aligned}\sum_{\ell,n\geq 0} p^\ell(n|m)\, x^\ell q^n &= \sum_{\ell,n\geq 0}\sum_{k=0}^{\ell} p_k(n|m) x^\ell q^n\\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_k(n|m) q^n \sum_{\ell=k}^{\infty} x^\ell\\ &= \frac{1}{1-x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_k(n|m) x^k q^n.\end{aligned}$$

The last expression leads us immediately to the second bivariate generating function (B4.1b) in view of the first generating function (B4.1a). □

B4.2. The Gauss q-binomial coefficients as generating functions. Let $p_\ell(n|m)$ and $p^\ell(n|m)$ be as in Lemma B4.1. The corresponding univariate generating functions read respectively as

$$\sum_{n\geq 0} p_\ell(n|m)\, q^n = \begin{bmatrix}\ell+m-1\\ m-1\end{bmatrix} q^\ell \tag{B4.2a}$$

$$\sum_{n\geq 0} p^\ell(n|m)\, q^n = \begin{bmatrix}\ell+m\\ m\end{bmatrix} \tag{B4.2b}$$

where the q-Gauss binomial coefficient is defined by

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{(q;\, q)_{m+n}}{(q;\, q)_m (q;\, q)_n}.$$

Proof. For these two formulae, it is sufficient to prove only one identity because

$$p_\ell(n|m) = p^\ell(n|m) - p^{\ell-1}(n|m).$$

In fact, supposing that (B4.2b) is true, then (B4.2a) follows in this manner:

$$\sum_{n\geq 0} p_\ell(n|m)\, q^n = \sum_{n\geq 0} p^\ell(n|m)\, q^n - \sum_{n\geq 0} p^{\ell-1}(n|m)\, q^n$$
$$= \begin{bmatrix} \ell+m \\ m \end{bmatrix}_q - \begin{bmatrix} \ell-1+m \\ m \end{bmatrix}_q = q^\ell \begin{bmatrix} \ell+m-1 \\ m-1 \end{bmatrix}_q.$$

Now we should prove (B4.2b). Extracting the coefficient of x^ℓ from the generation function (B4.1b), we get

$$\sum_{n=0}^{\infty} p^\ell(n|m)\, q^n = [x^\ell] \frac{1}{(x;q)_{m+1}}.$$

Observing that the function $1/(x;q)_{m+1}$ is analytic at $x = 0$ for $|q| < 1$, we can expand it into MacLaurin series:

$$\frac{1}{(x;q)_{m+1}} = \sum_{k=0}^{\infty} B_k(q) x^k$$

where the coefficients $\{B_k(q)\}$ are independent of x to be determinated. Reformulating it under replacement $x \to qx$ as

$$\frac{1}{(qx;q)_{m+1}} = \sum_{k=0}^{\infty} B_k(q) x^k q^k$$

and then noting further that both fractions just displayed differ in factors $(1-x)$ and $(1-xq^{m+1})$, we have accordingly the following:

$$(1-x)\sum_{k=0}^{\infty} B_k(q) x^k = (1-xq^{m+1})\sum_{k=0}^{\infty} B_k(q) x^k q^k.$$

Extracting the coefficient of x^ℓ from both sides we get

$$B_\ell(q) - B_{\ell-1}(q) = q^\ell\, B_\ell(q) - q^{m+\ell}\, B_{\ell-1}(q)$$

which is equivalent to the following recurrence relation

$$B_\ell(q) = B_{\ell-1}(q) \frac{1-q^{m+\ell}}{1-q^\ell} \quad \text{for} \quad \ell = 1, 2, \cdots.$$

Iterating this relation ℓ-times, we find that

$$B_\ell(q) \;=\; B_0(q)\frac{(q^{m+1};q)_\ell}{(q;q)_\ell} \;=\; \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_q$$

where $B_0(q)=1$ follows from setting $x=0$ in the generating function

$$\frac{1}{(xq;q)_{m+1}} \;=\; \sum_{k=0}^{\infty} B_k(q)x^k.$$

Therefore we conclude the proof. □

B4.3. Theorem. Classifying the partitions according to the number of parts, we derive immediately two q-binomial identities (*finite and infinite*):

$$\sum_{\ell=0}^{n} q^\ell \begin{bmatrix} \ell+m \\ m \end{bmatrix} \;=\; \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \tag{B4.3a}$$

$$\sum_{\ell=0}^{\infty} x^\ell \begin{bmatrix} \ell+m \\ m \end{bmatrix} \;=\; \prod_{k=0}^{m} \frac{1}{1-xq^k}. \tag{B4.3b}$$

Proof. In view of (B4.2a) and (B4.2b), the univariate generating functions for the partitions into parts $\le m+1$ with the lengths equal to ℓ and $\le n$ are respectively given by the q-binomial coefficients $q^\ell\begin{bmatrix} \ell+m \\ m \end{bmatrix}$ and $\begin{bmatrix} m+n+1 \\ n \end{bmatrix}$. Classifying the partitions enumerated by the latter according to the number of parts ℓ with $0 \le \ell \le n$, we establish the first identity.

By means of (B4.1b), we have

$$\prod_{k=0}^{m} \frac{1}{1-xq^k} \;=\; \sum_{\ell=0}^{\infty} x^\ell \sum_{n=0}^{\infty} p^\ell(n|m)\, q^n \;=\; \sum_{\ell=0}^{\infty} x^\ell \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}$$

which is the second q-binomial identity. □

B5. Partitions into distinct parts and finite q-differences

Similarly, let $Q_\ell(n|m)$ be the number of partitions of n into exactly ℓ distinct parts with each part $\leq m$. Then we have generating functions

$$\begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{1+\ell}{2}} = \sum_{n\geq 0} Q_\ell(n|m)\, q^n \tag{B5.1}$$

$$\prod_{k=1}^{m}(1+xq^k) = \sum_{\ell,n\geq 0} Q_\ell(n|m)\, x^\ell q^n \tag{B5.2}$$

whose combination leads us to Euler's finite q-differences

$$(x;q)_n = \prod_{\ell=0}^{n-1}(1-xq^\ell) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k. \tag{B5.3}$$

Following the second proof of Theorem B3.1, we can check without difficulty that the shifted Ferrers diagrams of the partitions into ℓ-parts $\leq m$ are unions of the same triangle of length ℓ enumerated by $q^{\binom{\ell+1}{2}}$ and the ordinary partitions into parts $\leq m-\ell$ with length $\leq \ell$ whose generating function reads as the q-binomial coefficient $\begin{bmatrix} m \\ \ell \end{bmatrix}$. The product of them gives the generating function for $\{Q_\ell(n|m)\}_n$.

The second formula is a particular case of (B1.2b). Its combination with the univariate generating function just proved leads us to the following:

$$(-qx;q)_m = \sum_{\ell=0}^{m}\sum_{n\geq 0} Q_\ell(n|m)\, q^n\, x^\ell = \sum_{\ell=0}^{m} x^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{1+\ell}{2}}.$$

Replacing x by $-x/q$ in the above, we get Euler's q-difference formula:

$$(x;q)_m = \sum_{\ell=0}^{m}(-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} x^\ell.$$

Remark The last formula is called the *Euler q-difference formula* because if we put $x := q^{-n}$, the finite sum results in

$$\sum_{\ell=0}^{m}(-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{\ell}{2}-\ell n} = (q^{-n};q)_m = \begin{cases} 0, & 0\leq n<m \\ (-1)^n q^{-\binom{n+1}{2}}(q;q)_n, & n=m \end{cases}$$

just like the ordinary finite differences of polynomials.

Keep in mind of the q-binomial limit

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{1+n-k};q)_k}{(q;q)_k} \longrightarrow \frac{1}{(q;q)_k} \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in Euler's q-finite differences, we recover again the Euler classical partition identity

$$(x;\ q)_\infty \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k\ x^k}{(q;\ q)_k} q^{\binom{k}{2}}$$

where Tannery's theorem has been applied for the limiting process.

CHAPTER C

Durfee Rectangles and Classical Partition Identities

For a partition λ, its Durfee square is the maximum square contained in the Ferrers diagram of λ. It can be generalized similarly to the Durfee rectangles. They will be used, in this chapter, to classify partitions and establish classical partition identities.

C1. q-Series identities of Cauchy and Kummer: Unification

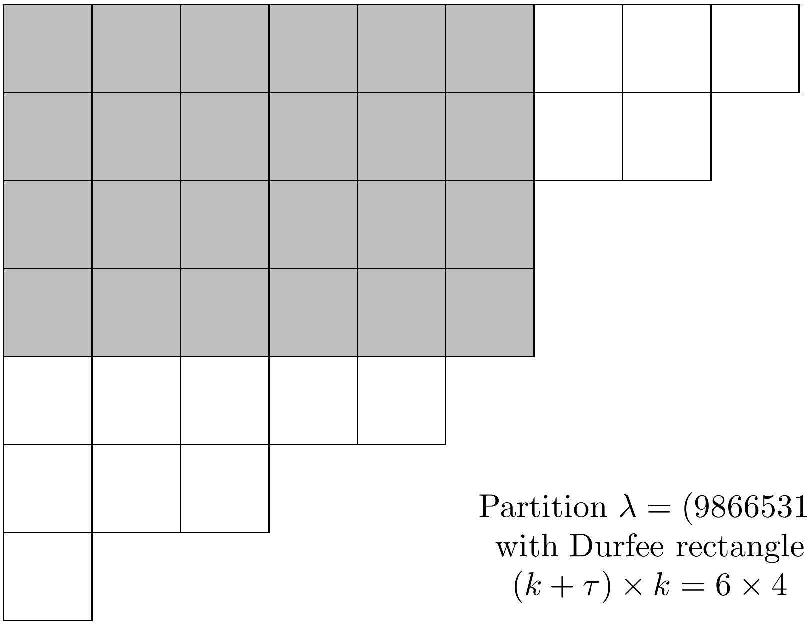
C1.1. Theorem. For the partitions into parts $\leq n$, classify them with respect to the Durfee rectangles of $(k+\tau)\times k$ for a fixed τ. We can derive the following

$$\frac{1}{(qx;\, q)_n} = \sum_{k=0}^{n-\tau} \begin{bmatrix} n-\tau \\ k \end{bmatrix} \frac{q^{k(k+\tau)}}{(qx;\, q)_{k+\tau}}\, x^k. \tag{C1.1}$$

Proof. The partitions into parts $\leq n$ with Durfee rectangles of $(k+\tau)\times k$ for a fixed τ are composed by three pieces. One of them is the Durfee rectangle $(k+\tau)\times k$ in common with enumerator $x^k q^{k(k+\tau)}$. Another is the piece right to Durfee rectangle which are partitions of length $\leq k$ with parts $\leq n-k-\tau$, whose univariate generating function is $\begin{bmatrix} n-\tau \\ k \end{bmatrix}$ in view of (B4.2b) (only the univariate function is considered because the length of partitions has been counted by the Durfee rectangle). The last piece corresponds to the partitions with parts $\leq k+\tau$ whose bivariate generating function is $1/(qx;q)_{k+\tau}$. Classifying the partitions into parts $\leq n$ with respect to Durfee rectangles of $(k+\tau)\times k$ with $0\leq k\leq n-\tau$, we find

$$\frac{1}{(qx;q)_n} = \sum_{k=0}^{n-\tau} \begin{bmatrix} n-\tau \\ k \end{bmatrix} \frac{x^k q^{k(k+\tau)}}{(qx;q)_{k+\tau}}$$

which is exactly the identity required in the theorem. □



Partition $\lambda = (9866531)$
with Durfee rectangle
$(k+\tau) \times k = 6 \times 4$

C1.2. Corollary. The formula just established contains the following known results as special cases:

- The finite version of Kummer's theorem ($\tau = 0$)

$$\frac{1}{(qx;\, q)_n} \quad = \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k\, q^{k^2}}{(qx;\, q)_k}. \tag{C1.2}$$

- The identity due to Gordon and Houten [1968] ($n \to \infty$)

$$\frac{1}{(qx;\, q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k\, q^{k(k+\tau)}}{(q;\, q)_k (qx;\, q)_{k+\tau}} \tag{C1.3}$$

which reduces further to the Cauchy formula with $\tau = 0$.

C2. q-Binomial convolutions and the Jacobi triple product

C2.1. Theorem. For the partitions into parts $\le n$, with at most $\alpha+\gamma-n$ parts, classify them according to the Durfee rectangles of $(n-k)\times(\alpha-k)$. We obtain the first q-Vandermonde convolution formula

$$\begin{bmatrix} \alpha+\gamma \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} \alpha \\ k \end{bmatrix} \begin{bmatrix} \gamma \\ n-k \end{bmatrix} q^{(\alpha-k)(n-k)}. \tag{C2.1}$$

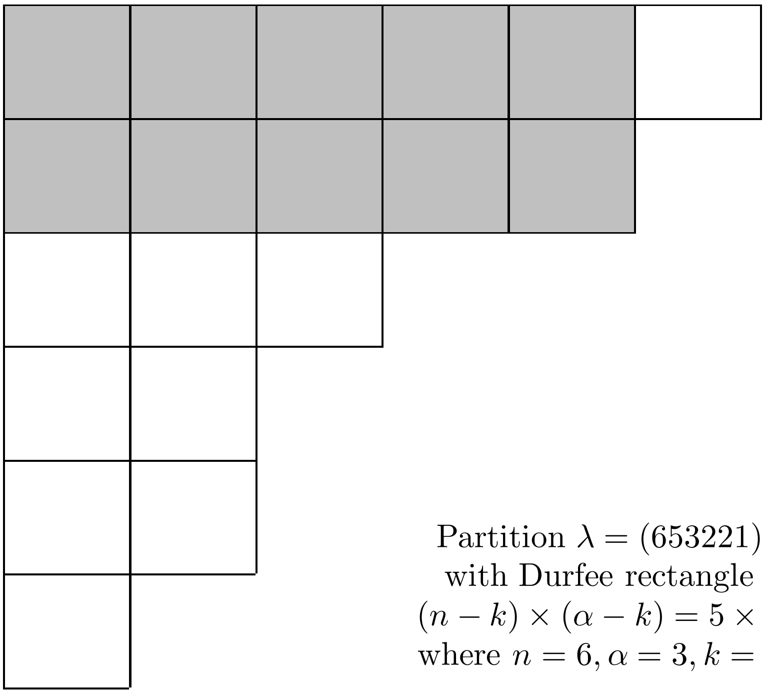
Proof. The univariate generating function of the partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts is equal to $\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix}$ by (B4.2b). Fixing the Durfee rectangle of $(n-k)\times(\alpha-k)$ we see that the corresponding partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts consist of three pieces. The first piece is the rectangle of $(n-k)\times(\alpha-k)$ on the top-left with univariate enumerator $q^{(\alpha-k)(n-k)}$. The second piece right to the rectangle is a partition into parts $\leq k$ with at most $\alpha-k$ parts enumerated by $\begin{bmatrix}\alpha\\ k\end{bmatrix}$. The third and the last piece under the rectangle is a partition into parts $\leq n-k$ with at most $\gamma-n+k=(\alpha+\gamma-n)-(\alpha-k)$ parts enumerated by $\begin{bmatrix}\gamma\\ n-k\end{bmatrix}$. Classifying the partitions according to the Durfee rectangles of $(n-k)\times(\alpha-k)$ and summing the product of three generating functions over $0\leq k\leq n$, we find the following identity:

$$\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix} = \sum_{k=0}^{n}\begin{bmatrix}\alpha\\ k\end{bmatrix}\begin{bmatrix}\gamma\\ n-k\end{bmatrix} q^{(\alpha-k)(n-k)}.$$

Its limiting case $q\to 1$ reduces to

$$\binom{\alpha+\gamma}{n} = \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\gamma}{n-k}$$

which is the well-known Chu-Vandermonde convolution formula. □



Partition $\lambda = (653221)$
with Durfee rectangle
$(n-k)\times(\alpha-k) = 5\times 2$
where $n = 6, \alpha = 3, k = 1$

C2.2. Proposition. Instead, considering the Durfee rectangle of $k\times(\gamma-n)$ for the same partitions, we derive the second q-Vandermonde convolution formula

$$\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix} = \sum_{k=0}^{n}\begin{bmatrix}\alpha+k\\ k\end{bmatrix}\begin{bmatrix}\gamma-k-1\\ n-k\end{bmatrix} q^{k(\gamma-n)}. \tag{C2.2}$$

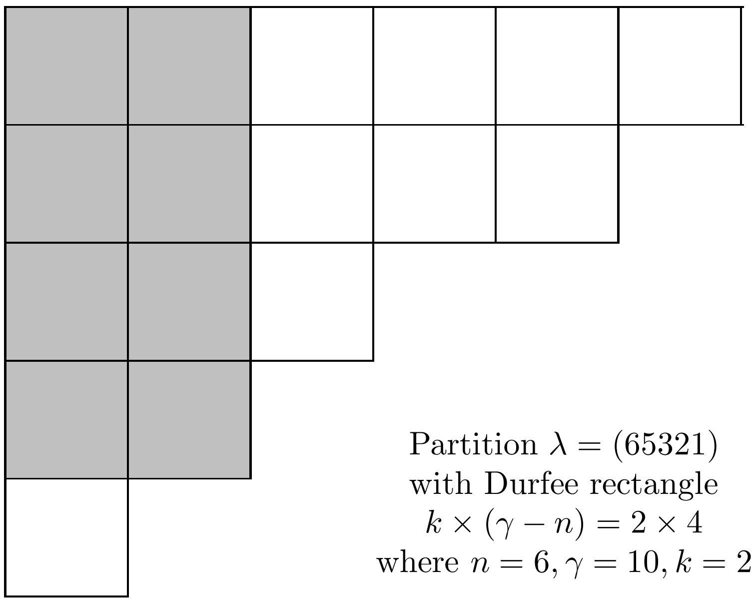
Proof. The univariate generating function of the partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts is equal to $\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix}$ by (B4.2b). For a fixed Durfee rectangle of $k\times(\gamma-n)$ the corresponding partition into parts $\leq n$ with at most $\alpha+\gamma-n$ parts consists of three pieces: the first piece is the rectangle of $k\times(\gamma-n)$ on the top-left with univariate enumerator $q^{k(\gamma-n)}$, the second piece right to the rectangle is a partition into parts $\leq n-k$ with at most $\gamma-n-1$ parts enumerated by $\begin{bmatrix}\gamma-k-1\\ n-k\end{bmatrix}$, where we can easily justify that the partition length can not be $\gamma-n$, otherwise, we would have a larger Durfee rectangle $(k+1)\times(\gamma-n)$, and the third part under the rectangle is a partition into parts $\leq k$ with at most α parts enumerated by $\begin{bmatrix}\alpha+k\\k\end{bmatrix}$. Classifying the partitions with respect to Durfee rectangles of $k\times(\gamma-n)$ and then summing the product of three generating functions over $0\le k\le n$, we find the following identity:

$$\begin{bmatrix}\alpha+\gamma\\n\end{bmatrix}=\sum_{k=0}^{n}\begin{bmatrix}\alpha+k\\k\end{bmatrix}\begin{bmatrix}\gamma-k-1\\n-k\end{bmatrix}q^{k(\gamma-n)}.$$

For $q\to1$, the limiting case reads as

$$\binom{\alpha+\gamma}{n}=\sum_{k=0}^{n}\binom{\alpha+k}{k}\binom{\gamma-k-1}{n-k}$$

which is another binomial convolution formula. □



Partition $\lambda=(65321)$
with Durfee rectangle
$k\times(\gamma-n)=2\times4$
where $n=6,\gamma=10,k=2$

C2.3. Corollary. Given the diagram of $(m-\tau)\times(n+\tau)$, consider the partitions contained in it. The classification with respect to Durfee rectangles of $k\times(k+\tau)$ leads us to the following finite summation formula

$$\begin{bmatrix}m+n\\n+\tau\end{bmatrix}=\sum_{k=0}^{n}\begin{bmatrix}m\\k+\tau\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k+\tau)} \tag{C2.3}$$

which is a special case of the first q-Chu-Vandermonde convolution formula.

Proof. For the partitions into parts $\leq m-\tau$ with at most $n+\tau$ parts, the univariate generating function is equal to $\begin{bmatrix} m+n \\ n+\tau \end{bmatrix}$ by (B4.2b). Fixing a Durfee rectangle of $k \times (k+\tau)$, we observe that the partitions into parts $\leq m-\tau$ with at most $n+\tau$ parts consist of three pieces. The first piece is the rectangle of $k \times (k+\tau)$ on the top-left with univariate enumerator $q^{k(k+\tau)}$. The second piece right to the rectangle is a partition into parts $\leq m-\tau-k$ with at most $k+\tau$ parts enumerated by $\begin{bmatrix} m \\ k+\tau \end{bmatrix}$ and the third one under the rectangle is a partition into parts $\leq k$ with at most $n-k$ parts enumerated by $\begin{bmatrix} n \\ k \end{bmatrix}$. Classifying the partitions according to the Durfee rectangles of $k \times (k+\tau)$ for $0 \leq k \leq n$ and then summing the product of three generating functions over $0 \leq k \leq n$, we find the following identity:

$$\begin{bmatrix} m+n \\ n+\tau \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} m \\ k+\tau \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+\tau)}$$

which is exactly the identity stated in the theorem.

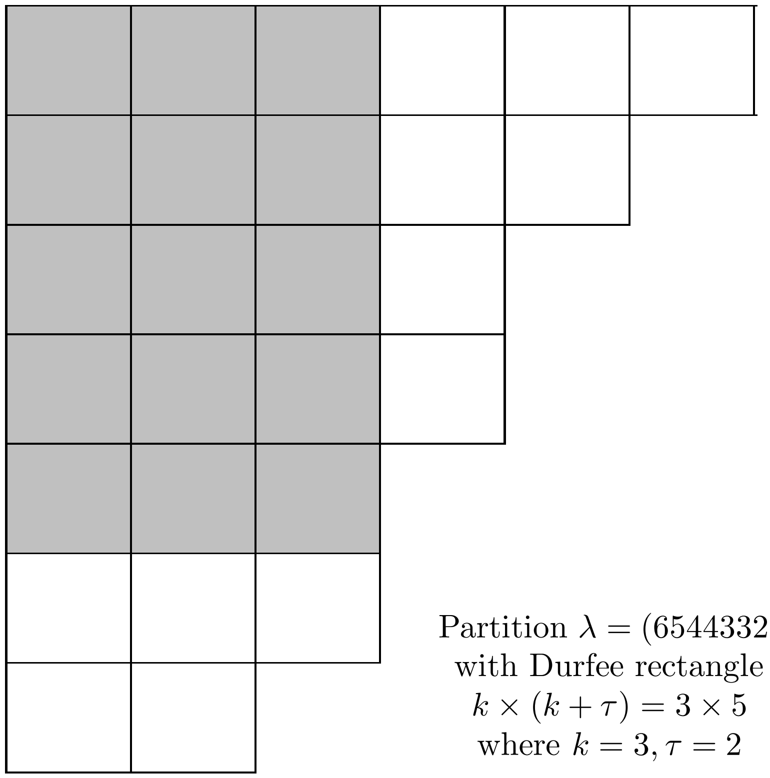
We remark that this identity is a special case of the first q-Vandermonde convolution formula stated in Theorem C2.1. In fact replacing n with ℓ, we can state the reversal of the q-Vandermonde convolution formula in Theorem C2.1 as follows:

$$\begin{bmatrix} \alpha+\gamma \\ \ell \end{bmatrix} = \sum_{k=0}^{\ell} \begin{bmatrix} \alpha \\ \ell-k \end{bmatrix} \begin{bmatrix} \gamma \\ k \end{bmatrix} q^{k(\alpha+k-\ell)}.$$

Performing parameter replacements

$$\alpha \to m, \quad \gamma \to n \quad \text{and} \quad \ell \to m-\tau$$

we obtain immediately the identity stated in Corollary C2.3. □



Partition $\lambda = (6544332)$
with Durfee rectangle
$k \times (k+\tau) = 3 \times 5$
where $k = 3, \tau = 2$

C2.4. The Jacobi-triple product identity. From the last q-binomial convolution identity, we can derive the following bilateral summation formula

$$(x;\, q)_m\, (q/x;\, q)_n \quad = \quad \sum_{k=-n}^{m} (-1)^k\, q^{\binom{k}{2}} \begin{bmatrix} m+n \\ n+k \end{bmatrix} x^k. \tag{C2.4}$$

It can be considered as a finite form of the well-known Jacobi triple product identity

$$(q;\, q)_\infty\, (x;\, q)_\infty\, (q/x;\, q)_\infty \quad = \quad \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\binom{n}{2}}\, x^n \tag{C2.5}$$

whose limiting case $x \to 1$ reads as the cubic form of the triple product (Jacobi):

$$(q;\, q)_\infty^3 \quad = \quad \sum_{n=0}^{\infty} (-1)^n \{1+2n\} q^{\binom{1+n}{2}}. \tag{C2.6}$$

PROOF. According to the Euler q-finite differences (B5.3), we have two finite expansions

$$\begin{aligned}(x,q)_m &= \sum_{i=0}^{m}(-1)^i \begin{bmatrix} m \\ i \end{bmatrix} q^{\binom{i}{2}} x^i \\ (q/x,q)_n &= \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{1+j}{2}} x^{-j}.\end{aligned}$$

Their product reads as the following double sum

$$\begin{aligned}(x,q)_m(q/x,q)_n &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{i}{2}+\binom{1+j}{2}} x^{i-j} \\ &= \sum_{k=-n}^{m}(-1)^k x^k \sum_{j=0}^{n} \begin{bmatrix} m \\ k+j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{k+j}{2}+\binom{1+j}{2}}\end{aligned}$$

where the last line is justified by the replacement $k = i - j$. Observe that

$$\binom{k+j}{2} + \binom{1+j}{2} = \binom{k}{2} + \binom{j}{2} + kj + \binom{1+j}{2} = \binom{k}{2} + j(j+k).$$

Reformulating the double sum and then applying the convolution formula stated in Corollary C2.3, we derive the finite bilateral summation formula (C2.4)

$$\begin{aligned}(x,q)_m(q/x,q)_n &= \sum_{k=-n}^{m}(-1)^k q^{\binom{k}{2}} x^k \sum_{j=0}^{n} \begin{bmatrix} m \\ k+j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+k)} \\ &= \sum_{k=-n}^{m}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} m+n \\ n+k \end{bmatrix} x^k.\end{aligned}$$

When m and n tend to infinity, the limit of q-binomial coefficient reads as

$$\begin{bmatrix} m+n \\ n+k \end{bmatrix} = \frac{(q;q)_{m+n}}{(q;q)_{n+k}(q;q)_{m-k}} \to \frac{1}{(q;q)_\infty}.$$

Applying the Tannery Theorem, we therefore have

$$(x,q)_\infty(q/x,q)_\infty = \sum_{k=-\infty}^{+\infty}(-1)^k \frac{q^{\binom{k}{2}} x^k}{(q;q)_\infty}$$

which is equivalent to the Jacobi-triple product identity (C2.5).

In order to prove (C2.6), we rewrite the Jacobi triple product identity as

$$\begin{aligned}(q;\, q)_\infty (x;\, q)_\infty (q/x;\, q)_\infty &= \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\binom{1+n}{2}}\, x^{-n} \\ &= \sum_{n=0}^{+\infty} (-1)^n\, q^{\binom{1+n}{2}}\, x^{-n} \\ &+ \sum_{n=1}^{+\infty} (-1)^n\, q^{\binom{1-n}{2}}\, x^{n}.\end{aligned}$$

Replacing the summation index n by $1+m$ in the last sum:

$$\sum_{n=1}^{\infty} (-1)^n\, q^{\binom{n}{2}}\, x^n = -\sum_{m=0}^{\infty} (-1)^m\, q^{\binom{1+m}{2}}\, x^{m+1}$$

we can combine two sums into one unilateral sum

$$(q;\, q)_\infty (x;\, q)_\infty (q/x;\, q)_\infty = \sum_{n=0}^{\infty} (-1)^n\, q^{\binom{1+n}{2}} \big\{x^{-n} - x^{n+1}\big\}.$$

Dividing both sides by $1-x$, we get

$$(q;\, q)_\infty (qx;\, q)_\infty (q/x;\, q)_\infty = \sum_{n=0}^{\infty} (-1)^n\, q^{\binom{1+n}{2}} \frac{x^{-n} - x^{n+1}}{1-x}.$$

Applying L'Hôspital's rule for the limit, we have

$$\lim_{x\to 1} \frac{x^{-n} - x^{n+1}}{1-x} = 2n+1.$$

Considering that the series is uniformly convergent and then evaluating the limit $x \to 1$ term by term, we establish

$$(q;\, q)^3_\infty = \sum_{n=0}^{\infty} (-1)^n \big\{2n+1\big\} q^{\binom{1+n}{2}}$$

which is the cubic form of triple product. □

Remark The shortest proof of the Jacobi triple product identity is due to Cauchy (1843) and Gauss (1866). It can be reproduced in the sequel.

Recall the q-binomial theorem (finite q-differences) displayed in (B5.3)

$$(x; q)_\ell = \sum_{k=0}^{\ell} (-1)^k \begin{bmatrix} \ell \\ k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

Replacing ℓ by $m+n$ and x by xq^{-n} respectively, and then noting the relation

$$(q^{-n}x;q)_{m+n} \;=\; (q^{-n}x;q)_n(x;q)_m \;=\; (-1)^n q^{-\binom{1+n}{2}} x^n (q/x;q)_n(x;q)_m$$

we can reformulate the q-binomial theorem as

$$(x;q)_m(q/x;q)_n \;=\; \sum_{k=0}^{m+n} (-1)^{k-n} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\binom{k-n}{2}} x^{k-n}$$

which becomes, under summation index substitution $k \to n+k$, the following finite form of the Jacobi triple product identity

$$(x;q)_m(q/x;q)_n \;=\; \sum_{k=-n}^{m} (-1)^k \begin{bmatrix} m+n \\ n+k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

This is exactly the finite form (C2.4) of the Jacobi triple product identity.

C2.5. Corollary. From Jacobi's triple product identity, we may further derive the following infinite series identities:

- Triangle number theorem (Gauss)

$$\frac{(q^2;\,q^2)_\infty}{(q;\,q^2)_\infty} \;=\; \sum_{n=0}^{\infty} q^{\binom{1+n}{2}}.$$

- Pentagon number theorem (Euler)

$$(q;\,q)_\infty \;=\; \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\frac{n}{2}(3n+1)}.$$

Proof. Reformulate the factorial fraction in this way:

$$\begin{aligned}
\frac{(q^2;\,q^2)_\infty}{(q;\,q^2)_\infty} \;&=\; \frac{(q;\,q)_\infty(-q;\,q)_\infty}{(q;\,q^2)_\infty} \;=\; (q^2;\,q^2)_\infty(-q;\,q)_\infty \\
&=\; (q;\,q)_\infty(-q;\,q)_\infty(-q;\,q)_\infty \\
&=\; \frac{1}{2}(q;\,q)_\infty(-1;\,q)_\infty(-q;\,q)_\infty.
\end{aligned}$$

Applying the Jacobi triple product identity, we have

$$\begin{aligned}
\frac{(q^2;\,q^2)_\infty}{(q;\,q^2)_\infty} \;=\; \frac{1}{2}\sum_{n=-\infty}^{+\infty} q^{\binom{n}{2}} \;&=\; \frac{1}{2}\Big\{\sum_{n=1}^{+\infty} q^{\binom{n}{2}} + \sum_{n=0}^{+\infty} q^{\binom{-n}{2}}\Big\} \\
&=\; \frac{1}{2}\Big\{\sum_{n=0}^{+\infty} q^{\binom{1+n}{2}} + \sum_{n=0}^{+\infty} q^{\binom{n+1}{2}}\Big\}
\end{aligned}$$

where the substitution $n \to 1+n$ has been made for the first sum and $\binom{-n}{2} = \binom{1+n}{2}$ for the second sum. Canceling the factor 1/2 by two times of the same sum, we have the triangle number theorem.

Now, we prove pentagon number theorem. Classifying the factors of product $(q;q)_\infty$ according to the residues of the indices modulo 3, we have

$$(q;\, q)_\infty \;=\; (q^3;\, q^3)_\infty (q;\, q^3)_\infty (q^2;\, q^3)_\infty.$$

Then the Jacobi triple product identity (C2.5) yields

$$(q;\, q)_\infty \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{3\binom{n}{2}+n} \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n}{2}(3n+1)}$$

which is Euler's pentagon number theorem. □

C2.6. The quintuple product identity. Furthermore, we can derive the quintuple product identity

$$\begin{aligned} [q,\, z,\, q/z; q]_\infty \; \left[qz^2,\, q/z^2; q^2\right]_\infty &= \sum_{n=-\infty}^{+\infty} \left\{1 - zq^n\right\} q^{3\binom{n}{2}} \left(qz^3\right)^n \\ &= \sum_{n=-\infty}^{+\infty} \left\{1 - z^{1+6n}\right\} q^{3\binom{n}{2}} \left(q^2/z^3\right)^n \end{aligned}$$

and its limit form

$$(q;\, q)_\infty^3 \, (q;\, q^2)_\infty^2 \quad = \quad \sum_{n=-\infty}^{+\infty} \{1+6n\} q^{\frac{n}{2}(3n+1)}.$$

C2.7. Proof. Multiplying two copies of the Jacobi triple products

$$\begin{aligned} [q,\, z,\, q/z; q]_\infty \quad &= \quad \sum_{i=-\infty}^{+\infty} (-1)^i q^{\binom{i}{2}} z^i \\ \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \quad &= \quad \sum_{j=-\infty}^{+\infty} (-1)^j q^{j^2} z^{2j} \end{aligned}$$

we have the double sum expression

$$[q,\, z,\, q/z; q]_\infty \; \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \;=\; \sum_{i,\, j=-\infty}^{+\infty} (-1)^{i+j} q^{\binom{i}{2}+j^2} z^{i+2j}.$$

Defining a new summation index $k = i + 2j$ and then rearranging the double sum, we can write

$$[q,\, z,\, q/z; q]_\infty \; \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \;=\; \sum_{k=-\infty}^{+\infty} (-1)^k z^k \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{k-2j}{2}+j^2}.$$

Noting the binomial relation

$$\binom{k-2j}{2} \;=\; \binom{k}{2} + \binom{2j+1}{2} - 2kj \;=\; \binom{k}{2} + 2j^2 + j - 2kj$$

we find that

$$\begin{aligned}[q,\, z,\, q/z; q]_\infty \; \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \;&=\; \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} z^k \\ &\times\; \sum_{j=-\infty}^{+\infty} (-1)^j q^{3j^2+j-2kj}.\end{aligned}$$

Applying the Jacobi product identity to the inner sum, we get

$$\begin{aligned}\sum_{j=-\infty}^{+\infty} (-1)^j q^{3j^2+j-2kj} \;&=\; \sum_{j=-\infty}^{+\infty} (-1)^j q^{6\binom{j}{2}+2(2-k)j} \\ &=\; [q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty.\end{aligned}$$

This product can be simplified according to the residues of k modulo 3.

- $k = 3m$ with $m \in \mathbb{Z}$:

$$\begin{aligned}[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;&=\; [q^6, q^{2+6m}, q^{4-6m}, q^6]_\infty \\ &=\; \frac{(q^{4-6m}; q^6)_m}{(q^2; q^6)_m}[q^6, q^2, q^4, q^6]_\infty \\ &=\; (-1)^m\, (q^2; q^2)_\infty\, q^{m-3m^2}.\end{aligned}$$

- $k = 1 + 3m$ with $m \in \mathbb{Z}$:

$$\begin{aligned}[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;&=\; [q^6, q^{4+6m}, q^{2-6m}, q^6]_\infty \\ &=\; \frac{(q^{2-6m}; q^6)_m}{(q^4; q^6)_m}[q^6, q^2, q^4, q^6]_\infty \\ &=\; (-1)^m\, (q^2; q^2)_\infty\, q^{-m-3m^2}.\end{aligned}$$

- $k = 2 + 3m$ with $m \in \mathbb{Z}$:

$$[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;=\; [q^6, q^{6+6m}, q^{-6m}, q^6]_\infty \;=\; 0$$

because of the presence of zero-factor:

$$(q^{-6m};\ q)_\infty = 0, \quad m \geq 0$$
$$(q^{6+6m}; q)_\infty = 0, \quad m < 0.$$

Substituting these results into the infinity series expression, we obtain

$$\begin{aligned}
&[q,\ z,\ q/z; q]_\infty \ \left[q^2,\ qz^2,\ q/z^2; q^2\right]_\infty \\
=\ & \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} z^k [q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \\
=\ & (q^2; q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\binom{3m}{2}+m-3m^2}\ z^{3m} \\
-\ & (q^2; q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\binom{1+3m}{2}-m-3m^2} z^{1+3m} \\
=\ & (q^2; q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\frac{3m^2-m}{2}} \big\{1 - zq^m\big\} z^{3m}.
\end{aligned}$$

Dividing both sides by $(q^2; q^2)_\infty$, we get the quintuple product identity:

$$[q,\ z,\ q/z; q]_\infty \ \left[qz^2,\ q/z^2; q^2\right]_\infty = \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}} \big\{1 - zq^m\big\} (qz^3)^m.$$

Splitting the last sum into two and then reverse the first sum, we have

$$\begin{aligned}
&[q,\ z,\ q/z; q]_\infty \times \left[qz^2,\ q/z^2; q^2\right]_\infty \\
=\ & \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}} \big\{1 - zq^m\big\} (qz^3)^m \\
=\ & \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+m} z^{3m} - \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+2m} z^{1+3m} \\
=\ & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}+2n} z^{-3n} - \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+2m} z^{1+3m} \\
=\ & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \big\{1 - z^{1+6n}\big\} \left(q^2/z^3\right)^n
\end{aligned}$$

which is exactly the second version of the quintuple product identity.

Finally, dividing both sides by $1-z$

$$[q,\,qz,\,q/z;q]_\infty\ \left[qz^2,\,q/z^2;q^2\right]_\infty \;=\; \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\frac{1-z^{1+6n}}{1-z}\left(q^2/z^3\right)^n$$

and then letting $z\to 1$, we get the limiting case of the quintuple product identity

$$(q;\,q)^3_\infty\,(q;\,q^2)^2_\infty \;=\; \sum_{n=-\infty}^{+\infty}\{1+6n\}\,q^{\frac{n}{2}(3n+1)}.$$

C3. The finite form of Euler's pentagon number theorem

C3.1. Theorem. The classification of partitions enumerated by $(-qx;\,q)_n$ with respect to the Durfee rectangles of $(k+\epsilon)\times k$ leads us to the following finite form of the Euler pentagon number theorem.

Denote by $[\theta]$ the integral part of real number θ. Then there holds

$$\begin{aligned}(-qx;\,q)_n \;&=\; \sum_{k=0}^{[\frac{n-\epsilon}{2}]} q^{k(k+\epsilon)+\binom{k}{2}}\begin{bmatrix}n-k-\epsilon\\ k\end{bmatrix}(-qx;\,q)_{k+\epsilon}\\ &\times\;\frac{1+xq^{2k+\epsilon}-q^{1+n-k-\epsilon}(1+xq^{k+\epsilon})}{(1+xq^{k+\epsilon})\,(1-q^{1+n-2k-\epsilon})}\,x^k.\end{aligned}$$

C3.2. Proof. For the partitions into distinct parts $\le n$ enumerated by $(-qx;\,q)_n$, they are divided by the Durfee rectangles of $(k+\epsilon)\times k$ into three pieces:

A: the Durfee rectangle $(k+\epsilon)\times k$ itself with enumerator $x^kq^{k(k+\epsilon)}$.

B: the piece of partitions right to the Durfee rectangle counted by

$$\begin{cases}\begin{bmatrix}n-k-\epsilon\\ k\end{bmatrix}q^{\binom{1+k}{2}}, & \text{with } k \text{ parts},\\ \begin{bmatrix}n-k-\epsilon\\ k-1\end{bmatrix}q^{\binom{k}{2}}, & \text{with } k-1 \text{ parts}.\end{cases}$$

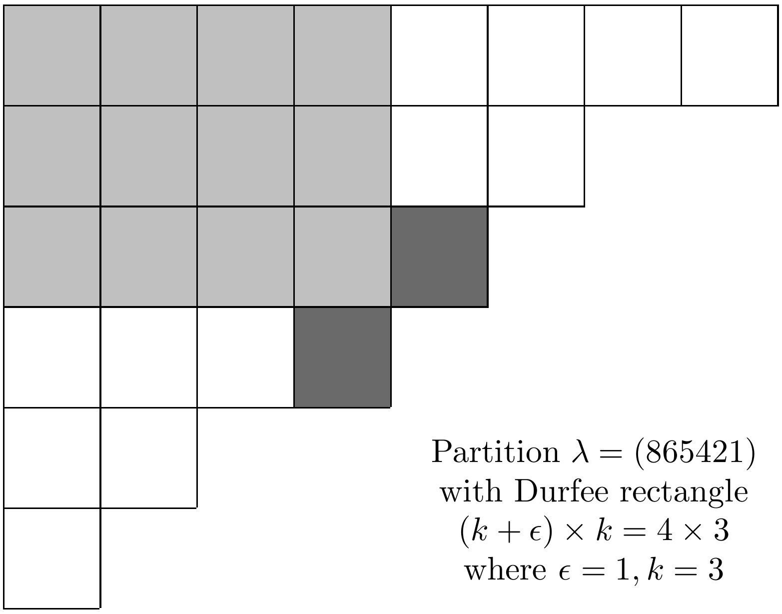
C: the piece of partitions below the Durfee rectangle enumerated by

$$\begin{cases}(-qx;\,q)_{k+\epsilon}, & \text{when } \mathbf{B} \text{ has } k \text{ parts},\\ (-qx;\,q)_{k+\epsilon-1}, & \text{when } \mathbf{B} \text{ has } k-1 \text{ parts}.\end{cases}$$

Therefore for the fixed Durfee rectangle **A**, the enumerator for the rest of partitions is given by the combination of **B** and **C** as follows

$$\begin{aligned}
& q^{\binom{1+k}{2}} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} (-qx;\, q)_{k+\epsilon} + q^{\binom{k}{2}} \begin{bmatrix} n-k-\epsilon \\ k-1 \end{bmatrix} (-qx;\, q)_{k+\epsilon-1} \\
= \quad & q^{\binom{k}{2}} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} \frac{1+xq^{\epsilon+2k} - q^{1+n-k-\epsilon}(1+xq^{k+\epsilon})}{(1+xq^{k+\epsilon})\,(1-q^{1+n-2k-\epsilon})} (-qx;\, q)_{k+\epsilon}.
\end{aligned}$$

Summing the last expression over $0 \le k \le [(n-\epsilon)/2]$, we get the identity stated in Theorem C3.1.



Partition $\lambda = (865421)$
with Durfee rectangle
$(k+\epsilon) \times k = 4 \times 3$
where $\epsilon = 1, k = 3$

C3.3. Corollary. This formula contains the following well-known results as special cases:

- The limiting version with two parameters ($n \to \infty$)

$$(-qx;\, q)_\infty \;=\; \sum_{n=0}^{\infty} q^{n(n+\epsilon)+\binom{n}{2}} \frac{1+xq^{2n+\epsilon}}{1+xq^{n+\epsilon}} \frac{(-qx;\, q)_{n+\epsilon}}{(q;\, q)_n} x^n.$$

- The Sylvester formula ($\epsilon = 1$, $x = -y/q$ and $n \to \infty$)

$$(y;\, q)_\infty \quad = \quad \sum_{n=0}^{\infty} (-y)^n \left\{1 - yq^{2n}\right\} \frac{(y;\, q)_n}{(q;\, q)_n} q^{\frac{3n^2-n}{2}}.$$

- The Euler pentagon number theorem ($\epsilon = 0$, $x = -1$ and $n \to \infty$)

$$(q;\, q)_\infty \quad = \quad 1 + \sum_{n=1}^{\infty} (-1)^n \left\{1 + q^n\right\} q^{\frac{3n^2-n}{2}}.$$

Remark The Euler pentagon number theorem is also a particular case of the Sylvester formula. In fact, for $y \to 1$, the limit can be computed term by term as follows:

$$\begin{aligned}
(q;\, q)_\infty &= \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\, q)_n} \lim_{y\to 1} (-y)^n \frac{(1-yq^{2n})(y;\, q)_n}{1-y} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \big\{1 - q^{2n}\big\} \frac{(q;\, q)_{n-1}}{(q;\, q)_n} q^{\frac{3n^2-n}{2}} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \big\{1 + q^n\big\} q^{\frac{3n^2-n}{2}}.
\end{aligned}$$

CHAPTER D

The Carlitz Inversions and Rogers-Ramanujan Identities

According to the Jacobi triple product identity, we have

$$\big[q^4, \pm q, \pm q^3;\ q^4\big] \quad = \quad \sum_{k=-\infty}^{+\infty} (\mp 1)^k q^{2k^2+k}.$$

The sum of both triple products can be evaluated as a single triple product:

$$\begin{aligned} &\big[q^4,\ -q,\ -q^3;\ q^4\big]_\infty + \big[q^4,\ q,\ q^3;\ q^4\big]_\infty \\ =\ & 2\sum_{n=-\infty}^{+\infty} q^{8n^2+2n} \ =\ 2\sum_{n=-\infty}^{+\infty} q^{16\binom{n}{2}+10n} \\ =\ & 2\big[q^{16}, -q^6, -q^{10};\ q^{16}\big]_\infty. \end{aligned}$$

We can similarly treat their difference as follows:

$$\begin{aligned} &\big[q^4,\ -q,\ -q^3;\ q^4\big]_\infty - \big[q^4,\ q,\ q^3;\ q^4\big]_\infty \\ =\ & 2\sum_{n=-\infty}^{+\infty} q^{8n^2-6n+1} \ =\ 2\sum_{n=-\infty}^{+\infty} q^{16\binom{n}{2}+2n+1} \\ =\ & 2q\big[q^{16}, -q^2, -q^{14};\ q^{16}\big]_\infty. \end{aligned}$$

Dividing both equations by $(q^4; q^4)_\infty$ and noting the fact that the odd natural numbers are congruent to 1 or to 3 modulo 4, we get two q-difference equations:

$$(-q;\ q^2)_\infty \ +\ (q;\ q^2)_\infty = \frac{2}{(q^4;\ q^4)_\infty}\sum_n q^{8n^2+2n} \tag{D0.1a}$$

$$= 2\frac{\big[q^{16}, -q^6, -q^{10};\ q^{16}\big]}{(q^4;\ q^4)_\infty} \tag{D0.1b}$$

$$(-q;\ q^2)_\infty \ -\ (q;\ q^2)_\infty = \frac{2q}{(q^4;\ q^4)_\infty}\sum_n q^{8n^2-6n} \tag{D0.2a}$$

$$= 2q\frac{\big[q^{16}, -q^2, -q^{14};\ q^{16}\big]}{(q^4;\ q^4)_\infty}. \tag{D0.2b}$$

Further, if we specify with $x \mapsto \pm q^{1/2}$ in Euler's q-difference formula

$$(x;\ q)_\infty \quad = \quad \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(q;\ q)_m} q^{\binom{m}{2}}$$

then we find that

$$(\pm q^{1/2};\ q)_\infty \quad = \quad \sum_{m=0}^{\infty} \frac{(\mp 1)^m q^{m^2/2}}{(q;\ q)_m}$$

whose linear combinations lead us to two summation formulae as follows:

$$(-q^{1/2};\ q)_\infty + (q^{1/2};\ q)_\infty = 2 \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;\ q)_{2n}} \tag{D0.3a}$$

$$(-q^{1/2};\ q)_\infty - (q^{1/2};\ q)_\infty = 2q^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n(2n+1)}}{(q;\ q)_{2n+1}}. \tag{D0.3b}$$

Replacing the base q by $q^{1/2}$ in (D0.1a-D0.1b) and (D0.2a-D0.2b), we can reformulate the left hand sides of both equations just displayed respectively as follows:

$$(-q^{1/2};\ q)_\infty + (q^{1/2};\ q)_\infty = 2 \frac{\left[q^8, -q^3, -q^5;\ q^8\right]}{(q^2;\ q^2)_\infty} \tag{D0.4a}$$

$$(-q^{1/2};\ q)_\infty - (q^{1/2};\ q)_\infty = 2q^{1/2} \frac{\left[q^8, -q, -q^7;\ q^8\right]}{(q^2;\ q^2)_\infty}. \tag{D0.4b}$$

Combining (D0.3a) and (D0.3b) respectively with (D0.4a) and (D0.4b), we establish two infinite series identities:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;\ q)_{2n}} \quad = \quad \frac{\left[q^8, -q^3, -q^5;\ q^8\right]}{(q^2;\ q^2)_\infty} \tag{D0.5a}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;\ q)_{2n+1}} \quad = \quad \frac{\left[q^8, -q, -q^7;\ q^8\right]}{(q^2;\ q^2)_\infty}. \tag{D0.5b}$$

They are only very simple examples of classical partition identities of Roger-Ramanujan's type. By means of inverse series relations, we establish a finite series transformation, which leads us to an elementary derivation to the celebrated Rogers-Ramanujan identities and their finite forms.

D1. Combinatorial inversions and series transformations

D1.1. The Carlitz inversions. Let $\{a_i\}$ and $\{b_j\}$ be two complex sequences such that the polynomials defined by

$$\phi(x;\,0)=1 \quad \text{and} \quad \phi(x;\,n)=\prod_{k=0}^{n-1}(a_k+xb_k), \quad \text{for } n=1,2,\cdots$$

differ from zero for $x=q^n$ with n being non-negative integers. Then we have the following inverse series relations due to Carlitz (1973)

$$\begin{cases} F(n) = \displaystyle\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}\phi(q^k;n)\,G(k), & n=0,1,2,\cdots \quad \text{(D1.1a)} \\ G(n) = \displaystyle\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_k+q^kb_k}{\phi(q^n;k+1)}\,F(k), & n=0,1,2,\cdots \quad \text{(D1.1b)} \end{cases}$$

which may be considered as q-analogue of Gould-Hsu Inversions (1973).

Proof. To prove the bilateral implications (D1.1a) $\rightleftharpoons$ (D1.1b), it is sufficient to verify one implication because one system of equations with $F(n)$ in terms of $G(k)$ can be considered as the (unique) solution of another system with $G(n)$ in terms of $F(k)$, and vice versa.

$\Longleftarrow$ We first reproduce the original proof due to Carlitz. Suppose that the relations of $G(n)$ in terms of $F(k)$ are valid. We have to verify the relations of $F(n)$ in terms of $G(k)$.

Substituting the relations of $G(n)$ in terms of $F(k)$ into the right hand sides of those of $F(n)$ in terms of $G(k)$ and observing that

$$\begin{bmatrix} n \\ k \end{bmatrix} \times \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} \times \begin{bmatrix} n-i \\ k-i \end{bmatrix}$$

we get the double sum

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \phi(q^k;n)\, G(k) \\
= &\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k;n) \sum_{i=0}^{k}(-1)^i \begin{bmatrix} k \\ i \end{bmatrix} \frac{a_i+q^ib_i}{\phi(q^k;i+1)} F(i) \\
= &\sum_{i=0}^{n}(a_i+q^ib_i) \begin{bmatrix} n \\ i \end{bmatrix} F(i) \sum_{k=i}^{n}(-1)^{k+i} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^k;n)}{\phi(q^k;i+1)} q^{\binom{n-k}{2}} \\
= &\sum_{i=0}^{n}(a_i+q^ib_i) \begin{bmatrix} n \\ i \end{bmatrix} F(i) \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix} \frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)} q^{\binom{n-i-\ell}{2}}.
\end{aligned}$$

Let $S(i,n)$ stand for the inner sum with respect to ℓ:

$$S(i,n) := \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix} q^{\binom{n-i-\ell}{2}} \frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)}.$$

It is trivial to see that

$$S(n,n) = \frac{\phi(q^n;n)}{\phi(q^n;n+1)} = \frac{1}{a_n+q^nb_n}$$

which implies that the double sum reduces to $F(n)$ when $i=n$.

In order to prove that the double sum is equal to $F(n)$, it suffices for us to verify that $S(i,n)=0$ for $0 \le i < n$.

Noting that $\frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)}$ is a polynomial of degree $n-i-1$ in q^{ℓ}, we can write it formally as

$$\frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)} = \sum_{j=0}^{n-i-1} C_j\, q^{\ell(n-i-j-1)}$$

where $\{C_j\}$ are constants independent of ℓ. Therefore the sum $S(i,n)$ can be reformulated accordingly as follows:

$$\begin{aligned}
S(i,n) &= \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i \\ \ell \end{bmatrix} q^{\binom{n-i-\ell}{2}} \sum_{j=0}^{n-i-1} C_j\, q^{\ell(n-i-j-1)} \\
&= \sum_{j=0}^{n-i-1} C_j\, q^{\binom{n-i}{2}} \sum_{\ell=0}^{n-i}(-1)^{\ell} q^{\binom{\ell}{2}} \begin{bmatrix} n-i \\ \ell \end{bmatrix} q^{-\ell j}
\end{aligned}$$

where we have applied the binomial relation

$$\binom{n-i-\ell}{2} = \binom{n-i}{2} + \binom{\ell}{2} - \ell(n-i-1).$$

Evaluating the sum with respect to ℓ by Euler's q-difference formula (B5.3)

$$\sum_{\ell=0}^{n-i}(-1)^\ell \begin{bmatrix} n-i \\ \ell \end{bmatrix} q^{\binom{\ell}{2}-\ell j} \;=\; (q^{-j};q)_{n-i}$$

which vanishes for $0 \le j < n-i$.

This completes the proof of the Carlitz inversions stated in D1.1. □

$\Longrightarrow$ An alternative proof is worth to be included. Assuming that (D1.1a) is true for all $n \in \mathbb{N}_0$, we should verify the truth of (D1.1b).

In fact, substituting the first relation into the second, we reduce the question to the confirmation of the following orthogonal relation:

$$\sum_{k=i}^{n}(-1)^{k+i}\big\{a_k + q^k b_k\big\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k+1)} q^{\binom{k-i}{2}} \;=\; \begin{cases} 1, & i=n \\ 0, & i \neq n. \end{cases} \tag{D1.2}$$

It is obvious that the relation is valid for $i = n$. We therefore need to verify it only when $i < n$. For that purpose, we introduce the sequence

$$\tau_k := \begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k)} q^{\binom{k-i}{2}}.$$

Then it is not hard to check that the summand in (D1.2) can be expressed as follows:

$$\tau_k + \tau_{k+1} \;=\; \big\{a_k + q^k b_k\big\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k+1)} q^{\binom{k-i}{2}}.$$

Separating the two extreme terms indexed with $k = i$ and $k = n$ from the sum displayed in (D1.2)

$$\tau_{i+1} \;=\; \frac{\phi(q^i;i+1)}{\phi(q^n;i+1)}$$

$$\tau_n \;=\; \frac{\phi(q^i;n)}{\phi(q^n;n)} q^{\binom{n-i}{2}}$$

and then appealing for the telescoping method, we find that

$$\begin{aligned} \mathrm{LHS(D1.2)} \;&=\; \tau_{i+1} + (-1)^{n+i}\tau_n \;+\; \sum_{i<k<n}(-1)^{k+i}\big\{\tau_k + \tau_{k+1}\big\} \\ &=\; \big\{\tau_{i+1} + (-1)^{n+i}\tau_n\big\} - \big\{\tau_{i+1} + (-1)^{n+i}\tau_n\big\} \;=\; 0. \end{aligned}$$

This completes the proof of (D1.2). □

D1.2. Series transformation. For the polynomials $\phi(x;n) = (\lambda x;q)_n$ specified with $a_k = 1$ and $b_k = -q^k\lambda$, the inverse series relations displayed in D1.1 become the following:

$$f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k\lambda;q)_n\, g(k), \quad n = 0,1,2,\cdots \tag{D1.3a}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^n\lambda;\ q)_{k+1}}\ f(k), \quad n = 0,1,2,\cdots. \tag{D1.3b}$$

By means of the finite version of Kummer's theorem and rearrangement of double sums, we may establish finite and infinite series transformations

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_n}\, g(n) = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{m+k+1}} \lambda^k q^{k^2} f(k) \tag{D1.4a}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q;q)_n(\lambda;q)_n} g(n) = \sum_{k=0}^{\infty} (-1)^k\ \frac{1-q^{2k}\lambda}{(\lambda;q)_\infty}\ \frac{\lambda^k q^{k^2}}{(q;q)_k}\ f(k). \tag{D1.4b}$$

Proof. By means of (D1.3b), we can express the left member of (D1.4a) as the following double sum

$$\begin{aligned} \text{LHS(D1.4a)} &= \sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_n} \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^n\lambda;\ q)_{k+1}} f(k) \\ &= \sum_{k=0}^{m}(-1)^k(1-q^{2k}\lambda) \begin{bmatrix} m \\ k \end{bmatrix} f(k) \sum_{n=k}^{m} \begin{bmatrix} m-k \\ n-k \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_{n+k+1}} \\ &= \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{2k+1}} \lambda^k q^{k^2} f(k) \sum_{j=0}^{m-k} \begin{bmatrix} m-k \\ j \end{bmatrix} \frac{\lambda^j q^{j(j+2k)}}{(q^{2k+1}\lambda;q)_j} \end{aligned}$$

where we have applied relations on shifted factorials

$$(\lambda;q)_{n+k+1} = (\lambda;q)_n(q^n\lambda;q)_{k+1} = (\lambda;q)_{2k+1}(q^{2k+1}\lambda;q)_{n-k} \tag{D1.5}$$

and the substitution $j := n-k$ on summation indices.

In view of the finite version of Kummer's theorem stated in Corollary C1.2

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k\, q^{k^2}}{(qx;\ q)_k} = \frac{1}{(qx;\ q)_n}$$

we can evaluate the inner sum as the following closed form:

$$\sum_{j=0}^{m-k} \begin{bmatrix} m-k \\ j \end{bmatrix} \frac{\lambda^j\, q^{j(j+2k)}}{(q^{2k+1}\lambda;q)_j} = \frac{1}{(q^{2k+1}\lambda;q)_{m-k}}.$$

Recalling (D1.5), we derive finally the following

$$\begin{aligned} \text{LHS(D1.4a)} &= \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{2k+1}} \frac{\lambda^k q^{k^2}}{(q^{2k+1}\lambda;q)_{m-k}} f(k) \\ &= \sum_{k=0}^{m}(-1)^k \{1-q^{2k}\lambda\} \begin{bmatrix} m \\ k \end{bmatrix} \frac{\lambda^k q^{k^2}}{(\lambda;q)_{m+k+1}} f(k) \end{aligned}$$

which is the first identity (D1.4a).

The second identity (D1.4b) follows from the limit $m \to \infty$ of (D1.4a). □

D2. Finite q-differences and further transformation

On account of the inverse series relations

$$(x;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k \tag{D2.1a}$$

$$q^{\binom{n}{2}} x^n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (x;q)_k \tag{D2.1b}$$

we may determine, as an example of (D1.3a-D1.3b), two sequences as follows:

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}}(\lambda;\, q)_n \quad \rightleftharpoons \quad g(n) = (\lambda;\, q)_n.$$

They may be used to reformulate the finite series transformation (D1.4a) explicitly

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \lambda^n q^{n^2} = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^k\lambda;\, q)_{m+1}} \lambda^{2k} q^{2k^2+\binom{k}{2}}. \tag{D2.2}$$

Proof. The first relation (D2.1a) is a restatement of Euler's q-finite difference formula (B5.3). Specifying the Carlitz inversions stated in D1.1 with

$$\phi(x;n) = 1, \quad f(n) = x^n q^{\binom{n}{2}}, \quad g(n) = (x;q)_n$$

we get the second relation (D2.1b) which is dual to the first one.

In order to verify that two sequences

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}}(\lambda;\, q)_n \quad \rightleftharpoons \quad g(n) = (\lambda;\, q)_n$$

satisfy (D1.3a-D1.3b), it is sufficient to show that

$$\begin{aligned}\lambda^n q^{n^2+\binom{n}{2}}(\lambda;\,q)_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix}n\\k\end{bmatrix} q^{\binom{n-k}{2}}(\lambda;\,q)_k(q^k\lambda;q)_n\\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix}n\\k\end{bmatrix} q^{\binom{n-k}{2}}(\lambda;q)_{n+k}\end{aligned}$$

in view of the inverse series relations specified with $\phi(x;n) = (\lambda x;q)_n$.

Applying (D2.1b) with $x = q^n\lambda$, we confirm the last summation identity:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \begin{bmatrix}n\\k\end{bmatrix} q^{\binom{n-k}{2}}(\lambda;q)_{n+k} =&(\lambda;q)_n \sum_{k=0}^{n}(-1)^k \begin{bmatrix}n\\k\end{bmatrix} q^{\binom{n-k}{2}}(q^n\lambda;q)_k\\ =&(\lambda;q)_n q^{n^2+\binom{n}{2}}\lambda^n.\end{aligned}$$

The transformation (D2.2) follows from (D1.4a) with the $\{f(k), g(n)\}$ sequences just displayed explicitly. □

D3. Rogers-Ramanujan identities and their finite forms

D3.1. Proposition. With the specifications $\lambda \mapsto 1$ and $\lambda \mapsto q$ in (D2.2), the finite forms of Rogers-Ramanujan identities can be derived as follows:

$$\sum_{n=0}^{m} \begin{bmatrix}m\\n\end{bmatrix} q^{n^2} = \frac{(q;q)_m}{(q;q)_{2m}} \sum_{k=-m}^{m} (-1)^k \begin{bmatrix}2m\\m+k\end{bmatrix} q^{\binom{k}{2}+2k^2} \tag{D3.1a}$$

$$\sum_{n=0}^{m} \begin{bmatrix}m\\n\end{bmatrix} q^{n^2+n} = \frac{(q;q)_m}{(q;q)_{2m+1}} \sum_{k=-m}^{m+1} (-1)^k \begin{bmatrix}2m+1\\m+k\end{bmatrix} q^{\binom{k}{2}+2k^2-k}. \tag{D3.1b}$$

Proof. Separating the first term from (D2.2), we have

$$\sum_{n=0}^{m} \begin{bmatrix}m\\n\end{bmatrix} \lambda^n q^{n^2} = \frac{1-\lambda}{(\lambda;q)_{m+1}} + \sum_{k=1}^{m}(-1)^k \begin{bmatrix}m\\k\end{bmatrix} \frac{1-q^{2k}\lambda}{(q^k\lambda;\,q)_{m+1}} \lambda^{2k} q^{2k^2+\binom{k}{2}}.$$

Its limiting case $\lambda \to 1$ may be manipulated as follows:

$$\begin{aligned}\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n^2} &= \frac{1}{(q;q)_m}+\sum_{k=1}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\frac{1-q^{2k}}{(q^k;\ q)_{m+1}}q^{2k^2+\binom{k}{2}}\\ &= \frac{1}{(q;q)_m}+\sum_{k=1}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\frac{1+q^{k}}{(q^{k+1};\ q)_{m}}q^{2k^2+\binom{k}{2}}.\end{aligned}$$

In view of the definition of q-Gauss binomial coefficient and the relation

$$(q;\ q)_{m+k} = (q;\ q)_k(q^{k+1};\ q)_m$$

we can further reformulate the sum as

$$\begin{aligned}\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n^2} &= \frac{1}{(q;q)_m}+\sum_{k=1}^{m}(-1)^k\frac{(q;\ q)_m}{(q;\ q)_{m-k}}\frac{1+q^{k}}{(q;\ q)_{m+k}}\,q^{2k^2+\binom{k}{2}}\\ &= \frac{1}{(q;q)_m}+\frac{(q;\ q)_m}{(q;\ q)_{2m}}\sum_{k=1}^{m}(-1)^k\begin{bmatrix}2m\\m+k\end{bmatrix}q^{2k^2+\binom{k}{2}}\\ &+ \frac{(q;\ q)_m}{(q;\ q)_{2m}}\sum_{k=1}^{m}(-1)^k\begin{bmatrix}2m\\m+k\end{bmatrix}q^{2k^2+\binom{k+1}{2}}.\end{aligned}$$

Performing the replacement $k \to -k$ in the last sum and noting that

$$\begin{bmatrix}2m\\m-k\end{bmatrix} = \begin{bmatrix}2m\\m+k\end{bmatrix}$$

we can combine the last three expressions as a single one:

$$\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n^2} = \sum_{k=-m}^{m}(-1)^k\begin{bmatrix}2m\\m+k\end{bmatrix}q^{2k^2+\binom{k}{2}}$$

which is the finite form of the first Rogers-Ramanujan identity (D3.1a).

Similarly, specifying (D2.2) with $\lambda \to q$, we have

$$\begin{aligned}\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n+n^2} &= \sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\frac{1-q^{2k+1}}{(q^{k+1};\ q)_{m+1}}q^{2k^2+\binom{k}{2}+2k}\\ &= \frac{(q;q)_m}{(q;q)_{2m+1}}\sum_{k=0}^{m}(-1)^k\begin{bmatrix}2m+1\\m-k\end{bmatrix}(1-q^{2k+1})\,q^{2k^2+\binom{k}{2}+2k}\\ &= \frac{(q;q)_m}{(q;q)_{2m+1}}\sum_{k=0}^{m}(-1)^k\begin{bmatrix}2m+1\\m-k\end{bmatrix}q^{2k^2+\binom{k}{2}+2k}\\ &- \frac{(q;q)_m}{(q;q)_{2m+1}}\sum_{k=0}^{m}(-1)^k\begin{bmatrix}2m+1\\m-k\end{bmatrix}q^{2k^2+\binom{k}{2}+4k+1}.\end{aligned}$$

Replacing the summation index k by $-1-k$ in the second sum and then combining the result with the first one, we get the following simplified transformation

$$\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n+n^2} = \frac{(q;q)_m}{(q;q)_{2m+1}}\sum_{k=-m-1}^{m}(-1)^k\begin{bmatrix}2m+1\\m-k\end{bmatrix}q^{2k^2+\binom{k}{2}+2k}$$

which is equivalent to the second finite form (D3.1b) of Rogers-Ramanujan identities under parameter replacement $k\to -k$. □

D3.2. Theorem. Their limiting cases give rise, with the help of the Jacobi-triple product identity, to the celebrated Rogers-Ramanujan identities:

$$\frac{1}{(q;\ q^5)_\infty\ (q^4;q^5)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\ q)_n} = \prod_{k=0}^{\infty}\frac{1}{(1-q^{1+5k})(1-q^{4+5k})} \qquad \text{(D3.2a)}$$

$$\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;\ q)_n} = \prod_{k=0}^{\infty}\frac{1}{(1-q^{2+5k})(1-q^{3+5k})}. \qquad \text{(D3.2b)}$$

Proof. Letting $m\to\infty$, we can state (D3.1a) as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q,q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}+2k^2}\\ &= \frac{1}{(q,q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k q^{5\binom{k}{2}+2k}.\end{aligned}$$

The sum on the right hand side can be evaluated, by means of Jacobi triple product identity, as

$$\sum_{k=-\infty}^{+\infty}(-1)^k q^{5\binom{k}{2}+2k} = \left[q^5,q^2,q^3;q^5\right]_\infty.$$

Therefore the first identity (D3.2a) follows consequently:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{\left[q^5,q^2,q^3;q^5\right]_\infty}{(q;\ q)_\infty} = \frac{1}{\left[q,q^4;q^5\right]_\infty}.$$

If we let $m\to\infty$ in (D3.1b), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\ q^{\binom{k}{2}+2k^2-k}\\ &= \frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\ q^{5\binom{k}{2}+k}.\end{aligned}$$

The sum on the right hand side reads as

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{5\binom{k}{2}+k} \;=\; \big[q^5, q, q^4; q^5\big]_\infty$$

in view of Jacobi triple product identity.

Hence we have established the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} \;=\; \frac{\big[q^5, q, q^4; q^5\big]_\infty}{(q;\;q)_\infty} \;=\; \frac{1}{\big[q^2, q^3; q^5\big]_\infty}$$

which is the second identity (D3.2b). □

Up to now, about ten proofs have been provided for this beautiful pair of identities. The most recent ones are, respectively, due to Baxter (1982) based on the statistical mechanics and Lepowsky-Milne (1978) through the character formula on infinite dimensional Lie algebra (Kac-Moody algebra [**45**, 1985]).

CHAPTER E

Basic Hypergeometric Series

This chapter introduces the basic hypergeometric series. Its convergence condition will be determined. The fundamental transformations and summation formulae will be covered briefly.

E1. Introduction and notation

E1.1. Definition. Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq q^{-n}$ with $n \in \mathbb{N}_0$ for all $j = 1, 2, \cdots, s$. Then the basic hypergeometric series with variable z is defined by

$$ {}_{1+r}\phi_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\middle| q; z\right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} z^n \big\{(-1)^n q^{\binom{n}{2}}\big\}^{s-r}. $$

Remark If there is a numerator parameter $a_i = q^{-k}$ with $k \in \mathbb{N}_0$, then the q-hypergeometric series is terminating, which is in fact a polynomial of z. When the series is nonterminating, we assume that $|q| < 1$ for convenience.

E1.2. Convergence condition. For the q-hypergeometric series just defined, the convergence conditions are as follows:

(A) If $s > r$, the series is convergent for all $z \in \mathbb{C}$;
(B) If $s < r$, the series is convergent only when $z = 0$;
(C) If $s = r$, the series is convergent for $|z| < 1$.

Proof. Denote by T_n the summand of q-hypergeometric series

$$ T_n := \big\{(-1)^n q^{\binom{n}{2}}\big\}^{s-r} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} z^n. $$

To determine the convergence conditions, we consider the term-ratio:

$$\frac{T_{n+1}}{T_n} = z\frac{(1-q^n a_0)(1-q^n a_1)\cdots(1-q^n a_r)}{(1-q^{n+1})(1-q^n b_1)\cdots(1-q^n a_s)}(-q^n)^{s-r}.$$

On account of $|q| < 1$, we have $|q^n| \to 0$ as $n \to +\infty$. Hence we get the following limit:

$$\lim_{n\to+\infty}\left|\frac{T_{n+1}}{T_n}\right| = \begin{cases} 0, & r < s \\ +\infty, & r > s \text{ and } z \neq 0 \\ |z|, & r = s. \end{cases}$$

According to the D'Alembert ratio test, the convergence conditions stated in the Theorem follow immediately. □

E1.3. Classification. For the basic hypergeometric series, suppose $r = s$, the very important case. If the product of denominator parameters is equal to the base q times the product of numerator parameters, i.e.,

$$qa_0 a_1 \cdots a_r = b_1 b_2 \cdots b_r$$

then the ${}_{1+r}\phi_r$-series is called *balanced* or *Saalschützian*.

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same product:

$$qa_0 = a_1 b_1 = \cdots = a_r b_r$$

then we say that the ${}_{1+r}\phi_r$-series is *well-poised*. In particular, it is said to be *very-well-poised* if we have $a_1 = -a_2 = q\sqrt{a_0}$ in addition. These pairs of parameters appear in the basic hypergeometric sum as a linear fraction

$$\frac{1-a_0 q^{2k}}{1-a_0} = \frac{(q\sqrt{a_0};q)_k}{(\sqrt{a_0};q)_k} \times \frac{(-q\sqrt{a_0};q)_k}{(-\sqrt{a_0};q)_k}.$$

E1.4. Examples. In terms of q-series, we can reformulate the Euler and Gauss summation formulae as follows:

$$(z;\, q)_\infty = {}_1\phi_1\left[\begin{matrix} - \\ - \end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty} \frac{(-z)^k}{(q;\, q)_k} q^{\binom{k}{2}}$$

$$\frac{1}{(z;\, q)_\infty} = {}_1\phi_0\left[\begin{matrix} 0 \\ - \end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty} \frac{z^k}{(q;\, q)_k}.$$

They will be used to demonstrate the q-binomial theorem.

E1.5. Ordinary hypergeometric series. In comparison with the basic hypergeometric series, we present here briefly the ordinary hypergeometric series, its convergence condition and classification. The details can be found in the book by Bailey (1935).

Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N}_0$ for $j = 1, 2, \cdots, s$. Then the ordinary hypergeometric series with variable z is defined by

$$ {}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\Big|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{n!\,(b_1)_n \cdots (b_s)_n}\, z^n $$

where the (rising) shifted factorial is defined by

$$ (c)_0 = 1 \quad \textit{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \textit{for} \quad n = 1, 2, \cdots. $$

Classification Similar to basic hypergeometric series, we consider the case $r = s$ for ordinary hypergeometric series. If the sum of denominator parameters is equal to one plus the sum of numerator parameters, i.e.,

$$ 1 + a_0 + a_1 + \cdots + a_r = b_1 + b_2 + \cdots + b_r $$

then the ${}_{1+r}F_r$-series is called *balanced* or *Saalschützian*.

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same sum:

$$ 1 + a_0 = a_1 + b_1 = \cdots = a_r + b_r $$

then we say that the ${}_{1+r}F_r$-series is *well-poised*. In particular, it is said to be *very-well-poised* if we have $a_1 = 1 + a_0/2$ in addition. The last pair of parameters appear in the (ordinary) hypergeometric sum as a linear fraction

$$ \frac{a_0 + 2k}{a_0} \quad = \quad \frac{(1 + a_0/2)_k}{(a_0/2)_k}. $$

Convergence condition for the (ordinary) hypergeometric series is determined as follows:

- if $r < s$, the ${}_{1+r}F_s$-series converges for all $z \in \mathbb{C}$;
- if $r > s$, the ${}_{1+r}F_s$-series diverges for all $z \in \mathbb{C}$ except for $z = 0$;
- if $r = s$, the ${}_{1+r}F_r$-series converges for $|z| < 1$, and when

$$ \begin{aligned} z &= +1 \quad \text{if} \quad \Re(B - A) > 0 \\ z &= -1 \quad \text{if} \quad \Re(B - A) > -1 \end{aligned} $$

  where A and B are defined respectively by

$$A = \sum_{i=0}^{r} a_i \quad \text{and} \quad B = \sum_{j=1}^{r} b_j.$$

Remark Noting that the limit relation between ordinary and q-shifted factorials

$$\lim_{q \to 1} \frac{(q^c;\, q)_k}{(1-q)^k} \quad = \quad (c)_k$$

we can consider the (ordinary) hypergeometric series as the limit of the basic hypergeometric series:

$${}_{1+r}F_s \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, z \right] = \lim_{q \to 1} {}_{1+r}\phi_s \left[\begin{matrix} q^{a_0}, q^{a_1}, \cdots, q^{a_r} \\ q^{b_1}, \cdots, q^{b_s} \end{matrix} \,\middle|\, q; \frac{(-1)^{r-s}\, z}{(1-q)^{r-s}} \right].$$

This explains why there exist generally the q-counterparts for the (ordinary) hypergeometric series identities.

E2. The q-Gauss summation formula

This section will prove the q-binomial theorem, the q-Gauss summation formula as well as the q-Chu-Vandermonde convolution.

E2.1. The q-binomial theorem. In terms of hypergeometric series, the classical binomial theorem reads as follows:

$${}_1F_0 \left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, z \right] = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} z^n = \frac{1}{(1-z)^c}, \quad (|z| < 1).$$

Its q-analog is given by the following q-binomial theorem:

$${}_1\phi_0 \left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, q;\, z \right] = \frac{(cz;\, q)_\infty}{(z;\, q)_\infty} = \sum_{n=0}^{\infty} \frac{(c;\, q)_n}{(q;\, q)_n} z^n, \quad (|z| < 1). \tag{E2.1}$$

For $c = 0$ this identity reduces to Gauss summation formula. Replacing z by z/c and then letting $c \to \infty$, we recover from it the Euler formula.

PROOF. In fact, expanding the numerator and the denominator respectively according to Euler and Gauss summation formulae, we have

$$\begin{aligned}\frac{(cz;\ q)_\infty}{(z;\ q)_\infty} &= \sum_{i=0}^{\infty}(-1)^i\frac{c^iz^i}{(q;\ q)_i}q^{\binom{i}{2}}\sum_{j=0}^{\infty}\frac{z^j}{(q;\ q)_j}\\ &= \sum_{n=0}^{\infty}\frac{z^n}{(q;\ q)_n}\sum_{i=0}^{n}(-1)^iq^{\binom{i}{2}}\begin{bmatrix}n\\ i\end{bmatrix}c^i = \sum_{n=0}^{\infty}\frac{(c;\ q)_n}{(q;\ q)_n}z^n\end{aligned}$$

where the last line follows from the finite q-differences. □

E2.2. The q-Gauss summation formula. The q-binomial theorem can be generalized to the following theorem.

For three complex numbers a, b and c with $|c/ab| < 1$, there holds

$$_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ c/ab\right] = \sum_{n=0}^{\infty}\frac{(a;\ q)_n(b;\ q)_n}{(q;\ q)_n(c;\ q)_n}\big(c/ab\big)^n = \frac{(c/a;\ q)_\infty(c/b;\ q)_\infty}{(c;\ q)_\infty(c/ab;\ q)_\infty}.$$

PROOF. We can manipulate, by means of the q-binomial theorem (E2.1), the infinite series as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\Big(\frac{c}{ab}\Big)^n &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(q^nc;q)_\infty}{(q;q)_n(q^nb;q)_\infty}\Big(\frac{c}{ab}\Big)^n\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{c}{ab}\Big)^n\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}\big(q^nb\big)^k\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{q^kc}{ab}\Big)^n\\ &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\frac{(q^kc/b;q)_\infty}{(q^kc/ab;q)_\infty}\\ &= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/ab;q)_k}{(q;q)_k}b^k\\ &= \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\end{aligned}$$

which establishes the q-Gauss summation formula. □

E2.3. The q-analog of Chu-Vandermonde convolution. The terminating case of the q-Gauss summation formula can be reformulated as the

q-analogues of the Chu-Vandermonde convolution:

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\ q^n\, c/b\right] = \frac{(c/b;\ q)_n}{(c;\ q)_n} \tag{E2.2a} $$

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\ q\right] = \frac{(c/b;\ q)_n}{(c;\ q)_n} b^n \tag{E2.2b} $$

$$ \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \begin{bmatrix} x+y \\ n \end{bmatrix}. \tag{E2.2c} $$

Proof. The first formula is the case $a = q^{-n}$ of the q-Gauss theorem, which can be reformulated to other two identities.

By definition of q-hypergeometric series, rewrite (E2.2a) explictly as

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} \,\middle|\, q; q^n c/b\right] = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k (c; q)_k} (q^n c/b)^k = \frac{(c/b;\ q)_n}{(c;\ q)_n}. $$

Considering that

$$ (x; q)_{n-k} = (-1)^k x^{-k} q^{\binom{k+1}{2} - nk} \frac{(x; q)_n}{(q^{1-n}/x; q)_k} $$

we can manipulate the reversed series as follows:

$$ \begin{aligned} \frac{(c/b;\ q)_n}{(c;\ q)_n} &= \sum_{k=0}^{n} \frac{(q^{-n}; q)_{n-k} (b; q)_{n-k}}{(q; q)_{n-k} (c; q)_{n-k}} (q^n c/b)^{n-k} \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{1-n}/c; q)_k}{(q; q)_k (q^{1-n}/b; q)_k} q^k \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\ q\right] \end{aligned} $$

which is equivalent to

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\ q\right] = (-1)^n q^{-\binom{n}{2}} \left(\frac{b}{c}\right)^n \frac{(c/b; q)_n}{(b; q)_n} $$

in view of

$$ (q^{-n}; q)_n = (-1)^n q^{-\binom{n+1}{2}} (q; q)_n. $$

Performing the parameter replacements

$$ \begin{aligned} B &\to q^{1-n}/c \\ C &\to q^{1-n}/b \end{aligned} $$

and then applying the relation

$$ (q^{1-n}/C; q)_n = (-1)^n q^{-\binom{n}{2}} C^{-n} (C; q)_n $$

we can restate the last formula as:

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & B \\ & C \end{matrix}\,\middle|\, q;\ q\right] = B^n \frac{(C/B;q)_n}{(C;q)_n} $$

which is the second formula (E2.2b).

Writing the q-binomial coefficients in terms of q-shifted factorials

$$ \begin{aligned} \begin{bmatrix} x \\ k \end{bmatrix} &= \tfrac{(q^{x-k+1};q)_k}{(q;q)_k} &&= (-1)^k q^{xk-\binom{k}{2}} \frac{(q^{-x};q)_k}{(q;q)_k} \\ \begin{bmatrix} y \\ n-k \end{bmatrix} &= \tfrac{(q^{y-n+k+1};q)_{n-k}}{(q;q)_{n-k}} &&= (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_n} \frac{(q^{y-n+1};q)_n}{(q^{y-n+1};q)_k} \end{aligned} $$

we can express the q-binomial sum in terms of q-series:

$$ \begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k}{(q;q)_k (q^{y-n+1};q)_k} q^k \\ &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx}\ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{-x} \\ & q^{y-n+1} \end{matrix}\,\middle|\, q;q\right]. \end{aligned} $$

Evaluate the last q-series by (E2.2b):

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{-x} \\ & q^{y-n+1} \end{matrix}\,\middle|\, q;\ q\right] = q^{-nx} \frac{(q^{x+y-n+1};q)_n}{(q^{y-n+1};q)_n} $$

we find consequently the following q-binomial identity

$$ \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \frac{(q^{x+y-n+1};q)_n}{(q;q)_n} = \begin{bmatrix} x+y \\ n \end{bmatrix} $$

which is, in fact, the convolution formula (E2.2c). □

E3. Transformations of Heine and Jackson

E3.1. Jackson's ${}_2\phi_2$-series transformation.

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} \times {}_2\phi_2\left[\begin{matrix} a, & c/b \\ c, & az \end{matrix}\,\middle|\, q;\ bz\right]. $$

Proof. According to the q-Chu-Vandermonde formula, we have

$$ \frac{(b;q)_n}{(c;q)_n} = {}_2\phi_1\left[\begin{matrix} q^{-n}, & c/b \\ & c \end{matrix}\,\middle|\, q;\ q^n b\right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (c/b;q)_k}{(q;q)_k (c;q)_k} (bq^n)^k. $$

Then the q-hypergeometric series in question can be expressed as a double sum:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\frac{(b;q)_n}{(c;q)_n}z^n\\
&= \sum_{n=0}^{\infty} z^n\frac{(a;q)_n}{(q;q)_n}\times\sum_{k=0}^{n}\frac{(q^{-n};q)_k(c/b;q)_k}{(q;q)_k(c;q)_k}(bq^n)^k\\
&= \sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k(c;q)_k}b^k\times\sum_{n=k}^{\infty}\frac{(a;q)_n(q^{-n};q)_k}{(q;q)_n}\big(zq^k\big)^n.
\end{aligned}
$$

For the last sum with respect to n, changing by $j := n-k$ on the summation index and then applying transformations

$$
\begin{aligned}
(a;q)_{j+k} &= (a;q)_k(aq^k;q)_j\\
\frac{(q^{-j-k};q)_k}{(q;q)_{j+k}} &= \frac{(-1)^kq^{-k(j+k)+\binom{k}{2}}}{(q;q)_j}
\end{aligned}
$$

we can evaluate it, by means of (E2.1) with $c \to aq^k$, as follows:

$$
\begin{aligned}
\sum_{n=k}^{\infty}\frac{(a;q)_n(q^{-n};q)_k}{(q;q)_n}\big(zq^k\big)^n &= (-z)^kq^{\binom{k}{2}}(a;q)_k\times\sum_{j=0}^{\infty}\frac{(aq^k;q)_j}{(q;q)_j}z^j\\
&= (-z)^kq^{\binom{k}{2}}(a;q)_k\times{}_1\phi_0\left[\begin{matrix}aq^k\\ -\end{matrix}\,\middle|\, q;\ z\right]\\
&= (-z)^kq^{\binom{k}{2}}(a;q)_k\times\frac{(q^kaz;q)_k}{(z;q)_k}\\
&= (-z)^kq^{\binom{k}{2}}\frac{(a;q)_k}{(az;q)_k}\frac{(az;q)_\infty}{(z;q)_\infty}.
\end{aligned}
$$

We have therefore established

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(az;q)_\infty}{(z;q)_\infty}\sum_{k=0}^{\infty}(-1)^kq^{\binom{k}{2}}\frac{(a;q)_k(c/b;q)_k}{(q;q)_k(az;q)_k(c;q)_k}(bz)^k\\
&= \frac{(az;q)_\infty}{(z;q)_\infty}\,{}_2\phi_2\left[\begin{matrix}a, & c/b\\ c, & az\end{matrix}\,\middle|\, q;\ bz\right]
\end{aligned}
$$

which is Jackson's transformation. $\square$

E3.2. Heine's q-Euler transformations.

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}\,\middle|\, q;\ b\right] \tag{E3.1a}
$$

$$
= \frac{[c/b, bz; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\middle|\, q; c/b\right] \tag{E3.1b}
$$

$$
= \frac{(abz/c; q)_\infty}{(z; q)_\infty} \times {}_2\phi_1\left[\begin{matrix} c/a, \ c/b \\ c \end{matrix}\,\middle|\, q;\ abz/c\right]. \tag{E3.1c}
$$

Proof. Substituting the q-factorial fraction

$$
\frac{(b;q)_n}{(c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \times \frac{(q^n c; q)_\infty}{(q^n b; q)_\infty}
$$

into the q-hypergeometric series

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n
$$

and then applying the q-binomial theorem (E2.1):

$$
\frac{(q^n c; q)_\infty}{(q^n b; q)_\infty} = {}_1\phi_0\left[\begin{matrix} c/b \\ - \end{matrix}\,\middle|\, q;\ q^n b\right] = \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} q^{nk} b^k
$$

we can manipulate the q-series as follows:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} q^{nk} b^k \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} b^k \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^k)^n.
\end{aligned}
$$

Again by means of (E2.1), evaluating the last sum with respect to n as

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^k)^n = \frac{(q^k az; q)_\infty}{(q^k z; q)_\infty}
$$

and then simplifying the series with

$$
\begin{aligned}
(q^k az; q)_\infty &= \frac{(az;q)_\infty}{(az;q)_k} \\
(q^k z; q)_\infty &= \frac{(z;q)_\infty}{(z;q)_k}
\end{aligned}
$$

we derive the following expression

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k(z;q)_k}{(q;q)_k(az;q)_k}b^k \\
&= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}\,\middle|\, q;\ b\right]
\end{aligned}$$

which is the first transformation (E3.1a).

Applying the transformation just established to the series on the right hand side, we have

$${}_2\phi_1\left[\begin{matrix} z, & c/b \\ & az \end{matrix}\,\middle|\, q;\ b\right] = \frac{(c/b;q)_\infty(bz;q)_\infty}{(az;q)_\infty(b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\middle|\, q;\ c/b\right]$$

whose combination with the first one result in

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}\,\middle|\, q;\ b\right] \\
&= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\middle|\, q; c/b\right].
\end{aligned}$$

This is the second transformation (E3.1b).

Applying again the first transformation, we get

$${}_2\phi_1\left[\begin{matrix} b, abz/c \\ bz \end{matrix}\,\middle|\, q; c/b\right] = \frac{(abz/c;q)_\infty(c;q)_\infty}{(bz;q)_\infty(c/b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix}\,\middle|\, q; abz/c\right].$$

This leads us to the following

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] &= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\middle|\, q;\ c/b\right] \\
&= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix}\,\middle|\, q;\ abz/c\right]
\end{aligned}$$

which is exactly the third transformation (E3.1c).

The last transformation can also be derived by means of the Jackson transformation stated in E3.1. In fact, interchanging a and b in the Jackson formula, we have

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \frac{(bz;q)_\infty}{(z;q)_\infty}\times{}_2\phi_2\left[\begin{matrix} b, & c/a \\ c, & bz \end{matrix}\,\middle|\, q;\ az\right].$$

While the q-series on the right hand side of (E3.1c) can be transformed, by means of the Jackson identity, into the following

$${}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\middle|\, q;\ abz/c\right] = \frac{(bz;q)_\infty}{(abz/c;q)_\infty}\times{}_2\phi_2\left[\begin{matrix} c/a, & b \\ c, & bz \end{matrix}\,\middle|\, q;\ az\right].$$

Equating both expressions, we have

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \,\middle|\, q;\ abz/c\right]. $$

The proof of (E3.1c) is therefore completed again. □

Remark The Heine's transformations (E3.1a-E3.1b-E3.1c) may be considered as q-analogues of the Pfaff-Euler Transformations for the (ordinary) hypergeometric series:

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, z\right] &= (1-z)^{-a} \quad {}_2F_1\left[\begin{matrix} a, & c-b \\ & c \end{matrix} \,\middle|\, \frac{z}{z-1}\right] \\ &= (1-z)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a, & c-b \\ & c \end{matrix} \,\middle|\, z\right]. \end{aligned} $$

E3.3. The Bailey-Daum summation formula.

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & qa/b \end{matrix} \,\middle|\, q;\ -q/b\right] = (-q;q)_\infty \frac{\left[qa, q^2a/b^2; q^2\right]_\infty}{\left[qa/b, -q/b; q\right]_\infty}, \quad (|q/b| < 1). $$

Proof. Applying the Heine transformation (E3.1a)

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix} \,\middle|\, q;\ b\right] $$

we can proceed as follows:

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} b,\ a \\ qa/b \end{matrix} \,\middle|\, q; -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} q/b, & -q/b \\ & -q \end{matrix} \,\middle|\, q;\ a\right] \\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n} a^n. \end{aligned} $$

Simplifying the last sum with relations

$$ \begin{aligned} (q^2/b^2;q^2)_n &= (q/b;q)_n(-q/b;q)_n \\ (q^2;q^2)_n &= (q;q)_n(-q;q)_n \end{aligned} $$

and then evaluating it by means of the q-binomial theorem (E2.1), we have

$$ \sum_{n=0}^{\infty} \frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n} a^n = \sum_{n=0}^{\infty} \frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n} a^n = \frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty} $$

which results consequently in the following

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} a, & b \\ & qa/b \end{matrix} \,\middle|\, q;\ -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty} \times \frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty} \\ &= \frac{(-q;q)_\infty(aq;q^2)_\infty(q^2a/b^2;q^2)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty} \end{aligned} $$

thanks to the shifted factorial relation

$$(a;q)_\infty \;=\; (aq;q^2)_\infty(a;q^2)_\infty.$$

This proves the Bailey-Daum summation theorem. □

E3.4. Infinite series transformation. Armed with the q-series transformation formulae, we apply again inverse series relations (D1.3a-D1.3b) to establish another infinite series transformation, which will be used in turn to prove two infinite series identities of Rogers-Ramanujan type.

Recalling the inverse series relations (D1.3a-D1.3b), if we take the g-sequence

$$g(n) \;=\; \frac{(\lambda;q)_n}{(q\lambda;q^2)_n}q^{\binom{n}{2}} \quad \text{with} \quad n=0,1,2,\cdots$$

then the dual sequence will be determined by

$$f(n) \;=\; \begin{cases} 0, & n-\text{odd} \\ (-1)^m \left[q,\lambda;q^2\right]_m q^{m^2-m}, & n=2m. \end{cases}$$

We have accordingly from (D1.4b) the infinite series transformation:

$$\sum_{n=0}^{\infty}\frac{\lambda^n q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q\lambda;\,q^2)_n} \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{1-q^{4k}\lambda}{(\lambda;q)_\infty}\frac{(\lambda;\,q^2)_k}{(q^2;\,q^2)_k}q^{5k^2-k}\lambda^{2k}. \qquad \text{(E3.2)}$$

Proof. Substituting $g(k)$ into (D1.3a) and then rewriting the q-Gauss binomial coefficient, we have

$$\begin{aligned} f(n) \;&=\; \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}(q^k\lambda;\,q)_n\,g(k) \\ &=\; \sum_{k=0}^{n}\frac{(q^{-n};\,q)_k}{(q;\,q)_k}q^{\binom{n}{2}+\binom{k+1}{2}}\frac{(\lambda;\,q)_{n+k}}{(q\lambda;\,q^2)_k}. \end{aligned}$$

By means of factorization

$$(q\lambda;\,q^2)_k \;=\; (\sqrt{q\lambda};\,q)_k\times(-\sqrt{q\lambda};\,q)_k$$

we can express $f(n)$ in terms of a terminating q-hypergeometric series

$$\begin{aligned} f(n) \;&=\; q^{\binom{n}{2}}(\lambda;\,q)_n\;\times\;\sum_{k=0}^{n}\left[\begin{matrix}q^{-n}, & q^n\lambda \\ q, & \pm\sqrt{q\lambda}\end{matrix}\,\Big|\,q\right]_k q^{\binom{k+1}{2}} \\ &=\; q^{\binom{n}{2}}(\lambda;\,q)_n\;\times\;{}_2\phi_2\left[\begin{matrix}q^{-n}, & q^n\lambda \\ \sqrt{q\lambda}, & -\sqrt{q\lambda}\end{matrix}\,\Big|\,q;\,-q\right]. \end{aligned}$$

Rewriting Jackson's transformation formula stated in E3.1

$$ {}_2\phi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix} \,\middle|\, q;\ \frac{bd}{ac}\right] = \frac{(d/a;q)_\infty}{(d;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} a, & b/c \\ & b \end{matrix} \,\middle|\, q;\ d/a\right] $$

we can further reformulate $f(n)$ as follows:

$$ f(n) = q^{\binom{n}{2}} \frac{(\lambda;\ q)_n}{(-\sqrt{q\lambda};\ q)_n} \times {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{-n}\sqrt{q/\lambda} \\ & \sqrt{q\lambda} \end{matrix} \,\middle|\, q;\ -q^n\sqrt{q\lambda}\right]. $$

Evaluating the last series by means of the Bailey-Daum formula stated in E3.3:

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & qa/b \end{matrix} \,\middle|\, q;\ -q/b\right] = (-q;q)_\infty \frac{[qa, q^2a/b^2; q^2]_\infty}{[qa/b, -q/b; q]_\infty}, \quad (|q/b| < 1) $$

we find that

$$ f(n) = q^{\binom{n}{2}} (\lambda;\ q)_n \left[\begin{matrix} q^{1-n}, & q^{1+n}\lambda \\ q, & q\lambda \end{matrix} \,\middle|\, q^2\right]_\infty . $$

If n is odd, we have $f(n) = 0$ for $(q^{1-n};\ q^2)_\infty = 0$. Suppose $n = 2m$ instead, we have the following reduction

$$ \begin{aligned} f(n) &= q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \left[\begin{matrix} q^{1-2m}, & q^{1+2m}\lambda \\ q, & q\lambda \end{matrix} \,\middle|\, q^2\right]_\infty \\ &= q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \frac{(q^{1-2m};\ q^2)_m}{(q\lambda;\ q^2)_m} \\ &= (-1)^m q^{m^2-m} \left[q,\ \lambda;\ q^2\right]_m . \end{aligned} $$

Substituting $g(n)$ and $f(k)$ into (D1.4b), we establish (E3.2). □

E3.5. Two further identities of Rogers-Ramanujan type. Specifying with $\lambda \to 1$ and $\lambda \to q^2$ in (E3.2), we derive the following identities of Rogers-Ramanujan type:

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\ q)_n\ (q;\ q^2)_n} &= \frac{[q^{10}, q^4, q^6; q^{10}]_\infty}{(q;\ q)_\infty} \\ \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2+3n}{2}}}{(q;\ q)_n (q;\ q^2)_{n+1}} &= \frac{[q^{10}, q^2, q^8; q^{10}]_\infty}{(q;\ q)_\infty}. \end{aligned} $$

PROOF. Putting $\lambda \to 1$ in (E3.2) and then separating the first term from the right hand side, we derive

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q;\,q^2)_n} &= \frac{1}{(q;q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k\left\{1+q^{2k}\right\}q^{5k^2-k}\Big\}\\ &= \frac{1}{(q;q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5k^2-k}+\sum_{k=1}^{\infty}(-1)^kq^{5k^2+k}\Big\}.\end{aligned}$$

Performing replacement $k \to -k$ in the last sum and then applying the Jacobi triple product identity, we reduce the sum inside $\{\cdots\}$ as

$$\sum_{k=-\infty}^{\infty}(-1)^kq^{10\binom{k}{2}+4k} = \left[q^{10},\,q^4,\,q^6;\,q^{10}\right]_\infty$$

which leads us to the first identity:

$$\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q;\,q^2)_n} = \frac{\left[q^{10},\,q^4,\,q^6;\,q^{10}\right]_\infty}{(q;q)_\infty}.$$

When $\lambda \to q^2$, we can similarly write (E3.2) as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2+3n}{2}}}{(q;\,q)_n(q;\,q^2)_{n+1}} &= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\left\{1-q^{4k+2}\right\}q^{5k^2+3k}\\ &= \frac{1}{(q;q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5k^2+3k}+\sum_{k=0}^{\infty}(-1)^{k+1}q^{5k^2+7k+2}\Big\}.\end{aligned}$$

Replacing k by $-k-1$ in the second sum and then applying the Jacobi triple product identity, we find that the sum inside $\{\cdots\}$ equals

$$\sum_{k=-\infty}^{\infty}(-1)^kq^{10\binom{k}{2}+8k} = \left[q^{10},\,q^2,\,q^8;\,q^{10}\right]_\infty$$

which results in the second identity:

$$\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2+3n}{2}}}{(q;\,q)_n(q;\,q^2)_{n+1}} = \frac{\left[q^{10},\,q^2,\,q^8;\,q^{10}\right]_\infty}{(q;q)_\infty}.$$

This completes proofs of two infinite series identities of Rogers-Ramanujan type. □

E4. The q-Pfaff-Saalschütz summation theorem

The formula under the title reads as the following

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\ q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\,\middle|\, q;\ q\right]_n. \tag{E4.1} $$

E4.1. Proof. Recall the q-Euler transformation (E3.1c):

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\middle|\, q;\ abz/c\right] = \frac{(z;q)_\infty}{(abz/c;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] $$

which can be reformulated through the q-binomial theorem (E2.1), as a product of two basic hypergeometric series:

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\middle|\, q;\ abz/c\right] = {}_1\phi_0\left[\begin{matrix} c/ab, \\ - \end{matrix}\,\middle|\, q;\ abz/c\right] \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right]. $$

Extracting the coefficient of z^n from both members, we have

$$ \frac{(c/a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(ab/c)^n = \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k} \frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}}(ab/c)^{n-k} $$

which can be restated equivalently as

$$ \begin{aligned} \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} &= \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k} \frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k} q^k \\ &= {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\ q\right] \end{aligned} $$

in view of shifted factorial fraction

$$ \begin{aligned} \frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}} &= \frac{(c/ab;q)_n}{(q;q)_n} \frac{(q^{n-k+1};q)_k}{(q^{n-k}c/ab;q)_k} \\ &= \frac{(c/ab;q)_n}{(q;q)_n} \frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k}(qab/c)^k. \end{aligned} $$

This completes the proof of the q-Saalschütz formula. □

E4.2. The formula (E4.1) can also be proved by means of series rearrangement.

Recalling the q-Chu-Vandermonde formula (E2.2a), we have

$$ \frac{(a;q)_k}{(c;q)_k} = {}_2\phi_1\left[\begin{matrix} q^{-k}, & c/a \\ & c \end{matrix}\,\middle|\, q;\ q^k a\right] = \sum_{j=0}^{k} \frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j} q^{kj} a^j. $$

Then the q-hypergeometric series in (E4.1) can be written as a double sum:

$$\begin{aligned}
\text{LHS(E4.1)} &= \sum_{k=0}^{n}\frac{(q^{-n};q)_k(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k(q^{1-n}ab/c;q)_k}q^k\\
&= \sum_{k=0}^{n}\frac{(q^{-n};q)_k(b;q)_k}{(q;q)_k(q^{1-n}ab/c;q)_k}q^k\sum_{j=0}^{k}\frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j}q^{kj}a^j\\
&= \sum_{j=0}^{n}\frac{(c/a;q)_j}{(q;q)_j(c;q)_j}a^j\sum_{k=j}^{n}\frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k}q^{k(j+1)}
\end{aligned}$$

where we have changed the summation order.

Denote by $\Omega(j)$ the last sum with respect to k. Changing the summation index with $i := k-j$ and then applying relations

$$\begin{aligned}
(x;q)_{i+j} &= (x;q)_i(q^ix;q)_j = (x;q)_j(q^jx;q)_i\\
(q^{-i-j};q)_j &= (-1)^jq^{-j(i+j)+\binom{j}{2}}(q^{i+1};q)_j
\end{aligned}$$

we can reduce $\Omega(j)$ as follows:

$$\begin{aligned}
\Omega(j) &= \sum_{k=j}^{n}\frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k}q^{k(j+1)}\\
&= \sum_{i=0}^{n-j}\frac{(q^{-n};q)_{i+j}(b;q)_{i+j}(q^{-i-j};q)_j}{(q;q)_{i+j}(q^{1-n}ab/c;q)_{i+j}}q^{(i+j)(j+1)}\\
&= (-1)^jq^{\binom{j+1}{2}}\frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j}\sum_{i=0}^{n-j}\frac{(q^{-n+j};q)_i(q^jb;q)_i}{(q;q)_i(q^{1-n+j}ab/c;q)_i}q^i\\
&= (-1)^jq^{\binom{j+1}{2}}\frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j}\,{}_2\phi_1\left[\begin{matrix}q^{-n+j},\ q^jb\\ q^{1-n+j}ab/c\end{matrix}\,\middle|\,q;\ q\right].
\end{aligned}$$

Applying now the q-Chu-Vandermonde formula (E2.2b), we can evaluate the q-series on the right hand side as

$$ {}_2\phi_1\left[\begin{matrix}q^{-n+j}, & bq^j\\ & q^{1-n+j}ab/c\end{matrix}\,\middle|\,q;\ q\right] = \frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}}\left(bq^j\right)^{n-j}$$

which results consequently in

$$\begin{aligned}
\Omega(j) &= (-1)^jq^{\binom{j+1}{2}}\frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j}\times\frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}}\left(bq^j\right)^{n-j}\\
&= (-1)^jq^{\binom{j+1}{2}}\frac{(q^{-n};q)_j(b;q)_j}{(q^{1-j}a/c;q)_j}\times\frac{(q^{1-n}a/c;q)_n}{(q^{1-n}ab/c;q)_n}\left(bq^j\right)^{n-j}\\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\frac{(q^{-n};q)_j(b;q)_j}{(c/a;q)_j}\left(q^nc/ab\right)^j.
\end{aligned}$$

Substituting the last expression of $\Omega(j)$ into the ${}_3\phi_2$-series and then applying the q-Chu-Vandermonde formula (E2.2a), we get the following evaluation

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\, q\right] &= \sum_{j=0}^{n} a^j \frac{(c/a;q)_j}{(q;q)_j(c;q)_j}\Omega(j) \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\,{}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}\,\middle|\, q; q^n c/b\right] \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\frac{(c/b;q)_n}{(c;q)_n}
\end{aligned}$$

which is equivalent to the q-Pfaff-Saalschütz formula (E4.1).

E5. The terminating q-Dougall-Dixon formula

It is, in fact, a very-well-poised terminating series identity

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a \end{matrix}\,\middle|\, q; \frac{q^{1+n}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\,\middle|\, q\right]_n. \tag{E5.1}$$

Proof. Based on the Carlitz inversions presented in (D1.1), we can derive the identity directly as the dual relation of the q-Pfaff-Saalschütz formula (E4.1).

Recalling the q-Pfaff-Saalschütz theorem (E4.1)

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\, q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\,\middle|\, q;\, q\right]_n $$

we can restate it under parameter replacements as

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix}\,\middle|\, q;\, q\right] &= \frac{(q^{1-n}/b;q)_n(c;q)_n}{(qa/b;q)_n(q^{-n}c/a;q)_n} \\
&= \frac{(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n.
\end{aligned}$$

In order to apply the Carlitz inversions, we reformulate the q-series

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix}\,\middle|\, q;\, q\right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^n a;q)_k(qa/bc;q)_k}{(q;q)_k(qa/b;q)_k(qa/c;q)_k} q^k $$

in terms of the q-binomial sum

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a;q)_n \frac{(a;q)_k(qa/bc;q)_k}{(qa/b;q)_k(qa/c;q)_k} \\
&= q^{\binom{n}{2}} \frac{(a;q)_n(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n
\end{aligned}$$

where we have used the following transformations:

$$\begin{aligned}\frac{(q^{-n};q)_k}{(q;q)_k} &= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} q^{-\binom{n}{2}-k} \\ (q^n a;q)_k &= \frac{(a;q)_{n+k}}{(a;q)_n} = \frac{(a;q)_k (q^k a;q)_n}{(a;q)_n}.\end{aligned}$$

Specifying the ϕ-polynomials with $a_k = 1$ and $b_k = -q^k a$ in the Carlitz inversions (D1.1a-D1.1b), which implies

$$\phi(x;n) := (ax;q)_n = \prod_{i=0}^{n-1}(1-axq^i)$$

and then choosing two sequences

$$\begin{aligned}f(n) &:= q^{\binom{n}{2}} \frac{(a;q)_n(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n \\ g(k) &:= \frac{(a;q)_k(qa/bc;q)_k}{(qa/b;q)_k(qa/c;q)_k}\end{aligned}$$

we write down directly the dual relation

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}a}{(q^n a;q)_{k+1}} q^{\binom{k}{2}} \frac{(a;q)_k(b;q)_k(c;q)_k}{(qa/b;q)_k(qa/c;q)_k}\left(\frac{qa}{bc}\right)^k \\ &= \frac{(a;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n}.\end{aligned}$$

Feeding back the q-binomial coefficient to factorial fraction

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{(q^{n-k+1};q)_k}{(q;q)_k} = (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k} \\ \frac{1-q^{2k}a}{1-a} &= \frac{(q\sqrt{a};q)_k}{(\sqrt{a};q)_k} \frac{(-q\sqrt{a};q)_k}{(-\sqrt{a};q)_k}\end{aligned}$$

we reformulate the dual relation in terms of q-series

$$\frac{(qa;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n} = {}_6\phi_5\left[\begin{matrix} a, \ q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, \ qa/c, & q^{1+n}a \end{matrix} \,\middle|\, q; \frac{q^{1+n}a}{bc}\right]$$

which is the terminating q-Dixon formula (E5.1). □

E6. The Sears balanced transformations

Replacing the base q with its inverse $1/q$ and then observing that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \times q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}$$

we can restate the Carlitz inversions in an equivalent form

$$\begin{cases} f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \phi(q^{-k};n)\, g(k), & n = 0,1,2,\cdots \quad \text{(E6.1a)} \\ g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \dfrac{a_k + q^{-k}b_k}{\phi(q^{-n};k+1)}\, f(k), & n = 0,1,2,\cdots \quad \text{(E6.1b)} \end{cases}$$

which will be used in this section to prove the Sears transformations on balanced basic hypergeometric series.

E6.1. The Sears balanced transformations.

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q\right] && \text{(E6.2a)} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} && \text{(E6.2b)} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n}. && \text{(E6.2c)} \end{aligned}$$

E6.2. Proof of (E6.2a-E6.2c). The second transformation formula is a consequence of the first. In fact, applying the symmetric property to (E6.2b) and then transform it by the first transformation (E6.2a-E6.2b), we have

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & a, & b/e \\ & bd/ce, & b, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & bd/ace, & d/c \\ & bd/ce, & bd/ac, & q^{1-n}/c \end{matrix} \,\middle|\, q;\ q\right] \times \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n}. \end{aligned}$$

Substituting this result into (E6.2b), we find the transformation

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n} \end{aligned}$$

which is the second formula (E6.2a-E6.2c).

E6.3. Proof of (E6.2a-E6.2b). Let the ϕ-polynomials be defined by

$$\phi(x;n) = (acex/bd;q)_n \quad \rightleftharpoons \quad a_k = 1 \text{ and } b_k = -q^k ace/bd.$$

Then the corresponding inversions (E6.1a-E6.1b) become the following:

$$f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{-k}ace/bd;q)_n \, g(k) \tag{E6.3a}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} f(k). \tag{E6.3b}$$

By means of two q-shifted factorial relations

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}-\binom{n}{2}-k}$$

$$(q^{1-n}ace/bd;q)_k = \frac{(q^{-n}ace/bd;q)_{k+1}}{1-q^{-n}ace/bd}$$

we can rewrite the ${}_4\phi_3$-series displayed in (E6.2a) as a q-binomial sum

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q\right] \frac{1-bd/ace}{1-q^n bd/ace} q^{\binom{n+1}{2}} \\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e \\ b,\ d \end{matrix} \,\middle|\, q\right]_k. \end{aligned}$$

Then the first transformation of Sears (E6.2a-E6.2b) can be stated equivalently as

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e \\ b,\ d \end{matrix} \,\middle|\, q\right]_k \tag{E6.4a}$$

$$= {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e \\ b,\ bd/ce,\ q^{1-n}a/d \end{matrix} \,\middle|\, q;q\right] \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix} \,\middle|\, q\right]_n q^{\binom{n+1}{2}}. \tag{E6.4b}$$

This expression matches perfectly with the relation (E6.3b), where two sequences have been specified by

$$f(k) := \left[\begin{matrix} a,\ c,\ e \\ b,\ d \end{matrix} \,\middle|\, q\right]_k \tag{E6.5a}$$

$$g(n) := \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix} \,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e \\ b,\ bd/ce,\ q^{1-n}a/d \end{matrix} \,\middle|\, q;q\right] q^{\binom{n+1}{2}}. \tag{E6.5b}$$

Therefore in order to demonstrate the first transformation (E6.4a-E6.4b) of Sears, it suffices to prove the following dual relation, which corresponds to the relation (E6.3a):

$$\left[\begin{matrix} a, c, e \\ b,\ d \end{matrix}\,\middle|\, q\right]_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{-k}ace/bd;q)_n \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix}\,\middle|\, q\right]_k q^{\binom{k+1}{2}} \tag{E6.6a}$$

$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-k}a/d \end{matrix}\,\middle|\, q;\ q\right]. \tag{E6.6b}$$

Let Ξ stand for the double sum on the right. We should therefore verify that Ξ reduces to the factorial fraction on the left.

Recalling the definition of q-hypergeometric series

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix}\,\middle|\, q;\ q\right] \\ = & \sum_{i=0}^{k} \left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ q, & b, & bd/ce, & q^{1-k}a/d \end{matrix}\,\middle|\, q\right]_i q^i \end{aligned}$$

and the relation of q-binomial coefficient in terms of factorial fraction

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \frac{(q^{-n};q)_k}{(q;q)_k} q^{nk-\binom{k}{2}}$$

we can rearrange the double sum as follows:

$$\begin{aligned} \Xi = & \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{-k}ace/bd;q)_n \left[\begin{matrix} d/a, bd/ce \\ d, qbd/ace \end{matrix}\,\middle|\, q\right]_k q^{\binom{k+1}{2}} \\ \times & \sum_{i=0}^{k} \left[\begin{matrix} q^{-k}, & a, & b/c, & b/e \\ q, & b, & bd/ce, & q^{1-k}a/d \end{matrix}\,\middle|\, q\right]_i q^i \\ = & \sum_{i=0}^{n} \left[\begin{matrix} a, b/c, b/e \\ q, b, bd/ce \end{matrix}\,\middle|\, q\right]_i q^i \sum_{k=i}^{n} \left[\begin{matrix} q^{-n}, d/a, bd/ce \\ q, d, qbd/ace \end{matrix}\,\middle|\, q\right]_k \\ \times & (q^{-k}ace/bd;q)_n \frac{(q^{-k};q)_i}{(q^{1-k}a/d;q)_i} q^{k(n+1)}. \end{aligned}$$

For the inner sum, performing the replacement $j := k - i$ on summation index and then applying relations

$$\begin{aligned} \frac{(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i} &= \left(\frac{d}{qa}\right)^i \frac{(q^{1+j};q)_i}{(q^j d/a;q)_i} = \left(\frac{d}{qa}\right)^i \frac{(q;q)_{i+j}}{(d/a;q)_{i+j}} \frac{(d/a;q)_j}{(q;q)_j} \\ (q^{-i-j}ace/bd;q)_n &= \frac{(q^{-i-j}ace/bd;q)_{i+j}}{(q^{n-i-j}ace/bd;q)_{i+j}} (ace/bd;q)_n \\ &= \frac{(qbd/ace;q)_{i+j}}{(q^{1-n}bd/ace;q)_i} \frac{(ace/bd;q)_n}{(q^{1+i-n}bd/ace;q)_j} q^{-n(i+j)} \end{aligned}$$

we can reduce it to the following

$$\begin{aligned}
&\sum_{k=i}^{n}\left[\begin{matrix}q^{-n}, & d/a, & bd/ce\\ q, & d, & qbd/ace\end{matrix}\Big| q\right]_k \frac{(q^{-k}ace/bd;q)_n(q^{-k};q)_i}{(q^{1-k}a/d;q)_i}q^{k(n+1)}\\
=&\sum_{j=0}^{n-i}\left[\begin{matrix}q^{-n}, & d/a, & bd/ce\\ q, & d, & qbd/ace\end{matrix}\Big| q\right]_{i+j} \frac{(q^{-i-j}ace/bd;q)_n(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i}q^{(i+j)(n+1)}\\
=&(ace/bd;q)_n\left[\begin{matrix}q^{-n},\ bd/ce\\ d,\ q^{1-n}bd/ace\end{matrix}\Big| q\right]_i\left(\frac{d}{a}\right)^i\sum_{j=0}^{n-i}\left[\begin{matrix}q^{i-n},\ d/a,\ q^ibd/ce\\ q,\ q^id,\ q^{1+i-n}bd/ace\end{matrix}\Big| q\right]_j q^j.
\end{aligned}$$

The last sum with respect to j can be evaluated by means of the q-Saalschütz formula as follows:

$$ {}_3\phi_2\left[\begin{matrix}q^{i-n}, & d/a, & q^ibd/ce\\ & q^id, & q^{1+i-n}bd/ace\end{matrix}\Big|\ q;\ q\right] = \left[\begin{matrix}q^ia, & ce/b\\ q^id, & ace/bd\end{matrix}\Big| q\right]_{n-i}.$$

Substituting this result into the double sum expression of Ξ and then applying transformation

$$\frac{(ce/b;q)_{n-i}}{(ace/bd;q)_{n-i}} = \frac{(ce/b;q)_n}{(ace/bd;q)_n}\frac{(q^{1-n}bd/ace;q)_i}{(q^{1-n}b/ce;q)_i}\left(\frac{a}{d}\right)^i$$

we reduce the double sum to a single ${}_3\phi_2$-series:

$$\begin{aligned}
\Xi &= \sum_{i=0}^{n}\left[\begin{matrix}a,\ b/c,\ b/e\\ q,\ b,\ bd/ce\end{matrix}\Big| q\right]_i\left(\frac{qd}{a}\right)^i\left[\begin{matrix}q^{-n},\ bd/ce\\ d,\ q^{1-n}bd/ace\end{matrix}\Big| q\right]_i\\
&\times\ (ace/bd;q)_n\left[\begin{matrix}q^ia, & ce/b\\ q^id, & ace/bd\end{matrix}\Big| q\right]_{n-i}\\
&= (ce/b;q)_n\frac{(a;q)_n}{(d;q)_n}\sum_{i=0}^{n}q^i\left[\begin{matrix}b/c,\ b/e,\ q^{-n}\\ q,\ b,\ q^{1-n}b/ce\end{matrix}\Big| q\right]_i.
\end{aligned}$$

Evaluating the last sum with respect to i through the q-Saalschütz formula

$$ {}_3\phi_2\left[\begin{matrix}q^{-n}, & b/c, & b/e\\ & b, & q^{1-n}b/ce\end{matrix}\Big| q;q\right] = \left[\begin{matrix}c, & e\\ b, & ce/b\end{matrix}\Big| q\right]_n$$

which is equivalent to

$$\Xi = \left[\begin{matrix}a,\ c,\ e\\ b,\ d\end{matrix}\Big| q\right]_n.$$

This completes the proof of (E6.2a-E6.2b). □

E6.4. The q-Kummer-Thomae-Whipple's formulae. As the limiting cases $n\to\infty$ of Sears' transformations, we have the non-terminating

q-Kummer-Thomae-Whipple's formulae:

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a,\, c,\, e \\ b,\, d \end{matrix} \,\middle|\, q;\, \frac{bd}{ace}\right] &= {}_3\phi_2\left[\begin{matrix} a, & b/c, & b/e \\ & b, & bd/ce \end{matrix} \,\middle|\, q;\, \frac{d}{a}\right] \times \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} \\
&= {}_3\phi_2\left[\begin{matrix} b/c,\, d/c,\, bd/ace \\ bd/ac,\, bd/ce \end{matrix} \,\middle|\, q;\, c\right] \times \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty}.
\end{aligned}
$$

Other transformations on terminating series derived from Sears' transformation may be displayed as follows:

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q; q\right] &\underset{e\to 0}{\overset{a\rightleftharpoons e}{=\!=\!=}} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/a,\, b/c \\ & b,\, bd/ac \end{matrix} \,\middle|\, q; q\right] \frac{(bd/ac; q)_n}{(d; q)_n} \left(\frac{qc}{b}\right)^n \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q; q\right] &\underset{e\to 0}{=\!=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n},\, a, & b/c \\ & b,\, q^{1-n}a/d \end{matrix} \,\middle|\, q;\, \frac{qc}{d}\right] \frac{(d/a; q)_n}{(d; q)_n} a^n \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q; q\right] &\underset{e\to 0}{=\!=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c, d/c \\ & q^{1-n}/c, bd/ac \end{matrix} \,\middle|\, q;\, \frac{q}{a}\right] \frac{[c, bd/ac; q]_n}{[b, d; q]_n} a^n.
\end{aligned}
$$

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to\infty}{\overset{a\rightleftharpoons e}{=\!=\!=}} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/a,\, b/c \\ & b,\, bd/ac \end{matrix} \,\middle|\, q;\, q^n d\right] \frac{(bd/ac; q)_n}{(d; q)_n} \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to\infty}{=\!=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b/c \\ & b, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \frac{(d/a; q)_n}{(d; q)_n} \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix} \,\middle|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to\infty}{=\!=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c,\, d/c \\ & q^{1-n}/c,\, bd/ac \end{matrix} \,\middle|\, q; q\right] \frac{[c, bd/ac; q]_n}{[b, d; q]_n}.
\end{aligned}
$$

E7. Watson's q-Whipple transformation

E7.1. The Watson transformation. One of the most important basic hypergeometric transformations reads as

$$
{}_8\phi_7\left[\begin{matrix} a,\, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix} \,\middle|\, q;\, \frac{q^{2+n}a^2}{bcde}\right] \tag{E7.1a}
$$

$$
= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \,\middle|\, q; q\right]. \tag{E7.1b}
$$

Proof. In view of the definition of q-hypergeometric series, we can write (E7.1a) explicitly as

$$\begin{aligned}\mathrm{Eq(E7.1a)} &= \sum_{k=0}^{n}\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n}\\ \sqrt{a}, & -\sqrt{a}, qa/b, & qa/c, & aq^{n+1}\end{matrix}\,\Big|\, q\right]_k \left(\frac{q^{1+n}a}{bc}\right)^k\\ &\times \left[\begin{matrix}d, & e\\ qa/d, & qa/e\end{matrix}\,\Big|\, q\right]_k \left(\frac{qa}{de}\right)^k.\end{aligned}$$

Recalling the q-Paff-Saalschütz theorem, we have

$$\begin{aligned}\left[\begin{matrix}d, & e\\ qa/d, & qa/e\end{matrix}\,\Big|\, q\right]_k \left(\frac{qa}{de}\right)^k &= {}_3\phi_2\left[\begin{matrix}q^{-k}, & q^k a, & qa/de\\ & qa/d, & qa/e\end{matrix}\,\Big|\, q\,;\, q\right]\\ &= \sum_{i=0}^{k}\left[\begin{matrix}q^{-k}, & q^k a, & qa/de\\ q, & qa/d, & qa/e\end{matrix}\,\Big|\, q\right]_i q^i.\end{aligned}$$

Therefore substituting this result into Eq(E7.1a) and changing the order of the double sum, we obtain

$$\begin{aligned}\mathrm{Eq(E7.1a)} &= \sum_{i=0}^{n}\left[\begin{matrix}qa/de\\ q, qa/d, qa/e\end{matrix}\,\Big|\, q\right]_i q^i\\ &\times \sum_{k=i}^{n}\frac{1-q^{2k}a}{1-a}\left[\begin{matrix}b, & c, & q^{-n}\\ qa/b, & qa/c, & q^{n+1}a\end{matrix}\,\Big|\, q\right]_k\\ &\times \frac{(a;\,q)_{k+i}(q^{-k};\,q)_i}{(q;\,q)_k}\left(\frac{q^{1+n}a}{bc}\right)^k.\end{aligned}$$

Indicate with Ω the inner sum with respect to k. Putting $k-i=j$ and observing that

$$\frac{(q^{-i-j};q)_i}{(q;q)_{i+j}} = \frac{(-1)^i q^{-\binom{i+1}{2}-ij}}{(q;q)_j}$$

we have

$$\begin{aligned}\Omega &= (-1)^i\, q^{-\binom{i+1}{2}}\,(a;q)_{2i}\,\frac{1-q^{2i}a}{1-a}\left[\begin{matrix}b, & c, & q^{-n}\\ qa/b, & qa/c, & q^{n+1}a\end{matrix}\,\Big|\, q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i\\ &\times \sum_{j=0}^{n-i}\frac{1-q^{2i+2j}a}{1-q^{2i}a}\left[\begin{matrix}q^{2i}a, & q^i b, & q^i c, & q^{-n+i}\\ q, & q^{1+i}a/b, & q^{1+i}a/c, & q^{1+n+i}a\end{matrix}\,\Big|\, q\right]_j \left(\frac{q^{1+n-i}a}{bc}\right)^j\\ &= (-1)^i\, q^{-\binom{i+1}{2}}\,(qa;q)_{2i}\left[\begin{matrix}b, & c, & q^{-n}\\ qa/b, & qa/c, & q^{n+1}a\end{matrix}\,\Big|\, q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i\\ &\times\ {}_6\phi_5\left[\begin{matrix}q^{2i}a, q^{1+i}\sqrt{a}, & -q^{1+i}\sqrt{a}, & q^i b, & q^i c, & q^{-n+i}\\ q^i\sqrt{a}, & -q^i\sqrt{a}, & q^{1+i}a/b, & q^{1+i}a/c, & q^{1+n+i}a\end{matrix}\,\Big|\, q;\frac{q^{1+n-i}a}{bc}\right].\end{aligned}$$

Evaluating the last series by the terminating q-Dougall-Dixon formula (E5.1), we obtain

$$
{}_6\phi_5\left[\begin{matrix} q^{2i}a,\ q^{1+i}\sqrt{a},\ -q^{1+i}\sqrt{a},\ \ q^ib,\ \ q^ic,\ \ q^{-n+i} \\ q^i\sqrt{a},\ \ -q^i\sqrt{a},\ q^{i+1}a/b,\ q^{i+1}a/c,\ q^{1+n+i}a \end{matrix}\ \middle|\ q;\frac{q^{1+n-i}a}{bc}\right]
= \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix}\ \middle|\ q\right]_{n-i}
$$

which implies the following:

$$
\begin{aligned}
\mathrm{Eq(E7.1a)} \;=\; & \sum_{i=0}^{n}\left[\begin{matrix} qa/de \\ q,\ qa/d,\ qa/e \end{matrix}\ \middle|\ q\right]_i q^i \times \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix}\ \middle|\ q\right]_{n-i} \\
\times\; & (-1)^i q^{-\binom{i+1}{2}}(qa;q)_{2i}\left[\begin{matrix} b, & c, & q^{-n} \\ qa/b, & qa/c, & q^{n+1}a \end{matrix}\ \middle|\ q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i.
\end{aligned}
$$

Noting that for the shifted factorials, there hold relations:

$$
\begin{aligned}
(qa/bc;q)_{n-i} &= (-1)^i q^{\binom{i}{2}-ni}\left(\frac{bc}{a}\right)^i\frac{(qa/bc;q)_n}{(q^{-n}bc/a;q)_i} \\
(qa;q)_{2i}\frac{(q^{1+2i}a;q)_{n-i}}{(q^{1+n}a;q)_i} &= (qa;q)_n.
\end{aligned}
$$

Consequently, we have the following expression

$$
\begin{aligned}
\mathrm{Eq(E7.1a)} &= \left[\begin{matrix} qa,qa/bc \\ qa/b,qa/c \end{matrix}\ \middle|\ q\right]_n \sum_{i=0}^{n}\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ q, & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\ \middle|\ q\right]_i q^i \\
&= \left[\begin{matrix} qa,qa/bc \\ qa/b,qa/c \end{matrix}\ \middle|\ q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ b,\ \ c,\ \ qa/de \\ qa/d,\ qa/e,\ q^{-n}bc/a \end{matrix}\ \middle|\ q;q\right]
\end{aligned}
$$

which is exactly (E7.1b). □

E7.2. Rogers-Ramanujan identities. In view of $|q| < 1$ and

$$
x \to \infty \qquad \Longrightarrow \qquad (x;\,q)_k \sim (-1)^k q^{\binom{k}{2}} x^k
$$

the limiting case $b, c, d, e, n \to \infty$ of the Watson transformation reads as:

$$
\sum_{m=0}^{\infty}\frac{q^{m^2}a^m}{(q;\,q)_m} = \frac{1}{(qa;\,q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,\frac{1-q^{2k}a}{1-a}\,\frac{(a;\,q)_k}{(q;\,q)_k}q^{5\binom{k}{2}+2k}a^{2k}. \tag{E7.2}
$$

This transformation can provide us an alternative demonstration of the well-known Rogers-Ramanujan identities (D3.2a) and (D3.2b):

$$
\begin{aligned}
\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} &= \frac{1}{(q;\,q^5)_\infty\,(q^4;\,q^5)_\infty} \\
\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} &= \frac{1}{(q^2;\,q^5)_\infty\,(q^3;\,q^5)_\infty}.
\end{aligned}
$$

In fact, observe first that

$$\frac{1-q^{2k}a}{1-a}\frac{(a;\,q)_k}{(q;\,q)_k} = \frac{1-q^{2k}a}{1-q^ka}\frac{(qa;\,q)_k}{(q;\,q)_k} \overset{a\to 1}{=\!=\!=} \begin{cases}1, & k=0\\ 1+q^k, & k>0.\end{cases}$$

Then letting $a\to 1$, we can restate the transformation (E7.2) as

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} &= \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k(1+q^k)q^{5\binom{k}{2}+2k}\Big\}\\ = \quad \frac{1}{(q;\,q)_\infty}&\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+3k}\Big\}\\ = \quad \frac{1}{(q;\,q)_\infty}&\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-2k}\Big\}.\end{aligned}$$

Applying the Jacobi-triple product identity, we therefore establish the first Rogers-Ramanujan identity:

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+2k} = \frac{[q^5,q^2,q^3;q^5]}{(q;\,q)_\infty}.$$

Letting $a\to q$ instead, we can write the transformation (E7.2) as

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} &= \frac{1}{(q;\,q)_\infty}\sum_{k=0}^{\infty}(-1)^k(1-q^{1+2k})q^{5\binom{k}{2}+4k}\\ = \quad \frac{1}{(q;\,q)_\infty}&\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=0}^{\infty}(-1)^{k+1}q^{5\binom{k}{2}+6k+1}\Big\}\\ = \quad \frac{1}{(q;\,q)_\infty}&\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-4k}\Big\}\end{aligned}$$

where the last line is justified by $k\to k-1$ in the second sum. It leads us to the second Rogers-Ramanujan identity

$$\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+4k} = \frac{[q^5,q^4,q;q^5]}{(q;\,q)_\infty}$$

thanks again to the Jacobi-triple product identity. □

E7.3. Jackson's q-Dougall-Dixon formula.

$$ {}_8\phi_7\left[\begin{matrix}a,\,q\sqrt{a},\,-q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ \sqrt{a},\;-\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix}\,\middle|\,q;q\right] \tag{E7.3a}$$

$$= \left[\begin{matrix}qa,qa/bc,qa/bd,qa/cd\\ qa/b,qa/c,qa/d,qa/bcd\end{matrix}\,\middle|\,q\right]_n, \text{ where } \; q^{n+1}a^2=bcde. \tag{E7.3b}$$

PROOF. When $q^{1+n}a^2 = bcde$ or equivalently $qa/de = q^{-n}bc/a$, the ${}_4\phi_3$-series in the Watson transformation reduces to a balanced ${}_3\phi_2$-series. Therefore, we have in this case the simplified form:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, \ -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix} \,\middle|\, q; q\right] \\
&= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c\end{matrix} \,\middle|\, q\right]_n {}_3\phi_2\left[\begin{matrix} q^{-n}, & b, & c \\ & qa/d, & qa/e\end{matrix} \,\middle|\, q\,;\, q\right].
\end{aligned}
$$

Evaluating the balanced series by the q-Pfaff-Saalschütz theorem, we have

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, \ -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix} \,\middle|\, q;\, q\right] \\
&= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c\end{matrix} \,\middle|\, q\right]_n \times \left[\begin{matrix} qa/bd, & qa/cd \\ qa/d, & qa/bcd\end{matrix} \,\middle|\, q\right]_n
\end{aligned}
$$

which is essentially the same as Jackson's q-Dougall-Dixon formula. □

E7.4. The non-terminating ${}_6\phi_5$-summation formula.

$$
{}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d\end{matrix} \,\middle|\, q;\, \frac{qa}{bcd}\right] \tag{E7.4a}
$$

$$
= \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd\end{matrix} \,\middle|\, q\right]_\infty, \qquad \left|\frac{qa}{bcd}\right| < 1. \tag{E7.4b}
$$

PROOF. Substituting $e = q^{1+n}a^2/bcd$ in the Jackson's q-Dougall-Dixon formula explicitly, we have

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & q^{1+n}a^2/bcd, & q^{-n} \\ \sqrt{a}, \ -\sqrt{a}, & qa/b, & qa/c, & qa/d, & q^{-n}bcd/a, & q^{n+1}a\end{matrix} \,\middle|\, q;\, q\right] \\
&= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd\end{matrix} \,\middle|\, q\right]_n .
\end{aligned}
$$

For $n \to \infty$, recalling the limit relations

$$
\frac{(q^{1+n}a^2/bcd;\, q)_k}{(q^{1+n}a;\, q)_k} \sim 1 \quad \text{and} \quad \frac{(q^{-n};\, q)_k}{(q^{-n}bcd/a;\, q)_k} \sim \left(\frac{a}{bcd}\right)^k
$$

and then applying the Tannery limiting theorem, we get the non-terminating q-Dougall-Dixon formula (E7.4a-E7.4b). □

We remark that when $d = q^{-n}$, the formula (E7.4a-E7.4b) reduces to the terminating q-Dougall-Dixon summation identity (E5.1).

CHAPTER E

Basic Hypergeometric Series

This chapter introduces the basic hypergeometric series. Its convergence condition will be determined. The fundamental transformations and summation formulae will be covered briefly.

E1. Introduction and notation

E1.1. Definition. Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq q^{-n}$ with $n \in \mathbb{N}_0$ for all $j = 1, 2, \cdots, s$. Then the basic hypergeometric series with variable z is defined by

$$_{1+r}\phi_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\,\middle|\, q; z\right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n}\, z^n \Big\{(-1)^n q^{\binom{n}{2}}\Big\}^{s-r}.$$

Remark If there is a numerator parameter $a_i = q^{-k}$ with $k \in \mathbb{N}_0$, then the q-hypergeometric series is terminating, which is in fact a polynomial of z. When the series is nonterminating, we assume that $|q| < 1$ for convenience.

E1.2. Convergence condition. For the q-hypergeometric series just defined, the convergence conditions are as follows:

(A) If $s > r$, the series is convergent for all $z \in \mathbb{C}$;
(B) If $s < r$, the series is convergent only when $z = 0$;
(C) If $s = r$, the series is convergent for $|z| < 1$.

Proof. Denote by T_n the summand of q-hypergeometric series

$$T_n := \Big\{(-1)^n q^{\binom{n}{2}}\Big\}^{s-r} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n}\, z^n.$$

To determine the convergence conditions, we consider the term-ratio:

$$\frac{T_{n+1}}{T_n} = z\frac{(1-q^n a_0)(1-q^n a_1)\cdots(1-q^n a_r)}{(1-q^{n+1})(1-q^n b_1)\cdots(1-q^n a_s)}(-q^n)^{s-r}.$$

On account of $|q| < 1$, we have $|q^n| \to 0$ as $n \to +\infty$. Hence we get the following limit:

$$\lim_{n\to+\infty}\left|\frac{T_{n+1}}{T_n}\right| = \begin{cases} 0, & r < s \\ +\infty, & r > s \text{ and } z \neq 0 \\ |z|, & r = s. \end{cases}$$

According to the D'Alembert ratio test, the convergence conditions stated in the Theorem follow immediately. □

E1.3. Classification. For the basic hypergeometric series, suppose $r = s$, the very important case. If the product of denominator parameters is equal to the base q times the product of numerator parameters, i.e.,

$$qa_0 a_1 \cdots a_r = b_1 b_2 \cdots b_r$$

then the ${}_{1+r}\phi_r$-series is called *balanced* or *Saalschützian.*

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same product:

$$qa_0 = a_1 b_1 = \cdots = a_r b_r$$

then we say that the ${}_{1+r}\phi_r$-series is *well-poised.* In particular, it is said to be *very-well-poised* if we have $a_1 = -a_2 = q\sqrt{a_0}$ in addition. These pairs of parameters appear in the basic hypergeometric sum as a linear fraction

$$\frac{1-a_0 q^{2k}}{1-a_0} = \frac{(q\sqrt{a_0};q)_k}{(\sqrt{a_0};q)_k} \times \frac{(-q\sqrt{a_0};q)_k}{(-\sqrt{a_0};q)_k}.$$

E1.4. Examples. In terms of q-series, we can reformulate the Euler and Gauss summation formulae as follows:

$$(z;\, q)_\infty = {}_1\phi_1\left[\begin{matrix} - \\ - \end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty} \frac{(-z)^k}{(q;\, q)_k} q^{\binom{k}{2}}$$

$$\frac{1}{(z;\, q)_\infty} = {}_1\phi_0\left[\begin{matrix} 0 \\ - \end{matrix}\,\middle|\, q;\, z\right] = \sum_{k=0}^{\infty} \frac{z^k}{(q;\, q)_k}.$$

They will be used to demonstrate the q-binomial theorem.

E1.5. Ordinary hypergeometric series. In comparison with the basic hypergeometric series, we present here briefly the ordinary hypergeometric series, its convergence condition and classification. The details can be found in the book by Bailey (1935).

Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N}_0$ for $j = 1, 2, \cdots, s$. Then the ordinary hypergeometric series with variable z is defined by

$$ {}_{1+r}F_s \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, z \right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{n!\,(b_1)_n \cdots (b_s)_n}\, z^n $$

where the (rising) shifted factorial is defined by

$$ (c)_0 = 1 \quad \textit{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \textit{for} \quad n = 1, 2, \cdots. $$

Classification Similar to basic hypergeometric series, we consider the case $r = s$ for ordinary hypergeometric series. If the sum of denominator parameters is equal to one plus the sum of numerator parameters, i.e.,

$$ 1 + a_0 + a_1 + \cdots + a_r = b_1 + b_2 + \cdots + b_r $$

then the ${}_{1+r}F_r$-series is called *balanced* or *Saalschützian.*

Instead, if the numerator parameters and the denominator parameters can be paired up so that each column has the same sum:

$$ 1 + a_0 = a_1 + b_1 = \cdots = a_r + b_r $$

then we say that the ${}_{1+r}F_r$-series is *well-poised.* In particular, it is said to be *very-well-poised* if we have $a_1 = 1 + a_0/2$ in addition. The last pair of parameters appear in the (ordinary) hypergeometric sum as a linear fraction

$$ \frac{a_0 + 2k}{a_0} \quad = \quad \frac{(1 + a_0/2)_k}{(a_0/2)_k}. $$

Convergence condition for the (ordinary) hypergeometric series is determined as follows:

- if $r < s$, the ${}_{1+r}F_s$-series converges for all $z \in \mathbb{C}$;
- if $r > s$, the ${}_{1+r}F_s$-series diverges for all $z \in \mathbb{C}$ except for $z = 0$;
- if $r = s$, the ${}_{1+r}F_r$-series converges for $|z| < 1$, and when

$$ \begin{aligned} z &= +1 \quad \text{if} \quad \Re(B - A) > 0 \\ z &= -1 \quad \text{if} \quad \Re(B - A) > -1 \end{aligned} $$

where A and B are defined respectively by

$$A = \sum_{i=0}^{r} a_i \quad \text{and} \quad B = \sum_{j=1}^{r} b_j.$$

Remark Noting that the limit relation between ordinary and q-shifted factorials

$$\lim_{q\to 1} \frac{(q^c;\ q)_k}{(1-q)^k} \quad = \quad (c)_k$$

we can consider the (ordinary) hypergeometric series as the limit of the basic hypergeometric series:

$${}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, z\right] = \lim_{q\to 1} {}_{1+r}\phi_s\left[\begin{matrix} q^{a_0}, q^{a_1}, \cdots, q^{a_r} \\ q^{b_1}, \cdots, q^{b_s} \end{matrix} \,\middle|\, q; \frac{(-1)^{r-s}\, z}{(1-q)^{r-s}}\right].$$

This explains why there exist generally the q-counterparts for the (ordinary) hypergeometric series identities.

E2. The q-Gauss summation formula

This section will prove the q-binomial theorem, the q-Gauss summation formula as well as the q-Chu-Vandermonde convolution.

E2.1. The q-binomial theorem. In terms of hypergeometric series, the classical binomial theorem reads as follows:

$${}_1F_0\left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} z^n = \frac{1}{(1-z)^c}, \quad (|z| < 1).$$

Its q-analog is given by the following q-binomial theorem:

$${}_1\phi_0\left[\begin{matrix} c \\ - \end{matrix} \,\middle|\, q;\ z\right] = \frac{(cz;\ q)_\infty}{(z;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(c;\ q)_n}{(q;\ q)_n} z^n, \quad (|z| < 1). \tag{E2.1}$$

For $c = 0$ this identity reduces to Gauss summation formula. Replacing z by z/c and then letting $c \to \infty$, we recover from it the Euler formula.

PROOF. In fact, expanding the numerator and the denominator respectively according to Euler and Gauss summation formulae, we have

$$\begin{aligned}\frac{(cz;\ q)_\infty}{(z;\ q)_\infty} &= \sum_{i=0}^{\infty}(-1)^i \frac{c^i z^i}{(q;\ q)_i} q^{\binom{i}{2}} \sum_{j=0}^{\infty} \frac{z^j}{(q;\ q)_j} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{(q;\ q)_n} \sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} c^i = \sum_{n=0}^{\infty} \frac{(c;\ q)_n}{(q;\ q)_n} z^n\end{aligned}$$

where the last line follows from the finite q-differences. □

E2.2. The q-Gauss summation formula. The q-binomial theorem can be generalized to the following theorem.

For three complex numbers a, b and c with $|c/ab| < 1$, there holds

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ c/ab \right] = \sum_{n=0}^{\infty} \frac{(a;\ q)_n (b;\ q)_n}{(q;\ q)_n (c;\ q)_n} (c/ab)^n = \frac{(c/a;\ q)_\infty (c/b;\ q)_\infty}{(c;\ q)_\infty (c/ab;\ q)_\infty}.$$

PROOF. We can manipulate, by means of the q-binomial theorem (E2.1), the infinite series as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} \left(\frac{c}{ab}\right)^n &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n (q^n c;q)_\infty}{(q;q)_n (q^n b;q)_\infty} \left(\frac{c}{ab}\right)^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \left(\frac{c}{ab}\right)^n \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} (q^n b)^k \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} b^k \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \left(\frac{q^k c}{ab}\right)^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} b^k \frac{(q^k c/b;q)_\infty}{(q^k c/ab;q)_\infty} \\ &= \frac{(b;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/ab;q)_k}{(q;q)_k} b^k \\ &= \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}\end{aligned}$$

which establishes the q-Gauss summation formula. □

E2.3. The q-analog of Chu-Vandermonde convolution. The terminating case of the q-Gauss summation formula can be reformulated as the

q-analogues of the Chu-Vandermonde convolution:

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\, q^n\, c/b\right] = \frac{(c/b;\, q)_n}{(c;\, q)_n} \tag{E2.2a} $$

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \,\middle|\, q;\, q\right] = \frac{(c/b;\, q)_n}{(c;\, q)_n} b^n \tag{E2.2b} $$

$$ \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \begin{bmatrix} x+y \\ n \end{bmatrix}. \tag{E2.2c} $$

Proof. The first formula is the case $a = q^{-n}$ of the q-Gauss theorem, which can be reformulated to other two identities.

By definition of q-hypergeometric series, rewrite (E2.2a) explictly as

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix} \,\middle|\, q; q^n c/b\right] = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k (c; q)_k} (q^n c/b)^k = \frac{(c/b;\, q)_n}{(c;\, q)_n}. $$

Considering that

$$ (x; q)_{n-k} = (-1)^k x^{-k} q^{\binom{k+1}{2}-nk} \frac{(x; q)_n}{(q^{1-n}/x; q)_k} $$

we can manipulate the reversed series as follows:

$$ \begin{aligned} \frac{(c/b;\, q)_n}{(c;\, q)_n} &= \sum_{k=0}^{n} \frac{(q^{-n}; q)_{n-k} (b; q)_{n-k}}{(q; q)_{n-k} (c; q)_{n-k}} (q^n c/b)^{n-k} \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{1-n}/c; q)_k}{(q; q)_k (q^{1-n}/b; q)_k} q^k \\ &= \frac{(q^{-n}; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (q^n c/b)^n \; {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\, q\right] \end{aligned} $$

which is equivalent to

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \,\middle|\, q;\, q\right] = (-1)^n q^{-\binom{n}{2}} \Big(\frac{b}{c}\Big)^n \frac{(c/b; q)_n}{(b; q)_n} $$

in view of

$$ (q^{-n}; q)_n = (-1)^n q^{-\binom{n+1}{2}} (q; q)_n. $$

Performing the parameter replacements

$$ \begin{aligned} B &\to q^{1-n}/c \\ C &\to q^{1-n}/b \end{aligned} $$

and then applying the relation

$$ (q^{1-n}/C; q)_n = (-1)^n q^{-\binom{n}{2}} C^{-n} (C; q)_n $$

we can restate the last formula as:

$$ {}_2\phi_1\left[\begin{matrix}q^{-n}, & B\\ & C\end{matrix}\,\middle|\; q;\; q\right] = B^n \frac{(C/B;q)_n}{(C;q)_n} $$

which is the second formula (E2.2b).

Writing the q-binomial coefficients in terms of q-shifted factorials

$$ \begin{aligned} \begin{bmatrix}x\\ k\end{bmatrix} &= \tfrac{(q^{x-k+1};q)_k}{(q;q)_k} &&= (-1)^k q^{xk-\binom{k}{2}} \frac{(q^{-x};q)_k}{(q;q)_k} \\ \begin{bmatrix}y\\ n-k\end{bmatrix} &= \tfrac{(q^{y-n+k+1};q)_{n-k}}{(q;q)_{n-k}} &&= (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_n} \frac{(q^{y-n+1};q)_n}{(q^{y-n+1};q)_k} \end{aligned} $$

we can express the q-binomial sum in terms of q-series:

$$ \begin{aligned} \sum_{k=0}^{n} \begin{bmatrix}x\\ k\end{bmatrix}\begin{bmatrix}y\\ n-k\end{bmatrix} q^{(x-k)(n-k)} &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k}{(q;q)_k (q^{y-n+1};q)_k} q^k \\ &= \frac{(q^{y-n+1};q)_n}{(q;q)_n} q^{nx}\; {}_2\phi_1\left[\begin{matrix}q^{-n}, & q^{-x}\\ & q^{y-n+1}\end{matrix}\,\middle|\; q;q\right]. \end{aligned} $$

Evaluate the last q-series by (E2.2b):

$$ {}_2\phi_1\left[\begin{matrix}q^{-n}, & q^{-x}\\ & q^{y-n+1}\end{matrix}\,\middle|\; q;\; q\right] = q^{-nx} \frac{(q^{x+y-n+1};q)_n}{(q^{y-n+1};q)_n} $$

we find consequently the following q-binomial identity

$$ \sum_{k=0}^{n} \begin{bmatrix}x\\ k\end{bmatrix}\begin{bmatrix}y\\ n-k\end{bmatrix} q^{(x-k)(n-k)} = \frac{(q^{x+y-n+1};q)_n}{(q;q)_n} = \begin{bmatrix}x+y\\ n\end{bmatrix} $$

which is, in fact, the convolution formula (E2.2c). □

E3. Transformations of Heine and Jackson

E3.1. Jackson's ${}_2\phi_2$-series transformation.

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\; q;\; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} \times {}_2\phi_2\left[\begin{matrix}a, & c/b\\ c, & az\end{matrix}\,\middle|\; q;\; bz\right]. $$

Proof. According to the q-Chu-Vandermonde formula, we have

$$ \frac{(b;q)_n}{(c;q)_n} = {}_2\phi_1\left[\begin{matrix}q^{-n}, & c/b\\ & c\end{matrix}\,\middle|\; q;\; q^n b\right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (c/b;q)_k}{(q;q)_k (c;q)_k} (bq^n)^k. $$

Then the q-hypergeometric series in question can be expressed as a double sum:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \frac{(b;q)_n}{(c;q)_n} z^n \\
&= \sum_{n=0}^{\infty} z^n \frac{(a;q)_n}{(q;q)_n} \times \sum_{k=0}^{n} \frac{(q^{-n};q)_k (c/b;q)_k}{(q;q)_k (c;q)_k} (bq^n)^k \\
&= \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k (c;q)_k} b^k \times \sum_{n=k}^{\infty} \frac{(a;q)_n (q^{-n};q)_k}{(q;q)_n} \left(zq^k\right)^n.
\end{aligned}
$$

For the last sum with respect to n, changing by $j := n-k$ on the summation index and then applying transformations

$$
\begin{aligned}
(a;q)_{j+k} &= (a;q)_k (aq^k;q)_j \\
\frac{(q^{-j-k};q)_k}{(q;q)_{j+k}} &= \frac{(-1)^k q^{-k(j+k)+\binom{k}{2}}}{(q;q)_j}
\end{aligned}
$$

we can evaluate it, by means of (E2.1) with $c \to aq^k$, as follows:

$$
\begin{aligned}
\sum_{n=k}^{\infty} \frac{(a;q)_n (q^{-n};q)_k}{(q;q)_n} \left(zq^k\right)^n &= (-z)^k q^{\binom{k}{2}} (a;q)_k \times \sum_{j=0}^{\infty} \frac{(aq^k;q)_j}{(q;q)_j} z^j \\
&= (-z)^k q^{\binom{k}{2}} (a;q)_k \times {}_1\phi_0\left[\begin{matrix} aq^k \\ - \end{matrix} \,\middle|\, q;\ z\right] \\
&= (-z)^k q^{\binom{k}{2}} (a;q)_k \times \frac{(q^k az;q)_k}{(z;q)_k} \\
&= (-z)^k q^{\binom{k}{2}} \frac{(a;q)_k}{(az;q)_k} \frac{(az;q)_\infty}{(z;q)_\infty}.
\end{aligned}
$$

We have therefore established

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q;\ z\right] &= \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{(a;q)_k (c/b;q)_k}{(q;q)_k (az;q)_k (c;q)_k} (bz)^k \\
&= \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left[\begin{matrix} a, & c/b \\ c, & az \end{matrix} \,\middle|\, q;\ bz\right]
\end{aligned}
$$

which is Jackson's transformation. □

E3.2. Heine's q-Euler transformations.

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}\,\middle|\, q;\ b\right] \tag{E3.1a}$$

$$ = \frac{[c/b, bz; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}\,\middle|\, q; c/b\right] \tag{E3.1b}$$

$$ = \frac{(abz/c; q)_\infty}{(z; q)_\infty} \times {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\middle|\, q;\ abz/c\right]. \tag{E3.1c}$$

PROOF. Substituting the q-factorial fraction

$$\frac{(b;q)_n}{(c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \times \frac{(q^n c;q)_\infty}{(q^n b;q)_\infty}$$

into the q-hypergeometric series

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n$$

and then applying the q-binomial theorem (E2.1):

$$\frac{(q^n c;q)_\infty}{(q^n b;q)_\infty} = {}_1\phi_0\left[\begin{matrix} c/b \\ - \end{matrix}\,\middle|\, q;\ q^n b\right] = \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} q^{nk} b^k$$

we can manipulate the q-series as follows:

$$\begin{aligned} {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} q^{nk} b^k \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b;q)_k}{(q;q)_k} b^k \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^k)^n. \end{aligned}$$

Again by means of (E2.1), evaluating the last sum with respect to n as

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^k)^n = \frac{(q^k az;q)_\infty}{(q^k z;q)_\infty}$$

and then simplifying the series with

$$\begin{aligned} (q^k az;q)_\infty &= \frac{(az;q)_\infty}{(az;q)_k} \\ (q^k z;q)_\infty &= \frac{(z;q)_\infty}{(z;q)_k} \end{aligned}$$

we derive the following expression

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k(z;q)_k}{(q;q)_k(az;q)_k}b^k\\
&= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;\ b\right]
\end{aligned}$$

which is the first transformation (E3.1a).

Applying the transformation just established to the series on the right hand side, we have

$${}_2\phi_1\left[\begin{matrix}z, & c/b\\ & az\end{matrix}\,\middle|\, q;\ b\right] = \frac{(c/b;q)_\infty(bz;q)_\infty}{(az;q)_\infty(b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;\ c/b\right]$$

whose combination with the first one result in

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;\ b\right]\\
&= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;c/b\right].
\end{aligned}$$

This is the second transformation (E3.1b).

Applying again the first transformation, we get

$${}_2\phi_1\left[\begin{matrix}b, abz/c\\ bz\end{matrix}\,\middle|\, q;c/b\right] = \frac{(abz/c;q)_\infty(c;q)_\infty}{(bz;q)_\infty(c/b;q)_\infty}{}_2\phi_1\left[\begin{matrix}c/a, c/b\\ c\end{matrix}\,\middle|\, q;abz/c\right].$$

This leads us to the following

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] &= \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;\ c/b\right]\\
&= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a, c/b\\ c\end{matrix}\,\middle|\, q;\ abz/c\right]
\end{aligned}$$

which is exactly the third transformation (E3.1c).

The last transformation can also be derived by means of the Jackson transformation stated in E3.1. In fact, interchanging a and b in the Jackson formula, we have

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] = \frac{(bz;q)_\infty}{(z;q)_\infty}\times{}_2\phi_2\left[\begin{matrix}b, & c/a\\ c, & bz\end{matrix}\,\middle|\, q;\ az\right].$$

While the q-series on the right hand side of (E3.1c) can be transformed, by means of the Jackson identity, into the following

$${}_2\phi_1\left[\begin{matrix}c/a, & c/b\\ & c\end{matrix}\,\middle|\, q;\ abz/c\right] = \frac{(bz;q)_\infty}{(abz/c;q)_\infty}\times{}_2\phi_2\left[\begin{matrix}c/a, & b\\ c, & bz\end{matrix}\,\middle|\, q;\ az\right].$$

Equating both expressions, we have

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \times {}_2\phi_1\left[\begin{matrix}c/a, & c/b\\ & c\end{matrix}\,\middle|\, q;\ abz/c\right]. $$

The proof of (E3.1c) is therefore completed again. □

Remark The Heine's transformations (E3.1a-E3.1b-E3.1c) may be considered as q-analogues of the Pfaff-Euler Transformations for the (ordinary) hypergeometric series:

$$ \begin{aligned} {}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, z\right] &= (1-z)^{-a}\quad {}_2F_1\left[\begin{matrix}a, & c-b\\ & c\end{matrix}\,\middle|\, \frac{z}{z-1}\right]\\ &= (1-z)^{c-a-b}\ {}_2F_1\left[\begin{matrix}c-a, & c-b\\ & c\end{matrix}\,\middle|\, z\right]. \end{aligned} $$

E3.3. The Bailey-Daum summation formula.

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & qa/b\end{matrix}\,\middle|\, q;\ -q/b\right] = (-q;q)_\infty \frac{\left[qa, q^2a/b^2; q^2\right]_\infty}{\left[qa/b, -q/b; q\right]_\infty}, \quad (|q/b|<1). $$

Proof. Applying the Heine transformation (E3.1a)

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;\ z\right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} \times {}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;\ b\right] $$

we can proceed as follows:

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix}b,\ a\\ qa/b\end{matrix}\,\middle|\, q; -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/b, & -q/b\\ & -q\end{matrix}\,\middle|\, q;\ a\right]\\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n}a^n. \end{aligned} $$

Simplifying the last sum with relations

$$ \begin{aligned} (q^2/b^2;q^2)_n &= (q/b;q)_n(-q/b;q)_n\\ (q^2;q^2)_n &= (q;q)_n(-q;q)_n \end{aligned} $$

and then evaluating it by means of the q-binomial theorem (E2.1), we have

$$ \sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(q;q)_n(-q;q)_n}a^n = \sum_{n=0}^{\infty}\frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n}a^n = \frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty} $$

which results consequently in the following

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix}a, & b\\ & qa/b\end{matrix}\,\middle|\, q;\ -q/b\right] &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty} \times \frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty}\\ &= \frac{(-q;q)_\infty(aq;q^2)_\infty(q^2a/b^2;q^2)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty} \end{aligned} $$

thanks to the shifted factorial relation

$$(a;q)_\infty \;=\; (aq;q^2)_\infty(a;q^2)_\infty.$$

This proves the Bailey-Daum summation theorem. □

E3.4. Infinite series transformation. Armed with the q-series transformation formulae, we apply again inverse series relations (D1.3a-D1.3b) to establish another infinite series transformation, which will be used in turn to prove two infinite series identities of Rogers-Ramanujan type.

Recalling the inverse series relations (D1.3a-D1.3b), if we take the g-sequence

$$g(n) \;=\; \frac{(\lambda;q)_n}{(q\lambda;q^2)_n}q^{\binom{n}{2}} \quad \text{with} \quad n=0,1,2,\cdots$$

then the dual sequence will be determined by

$$f(n) \;=\; \begin{cases} 0, & n-\text{odd} \\ (-1)^m\left[q,\lambda;q^2\right]_m q^{m^2-m}, & n=2m. \end{cases}$$

We have accordingly from (D1.4b) the infinite series transformation:

$$\sum_{n=0}^{\infty}\frac{\lambda^n q^{\frac{3n^2-n}{2}}}{(q;\,q)_n(q\lambda;\,q^2)_n} \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{1-q^{4k}\lambda}{(\lambda;q)_\infty}\frac{(\lambda;\,q^2)_k}{(q^2;\,q^2)_k}q^{5k^2-k}\lambda^{2k}. \tag{E3.2}$$

Proof. Substituting $g(k)$ into (D1.3a) and then rewriting the q-Gauss binomial coefficient, we have

$$\begin{aligned} f(n) \;&=\; \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}(q^k\lambda;\,q)_n\,g(k) \\ &=\; \sum_{k=0}^{n}\frac{(q^{-n};\,q)_k}{(q;\,q)_k}q^{\binom{n}{2}+\binom{k+1}{2}}\frac{(\lambda;\,q)_{n+k}}{(q\lambda;\,q^2)_k}. \end{aligned}$$

By means of factorization

$$(q\lambda;\,q^2)_k \;=\; (\sqrt{q\lambda};\,q)_k\times(-\sqrt{q\lambda};\,q)_k$$

we can express $f(n)$ in terms of a terminating q-hypergeometric series

$$\begin{aligned} f(n) \;&=\; q^{\binom{n}{2}}(\lambda;\,q)_n\;\times\;\sum_{k=0}^{n}\left[\begin{matrix}q^{-n}, & q^n\lambda \\ q, & \pm\sqrt{q\lambda}\end{matrix}\,\middle|\,q\right]_k q^{\binom{k+1}{2}} \\ &=\; q^{\binom{n}{2}}(\lambda;\,q)_n\;\times\;{}_2\phi_2\left[\begin{matrix}q^{-n}, & q^n\lambda \\ \sqrt{q\lambda}, & -\sqrt{q\lambda}\end{matrix}\,\middle|\,q;\,-q\right]. \end{aligned}$$

Rewriting Jackson's transformation formula stated in E3.1

$$ {}_2\phi_2 \left[\begin{matrix} a, & c \\ b, & d \end{matrix} \,\middle|\, q;\ \frac{bd}{ac}\right] = \frac{(d/a;q)_\infty}{(d;q)_\infty} \times {}_2\phi_1 \left[\begin{matrix} a, & b/c \\ & b \end{matrix} \,\middle|\, q;\ d/a\right] $$

we can further reformulate $f(n)$ as follows:

$$ f(n) = q^{\binom{n}{2}} \frac{(\lambda;\ q)_n}{(-\sqrt{q\lambda};\ q)_n} \times {}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{-n}\sqrt{q/\lambda} \\ & \sqrt{q\lambda} \end{matrix} \,\middle|\, q;\ -q^n\sqrt{q\lambda}\right]. $$

Evaluating the last series by means of the Bailey-Daum formula stated in E3.3:

$$ {}_2\phi_1 \left[\begin{matrix} a, & b \\ & qa/b \end{matrix} \,\middle|\, q;\ -q/b\right] = (-q;q)_\infty \frac{\left[qa, q^2a/b^2; q^2\right]_\infty}{\left[qa/b, -q/b; q\right]_\infty}, \quad (|q/b| < 1) $$

we find that

$$ f(n) = q^{\binom{n}{2}} (\lambda;\ q)_n \left[\begin{matrix} q^{1-n}, & q^{1+n}\lambda \\ q, & q\lambda \end{matrix} \,\middle|\, q^2\right]_\infty. $$

If n is odd, we have $f(n) = 0$ for $(q^{1-n};\ q^2)_\infty = 0$. Suppose $n = 2m$ instead, we have the following reduction

$$ \begin{aligned} f(n) &= q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \left[\begin{matrix} q^{1-2m}, & q^{1+2m}\lambda \\ q, & q\lambda \end{matrix} \,\middle|\, q^2\right]_\infty \\ &= q^{\binom{2m}{2}} (\lambda;\ q)_{2m} \frac{(q^{1-2m};\ q^2)_m}{(q\lambda;\ q^2)_m} \\ &= (-1)^m q^{m^2-m} \left[q,\ \lambda;\ q^2\right]_m. \end{aligned} $$

Substituting $g(n)$ and $f(k)$ into (D1.4b), we establish (E3.2). □

E3.5. Two further identities of Rogers-Ramanujan type. Specifying with $\lambda \to 1$ and $\lambda \to q^2$ in (E3.2), we derive the following identities of Rogers-Ramanujan type:

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\ q)_n\ (q;\ q^2)_n} &= \frac{\left[q^{10}, q^4, q^6; q^{10}\right]_\infty}{(q;\ q)_\infty} \\ \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2+3n}{2}}}{(q;\ q)_n (q;\ q^2)_{n+1}} &= \frac{\left[q^{10}, q^2, q^8; q^{10}\right]_\infty}{(q;\ q)_\infty}. \end{aligned} $$

Proof. Putting $\lambda \to 1$ in (E3.2) and then separating the first term from the right hand side, we derive

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2-n}{2}}}{(q;\ q)_n(q;\ q^2)_n} &= \frac{1}{(q;q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k\left\{1+q^{2k}\right\}q^{5k^2-k}\Big\}\\ &= \frac{1}{(q;q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k q^{5k^2-k}+\sum_{k=1}^{\infty}(-1)^k q^{5k^2+k}\Big\}.\end{aligned}$$

Performing replacement $k \to -k$ in the last sum and then applying the Jacobi triple product identity, we reduce the sum inside $\{\cdots\}$ as

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{10\binom{k}{2}+4k} = \left[q^{10},\ q^4,\ q^6;\ q^{10}\right]_\infty$$

which leads us to the first identity:

$$\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2-n}{2}}}{(q;\ q)_n(q;\ q^2)_n} = \frac{\left[q^{10},\ q^4,\ q^6;\ q^{10}\right]_\infty}{(q;q)_\infty}.$$

When $\lambda \to q^2$, we can similarly write (E3.2) as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2+3n}{2}}}{(q;\ q)_n(q;\ q^2)_{n+1}} &= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\left\{1-q^{4k+2}\right\}q^{5k^2+3k}\\ &= \frac{1}{(q;q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^k q^{5k^2+3k}+\sum_{k=0}^{\infty}(-1)^{k+1}q^{5k^2+7k+2}\Big\}.\end{aligned}$$

Replacing k by $-k-1$ in the second sum and then applying the Jacobi triple product identity, we find that the sum inside $\{\cdots\}$ equals

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{10\binom{k}{2}+8k} = \left[q^{10},\ q^2,\ q^8;\ q^{10}\right]_\infty$$

which results in the second identity:

$$\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2+3n}{2}}}{(q;\ q)_n(q;\ q^2)_{n+1}} = \frac{\left[q^{10},\ q^2,\ q^8;\ q^{10}\right]_\infty}{(q;q)_\infty}.$$

This completes proofs of two infinite series identities of Rogers-Ramanujan type. □

E4. The q-Pfaff-Saalschütz summation theorem

The formula under the title reads as the following

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\Big|\, q;\ q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\,\Big|\, q;\ q\right]_n. \tag{E4.1} $$

E4.1. Proof. Recall the q-Euler transformation (E3.1c):

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\Big|\, q;\ abz/c\right] = \frac{(z;q)_\infty}{(abz/c;q)_\infty} \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\Big|\, q;\ z\right] $$

which can be reformulated through the q-binomial theorem (E2.1), as a product of two basic hypergeometric series:

$$ {}_2\phi_1\left[\begin{matrix} c/a, & c/b \\ & c \end{matrix}\,\Big|\, q;\ abz/c\right] = {}_1\phi_0\left[\begin{matrix} c/ab, \\ - \end{matrix}\,\Big|\, q;\ abz/c\right] \times {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\Big|\, q;\ z\right]. $$

Extracting the coefficient of z^n from both members, we have

$$ \frac{(c/a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(ab/c)^n = \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}\frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}}(ab/c)^{n-k} $$

which can be restated equivalently as

$$ \begin{aligned} \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} &= \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}\frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k}q^k \\ &= {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\Big|\, q;\ q\right] \end{aligned} $$

in view of shifted factorial fraction

$$ \begin{aligned} \frac{(c/ab;q)_{n-k}}{(q;q)_{n-k}} &= \frac{(c/ab;q)_n}{(q;q)_n}\frac{(q^{n-k+1};q)_k}{(q^{n-k}c/ab;q)_k} \\ &= \frac{(c/ab;q)_n}{(q;q)_n}\frac{(q^{-n};q)_k}{(q^{1-n}ab/c;q)_k}(qab/c)^k. \end{aligned} $$

This completes the proof of the q-Saalschütz formula. □

E4.2. The formula (E4.1) can also be proved by means of series rearrangement.

Recalling the q-Chu-Vandermonde formula (E2.2a), we have

$$ \frac{(a;q)_k}{(c;q)_k} = {}_2\phi_1\left[\begin{matrix} q^{-k}, & c/a \\ & c \end{matrix}\,\Big|\, q;\ q^k a\right] = \sum_{j=0}^{k} \frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j}q^{kj}a^j. $$

Then the q-hypergeometric series in (E4.1) can be written as a double sum:

$$\begin{aligned}
\text{LHS(E4.1)} &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k(q^{1-n}ab/c;q)_k} q^k \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(b;q)_k}{(q;q)_k(q^{1-n}ab/c;q)_k} q^k \sum_{j=0}^{k} \frac{(q^{-k};q)_j(c/a;q)_j}{(q;q)_j(c;q)_j} q^{kj} a^j \\
&= \sum_{j=0}^{n} \frac{(c/a;q)_j}{(q;q)_j(c;q)_j} a^j \sum_{k=j}^{n} \frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k} q^{k(j+1)}
\end{aligned}$$

where we have changed the summation order.

Denote by $\Omega(j)$ the last sum with respect to k. Changing the summation index with $i := k-j$ and then applying relations

$$\begin{aligned}
(x;q)_{i+j} &= (x;q)_i(q^ix;q)_j = (x;q)_j(q^jx;q)_i \\
(q^{-i-j};q)_j &= (-1)^j q^{-j(i+j)+\binom{j}{2}}(q^{i+1};q)_j
\end{aligned}$$

we can reduce $\Omega(j)$ as follows:

$$\begin{aligned}
\Omega(j) &= \sum_{k=j}^{n} \frac{(q^{-n};q)_k(b;q)_k(q^{-k};q)_j}{(q;q)_k(q^{1-n}ab/c;q)_k} q^{k(j+1)} \\
&= \sum_{i=0}^{n-j} \frac{(q^{-n};q)_{i+j}(b;q)_{i+j}(q^{-i-j};q)_j}{(q;q)_{i+j}(q^{1-n}ab/c;q)_{i+j}} q^{(i+j)(j+1)} \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} \sum_{i=0}^{n-j} \frac{(q^{-n+j};q)_i(q^jb;q)_i}{(q;q)_i(q^{1-n+j}ab/c;q)_i} q^i \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} \, {}_2\phi_1\left[\begin{matrix} q^{-n+j},\ q^jb \\ q^{1-n+j}ab/c \end{matrix} \,\Big|\, q;\ q\right].
\end{aligned}$$

Applying now the q-Chu-Vandermonde formula (E2.2b), we can evaluate the q-series on the right hand side as

$${}_2\phi_1\left[\begin{matrix} q^{-n+j}, & bq^j \\ & q^{1-n+j}ab/c \end{matrix} \,\Big|\, q;\ q\right] = \frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}} \big(bq^j\big)^{n-j}$$

which results consequently in

$$\begin{aligned}
\Omega(j) &= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-n}ab/c;q)_j} \times \frac{(q^{1-n}a/c;q)_{n-j}}{(q^{1-n+j}ab/c;q)_{n-j}} \big(bq^j\big)^{n-j} \\
&= (-1)^j q^{\binom{j+1}{2}} \frac{(q^{-n};q)_j(b;q)_j}{(q^{1-j}a/c;q)_j} \times \frac{(q^{1-n}a/c;q)_n}{(q^{1-n}ab/c;q)_n} \big(bq^j\big)^{n-j} \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n} \frac{(q^{-n};q)_j(b;q)_j}{(c/a;q)_j} \big(q^nc/ab\big)^j.
\end{aligned}$$

Substituting the last expression of $\Omega(j)$ into the ${}_3\phi_2$-series and then applying the q-Chu-Vandermonde formula (E2.2a), we get the following evaluation

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\, q\right] &= \sum_{j=0}^{n} a^j \frac{(c/a;q)_j}{(q;q)_j(c;q)_j}\Omega(j) \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\,{}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}\,\middle|\, q; q^n c/b\right] \\
&= \frac{(c/a;q)_n}{(c/ab;q)_n}\frac{(c/b;q)_n}{(c;q)_n}
\end{aligned}$$

which is equivalent to the q-Pfaff-Saalschütz formula (E4.1).

E5. The terminating q-Dougall-Dixon formula

It is, in fact, a very-well-poised terminating series identity

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a \end{matrix}\,\middle|\, q; \frac{q^{1+n}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\,\middle|\, q\right]_n. \tag{E5.1}$$

Proof. Based on the Carlitz inversions presented in (D1.1), we can derive the identity directly as the dual relation of the q-Pfaff-Saalschütz formula (E4.1).

Recalling the q-Pfaff-Saalschütz theorem (E4.1)

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\middle|\, q;\, q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\,\middle|\, q;\, q\right]_n $$

we can restate it under parameter replacements as

$$\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix}\,\middle|\, q;\, q\right] &= \frac{(q^{1-n}/b;q)_n(c;q)_n}{(qa/b;q)_n(q^{-n}c/a;q)_n} \\
&= \frac{(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n.
\end{aligned}$$

In order to apply the Carlitz inversions, we reformulate the q-series

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix}\,\middle|\, q;\, q\right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^n a;q)_k(qa/bc;q)_k}{(q;q)_k(qa/b;q)_k(qa/c;q)_k} q^k $$

in terms of the q-binomial sum

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a;q)_n \frac{(a;q)_k(qa/bc;q)_k}{(qa/b;q)_k(qa/c;q)_k} \\
&= q^{\binom{n}{2}} \frac{(a;q)_n(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n
\end{aligned}$$

where we have used the following transformations:

$$\begin{aligned}\frac{(q^{-n};q)_k}{(q;q)_k} &= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} q^{-\binom{n}{2}-k} \\ (q^n a;q)_k &= \frac{(a;q)_{n+k}}{(a;q)_n} = \frac{(a;q)_k(q^k a;q)_n}{(a;q)_n}.\end{aligned}$$

Specifying the ϕ-polynomials with $a_k = 1$ and $b_k = -q^k a$ in the Carlitz inversions (D1.1a-D1.1b), which implies

$$\phi(x;n) := (ax;q)_n = \prod_{i=0}^{n-1}(1-axq^i)$$

and then choosing two sequences

$$\begin{aligned}f(n) &:= q^{\binom{n}{2}} \frac{(a;q)_n(b;q)_n(c;q)_n}{(qa/b;q)_n(qa/c;q)_n}\left(\frac{qa}{bc}\right)^n \\ g(k) &:= \frac{(a;q)_k(qa/bc;q)_k}{(qa/b;q)_k(qa/c;q)_k}\end{aligned}$$

we write down directly the dual relation

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}a}{(q^n a;q)_{k+1}} q^{\binom{k}{2}} \frac{(a;q)_k(b;q)_k(c;q)_k}{(qa/b;q)_k(qa/c;q)_k}\left(\frac{qa}{bc}\right)^k \\ &= \frac{(a;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n}.\end{aligned}$$

Feeding back the q-binomial coefficient to factorial fraction

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{(q^{n-k+1};q)_k}{(q;q)_k} = (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k} \\ \frac{1-q^{2k}a}{1-a} &= \frac{(q\sqrt{a};q)_k}{(\sqrt{a};q)_k}\frac{(-q\sqrt{a};q)_k}{(-\sqrt{a};q)_k}\end{aligned}$$

we reformulate the dual relation in terms of q-series

$$\frac{(qa;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n} = {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a \end{matrix} \,\middle|\, q; \frac{q^{1+n}a}{bc}\right]$$

which is the terminating q-Dixon formula (E5.1). □

E6. The Sears balanced transformations

Replacing the base q with its inverse $1/q$ and then observing that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \times q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}$$

we can restate the Carlitz inversions in an equivalent form

$$\begin{cases} f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \phi(q^{-k};n)\, g(k), & n = 0,1,2,\cdots \quad \text{(E6.1a)} \\ g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \dfrac{a_k + q^{-k}b_k}{\phi(q^{-n};k+1)}\, f(k), & n = 0,1,2,\cdots \quad \text{(E6.1b)} \end{cases}$$

which will be used in this section to prove the Sears transformations on balanced basic hypergeometric series.

E6.1. The Sears balanced transformations.

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q\right] && \text{(E6.2a)} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} && \text{(E6.2b)} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n}. && \text{(E6.2c)} \end{aligned}$$

E6.2. Proof of (E6.2a-E6.2c). The second transformation formula is a consequence of the first. In fact, applying the symmetric property to (E6.2b) and then transform it by the first transformation (E6.2a-E6.2b), we have

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & a, & b/e \\ & bd/ce, & b, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & bd/ace, & d/c \\ & bd/ce, & bd/ac, & q^{1-n}/c \end{matrix} \,\middle|\, q;\ q\right] \times \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n}. \end{aligned}$$

Substituting this result into (E6.2b), we find the transformation

$$\begin{aligned} & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ & b, & d, & q^{1-n}ace/bd \end{matrix} \,\middle|\, q;\ q\right] \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b/c, & b/e \\ & b, & bd/ce, & q^{1-n}a/d \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[d/a, bd/ce; q]_n}{[d, bd/ace; q]_n} \frac{[c, bd/ac; q]_n}{[b, d/a; q]_n} \\ = \ & {}_4\phi_3\left[\begin{matrix} q^{-n}, & b/c, & d/c, & bd/ace \\ & bd/ac, & bd/ce, & q^{1-n}/c \end{matrix} \,\middle|\, q; q\right] \times \frac{[c, bd/ac, bd/ce; q]_n}{[b, d, bd/ace; q]_n} \end{aligned}$$

which is the second formula (E6.2a-E6.2c).

E6.3. Proof of (E6.2a-E6.2b). Let the ϕ-polynomials be defined by

$$\phi(x;n) \;=\; (acex/bd;q)_n \quad\rightleftharpoons\quad a_k = 1 \text{ and } b_k = -q^k ace/bd.$$

Then the corresponding inversions (E6.1a-E6.1b) become the following:

$$f(n) \;=\; \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} (q^{-k}ace/bd;q)_n\, g(k) \tag{E6.3a}$$

$$g(n) \;=\; \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}}\, f(k). \tag{E6.3b}$$

By means of two q-shifted factorial relations

$$\frac{(q^{-n};q)_k}{(q;q)_k} \;=\; (-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}-\binom{n}{2}-k}$$

$$(q^{1-n}ace/bd;q)_k \;=\; \frac{(q^{-n}ace/bd;q)_{k+1}}{1-q^{-n}ace/bd}$$

we can rewrite the ${}_4\phi_3$-series displayed in (E6.2a) as a q-binomial sum

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e\\ & b, & d, & q^{1-n}ace/bd\end{matrix}\;\middle|\; q;\, q\right] \frac{1-bd/ace}{1-q^n bd/ace} q^{\binom{n+1}{2}}\\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\middle|\; q\right]_k .\end{aligned}$$

Then the first transformation of Sears (E6.2a-E6.2b) can be stated equivalently as

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \frac{1-ace/bd}{(q^{-n}ace/bd;q)_{k+1}} \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\middle|\; q\right]_k \tag{E6.4a}$$

$$= {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e\\ b,\ bd/ce,\ q^{1-n}a/d\end{matrix}\;\middle|\; q;q\right] \left[\begin{matrix} d/a, bd/ce\\ d, qbd/ace\end{matrix}\;\middle|\; q\right]_n q^{\binom{n+1}{2}}. \tag{E6.4b}$$

This expression matches perfectly with the relation (E6.3b), where two sequences have been specified by

$$f(k) := \left[\begin{matrix} a,\ c,\ e\\ b,\ d\end{matrix}\;\middle|\; q\right]_k \tag{E6.5a}$$

$$g(n) := \left[\begin{matrix} d/a, bd/ce\\ d, qbd/ace\end{matrix}\;\middle|\; q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ b/c,\ b/e\\ b,\ bd/ce,\ q^{1-n}a/d\end{matrix}\;\middle|\; q;q\right] q^{\binom{n+1}{2}}. \tag{E6.5b}$$

Therefore in order to demonstrate the first transformation (E6.4a-E6.4b) of Sears, it suffices to prove the following dual relation, which corresponds to the relation (E6.3a):

$$\left[\begin{matrix}a,c,e\\b,\ d\end{matrix}\,\middle|\, q\right]_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix}n\\k\end{bmatrix}(q^{-k}ace/bd;q)_n \left[\begin{matrix}d/a, bd/ce\\d, qbd/ace\end{matrix}\,\middle|\, q\right]_k q^{\binom{k+1}{2}} \tag{E6.6a}$$

$$\times\ {}_4\phi_3\left[\begin{matrix}q^{-k}, & a, & b/c, & b/e\\ & b, & bd/ce, & q^{1-k}a/d\end{matrix}\,\middle|\, q;\ q\right]. \tag{E6.6b}$$

Let Ξ stand for the double sum on the right. We should therefore verify that Ξ reduces to the factorial fraction on the left.

Recalling the definition of q-hypergeometric series

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{-k}, & a, & b/c, & b/e\\ & b, & bd/ce, & q^{1-n}a/d\end{matrix}\,\middle|\, q;\ q\right]\\ =\ &\sum_{i=0}^{k}\left[\begin{matrix}q^{-k}, & a, & b/c, & b/e\\ q, & b, & bd/ce, & q^{1-k}a/d\end{matrix}\,\middle|\, q\right]_i q^i\end{aligned}$$

and the relation of q-binomial coefficient in terms of factorial fraction

$$\begin{bmatrix}n\\k\end{bmatrix} = (-1)^k\frac{(q^{-n};q)_k}{(q;q)_k}q^{nk-\binom{k}{2}}$$

we can rearrange the double sum as follows:

$$\begin{aligned}\Xi\ =\ &\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^{-k}ace/bd;q)_n\left[\begin{matrix}d/a, bd/ce\\d, qbd/ace\end{matrix}\,\middle|\, q\right]_k q^{\binom{k+1}{2}}\\ \times\ &\sum_{i=0}^{k}\left[\begin{matrix}q^{-k}, & a, & b/c, & b/e\\ q, & b, & bd/ce, & q^{1-k}a/d\end{matrix}\,\middle|\, q\right]_i q^i\\ =\ &\sum_{i=0}^{n}\left[\begin{matrix}a, b/c, b/e\\q, b, bd/ce\end{matrix}\,\middle|\, q\right]_i q^i\sum_{k=i}^{n}\left[\begin{matrix}q^{-n}, d/a, bd/ce\\q, d, qbd/ace\end{matrix}\,\middle|\, q\right]_k\\ \times\ &(q^{-k}ace/bd;q)_n\frac{(q^{-k};q)_i}{(q^{1-k}a/d;q)_i}q^{k(n+1)}.\end{aligned}$$

For the inner sum, performing the replacement $j := k-i$ on summation index and then applying relations

$$\begin{aligned}\frac{(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i} &= \left(\frac{d}{qa}\right)^i\frac{(q^{1+j};q)_i}{(q^jd/a;q)_i} = \left(\frac{d}{qa}\right)^i\frac{(q;q)_{i+j}}{(d/a;q)_{i+j}}\frac{(d/a;q)_j}{(q;q)_j}\\ (q^{-i-j}ace/bd;q)_n &= \frac{(q^{-i-j}ace/bd;q)_{i+j}}{(q^{n-i-j}ace/bd;q)_{i+j}}(ace/bd;q)_n\\ &= \frac{(qbd/ace;q)_{i+j}}{(q^{1-n}bd/ace;q)_i}\frac{(ace/bd;q)_n}{(q^{1+i-n}bd/ace;q)_j}q^{-n(i+j)}\end{aligned}$$

we can reduce it to the following

$$
\begin{aligned}
&\sum_{k=i}^{n}\left[\begin{matrix} q^{-n}, & d/a, & bd/ce \\ q, & d, & qbd/ace \end{matrix}\Big|\, q\right]_k \frac{(q^{-k}ace/bd;q)_n(q^{-k};q)_i}{(q^{1-k}a/d;q)_i}q^{k(n+1)}\\
=&\sum_{j=0}^{n-i}\left[\begin{matrix} q^{-n}, & d/a, & bd/ce \\ q, & d, & qbd/ace \end{matrix}\Big|\, q\right]_{i+j} \frac{(q^{-i-j}ace/bd;q)_n(q^{-i-j};q)_i}{(q^{1-i-j}a/d;q)_i}q^{(i+j)(n+1)}\\
=&(ace/bd;q)_n\left[\begin{matrix} q^{-n},\ bd/ce \\ d,\ q^{1-n}bd/ace \end{matrix}\Big|\, q\right]_i \Big(\frac{d}{a}\Big)^i\sum_{j=0}^{n-i}\left[\begin{matrix} q^{i-n},\ d/a,\ q^ibd/ce \\ q,\ q^id,\ q^{1+i-n}bd/ace \end{matrix}\Big|\, q\right]_j q^j.
\end{aligned}
$$

The last sum with respect to j can be evaluated by means of the q-Saalschütz formula as follows:

$$
{}_3\phi_2\left[\begin{matrix} q^{i-n}, & d/a, & q^ibd/ce \\ & q^id, & q^{1+i-n}bd/ace \end{matrix}\Big|\ q;\ q\right] = \left[\begin{matrix} q^ia, & ce/b \\ q^id, & ace/bd \end{matrix}\Big|\, q\right]_{n-i}.
$$

Substituting this result into the double sum expression of Ξ and then applying transformation

$$
\frac{(ce/b;q)_{n-i}}{(ace/bd;q)_{n-i}} = \frac{(ce/b;q)_n}{(ace/bd;q)_n}\frac{(q^{1-n}bd/ace;q)_i}{(q^{1-n}b/ce;q)_i}\Big(\frac{a}{d}\Big)^i
$$

we reduce the double sum to a single ${}_3\phi_2$-series:

$$
\begin{aligned}
\Xi &= \sum_{i=0}^{n}\left[\begin{matrix} a,\ b/c,\ b/e \\ q,\ b,\ bd/ce \end{matrix}\Big|\, q\right]_i \Big(\frac{qd}{a}\Big)^i \left[\begin{matrix} q^{-n},\ bd/ce \\ d,\ q^{1-n}bd/ace \end{matrix}\Big|\, q\right]_i\\
&\times\ (ace/bd;q)_n\left[\begin{matrix} q^ia, & ce/b \\ q^id, & ace/bd \end{matrix}\Big|\, q\right]_{n-i}\\
&= (ce/b;q)_n\frac{(a;q)_n}{(d;q)_n}\sum_{i=0}^{n}q^i\left[\begin{matrix} b/c,\ b/e,\ q^{-n} \\ q,\ b,\ q^{1-n}b/ce \end{matrix}\Big|\, q\right]_i.
\end{aligned}
$$

Evaluating the last sum with respect to i through the q-Saalschütz formula

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c, & b/e \\ & b, & q^{1-n}b/ce \end{matrix}\Big|\, q;q\right] = \left[\begin{matrix} c, & e \\ b, & ce/b \end{matrix}\Big|\, q\right]_n
$$

which is equivalent to

$$
\Xi = \left[\begin{matrix} a,\ c,\ e \\ b,\ d \end{matrix}\Big|\, q\right]_n.
$$

This completes the proof of (E6.2a-E6.2b). □

E6.4. The q-Kummer-Thomae-Whipple's formulae. As the limiting cases $n \to \infty$ of Sears' transformations, we have the non-terminating

q-Kummer-Thomae-Whipple's formulae:

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a,\, c,\, e \\ b,\, d \end{matrix}\,\Big|\, q;\, \frac{bd}{ace}\right] &= {}_3\phi_2\left[\begin{matrix} a, & b/c, & b/e \\ & b, & bd/ce \end{matrix}\,\Big|\, q;\, \frac{d}{a}\right] \times \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} \\
&= {}_3\phi_2\left[\begin{matrix} b/c,\, d/c,\, bd/ace \\ bd/ac,\, bd/ce \end{matrix}\,\Big|\, q;\, c\right] \times \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty}.
\end{aligned}
$$

Other transformations on terminating series derived from Sears' transformation may be displayed as follows:

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;q\right] &\underset{e\to 0}{\overset{a\rightleftharpoons e}{=\!=}} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/a,\, b/c \\ & b,\, bd/ac \end{matrix}\,\Big|\, q;q\right] \frac{(bd/ac;q)_n}{(d;q)_n}\left(\frac{qc}{b}\right)^n \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;q\right] &\underset{e\to 0}{=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n},\, a, & b/c \\ & b,\, q^{1-n}a/d \end{matrix}\,\Big|\, q;\, \frac{qc}{d}\right] \frac{(d/a;q)_n}{(d;q)_n} a^n \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;q\right] &\underset{e\to 0}{=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c, d/c \\ & q^{1-n}/c, bd/ac \end{matrix}\,\Big|\, q;\, \frac{q}{a}\right] \frac{[c, bd/ac; q]_n}{[b, d; q]_n} a^n.
\end{aligned}
$$

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to \infty}{\overset{a\rightleftharpoons e}{=\!=}} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/a,\, b/c \\ & b,\, bd/ac \end{matrix}\,\Big|\, q;\, q^n d\right] \frac{(bd/ac;q)_n}{(d;q)_n} \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to \infty}{=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b/c \\ & b, & q^{1-n}a/d \end{matrix}\,\Big|\, q;q\right] \frac{(d/a;q)_n}{(d;q)_n} \\
{}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, c \\ b,\, d \end{matrix}\,\Big|\, q;\, \frac{bd}{ac}q^n\right] &\underset{e\to \infty}{=\!=} {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c,\, d/c \\ & q^{1-n}/c,\, bd/ac \end{matrix}\,\Big|\, q;q\right] \frac{[c, bd/ac; q]_n}{[b, d; q]_n}.
\end{aligned}
$$

E7. Watson's q-Whipple transformation

E7.1. The Watson transformation. One of the most important basic hypergeometric transformations reads as

$$
{}_8\phi_7\left[\begin{matrix} a,\, q\sqrt{a},\, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix}\,\Big|\, q;\, \frac{q^{2+n}a^2}{bcde}\right] \tag{E7.1a}
$$

$$
= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\,\Big|\, q;q\right]. \tag{E7.1b}
$$

Proof. In view of the definition of q-hypergeometric series, we can write (E7.1a) explicitly as

$$\begin{aligned}\text{Eq(E7.1a)} &= \sum_{k=0}^{n} \left[\begin{matrix} a, \; q\sqrt{a}, \; -q\sqrt{a}, \; b, \; c, \; q^{-n} \\ \sqrt{a}, \; -\sqrt{a}, \; qa/b, \; qa/c, \; aq^{n+1} \end{matrix} \Big| \, q\right]_k \left(\frac{q^{1+n}a}{bc}\right)^k \\ &\times \left[\begin{matrix} d, \; e \\ qa/d, \; qa/e \end{matrix} \Big| \, q\right]_k \left(\frac{qa}{de}\right)^k.\end{aligned}$$

Recalling the q-Paff-Saalschütz theorem, we have

$$\begin{aligned}\left[\begin{matrix} d, \; e \\ qa/d, \; qa/e \end{matrix} \Big| \, q\right]_k \left(\frac{qa}{de}\right)^k &= {}_3\phi_2\left[\begin{matrix} q^{-k}, \; q^k a, \; qa/de \\ qa/d, \; qa/e \end{matrix} \Big| \, q\,;\, q\right] \\ &= \sum_{i=0}^{k} \left[\begin{matrix} q^{-k}, \; q^k a, \; qa/de \\ q, \; qa/d, \; qa/e \end{matrix} \Big| \, q\right]_i q^i.\end{aligned}$$

Therefore substituting this result into Eq(E7.1a) and changing the order of the double sum, we obtain

$$\begin{aligned}\text{Eq(E7.1a)} &= \sum_{i=0}^{n} \left[\begin{matrix} qa/de \\ q, \; qa/d, \; qa/e \end{matrix} \Big| \, q\right]_i q^i \\ &\times \sum_{k=i}^{n} \frac{1-q^{2k}a}{1-a} \left[\begin{matrix} b, \; c, \; q^{-n} \\ qa/b, \; qa/c, \; q^{n+1}a \end{matrix} \Big| \, q\right]_k \\ &\times \frac{(a;\, q)_{k+i}(q^{-k};\, q)_i}{(q;\, q)_k} \left(\frac{q^{1+n}a}{bc}\right)^k.\end{aligned}$$

Indicate with Ω the inner sum with respect to k. Putting $k-i=j$ and observing that

$$\frac{(q^{-i-j};q)_i}{(q;q)_{i+j}} = \frac{(-1)^i q^{-\binom{i+1}{2}-ij}}{(q;q)_j}$$

we have

$$\begin{aligned}\Omega &= (-1)^i \, q^{-\binom{i+1}{2}} \, (a;q)_{2i} \, \frac{1-q^{2i}a}{1-a} \left[\begin{matrix} b, \; c, \; q^{-n} \\ qa/b, \; qa/c, \; q^{n+1}a \end{matrix} \Big| \, q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i \\ &\times \sum_{j=0}^{n-i} \frac{1-q^{2i+2j}a}{1-q^{2i}a} \left[\begin{matrix} q^{2i}a, \; q^i b, \; q^i c, \; q^{-n+i} \\ q, \; q^{1+i}a/b, \; q^{1+i}a/c, \; q^{1+n+i}a \end{matrix} \Big| \, q\right]_j \left(\frac{q^{1+n-i}a}{bc}\right)^j \\ &= (-1)^i \, q^{-\binom{i+1}{2}} \, (qa;q)_{2i} \left[\begin{matrix} b, \; c, \; q^{-n} \\ qa/b, \; qa/c, \; q^{n+1}a \end{matrix} \Big| \, q\right]_i \left(\frac{q^{1+n}\,a}{b\,c}\right)^i \\ &\times \, {}_6\phi_5\left[\begin{matrix} q^{2i}a, \; q^{1+i}\sqrt{a}, \; -q^{1+i}\sqrt{a}, \; q^i b, \; q^i c, \; q^{-n+i} \\ q^i\sqrt{a}, \; -q^i\sqrt{a}, \; q^{1+i}a/b, \; q^{1+i}a/c, \; q^{1+n+i}a \end{matrix} \Big| \, q;\, \frac{q^{1+n-i}a}{bc}\right].\end{aligned}$$

Evaluating the last series by the terminating q-Dougall-Dixon formula (E5.1), we obtain

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} q^{2i}a,\ q^{1+i}\sqrt{a},\ -q^{1+i}\sqrt{a}, & q^ib, & q^ic, & q^{-n+i} \\ q^i\sqrt{a}, & -q^i\sqrt{a}, & q^{i+1}a/b,\ q^{i+1}a/c, & q^{1+n+i}a \end{matrix}\ \middle|\ q;\frac{q^{1+n-i}a}{bc}\right] \\
&= \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix}\ \middle|\ q\right]_{n-i}
\end{aligned}$$

which implies the following:

$$\begin{aligned}
\mathrm{Eq(E7.1a)} &= \sum_{i=0}^{n}\left[\begin{matrix} qa/de \\ q,\ qa/d,\ qa/e \end{matrix}\ \middle|\ q\right]_i q^i \times \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix}\ \middle|\ q\right]_{n-i} \\
&\times\ (-1)^i q^{-\binom{i+1}{2}}(qa;q)_{2i}\left[\begin{matrix} b, & c, & q^{-n} \\ qa/b, & qa/c, & q^{n+1}a \end{matrix}\ \middle|\ q\right]_i\left(\frac{q^{1+n}\,a}{b\,c}\right)^i.
\end{aligned}$$

Noting that for the shifted factorials, there hold relations:

$$\begin{aligned}
(qa/bc;q)_{n-i} &= (-1)^i q^{\binom{i}{2}-ni}\left(\frac{bc}{a}\right)^i\frac{(qa/bc;q)_n}{(q^{-n}bc/a;q)_i} \\
(qa;q)_{2i}\frac{(q^{1+2i}a;q)_{n-i}}{(q^{1+n}a;q)_i} &= (qa;q)_n.
\end{aligned}$$

Consequently, we have the following expression

$$\begin{aligned}
\mathrm{Eq(E7.1a)} &= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\ \middle|\ q\right]_n \sum_{i=0}^{n}\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ q, & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\ \middle|\ q\right]_i q^i \\
&= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\ \middle|\ q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\ \middle|\ q;q\right]
\end{aligned}$$

which is exactly (E7.1b). □

E7.2. Rogers-Ramanujan identities. In view of $|q| < 1$ and

$$x \to \infty \qquad \Longrightarrow \qquad (x;\ q)_k \sim (-1)^k q^{\binom{k}{2}} x^k$$

the limiting case $b, c, d, e,\ n \to \infty$ of the Watson transformation reads as:

$$\sum_{m=0}^{\infty}\frac{q^{m^2}a^m}{(q;\ q)_m} = \frac{1}{(qa;\ q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,\frac{1-q^{2k}a}{1-a}\,\frac{(a;\ q)_k}{(q;\ q)_k}q^{5\binom{k}{2}+2k}a^{2k}. \tag{E7.2}$$

This transformation can provide us an alternative demonstration of the well-known Rogers-Ramanujan identities (D3.2a) and (D3.2b):

$$\begin{aligned}
\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\ q)_m} &= \frac{1}{(q;\ q^5)_\infty\,(q^4;\ q^5)_\infty} \\
\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\ q)_m} &= \frac{1}{(q^2;\ q^5)_\infty\,(q^3;\ q^5)_\infty}.
\end{aligned}$$

In fact, observe first that

$$\frac{1-q^{2k}a}{1-a}\frac{(a;\,q)_k}{(q;\,q)_k} = \frac{1-q^{2k}a}{1-q^ka}\frac{(qa;\,q)_k}{(q;\,q)_k} \xlongequal{a\to 1} \begin{cases}1, & k=0\\ 1+q^k, & k>0.\end{cases}$$

Then letting $a\to 1$, we can restate the transformation (E7.2) as

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} &= \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^k(1+q^k)q^{5\binom{k}{2}+2k}\Big\}\\ = &\quad \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+3k}\Big\}\\ = &\quad \frac{1}{(q;\,q)_\infty}\Big\{1+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-2k}\Big\}.\end{aligned}$$

Applying the Jacobi-triple product identity, we therefore establish the first Rogers-Ramanujan identity:

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+2k} = \frac{[q^5,q^2,q^3;q^5]}{(q;\,q)_\infty}.$$

Letting $a\to q$ instead, we can write the transformation (E7.2) as

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} &= \frac{1}{(q;\,q)_\infty}\sum_{k=0}^{\infty}(-1)^k(1-q^{1+2k})q^{5\binom{k}{2}+4k}\\ = &\quad \frac{1}{(q;\,q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=0}^{\infty}(-1)^{k+1}q^{5\binom{k}{2}+6k+1}\Big\}\\ = &\quad \frac{1}{(q;\,q)_\infty}\Big\{\sum_{k=0}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}+\sum_{k=1}^{\infty}(-1)^kq^{5\binom{-k}{2}-4k}\Big\}\end{aligned}$$

where the last line is justified by $k\to k-1$ in the second sum. It leads us to the second Rogers-Ramanujan identity

$$\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;\,q)_m} = \frac{1}{(q;\,q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+4k} = \frac{[q^5,q^4,q;q^5]}{(q;\,q)_\infty}$$

thanks again to the Jacobi-triple product identity. □

E7.3. Jackson's q-Dougall-Dixon formula.

$${}_8\phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1}\end{matrix}\,\Big|\, q;q\right] \tag{E7.3a}$$

$$= \left[\begin{matrix}qa, qa/bc, qa/bd, qa/cd\\ qa/b, qa/c, qa/d, qa/bcd\end{matrix}\,\Big|\, q\right]_n, \quad\text{where}\quad q^{n+1}a^2=bcde. \tag{E7.3b}$$

Proof. When $q^{1+n}a^2 = bcde$ or equivalently $qa/de = q^{-n}bc/a$, the ${}_4\phi_3$-series in the Watson transformation reduces to a balanced ${}_3\phi_2$-series. Therefore, we have in this case the simplified form:

$$
{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix} \middle| q; q\right]
$$
$$
= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} \middle| q\right]_n {}_3\phi_2\left[\begin{matrix} q^{-n}, & b, & c \\ & qa/d, & qa/e \end{matrix} \middle| q\,;\, q\right].
$$

Evaluating the balanced series by the q-Pfaff-Saalschütz theorem, we have

$$
{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix} \middle| q;\, q\right]
$$
$$
= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} \middle| q\right]_n \times \left[\begin{matrix} qa/bd, & qa/cd \\ qa/d, & qa/bcd \end{matrix} \middle| q\right]_n
$$

which is essentially the same as Jackson's q-Dougall-Dixon formula. □

E7.4. The non-terminating ${}_6\phi_5$-summation formula.

$$
{}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix} \middle| q;\, \frac{qa}{bcd}\right] \tag{E7.4a}
$$
$$
= \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd \end{matrix} \middle| q\right]_\infty, \qquad \left|\frac{qa}{bcd}\right| < 1. \tag{E7.4b}
$$

Proof. Substituting $e = q^{1+n}a^2/bcd$ in the Jackson's q-Dougall-Dixon formula explicitly, we have

$$
{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & q^{1+n}a^2/bcd, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & qa/b, & qa/c, & qa/d, & q^{-n}bcd/a, & q^{n+1}a \end{matrix} \middle| q;\, q\right]
$$
$$
= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix} \middle| q\right]_n.
$$

For $n \to \infty$, recalling the limit relations

$$
\frac{(q^{1+n}a^2/bcd;\, q)_k}{(q^{1+n}a;\, q)_k} \sim 1 \quad \text{and} \quad \frac{(q^{-n};\, q)_k}{(q^{-n}bcd/a;\, q)_k} \sim \left(\frac{a}{bcd}\right)^k
$$

and then applying the Tannery limiting theorem, we get the non-terminating q-Dougall-Dixon formula (E7.4a-E7.4b). □

We remark that when $d = q^{-n}$, the formula (E7.4a-E7.4b) reduces to the terminating q-Dougall-Dixon summation identity (E5.1).

CHAPTER F

Bilateral Basic Hypergeometric Series

This chapter deals with the bilateral basic hypergeometric series. The Ramanujan ${}_1\psi_1$-summation formula and Bailey's very-well-poised ${}_6\psi_6$-series identity will be established. We shall also investigate the non-terminating bilateral q-analogue of Dixon's theorem on cubic-sum of binomial coefficients, partial fraction decomposition method on basic hypergeometric series with integral differences between numerator parameters and denominator parameters.

F1. Definition and notation

F1.1. Definition and convergence. Let $\{a_i\}_{i=1}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $a_i \neq q^m$ and $b_j \neq q^{-n}$ with $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for all $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Then the bilateral q-hypergeometric series with variable z is defined by

$$ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q;\ z \right] = \sum_{n=-\infty}^{+\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q \right]_n z^n. $$

For $m \in \mathbb{Z}$, we find by shifting the summation index $n \to m+n$, that the bilateral ${}_r\psi_s$-series satisfies relation

$$ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q;\ z \right] = \left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix} \,\middle|\, q \right]_m z^m \quad \text{(F1.1a)} $$

$$ \times\ {}_r\psi_s \left[\begin{matrix} q^m a_1, q^m a_2, \cdots, q^m a_r \\ q^m b_1, q^m b_2, \cdots, q^m b_s \end{matrix} \,\middle|\, q; z \right]. \quad \text{(F1.1b)} $$

When ${}_r\psi_s$ has no zero parameters, we can reverse the summation order and get another equivalent expression for the bilateral ${}_r\psi_s$-series:

$$ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q;\ z \right] = \sum_{m=-\infty}^{+\infty} \left[\begin{matrix} q/b_1, q/b_2, \cdots, q/b_s \\ q/a_1, q/a_2, \cdots, q/a_r \end{matrix} \,\middle|\, q \right]_m \left\{ \frac{\mathcal{B}}{\mathcal{A}z} \right\}^m \quad \text{(F1.2)} $$

where $\mathcal{A} := a_1a_2\cdots a_r$ and $\mathcal{B} := b_1b_2\cdots b_s$ have been defined for brevity, and the shifted factorial with negative integer order has been inverted as follows:

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(q^{-n}x;q)_\infty} = \frac{1}{(q^{-n}x;q)_n} = \frac{q^{\binom{n+1}{2}}(-1/x)^n}{(q/x;q)_n}, \quad (n\in\mathbb{N}_0).$$

Splitting the bilateral series ${}_r\phi_s$ into two infinite series:

$$\begin{aligned}{}_r\psi_s\left[\begin{matrix}a_1,\ a_2,\ \cdots,\ a_r\\ b_1,\ b_2,\ \cdots,\ b_s\end{matrix}\ \middle|\ q;\ z\right] &= \sum_{n=0}^{+\infty}\Big\{(-1)^n q^{\binom{n}{2}}\Big\}^{s-r}\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}\ \middle|\ q\right]_n z^n\\ &+ \sum_{m=1}^{+\infty}\left[\begin{matrix}q/b_1, q/b_2, \cdots, q/b_s\\ q/a_1, q/a_2, \cdots, q/a_r\end{matrix}\ \middle|\ q\right]_m \left\{\frac{\mathcal{B}}{\mathcal{A}z}\right\}^m\end{aligned}$$

we can check without difficulty that for $|q|<1$ and $\mathcal{R} = \big|\mathcal{B}/\mathcal{A}\big|$, the convergence condition of the bilateral series is determined as follows:

- if $r<s$, the series converges for $|z|>\mathcal{R}$;
- if $r>s$, the series diverges for all $z\in\mathbb{C}$ except for $z=0$;
- if $r=s$, which is the most important case, the series converges for $\mathcal{R}<|z|<1$.

F1.2. Ordinary bilateral hypergeometric series. Similarly, we can define the (ordinary) bilateral hypergeometric series.

Let $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ be complex numbers subject to the condition that $a_i\neq m$ and $b_j\neq -n$ with $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$ for all $i=1,2,\cdots,r$ and $j=1,2,\cdots,s$. Then the (ordinary) bilateral hypergeometric series with variable z is defined by

$${}_rH_s\left[\begin{matrix}a_1,\ a_2,\ \cdots,\ a_r\\ b_1,\ b_2,\ \cdots,\ b_s\end{matrix}\ \middle|\ z\right] = \sum_{n=-\infty}^{+\infty}\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}\right]_n z^n.$$

Only when $r=s$ and $|z|=1$, the bilateral ${}_rH_r$-series is of some interest. Writing it in the sum of two unilateral series

$$\begin{aligned}{}_rH_r\left[\begin{matrix}a_1,\ a_2,\ \cdots,\ a_r\\ b_1,\ b_2,\ \cdots,\ b_r\end{matrix}\ \middle|\ z\right] &= \sum_{n=0}^{+\infty}\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_r\end{matrix}\right]_n z^n\\ &+ \sum_{m=1}^{+\infty}\left[\begin{matrix}1-b_1, 1-b_2, \cdots, 1-b_r\\ 1-a_1, 1-a_2, \cdots, 1-a_r\end{matrix}\right]_m z^{-m}\end{aligned}$$

we can determine the convergence condition of ${}_rH_r$-series as follows:

- if $z=+1$, the series converges for $|\Re(B-A)|>1$;

- if $z = -1$, the series converges for $|\Re(B - A)| > 0$.

F1.3. Examples. Here we shall review the Jacobi triple and the quintuple product identities derived in C2.4 and C2.6 respectively.

- **The Jacobi triple product identity** can be stated in terms of bilateral series as follows:

$$ {}_0\psi_1\left[\begin{matrix}-\\0\end{matrix}\,\Big|\, q; x\right] = \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\binom{n}{2}}\, x^n = [q, x, q/x; q]_\infty . \tag{F1.3}$$

- **An alternative form** can be obtained by separating the sum into two according to the parity of summation index n:

$$\begin{aligned} \left[q^2,\, qy,\, q/y; q^2\right]_\infty &= \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{n^2}\, y^n && \text{(F1.4a)}\\ &= \sum_{n=-\infty}^{+\infty} q^{4n^2}\Big\{1 - yq^{1+4n}\Big\}\, y^{2n} && \text{(F1.4b)}\\ &= \sum_{n=-\infty}^{+\infty} q^{4n^2}\Big\{1 - (q/y)^{1+4n}\Big\}\, y^{2n}. && \text{(F1.4c)} \end{aligned}$$

- **Quintuple product identity** can have different forms such as

$$\begin{aligned} & [q,\, z,\, q/z; q]_\infty \left[qz^2,\, q/z^2; q^2\right]_\infty && \text{(F1.5a)}\\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \Big\{1 - zq^n\Big\} \left(qz^3\right)^n && \text{(F1.5b)}\\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \Big\{1 - z^{1+6n}\Big\} \left(q^2/z^3\right)^n && \text{(F1.5c)}\\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \Big\{1 - (q/z^2)^{1+3n}\Big\} \left(qz^3\right)^n && \text{(F1.5d)}\\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \Big\{1 - (qz^2)^{2+3n}\Big\} \left(q^2/z^3\right)^n && \text{(F1.5e)} \end{aligned}$$

as well as different limiting expressions:

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \big(1 + 6k\big)\, q^{3\binom{k}{2}+2k} &= [q,\, q,\, q; q]_\infty \left[q,\, q; q^2\right]_\infty && \text{(F1.6a)}\\ \sum_{k=-\infty}^{+\infty} \big(1 + 3k\big) q^{3\binom{k}{2}+\frac{5}{2}k} &= \left[q,\, q^{1/2}, q^{1/2}; q\right]_\infty \left[q^2, q^2; q^2\right]_\infty . && \text{(F1.6b)} \end{aligned}$$

Proof. We need to show only (F1.5d), (F1.5e) and (F1.6b) because the others have been demonstrated in C2.6.

It is obvious that (F1.6b) is the limiting case $z \to q^{1/2}$ of (F1.5a) and (F1.5d). Therefore only (F1.5d) and (F1.5e) remain to be confirmed.

Splitting (F1.5b) into two sums and reversing the later by $n \to -1-n$, we can manipulate the sum as follows:

$$\begin{aligned}\sum_n q^{3\binom{n}{2}}\Big\{1 - zq^n\Big\}\big(qz^3\big)^n &= \sum_n \Big\{q^{3\binom{n}{2}+n} z^{3n} - q^{3\binom{n}{2}+2n} z^{1+3n}\Big\}\\ &= \sum_n \Big\{q^{3\binom{n}{2}+n} z^{3n} - q^{3\binom{n}{2}+1+4n} z^{-2-3n}\Big\}\\ &= \sum_n q^{3\binom{n}{2}}\Big\{1-(q/z^2)^{1+3n}\Big\}\big(qz^3\big)^n\end{aligned}$$

which gives the bilateral sum stated in (F1.5d).

Similarly, splitting (F1.5c) into two sums and shifting the summation index $n \to 1+n$ for the latter, we can reformulate the sum as follows:

$$\begin{aligned}\sum_n q^{3\binom{n}{2}}\Big\{1 - z^{1+6n}\Big\}\big(q^2/z^3\big)^n &= \sum_n \Big\{q^{3\binom{n}{2}+2n} z^{-3n} - q^{3\binom{n}{2}+2n} z^{1+3n}\Big\}\\ &= \sum_n \Big\{q^{3\binom{n}{2}+2n} z^{-3n} - q^{3\binom{n}{2}+2+5n} z^{4+3n}\Big\}\\ &= \sum_n q^{3\binom{n}{2}}\Big\{1-(qz^2)^{2+3n}\Big\}\big(q^2/z^3\big)^n\end{aligned}$$

which is exactly the sum displayed in (F1.5e). $\square$

F2. Ramanujan's bilateral ${}_1\psi_1$-series identity

$$ {}_1\psi_1\left[\begin{matrix}a\\c\end{matrix}\,\Big|\,q;\,z\right] = \left[\begin{matrix}q, & c/a, & az, & q/az\\ c, & q/a, & z, & c/az\end{matrix}\,\Big|\,q\right]_\infty, \quad \big(|c/a|<|z|<1\big). \tag{F2.1}$$

Proof. For a large natural number M, choose three complex parameters

$$\begin{aligned} a &\to aq^{-M}\\ b &\to c/az\\ c &\to cq^{-M}.\end{aligned}$$

Then the q-Gauss theorem E2.2 can be restated as

$$ {}_2\phi_1\left[\begin{matrix} aq^{-M}, & c/az \\ & cq^{-M} \end{matrix} \,\Big|\, q;\ z\right] = \frac{(c/a;\ q)_\infty (q^{-M}az;\ q)_\infty}{(z;\ q)_\infty (q^{-M}c;\ q)_\infty}. $$

We can further reformulate it by shifting the summation index $n \to k+M$ as

$$ \sum_{k=-M}^{\infty} \frac{(a;q)_k (q^M c/az;q)_k}{(c;q)_k (q^{1+M};q)_k} z^k = z^{-M} \frac{(q;q)_M (q^{-M}c;q)_M}{(c/az;q)_M (q^{-M}a;q)_M} \frac{(c/a;q)_\infty (q^{-M}az;q)_\infty}{(z;q)_\infty (q^{-M}c;q)_\infty} $$

$$ = \frac{(q;\ q)_M (q/az;\ q)_M}{(q/a;\ q)_M (c/az;\ q)_M} \frac{(c/a;\ q)_\infty (az;\ q)_\infty}{(c;\ q)_\infty (z;\ q)_\infty}. $$

Letting $M \to \infty$, we get the Ramanujan ${}_1\psi_1$-bilateral series identity

$$ {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix} \,\Big|\, q; z\right] = \sum_{k=-\infty}^{+\infty} \frac{(a;q)_k}{(c;q)_k} z^k = \left[\begin{matrix} az,\ q/az,\ q,\ c/a \\ z,\ c/az,\ c,\ q/a \end{matrix} \,\Big|\, q\right]_\infty, \ |c/a| < |z| < 1 $$

where the convergence condition is figured out by analytical continuation.

In fact, the series along the positive direction

$$ \sum_{k=0}^{+\infty} \frac{(a;\ q)_k}{(c;\ q)_k} z^k $$

converges when $|z| < 1$. While the series along the negative direction

$$ \sum_{k=1}^{+\infty} \frac{(a;\ q)_{-k}}{(c;\ q)_{-k}} z^{-k} = \sum_{k=1}^{+\infty} \frac{(q/c;\ q)_k}{(q/a;\ q)_k} \Big(\frac{c}{az}\Big)^k $$

converges when $|z| > |c/a|$. □

F3. Bailey's bilateral ${}_6\psi_6$-series identity

For complex parameters a, b, c, d, e satisfying the condition $\left|qa^2/bcde\right| < 1$, there holds Bailey's very well-poised non-terminating bilateral series identity (cf. [**56**, §7.1]):

$$ {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\Big|\, q; \frac{qa^2}{bcde}\right] \tag{F3.1a} $$

$$ = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\Big|\, q\right]_\infty. \tag{F3.1b} $$

Here we reproduce a recent proof provided by Schlosser (2003).

F3.1. Lemma. For three complex parameters a, b and d with $|qa/bd| < 1$, there holds the following summation formula:

$$\Big(\frac{a}{b}\Big)^n \begin{bmatrix} q, & b \\ a, & qa/b \end{bmatrix} q \Big]_n = \begin{bmatrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{bmatrix} q \Big]_\infty$$
$$\times\ {}_6\phi_5 \left[\begin{matrix} d/a, q\sqrt{d/a}, -q\sqrt{d/a}, & b/a, & q^{-n}d/a, & q^n d \\ \sqrt{d/a}, & -\sqrt{d/a}, qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix} \,\Big|\, q; \frac{qa}{bd}\right].$$

Proof. According to the q-Dixon-Dougall formula, we have

$${}_6\phi_5 \left[\begin{matrix} d/a, q\sqrt{d/a}, -q\sqrt{d/a}, & b/a, & q^{-n}d/a, & q^n d \\ \sqrt{d/a}, & -\sqrt{d/a}, qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix} \,\Big|\, q; \frac{qa}{bd}\right]$$
$$= \left[\begin{matrix} q^{1+n}a/b, q^{1-n}/b, q/d, qd/a \\ q^{1+n}, q^{1-n}/a, qd/b, qa/bd \end{matrix} \,\Big|\, q\right]_\infty.$$

Separating the factorials dependent in n:

$$\frac{(q^{1+n}a/b;\, q)_\infty}{(q^{1+n};\, q)_\infty} = \frac{(q;\, q)_n}{(qa/b;\, q)_n} \times \frac{(qa/b;\, q)_\infty}{(q;\, q)_\infty}$$
$$\frac{(q^{1-n}/b;\, q)_\infty}{(q^{1-n}/a;\, q)_\infty} = \Big(\frac{a}{b}\Big)^n \frac{(b;\, q)_n}{(a;\, q)_n} \times \frac{(q/b;\, q)_\infty}{(q/a;\, q)_\infty}$$

we can restate the last series as

$${}_6\phi_5 \left[\begin{matrix} d/a, q\sqrt{d/a}, -q\sqrt{d/a}, & b/a, & q^{-n}d/a, & q^n d \\ \sqrt{d/a}, & -\sqrt{d/a}, qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix} \,\Big|\, q; \frac{qa}{bd}\right]$$
$$= \left[\begin{matrix} q/b, q/d, qa/b, qd/a \\ q, qd/b, q/a, qa/bd \end{matrix} \,\Big|\, q\right]_\infty \times \left[\begin{matrix} b, & q \\ qa/b, & a \end{matrix} \,\Big|\, q\right]_n \Big(\frac{a}{b}\Big)^n$$

which is equivalent to the formula displayed in Lemma F3.1. □

F3.2. Now we are ready to prove Bailey's very well-poised non-terminating bilateral ${}_6\psi_6$-series identity.

Recalling the definition of bilateral series

$$\mathrm{Eq(F3.1a)} = \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\Big|\, q\right]_n \Big(\frac{qa^2}{bcde}\Big)^n$$

and then replacing the factorial faction related to b and d by Lemma F3.1

$$\Big(\frac{a}{b}\Big)^n \left[\begin{matrix} b, & d \\ qa/b, & qa/d \end{matrix} \,\Big|\, q\right]_n = \left[\begin{matrix} a, & d \\ q, & qa/d \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} q, q/a, qd/b, qa/bd \\ q/b, qa/b, q/d, qd/a \end{matrix} \,\Big|\, q\right]_\infty$$
$$\times\ {}_6\phi_5 \left[\begin{matrix} d/a, q\sqrt{d/a}, -q\sqrt{d/a}, & b/a, & q^{-n}d/a, & q^n d \\ \sqrt{d/a}, & -\sqrt{d/a}, qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix} \,\Big|\, q; \frac{qa}{bd}\right]$$

we can express the ${}_6\psi_6$-series as the following double sum:

$$\begin{aligned}
\mathrm{Eq(F3.1a)} \;=\; & \left[\begin{matrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{matrix}\,\middle|\, q\right]_\infty \\
\times & \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} a, & c, & d, & e \\ q, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q\right]_n \left(\frac{qa}{cde}\right)^n \\
\times & \sum_{k=0}^{\infty} \frac{1-q^{2k}d/a}{1-d/a} \left[\begin{matrix} d/a, & b/a, & q^{-n}d/a, & q^n d \\ q, & qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix}\,\middle|\, q\right]_k \left(\frac{qa}{bd}\right)^k .
\end{aligned}$$

Interchanging the summation order and then combining the following factorial fractions

$$\begin{aligned}
\frac{(d;q)_n}{(q;q)_n} \times \frac{(q^n d;q)_k}{(q^{1+n};q)_k} &= \frac{(d;q)_{n+k}}{(q;q)_{n+k}} \\
\frac{(a;q)_n}{(qa/d;q)_n} \times \frac{(q^{-n}d/a;q)_k}{(q^{1-n}/a;q)_k} &= \left(\frac{d}{q}\right)^k \frac{(a;q)_{n-k}}{(qa/d;q)_{n-k}}
\end{aligned}$$

we can further reformulate the equation (F3.1a) as follows:

$$\begin{aligned}
\mathrm{Eq(F3.1a)} \;=\; & \left[\begin{matrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{matrix}\,\middle|\, q\right]_\infty \\
\times & \sum_{k=0}^{\infty} \frac{1-q^{2k}d/a}{1-d/a} \left[\begin{matrix} d/a, b/a \\ q, qd/b \end{matrix}\,\middle|\, q\right]_k \left(\frac{a}{b}\right)^k \\
\times & \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} c, & e \\ qa/c, & qa/e \end{matrix}\,\middle|\, q\right]_n \\
\times & \frac{(d;q)_{n+k}}{(q;q)_{n+k}} \frac{(a;q)_{n-k}}{(qa/d;q)_{n-k}} \left(\frac{qa}{cde}\right)^n .
\end{aligned}$$

F3.3. The last sum with respect to n begins in effect with $n = -k$ because the shifted factorial $1/(q;q)_{n+k}$ is equal to zero when $n < -k$. Indicate with Ω the last sum. Therefore it can be reformulated through $j := n+k$ as follows:

$$\Omega \;=\; \sum_{j=0}^{+\infty} \frac{1-q^{2j-2k}a}{1-a} \left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{j-k} \frac{(d;q)_j}{(q;q)_j} \frac{(a;q)_{j-2k}}{(qa/d;q)_{j-2k}} \left(\frac{qa}{cde}\right)^{j-k} .$$

By means of two relations

$$\begin{aligned}
\frac{(a;q)_{j-2k}}{(qa/d;q)_{j-2k}} &= \frac{(a;q)_{-2k}}{(qa/d;q)_{-2k}} \frac{(q^{-2k}a;q)_j}{(q^{1-2k}a/d;q)_j} \\
\left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{j-k} &= \left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{-k} \left[\begin{matrix} q^{-k}c,\ q^{-k}e \\ q^{1-k}a/c,\ q^{1-k}a/e \end{matrix}\,\middle|\, q\right]_j
\end{aligned}$$

we can express Ω in terms of the q-hypergeometric series:

$$\begin{aligned}
\Omega &= \frac{1-q^{-2k}a}{1-a}\frac{(a;q)_{-2k}}{(qa/d;q)_{-2k}}\left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\Big|\, q\right]_{-k}\left(\frac{cde}{qa}\right)^k \\
&\times \sum_{j=0}^{+\infty}\frac{1-q^{2j-2k}a}{1-q^{-2k}a}\left[\begin{matrix} q^{-k}c, & q^{-k}e, & d, & q^{-2k}a \\ q^{1-k}a/c, & q^{1-k}a/e, & q, & q^{1-2k}a/d \end{matrix}\,\Big|\, q\right]_j\left(\frac{qa}{cde}\right)^j \\
&= \frac{(qa;q)_{-2k}}{(qa/d;q)_{-2k}}\left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\Big|\, q\right]_{-k}\left(\frac{cde}{qa}\right)^k \\
&\times\ {}_6\phi_5\left[\begin{matrix} q^{-2k}a, & q^{1-k}\sqrt{a}, & -q^{1-k}\sqrt{a}, & q^{-k}c, & q^{-k}e, & d \\ & q^{-k}\sqrt{a}, & -q^{1-k}\sqrt{a}, & q^{1-k}a/c, & q^{1-k}a/e, & q^{1-2k}a/d \end{matrix}\,\Big|\, q;\frac{qa}{cde}\right].
\end{aligned}$$

When $|qa/cde|<1$, the last series can be evaluated by q-Dixon-Dougall formula (E7.4a-E7.4b) as follows:

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} q^{-2k}a, & q^{1-k}\sqrt{a}, & -q^{1-k}\sqrt{a}, & q^{-k}c, & q^{-k}e, & d \\ & q^{-k}\sqrt{a}, & -q^{1-k}\sqrt{a}, & q^{1-k}a/c, & q^{1-k}a/e, & q^{1-2k}a/d \end{matrix}\,\Big|\, q;\frac{qa}{cde}\right] \\
&= \left[\begin{matrix} q^{1-2k}a, & qa/ce, & q^{1-k}a/cd, & q^{1-k}a/de \\ q^{1-k}a/c, & q^{1-k}a/e, & q^{1-2k}a/d, & qa/cde \end{matrix}\,\Big|\, q\right]_\infty \\
&= \frac{(qa/d;q)_{-2k}}{(qa;q)_{-2k}}\left[\begin{matrix} qa/c, & qa/e \\ qa/cd, & qa/de \end{matrix}\,\Big|\, q\right]_{-k}\left[\begin{matrix} qa, & qa/cd, & qa/ce, & qa/de \\ qa/c, & qa/d, & qa/e, & qa/cde \end{matrix}\,\Big|\, q\right]_\infty
\end{aligned}$$

which leads us consequently to the closed form for Ω:

$$\begin{aligned}
\Omega &= \left(\frac{cde}{qa}\right)^k\left[\begin{matrix} c, & e \\ qa/cd, & qa/de \end{matrix}\,\Big|\, q\right]_{-k}\left[\begin{matrix} qa, & qa/cd, & qa/ce, & qa/de \\ qa/c, & qa/d, & qa/e, & qa/cde \end{matrix}\,\Big|\, q\right]_\infty \\
&= \left(\frac{cde}{qa}\right)^k\left[\begin{matrix} q^{1-k}a/cd, & q^{1-k}a/de \\ q^{-k}c, & q^{-k}e \end{matrix}\,\Big|\, q\right]_k\left[\begin{matrix} qa, & qa/cd, & qa/ce, & qa/de \\ qa/c, & qa/d, & qa/e, & qa/cde \end{matrix}\,\Big|\, q\right]_\infty \\
&= \left(\frac{qa}{cde}\right)^k\left[\begin{matrix} cd/a, & de/a \\ q/c, & q/e \end{matrix}\,\Big|\, q\right]_k\left[\begin{matrix} qa, & qa/cd, & qa/ce, & qa/de \\ qa/c, & qa/d, & qa/e, & qa/cde \end{matrix}\,\Big|\, q\right]_\infty.
\end{aligned}$$

F3.4. Summing up, we can state the bilateral ${}_6\psi_6$-series displayed in (F3.1a) in terms of another q-series:

$$\begin{aligned}
\text{Eq(F3.1a)} &= \left[\begin{matrix} q, qa, q/a, qd/b, qa/bd, qa/cd, qa/ce, qa/de \\ q/b, q/d, qd/a, qa/b, qa/c, qa/d, qa/e, qa/cde \end{matrix}\,\Big|\, q\right]_\infty \\
&\times \sum_{k=0}^{\infty}\frac{1-q^{2k}d/a}{1-d/a}\left[\begin{matrix} d/a, & b/a, & cd/a, & de/a \\ q, & qd/b, & q/c, & q/e \end{matrix}\,\Big|\, q\right]_k\left(\frac{qa^2}{bcde}\right)^k.
\end{aligned}$$

The last series can be again evaluated by the very well poised non-terminating ${}_6\phi_5$-summation formula as follows:

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} d/a, \ q\sqrt{d/a}, \ -q\sqrt{d/a}, \ b/a, \ cd/a, \ de/a \\ \sqrt{d/a}, \ -\sqrt{d/a}, \ qd/b, \ q/c, \ q/e \end{matrix} \,\middle|\, q; \ \frac{qa^2}{bcde}\right] \\
&= \left[\begin{matrix} qd/a, \ qa/bc, \ qa/be, \ qa/cde \\ qd/b, \ q/c, \ q/e, \ qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}$$

We therefore have established the following

$$\begin{aligned}
\mathrm{Eq(F3.1a)} &= \left[\begin{matrix} q, qa, q/a, qd/b, qa/bd, qa/cd, qa/ce, qa/de \\ qd/a, q/b, q/d, qa/b, qa/c, qa/d, qa/e, qa/cde \end{matrix} \,\middle|\, q\right]_\infty \\
&\times \left[\begin{matrix} qd/a, \ qa/bc, \ qa/be, \ qa/cde \\ qd/b, \ q/c, \ q/e, \ qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty \\
&= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty
\end{aligned}$$

which corresponds exactly to (F3.1b).

This completes the proof of Bailey's very-well-poised ${}_6\psi_6$-series identity. □

F3.5. The quintuple product identity. The identity displayed in (F1.5a) and (F1.5b) is a limiting case of Bailey's very-well-poised ${}_6\psi_6$-series identity.

In fact, letting $b = -\sqrt{a}$ and $c, d, e \to \infty$, we can state (F3.1a-F3.1b) as

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\left\{1 - q^n\sqrt{a}\right\}\left(qa^{3/2}\right)^n &= (1-\sqrt{a})\frac{[q, \ qa, \ q/a; q]_\infty}{[-q\sqrt{a}, \ -q/\sqrt{a}; q]_\infty} \\
&= (q;q)_\infty \frac{\left[qa, \ a, \ q/a, \ q^2/a; q^2\right]_\infty}{[-\sqrt{a}, \ -q/\sqrt{a}; q]_\infty} \\
&= \left[q, \ \sqrt{a}, \ q/\sqrt{a}; q\right]_\infty \left[qa, \ q/a; q^2\right]_\infty .
\end{aligned}$$

Replacing a by z^2, this becomes the quintuple product identity displayed in (F1.5a) and (F1.5b). □

F4. Bilateral q-analogue of Dixon's theorem

For the cubic-sums of binomial coefficients, there is Dixon's well-known theorem, which states that

$$\sum_{k=-n}^{n+\delta} (-1)^k \binom{2n+\delta}{n+k}^3 = \begin{cases} \binom{3n}{n,n,n}, & \delta = 0 \\ 0, & \delta = 1. \end{cases} \tag{F4.1}$$

Its terminating q-analogue was first found by Jackson [**44**] and subsequently generalized by Bailey [**8**]. Following Bailey's derivation, we will establish two general well-poised bilateral series identities:

$$ {}_4\psi_4\left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix}\,\Big|\, q;\ \frac{q}{bcd}\right] = \left[\begin{matrix} q,\ q/bc,\ q/bd,\ q/cd \\ q/b,\ q/c,\ q/d,\ q/bcd \end{matrix}\,\Big|\, q\right]_\infty \tag{F4.2}$$

$$ {}_5\psi_5\left[\begin{matrix} qu, qv, & b, & c, & d \\ u, & v, & 1/b, 1/c, 1/d \end{matrix}\,\Big|\, q;\ \frac{q^{-1}}{bcd}\right] = \left[\begin{matrix} q,\ 1/bc,\ 1/bd,\ 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix}\,\Big|\, q\right]_\infty \tag{F4.3}$$

$$ \times \quad \frac{1-1/quv}{(1-1/u)\,(1-1/v)}. \tag{F4.4}$$

Further bilateral identities of this type and applications can be found in Chu [**26**], where a systematic treatment of basic almost-poised hypergeometric series has been presented.

F4.1. Proof of (F4.2)**.** Recall the non-terminating very-well-poised ${}_6\phi_5$ summation identity (E7.4a-E7.4b):

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix}\,\Big|\, q;\ \frac{qa}{bcd}\right] $$
$$ = \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd \end{matrix}\,\Big|\, q\right]_\infty, \qquad \left|\frac{qa}{bcd}\right| < 1. $$

Letting $a \to 1$, we can restate the result as

$$ \begin{aligned} \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\,\Big|\, q\right]_\infty &= 1 + \sum_{k=1}^{+\infty}\left\{1+q^k\right\}\left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix}\,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k \\ &= 1 + \sum_{k=1}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix}\,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k \\ &+ \sum_{k=1}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix}\,\Big|\, q\right]_{-k} \left(\frac{q}{bcd}\right)^{-k} \\ &= \sum_{k=-\infty}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix}\,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k. \end{aligned} $$

In terms of bilateral series, this becomes the following well-poised summation identity

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}\,\Big|\, q;\ \frac{q}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\,\Big|\, q\right]_\infty \tag{F4.5}$$

whose reversal reads as

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}\,\Big|\, q;\ \frac{q^2}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\,\Big|\, q\right]_\infty. \tag{F4.6}$$

In view of the fact that

$$\frac{(qw;q)_k}{(w;q)_k} = \frac{1-wq^k}{1-w} = \frac{1}{1-w} - \frac{w}{1-w}q^k$$

the linear combination of (F4.5) and (F4.6) leads us to bilateral identity (F4.2) with an extra w-parameter:

$${}_4\psi_4\left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix}\,\middle|\, q;\ \frac{q}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\,\middle|\, q\right]_\infty .$$

F4.2. Proof of (F4.3-F4.4). Instead, if taking $a = q$ in non-terminating very-well-poised ${}_6\phi_5$ summation identity (E7.4a-E7.4b) and then multiplying both sides by $1-q$, we get

$$\begin{aligned}
\left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\,\middle|\, q\right]_\infty &= \sum_{k=0}^{+\infty}\{1-q^{1+2k}\}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k \left(\frac{q^2}{bcd}\right)^k \\
&= \sum_{k=0}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k \left(\frac{q^2}{bcd}\right)^k \\
&- \sum_{k=1}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_{-k} \left(\frac{q^2}{bcd}\right)^{-k} \\
&= \sum_{k=-\infty}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k \left(\frac{q^2}{bcd}\right)^k
\end{aligned}$$

where we have performed the summation index substitution $k \to k-1$ for the second sum.

In terms of bilateral series, this reads as the following well-poised summation identity

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix}\,\middle|\, q;\ \frac{q^2}{bcd}\right] = \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\,\middle|\, q\right]_\infty . \quad \text{(F4.7)}$$

Shifting the summation index by $k \to k-1$ and then performing parameter replacement $b \to qb$, $c \to qc$ and $d \to qd$, we can derive the following equivalent result:

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q;\ \frac{q^{-1}}{bcd}\right] = \frac{-1}{q}\left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ 1/b, 1/c, 1/d, q^{-1}/bcd \end{matrix}\,\middle|\, q\right]_\infty . \quad \text{(F4.8)}$$

Its reversal can be stated, after some little modification, as

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q;\ \frac{q}{bcd}\right] = \left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ 1/b, 1/c, 1/d, q^{-1}/bcd \end{matrix}\,\middle|\, q\right]_\infty . \quad \text{(F4.9)}$$

Note further that

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q; \frac{1}{bcd}\right] = 0 \tag{F4.10}$$

which is the case $\kappa = 1$ of the following general statement:

$$ {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \cdots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \cdots, & 1/c_{1+2\kappa} \end{matrix}\,\middle|\, q; \frac{1}{c_1 c_2 \cdots c_{1+2\kappa}}\right] = 0. \tag{F4.11}$$

In fact, denote by Θ the bilateral ψ-series on the left hand side. Its reversal with the summation index shifted by $k \to k-1$ can be stated as

$$\begin{aligned}
\Theta &= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} qc_1, & qc_2, \cdots, & qc_{1+2n} \\ q/c_1, & q/c_2, \cdots, & q/c_{1+2\kappa} \end{matrix}\,\middle|\, q; \prod_{\iota=1}^{1+2\kappa}\frac{1}{c_\iota}\right] \\
&= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \cdots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \cdots, & 1/c_{1+2\kappa} \end{matrix}\,\middle|\, q; \prod_{\iota=1}^{1+2\kappa}\frac{1}{c_\iota}\right] \\
&\times \prod_{\iota=1}^{1+2\kappa}\frac{1-1/c_\iota}{1-c_\iota}\left\{\prod_{\iota=1}^{1+2\kappa}\frac{1}{c_\iota}\right\}^{-1}.
\end{aligned}$$

Simplifying the last factor-product, we find that

$$\Theta = (-1)^{1+2\kappa}\Theta = 0$$

which is exactly (F4.11).

By means of three terms relation

$$\begin{aligned}
\frac{(qu;q)_k}{(u;q)_k}\frac{(qv;q)_k}{(v;q)_k} &= \frac{1-uq^k}{1-u}\frac{1-vq^k}{1-v} \\
= \frac{1}{(1-u)(1-v)} &- \frac{u+v}{(1-u)(1-v)}q^k + \frac{uv}{(1-u)(1-v)}q^{2k}
\end{aligned}$$

we can establish from the combination of (F4.8), (F4.9) and (F4.10) the following general identity with two extra free-parameters:

$$\begin{aligned}
{}_5\psi_5\left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q; \frac{q^{-1}}{bcd}\right] &= \frac{1-1/quv}{(1-1/u)\,(1-1/v)} \\
&\times \left[\begin{matrix} q,\, 1/bc,\, 1/bd,\, 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix}\,\middle|\, q\right]_\infty
\end{aligned}$$

which is the bilateral identity stated in (F4.3-F4.4).

F4.3. q-Analogue of Dixon's theorem. Putting $b = c = d = q^{-n}$ in (F4.5), we can state the result in terms of q-binomial sum

$$\sum_{k=-n}^{n}(-1)^k\left[\begin{matrix} 2n \\ n+k \end{matrix}\right]^3 q^{k(3k-1)/2} = \frac{(q;q)_{3n}}{(q;q)_n^3} \tag{F4.12}$$

which is the q-analogue of (F4.1) corresponding to $\delta = 0$.

Similarly, taking $b = c = d = q^{-n}$ in (F4.7), we can express the result as another q-binomial sum

$$\sum_{k=-n}^{1+n} (-1)^k \begin{bmatrix} 1+2n \\ n+k \end{bmatrix}^3 q^{k(3k-1)/2} = \frac{(q;q)_{1+3n}}{(q;q)_n^3} \tag{F4.13}$$

which is the q-analogue of (F4.1) corresponding to $\delta = 1$.

F5. Partial fraction decomposition method

For terminating hypergeometric series with integral differences between numerator parameters and denominator parameters, an interesting identity was first discovered by Minton (1970) and slightly extended by Karlsson (1971). Their q-analogue was then established by Gasper (1984). The bilateral non-terminating forms of these formulae have been found by Chu (1994), who has further generalized these results into basic bilateral very well-poised hypergeometric summation identities.

The purpose of this section is to present the formulae of Chu-Karlsson-Minton for basic hypergeometric series with integral parameter differences. For the terminating cases and their dual formulae, refer to Chu (1998).

F5.1. Theorem. For complex numbers a, c, d, and $\{x_k\}_{k=1}^{\ell}$ with λ and $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n = \sum_{k=1}^{\ell} n_k$, there holds the bilateral series identity with integral parameter differences

$${}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & d, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & qd, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\Big|\, q; q^{1-\lambda}/a\right] \tag{F5.1a}$$

$$= \left[\begin{matrix} q, & q, & c/d, & qd/a \\ q/a, & c, & qd, & q/d \end{matrix} \,\Big|\, q\right]_\infty d^\lambda \prod_{k=1}^{\ell} \frac{(x_k/d;q)_{n_k}}{(x_k;q)_{n_k}} \tag{F5.1b}$$

provided that $(|q/a| < |q^\lambda| < |q^n/c|)$. This identity contains, in particular, the results due to Gasper [**33**] on unilateral q-series as special cases.

Proof. Consider the function of complex variable z defined in terms of q-shifted factorials by

$$f(z) = z^{1+\lambda} \left[\begin{matrix} q, \ q, \ c/z, \ qz/a \\ c, \ q/a, \ z, \ q/z \end{matrix} \,\Big|\, q\right]_\infty \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}}. \tag{F5.2}$$

It is not hard to verify that $f(z)$ has simple poles $z=q^j$ for $j=0,\pm1,\pm2,\cdots$ with residues

$$-\Big\{q^{-1-\lambda}/a\Big\}^j\frac{(a;q)_j}{(c;q)_j}\prod_{k=1}^{\ell}\frac{(q^{n_k}x_k;q)_j}{(x_kc;q)_j}. \tag{F5.3}$$

Denote by δ_m and ∂_m the circles with the center at the origin and radii $|q|^{m+1/2}$ and $|q|^{-m-1/2}$, respectively. It is clear that there is no pole of $f(z)$ passing through δ_m or ∂_m. Then the residue theorem (cf. [**58**, §3.1]) states that

$$\frac{f(z)}{z}+\sum_{\tau}\frac{\operatorname{Res}\,[f(z)]_{z=\tau}}{\tau\,(\tau-z)} = \frac{1}{2\pi i}\int_{\partial_m}\frac{f(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.4a}$$

$$- \frac{1}{2\pi i}\int_{\delta_m}\frac{f(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.4b}$$

where the summation runs over all poles $\{\tau\}$ of $f(z)$ between contours δ_m and ∂_m. For sufficiently large m, we can estimate that

$$\int_{\delta_m}\frac{f(t)}{t(t-z)}\mathrm{dt} = \mathcal{O}\left\{q^{(1+\lambda-n)m}\frac{(q^{-m-1/2}c;q)_m}{(q^{-m+1/2};q)_m}\right\} \tag{F5.5a}$$

$$= \mathcal{O}\big(|q^{\lambda-n}c|^m\big) \tag{F5.5b}$$

$$\int_{\partial_m}\frac{F(t)}{t(t-z)}\mathrm{dt} = \mathcal{O}\left\{q^{-m\lambda}\frac{(q^{-m+1/2}/a;q)_m}{(q^{-m-1/2};q)_m}\right\} \tag{F5.5c}$$

$$= \mathcal{O}\big(|q^{1-\lambda}/a|^m\big). \tag{F5.5d}$$

When $|q/a|<|q^\lambda|<|q^n/c|$, both integrals displayed in (F5.4a-F5.4b) tend to zero as $m\to\infty$. Therefore we can express (F5.4a-F5.4b) as a bilateral summation identity:

$$\begin{aligned}&{}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix}a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell\\ c, & qz, & x_1, & x_2, & \cdots, & x_\ell\end{matrix}\;\Big|\; q;\, q^{1-\lambda}/a\right]\\ &= \left[\begin{matrix}q, & q, & c/z, & qz/a\\ q/a, & c, & qz, & q/z\end{matrix}\;\Big|\; q\right]_\infty z^\lambda\prod_{k=1}^{\ell}\frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}}\end{aligned}$$

whose convergent condition coincides with $|q/a|<|q^\lambda|<|q^n/c|$. Rewriting the last identity with z being replaced by d, we confirm the formula displayed in (F5.1a-F5.1b). □

Two interesting special cases of (F5.1a-F5.1b) are worth to mention:

$${}_3\psi_3\left[\begin{matrix}a, & qx, & z\\ c, & x, & qz\end{matrix}\;\Big|\; q;q/a\right] = \frac{1-x/z}{1-x}\left[\begin{matrix}q,\ q,\ c/z,\ qz/a\\ c,\ q/a,\ qz,\ q/z\end{matrix}\;\Big|\; q\right] \tag{F5.6a}$$

$${}_3\psi_3\left[\begin{matrix}a, & qx, & z\\ c, & x, & qz\end{matrix}\;\Big|\; q;1/a\right] = \frac{1-z/x}{1-1/x}\left[\begin{matrix}q,\ q,\ c/z,\ qz/a\\ c,\ q/a,\ qz,\ q/z\end{matrix}\;\Big|\; q\right] \tag{F5.6b}$$

where their convergence conditions are given respectively by $|c| < |q| < |a|$ and $|c| < 1 < |a|$.

F5.2. Corollary. Replacing each parameter by its q-exponential in equality (F5.1a-F5.1b) and then letting $q \to 1$, we can state the limit as bilateral hypergeometric summation formula $(n + \Re(a-c) < 0)$:

$$ {}_{\ell+2}H_{2+\ell}\left[\begin{matrix} a, & d, & x_1+n_1, & x_2+n_2, & \cdots, & x_\ell+n_\ell \\ c, & 1+d, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, 1\right] \tag{F5.7a}$$

$$ = \quad \Gamma\left[\begin{matrix} 1-a, \ c \\ 1-a+d, \ c-d \end{matrix}\right] \frac{\pi d}{\sin \pi d} \prod_{k=1}^{\ell} \frac{(x_k-d)_{n_k}}{(x_k)_{n_k}}. \tag{F5.7b}$$

This generalizes the identity due to Karlsson [**46**]

$$ {}_{\ell+2}F_{1+\ell}\left[\begin{matrix} a, & d, & x_1+n_1, & x_2+n_2, & \cdots, & x_\ell+n_\ell \\ & 1+d, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, 1\right] \tag{F5.8a}$$

$$ = \quad \frac{\Gamma(1-a)\Gamma(1+d)}{\Gamma(1-a+d)} \prod_{k=1}^{\ell} \frac{(x_k-d)_{n_k}}{(x_k)_{n_k}}, \quad (n+\Re(a) < 1) \tag{F5.8b}$$

which is a non-terminating extension of an earlier result due to Minton [**50**].

F5.3. Transformation of bilateral series into unilateral series. For $|w| < 1$, recalling the q-Gauss summation formula

$$\frac{(z;q)_k}{(wz;q)_k} = \frac{1}{1-zq^k}\left[\begin{matrix} z, w \\ q, wz \end{matrix} \,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} q/w, q^k z \\ q^{1+k} z \end{matrix} \,\middle|\, q; w\right]$$

we can consider the series composition $(|q/a| < |q^\lambda| < |q^{1+n}/cw|)$:

$$\begin{aligned} & {}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & wz, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, q;\ q^{1-\lambda}/a\right] \times \left[\begin{matrix} q, wz \\ z, w \end{matrix} \,\middle|\, q\right]_\infty \\ = \quad & \sum_{k=-\infty}^{+\infty} \frac{\left(q^{1-\lambda}/a\right)^k}{1-zq^k} \left[\begin{matrix} a, & \{q^{n_\kappa}x_\kappa\} \\ c, & \{x_\kappa\} \end{matrix} \,\middle|\, q\right]_k {}_2\phi_1\left[\begin{matrix} q/w, q^k z \\ q^{1+k} z \end{matrix} \,\middle|\, q; w\right] \\ = \quad & \sum_{i=0}^{+\infty} \frac{w^i}{1-zq^i} \frac{(q/w;q)_i}{(q;q)_i} \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} a, & q^i z, & \{q^{n_\kappa}x_\kappa\} \\ c, & q^{1+i}z, & \{x_\kappa\} \end{matrix} \,\middle|\, q\right]_k \left(q^{1-\lambda}/a\right)^k. \end{aligned}$$

Under the condition $|q/a| < |q^\lambda| < |q^n/c|$, the last series can be evaluated by (F5.1a-F5.1b) as

$$
\begin{aligned}
&{}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & q^i z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & q^{1+i}z, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, q;\, q^{1-\lambda}/a\right] \\
&= \left[\begin{matrix} q, & q, & q^{-i}c/z, & q^{1+i}z/a \\ q/a, & c, & q^{1+i}z, & q^{1-i}/z \end{matrix} \,\middle|\, q\right]_\infty q^{i\lambda} z^\lambda \prod_{k=1}^{\ell} \frac{(q^{-i}x_k/z;q)_{n_k}}{(x_k;q)_{n_k}} \\
&= \left[\begin{matrix} q, & q, & c/z, & qz/a \\ q/a, & c, & qz, & q/z \end{matrix} \,\middle|\, q\right]_\infty z^\lambda \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}} \\
&\times \left(q^{\lambda-1-n}c\right)^i \left[\begin{matrix} qz/c, qz \\ qz/a,\ z \end{matrix} \,\middle|\, q\right]_i \prod_{k=1}^{\ell} \frac{(qz/x_k;q)_i}{(q^{1-n_k}z/x_k;q)_i}.
\end{aligned}
$$

Substituting this result into the last expression, we get the following transformation:

$$
{}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & wz, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, q;\, q^{1-\lambda}/a\right] \tag{F5.9a}
$$

$$
= \left[\begin{matrix} q, & w, & c/z, & qz/a \\ q/a, & c, & wz, & q/z \end{matrix} \,\middle|\, q\right]_\infty z^\lambda \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}} \tag{F5.9b}
$$

$$
\times \quad {}_{\ell+2}\phi_{1+\ell}\left[\begin{matrix} q/w, & qz/c, & \{qz/x_\kappa\} \\ & qz/a, & \{q^{1-n_\kappa}z/x_\kappa\} \end{matrix} \,\middle|\, q;\, q^{\lambda-1-n}cw\right] \tag{F5.9c}
$$

provided that $|q/a| < |q^\lambda| < |q^{1+n}/cw|$, which guarantees that both non-terminating series are convergent.

When $q \to 1$, we write down the transformation for ordinary hypergeometric series $(1 + n + \Re(a-c) < 1 \leq \Re(w))$:

$$
\begin{aligned}
&{}_{\ell+2}H_{2+\ell}\left[\begin{matrix} a, & z, & n_1+x_1, & n_2+x_2, & \cdots, & n_\ell+x_\ell \\ c, & w+z, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, 1\right] \\
&= \Gamma\left[\begin{matrix} 1-a, c, w+z, 1-z \\ w, c-z, 1-a+z \end{matrix}\right] \prod_{k=1}^{\ell} \frac{(x_k-z)_{n_k}}{(x_k)_{n_k}} \\
&\times \quad {}_{\ell+2}F_{1+\ell}\left[\begin{matrix} 1-w, & 1-c+z, & \{1+z-x_\kappa\} \\ & 1-a+z, & \{1+z-x_\kappa-n_\kappa\} \end{matrix} \,\middle|\, 1\right].
\end{aligned}
$$

In particular, putting $n = 0$ and then evaluating the ${}_2\phi_1$-series by the Gauss summation theorem, we recover the well-known Dougall formula:

$$
{}_2H_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix} \,\middle|\, 1\right] = \Gamma\left[\begin{matrix} 1-a, 1-b, c, d, c+d-a-b-1 \\ c-a, c-b, d-a, d-b \end{matrix}\right]
$$

provided that $\Re(c+d-a-b) > 1$ for the convergence of the bilateral series.

F5.4. Bilateral basic very-well-poised summation formula. The partial decomposition method can further be applied to derive the following bilateral very-well-poised summation formula (Chu, 1998).

For complex numbers a, b, c, d, and $\{x_k,\ y_k\}_{k=1}^{\ell}$ satisfying $x_k y_k = aq^{1+n_k}$ $(k = 1, 2, \cdots, \ell)$ with $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n = \sum_{k=1}^{\ell} n_k$, there holds

$$
{}_{2\ell+6}\psi_{6+2\ell}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, & b,\ c, & d,\ a/d, & \{x_k, \quad y_k\} \\ \sqrt{a},\ -\sqrt{a}, & qa/b, qa/c, & qa/d, qd, & \{qa/x_k, qa/y_k\} \end{matrix}\Bigg| q; \frac{q^{1-n}a}{bc}\right] \tag{F5.10a}
$$

$$
= \left[\begin{matrix} q,\ q,\ qa,\ q/a,\ qa/bd,\ qa/cd,\ qd/b,\ qd/c \\ qa/d,\ qd/a,\ qd,\ q/d,\ q/b,\ q/c,\ qa/b,\ qa/c \end{matrix}\Bigg| q\right]_\infty \tag{F5.10b}
$$

$$
\times \quad (a/d)^n \prod_{k=1}^{\ell} \left[\begin{matrix} qd/x_k,\ qd/y_k \\ qa/x_k,\ qa/y_k \end{matrix}\Bigg| q\right]_{n_k} \tag{F5.10c}
$$

provided that $|q^{1-n}a/bc| < 1$ and the bilateral series is well-defined.

Proof. Consider the meromorphic function of complex variable z defined by

$$
F(z) = \left[\begin{matrix} q,\ q,\ a,\ q/a,\ qa/bz,\ qa/cz,\ qz/b,\ qz/c \\ a/z,\ qz/a,\ z,\ q/z,\ q/b,\ q/c,\ qa/b,\ qa/c \end{matrix}\Bigg| q\right]_\infty \tag{F5.11a}
$$

$$
\times \quad (a/z)^n \prod_{k=1}^{\ell} \left[\begin{matrix} qz/x_k,\ qz/y_k \\ qa/x_k,\ qa/y_k \end{matrix}\Bigg| q\right]_{n_k}. \tag{F5.11b}
$$

We can check without difficulty that $F(z)$ satisfies the multiplicative reflection property $F(z) = F(a/z)$. It has simple poles $z = q^j$ and $z = aq^j$ $(j = 0, \pm 1, \pm 2, \cdots)$ with residues

$$
\left\{\frac{qa}{bc}\right\}^j \left[\begin{matrix} b,\ c \\ qa/b, qa/c \end{matrix}\Bigg| q\right]_j \prod_{k=1}^{\ell} \left[\begin{matrix} x_k,\ y_k \\ qa/x_k, qa/y_k \end{matrix}\Bigg| q\right]_j \tag{F5.12a}
$$

$$
\times \quad \begin{cases} -q^{-j(1+n)} & \text{at} \quad z = q^{-j} \\ +aq^{j(1-n)} & \text{at} \quad z = aq^j. \end{cases} \tag{F5.12b}
$$

Write $|a| = |q|^A$ and $A = \epsilon \pmod 1$. Denote by δ_m and ∂_m the circles with the center at the origin and radii $|q|^{m+(1+\epsilon)/2}$ and $|q|^{-m+(1+\epsilon)/2}$, respectively. It is clear that there is no pole of $F(z)$ passing through δ_m or ∂_m. Then the residue theorem (cf. [**58**, §3.1]) states that

$$
\frac{F(z)}{z} + \sum_{\tau} \frac{\operatorname{Res}\,[F(z)]_{z=\tau}}{\tau\,(\tau - z)} = \frac{1}{2\pi i}\int_{\partial_m} \frac{F(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.13a}
$$

$$
- \quad \frac{1}{2\pi i}\int_{\delta_m} \frac{F(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.13b}
$$

where the summation runs over all poles $\{\tau\}$ of $F(z)$ between contours δ_m and ∂_m. For sufficiently large m, we can estimate that

$$\begin{aligned}
\int_{\delta_m}\frac{F(t)}{t(t-z)}\mathrm{dt} &= \mathcal{O}\left\{q^{-mn}\left[\begin{matrix}q^{-m+(1-\epsilon)/2}a/b, q^{-m+(1-\epsilon)/2}a/c\\ q^{-m+(1-\epsilon)/2}, q^{-m-(1+\epsilon)/2}a\end{matrix}\,\Big|\, q\right]_m\right\} &\text{(F5.14a)}\\
&= \mathcal{O}\left\{(q^{1-n}a/bc)^m\right\} &\text{(F5.14b)}\\
\int_{\partial_m}\frac{F(t)}{t(t-z)}\mathrm{dt} &= \mathcal{O}\left\{q^{m-mn}\left[\begin{matrix}q^{-m+(3+\epsilon)/2}/b, q^{-m+(3+\epsilon)/2}/c\\ q^{-m+(1+\epsilon)/2}, q^{-m+(3+\epsilon)/2}/a\end{matrix}\,\Big|\, q\right]_m\right\} &\text{(F5.14c)}\\
&= \mathcal{O}\left\{(q^{2-n}a/bc)^m\right\}. &\text{(F5.14d)}
\end{aligned}$$

Note first that for $m \to \infty$, both integrals displayed in (F5.13a-F5.13b) tend to zero under condition $|q^{1-n}a/bc| < 1$. Write then the residue-sum displayed on the left hand side of (F5.13a) explicitly

$$\begin{aligned}
\sum_\tau \frac{\mathrm{Res}\,[F(z)]_{z=\tau}}{\tau\,(\tau - z)} = \sum_{j=-\infty}^{+\infty}\left\{\frac{qa}{bc}\right\}^j \left[\begin{matrix}b,\ c\\ qa/b, qa/c\end{matrix}\,\Big|\, q\right]_j \prod_{k=1}^{\ell}\left[\begin{matrix}x_k,\ y_k\\ qa/x_k, qa/y_k\end{matrix}\,\Big|\, q\right]_j\\
\times\left\{\frac{aq^{j(1-n)}}{aq^j(aq^j-z)} - \frac{q^{-j(1+n)}}{q^{-j}(q^{-j}-z)}\right\}
\end{aligned}$$

where the difference of two fractions in the last line can be simplified as

$$\frac{aq^{j(1-n)}}{aq^j(aq^j-z)} - \frac{q^{-j(1+n)}}{q^{-j}(q^{-j}-z)} = \frac{-(1-q^{2j}a)q^{-jn}}{z(1-q^jz)(1-q^ja/z)}.$$

We can therefore reformulate the limit of integral-sum (F5.13a-F5.13b) as the following bilateral series identity:

$$\begin{aligned}
F(z)\frac{(1-z)(1-a/z)}{1-a} &= \sum_{j=-\infty}^{+\infty}\frac{1-z}{1-q^jz}\,\frac{1-a/z}{1-q^ja/z}\left[\begin{matrix}b,\ c\\ qa/b, qa/c\end{matrix}\,\Big|\, q\right]_j\\
&\times\ \frac{1-q^{2j}a}{1-a}\prod_{k=1}^{\ell}\left[\begin{matrix}x_k,\ y_k\\ qa/x_k, qa/y_k\end{matrix}\,\Big|\, q\right]_j\left\{\frac{q^{1-n}a}{bc}\right\}^j
\end{aligned}$$

where the bilateral series on the right converges under the same condition $|q^{1-n}a/bc| < 1$.

Replacing z by d, we see that this identity becomes exactly the basic very well-poised bilateral hypergeometric formula (F5.10a-F5.10b). Unfortunately, this very well-poised evaluation is not a proper extension of (F5.1), even though we have expected that. □

Remark When $n = 0$, we recover from the formula (F5.10a-F5.10b-F5.10c) a special case of Bailey's bilateral ${}_6\psi_6$-series identity ($\left|qa^2/bcde\right| < 1$):

$$
\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\middle|\, q; \frac{qa^2}{bcde}\right] \\
&\quad = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}
$$

F5.5. Ordinary hypergeometric counterparts. For $q \to 1$, we derive from the limit of the last bilateral basic hypergeometric series identity the ordinary hypergeometric summation formula.

For complex numbers a, b, c, d, and $\{x_k,\ y_k\}_{k=1}^{\ell}$ satisfying the condition $x_k + y_k = 1 + a + n_k$ $(k = 1, 2, \cdots, \ell)$ with $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n = \sum_{k=1}^{\ell} n_k$, there holds

$$
{}_{2\ell+5}H_{5+2\ell}\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & a-d, & \{x_k, \quad y_k\} \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+d, & \{1+a-x_k, 1+a-y_k\} \end{matrix} \,\middle|\, 1\right] \tag{F5.15a}
$$

$$
= \quad \frac{\sin\pi a}{\pi a}\,\frac{\pi d}{\sin\pi d}\,\frac{\pi(a-d)}{\sin\pi(a-d)}\prod_{k=1}^{\ell}\left[\begin{matrix} 1+d-x_k,\ 1+d-y_k \\ 1+a-x_k,\ 1+a-y_k \end{matrix}\right]_{n_k} \tag{F5.15b}
$$

$$
\times \quad \Gamma\left[\begin{matrix} 1+a-b, & 1+a-c, & 1-b, & 1-c \\ 1+a-b-d, & 1+a-c-d, & 1-b+d, & 1-c+d \end{matrix}\right] \tag{F5.15c}
$$

provided that $n + \Re(b + c - a) < 1$ and the bilateral series is well-defined.

The Chu-Karlsson-Minton formulae (F5.7-F5.8) may be regarded as its limiting case of $a \to \infty$ after replacing c by $1 + a - c$. When $n = 0$, it reduces to a special case of the Dougall formula (cf. [**56**, §6.1]):

$$
{}_5H_5\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & e \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} \,\middle|\, 1\right] \tag{F5.16a}
$$

$$
= \quad \Gamma\left[\begin{matrix} 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e \\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d,\ 1-a \end{matrix}\right] \tag{F5.16b}
$$

$$
\times \quad \Gamma\left[\begin{matrix} 1-b,\ 1-c,\ 1-d,\ 1-e,\ 1+2a-b-c-d-e \\ 1+a-b-e,\ 1+a-c-e,\ 1+a-d-e,\ 1+a \end{matrix}\right] \tag{F5.16c}
$$

where $\Re(1 + 2a - b - c - d - e) > 0$.

CHAPTER G

The Lagrange Four Square Theorem

Representing natural numbers as sums of squares is an important topic of number theory. Given a general natural number n, denote by $r_\ell(n)$ the number of integer solutions of Diophantine equation

$$n = x_1^2 + x_2^2 + \cdots + x_\ell^2$$

which counts the number of ways in which n can be written as sums of ℓ squares. In ℓ-dimensional space, $r_\ell(n)$ gives also the number of points with integer coordinates on the sphere.

When ℓ is odd, the problem is very difficult. However for the even case, the problem may be treated in a fairly reasonable manner. Combining Ramanujan's ${}_1\psi_1$-bilateral formula with the Jacobi-triple product identity, we present solutions for the two square and four square problems. The six and eight square problems are dealt with similarly by means of Bailey's bilateral ${}_6\psi_6$-series identity.

G1. Representations by two square sums

When $\ell = 2$, the result may be stated as the following q-series identity

$$\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^2 = 1 + 4\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}$$

and the corresponding formula for the numbers of representations by two squares

$$r_2(n) = 4\sum_{(1+2c)\,|\,n} (-1)^c = 4\sum_{2\,\nmid\,d\,|\,n} (-1)^{\binom{d}{2}}.$$

Proof. According to the Jacobi triple product identity

$$\sum_{n=-\infty}^{+\infty} q^{n^2} = \sum_{n=-\infty}^{+\infty} (-1)^n q^{2\binom{n}{2}}(-q)^n = \left[q^2, -q, -q;\, q^2\right]_\infty$$

we have shifted factorial product expression

$$\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^2 = \left[q^2, -q, -q;\ q^2\right]_\infty^2$$

which can be reformulated by means of the Euler formula as

$$\left[q^2, -q, -q;\ q^2\right]_\infty^2 = \frac{\left[q^2,\ q^2, -q, -q;\ q^2\right]_\infty}{\left[q,\ q, -q^2, -q^2;\ q^2\right]_\infty}.$$

Recalling Ramanujan's ${}_1\psi_1$-bilateral series identity

$${}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix} \,\middle|\, q;\ z\right] = \left[\begin{matrix} q, & c/a, & az, & q/az \\ c, & q/a, & z, & c/az \end{matrix} \,\middle|\, q\right]_\infty$$

we have

$$\begin{aligned}\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^2 &= {}_1\psi_1\left[\begin{matrix} -1 \\ -q^2 \end{matrix} \,\middle|\, q^2;\ q\right] \\ &= 1 + \sum_{k=1}^{\infty}\left\{\frac{(-1;q^2)_k}{(-q^2;q^2)_k} q^k + \frac{(-1;q^2)_{-k}}{(-q^2;q^2)_{-k}} q^{-k}\right\}.\end{aligned}$$

Noting further two relations on shifted factorial fractions:

$$\begin{aligned}\frac{(-1;q^2)_{-k}}{(-q^2;q^2)_{-k}} &= \frac{(-1;q^2)_k}{(-q^2;q^2)_k} q^{2k} \\ \frac{(-1;q^2)_k}{(-q^2;q^2)_k} &= \frac{2}{1+q^{2k}}\end{aligned}$$

we find the following simplified expression

$$\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^2 = 1 + 4\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}.$$

Extracting the coefficient of q^n, we establish

$$\begin{aligned} r_2(n) &= [q^n]\left\{1 + 4\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}\right\} \\ &= [q^n]\left\{1 + 4\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} (-1)^m q^{k(1+2m)}\right\} \\ &= 4\sum_{(1+2m)\mid n} (-1)^m.\end{aligned}$$

This completes the solution of representations by two square sums. □

G2. Representations by four square sums

The Lagrange four square theorem states that every natural number can be expressed as sum of four square numbers. More precisely, we have the following q-series identity

$$\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^4 = \left\{\frac{(-q;\,-q)_\infty}{(q;\,-q)_\infty}\right\}^4 = 1+\sum_{n=1}^{\infty}\frac{8nq^n}{1+(-q)^n}$$

and the corresponding formula for the numbers of representations by four squares

$$r_4(n) = 8\sum_{4\,\nmid\, d\,|\,n} d \quad\Rightarrow\quad r_4(n)\geq 1 \quad \text{for} \quad n=1,2,\cdots.$$

Its demonstration is similar to that for the case of two squares. Based on Ramanujan's bilateral sum, we have the following limiting relation

$$\begin{aligned}
\left\{\frac{(q;\,q)_\infty}{(-q;\,q)_\infty}\right\}^4 &= \lim_{z\to -q}\frac{2}{1+q/z}\left[\begin{matrix} q, & q, & -z, & -q/z \\ -q, & -q, & z, & q/z\end{matrix}\,\middle|\, q\right]_\infty \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\,{}_1\psi_1\left[\begin{matrix}-1\\-q\end{matrix}\,\middle|\, q;\, z\right] \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\left\{1+\sum_{n=1}^{\infty}\frac{(-1;q)_n}{(-q;q)_n}z^n+\sum_{n=1}^{\infty}\frac{(-1;q)_{-n}}{(-q;q)_{-n}}z^{-n}\right\} \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\left\{1+\sum_{n=1}^{\infty}\frac{2z^n}{1+q^n}+\sum_{n=1}^{\infty}\frac{2(q/z)^n}{1+q^n}\right\}.
\end{aligned}$$

Reformulating the last sum as

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{2(q/z)^n}{1+q^n} &= \sum_{n=1}^{\infty}\left\{\frac{2(q/z)^n(1+q^n)}{1+q^n}-\frac{2(q^2/z)^n}{1+q^n}\right\} \\
&= \frac{2q/z}{1-q/z}-\sum_{n=1}^{\infty}\frac{2(q^2/z)^n}{1+q^n}
\end{aligned}$$

we may compute the limit explicitly as

$$\begin{aligned}
\left\{\frac{(q;\,q)_\infty}{(-q;\,q)_\infty}\right\}^4 &= \lim_{z\to -q}\frac{2}{1+q/z}\left\{\frac{1+q/z}{1-q/z}+\sum_{n=1}^{\infty}\frac{2z^n}{1+q^n}\big[1-(q/z)^{2n}\big]\right\} \\
&= 1+\sum_{n=1}^{\infty}\frac{8n(-q)^n}{1+q^n}.
\end{aligned}$$

On the other hand, it is not hard to derive

$$\sum_{n=-\infty}^{+\infty}(-1)^n q^{n^2} \;=\; \big[q^2,q,q;\;q^2\big]_\infty \;=\; \frac{(q;\;q)_\infty}{(-q;\;q)_\infty}.$$

Performing the parameter replacement $q \to -q$, we find the following expression:

$$\Big\{\sum_{n=-\infty}^{+\infty} q^{n^2}\Big\}^4 \;=\; \frac{\big(-q;-q\big)_\infty^4}{\big(q;-q\big)_\infty^4} \;=\; 1+\sum_{n=1}^{\infty}\frac{8nq^n}{1+(-q)^n}$$

which is equivalent to the sum for $r_4(n)$. In fact, observe that

$$1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} \;=\; 1+8\sum_{k=1}^{\infty}\frac{2kq^{2k}}{1+q^{2k}}+8\sum_{k=1}^{\infty}\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}.$$

Noting that

$$\frac{2kq^{2k}}{1+q^{2k}} \;=\; \frac{2k(q^{2k}-q^{4k})}{1-q^{4k}} \;=\; \frac{2kq^{2k}}{1-q^{2k}}-\frac{4kq^{4k}}{1-q^{4k}}$$

we have

$$1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} \;=\; 1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\frac{kq^k}{1-q^k}.$$

Extracting the coefficient of q^n, we therefore have

$$r_4(n) \;=\; [q^n]\Big\{1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\frac{kq^k}{1-q^k}\Big\} = [q^n]\Big\{1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\sum_{m=1}^{\infty}kq^{km}\Big\} \;=\; 8\sum_{\substack{k|n\\ 4\nmid k}}k.$$

This completes the solution of representations by four square sums. □

By means of Bailey's bilateral ${}_6\psi_6$-series identity, we now investigate the representations by six and eight squares.

G3. Representations by six square sums

There hold the following q-series identity

$$\Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^6 = \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^6 \;=\; 1+16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}}$$
$$-\;4\sum_{n=0}^{\infty}(-1)^n\frac{(1+2n)^2q^{1+2n}}{1-q^{1+2n}}$$

and the corresponding formula for the numbers of representations by six squares

$$r_6(n) \;=\; 16\sum_{d\,|\,n} d^2\chi(n/d) - 4\sum_{d\,|\,n} d^2\chi(d)$$

where the quadratic Dirichlet character $\chi(d)$ is defined by

$$\chi(d) \;=\; \begin{cases} +1, & d \equiv_4 +1 \\ -1, & d \equiv_4 -1 \\ 0, & d \equiv_2 \ 0. \end{cases}$$

The proof will be fulfilled in three steps.

G3.1. Recall Bailey's very well-poised non-terminating bilateral series identity

$$\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\;\middle|\; q;\ \frac{qa^2}{bcde}\right] \\
&= \left[\begin{matrix} q,\ qa,\ q/a,\ qa/bc,\ qa/bd,\ qa/be,\ qa/cd,\ qa/ce,\ qa/de \\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q/b,\ q/c,\ q/d,\ q/e,\ qa^2/bcde \end{matrix}\;\middle|\; q\right]_\infty
\end{aligned}$$

provided that $\left|qa^2/bcde\right| < 1$.

Specifying with $b = c = d = -1$ and $e \to \infty$, we may restate it as

$$\begin{aligned}
\frac{(q;q)_\infty (q/a;q)_\infty (qa;q)_\infty^4}{(-q;q)_\infty^3 (-qa;q)_\infty^3} &= 1 + \sum_{k=1}^{\infty} a^{2k}\frac{1-q^{2k}a}{1-a}\,\frac{(-1;q)_k^3}{(-qa;q)_k^3}\,q^{\binom{1+k}{2}} \\
&\quad + \sum_{k=1}^{\infty} a^{-2k}\frac{1-q^{-2k}a}{1-a}\,\frac{(-1;q)_{-k}^3}{(-qa;q)_{-k}^3}\,q^{\binom{1-k}{2}} \\
&= 1 + \sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}}}{1-a}\left\{\left(a^{2k} - q^{2k}a^{1+2k}\right)\frac{(-1;q)_k^3}{(-qa;q)_k^3}\right. \\
&\qquad\qquad \left. + \left(q^{2k}a^k - a^{1+k}\right)\frac{(-1/a;q)_k^3}{(-q;q)_k^3}\right\}.
\end{aligned}$$

Letting $a \to 1$, we can compute, through L'Hôspital's rule, the limit of the summand as follows:

$$8\frac{1-q^{2k}}{(1+q^k)^3}q^{\binom{1+k}{2}}\left\{\frac{(1+2k)q^{2k}-2k}{1-q^{2k}}+3\sum_{i=1}^{k}\frac{q^i}{1+q^i}\right.$$
$$\left.+\frac{(1+k)-kq^{2k}}{1-q^{2k}}-3\sum_{j=0}^{k-1}\frac{q^j}{1+q^j}\right\}$$
$$=\frac{4q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}.$$

Therefore we have found the expression:

$$\left\{\frac{(q;q)_\infty}{(-q;q)_\infty}\right\}^6 = 1+4\sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}.$$

G3.2. Denoting the last sum with respect to k by $\clubsuit(q)$ and then recalling the binomial expansion

$$\frac{q^k}{(1+q^k)^3} = \sum_{\ell=1}^{\infty}(-1)^{1+\ell}\binom{1+\ell}{2}q^{k\ell}$$

we can manipulate $\clubsuit(q)$ in the following manner:

$$\begin{aligned}\clubsuit(q) &= \sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}\\ &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{1+\ell}\binom{1+\ell}{2}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}q^{\binom{1+k}{2}+k\ell}\\ &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\left\{6\binom{1+\ell}{2}+(1-2k)\binom{\ell}{2}+(1+2k)\binom{2+\ell}{2}\right\}q^{\binom{1+k}{2}+k\ell}.\end{aligned}$$

Rewriting the q-exponent by

$$\binom{1+k}{2}+k\ell = \frac{1}{2}\Big\{k(1+2\ell+k)\Big\}$$

and then simplifying the binomial sum

$$\begin{aligned}&6\binom{1+\ell}{2}+(1-2k)\binom{\ell}{2}+(1+2k)\binom{2+\ell}{2}\\ =\;& (1+2\ell)\times(1+2k+2\ell) = (1+k+2\ell)^2-k^2\end{aligned}$$

we can split ♣, according to the parity of k, into two double sums:

$$\begin{aligned}
\clubsuit(q) &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(1+k+2\ell)^2-k^2\Big\}q^{\binom{1+k}{2}+k\ell} && \text{(G3.1a)}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(1+2k+2\ell)^2-(2k)^2\Big\}q^{k(1+2k+2\ell)} && \text{(G3.1b)}\\
&+ \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(2k+2\ell)^2-(2k-1)^2\Big\}q^{(k+\ell)(2k-1)}. && \text{(G3.1c)}
\end{aligned}$$

G3.3. Putting $n := k+\ell$ and then applying the geometric series, we can reduce (G3.1b) as follows:

$$\begin{aligned}
\mathrm{Eq(G3.1b)} &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(-1)^{1+n-k}\Big\{(1+2n)^2-(2k)^2\Big\}q^{k(1+2n)}\\
&= \sum_{n=1}^{\infty}(-1)^{1+n}(1+2n)^2\sum_{k=1}^{n}(-1)^k q^{k(1+2n)}\\
&+ \sum_{k=1}^{\infty}(-1)^k(2k)^2 q^k\sum_{n=k}^{\infty}(-1)^n q^{2nk}\\
&= \sum_{n=0}^{\infty}\frac{(1+2n)^2}{1+q^{1+2n}}\Big\{(-1)^n q^{1+2n}-q^{(1+n)(1+2n)}\Big\}\\
&+ \sum_{k=1}^{\infty}\frac{4k^2}{1+q^{2k}}q^{k(1+2k)}.
\end{aligned}$$

We can also treat (G3.1c) analogously:

$$\begin{aligned}
\mathrm{Eq(G3.1c)} &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(-1)^{1+n-k}\Big\{(2n)^2-(2k-1)^2\Big\}q^{n(2k-1)}\\
&= \sum_{n=1}^{\infty}(-1)^{1+n}(2n)^2 q^{-n}\sum_{k=1}^{n}(-1)^k q^{2nk}\\
&+ \sum_{k=1}^{\infty}(-1)^k(2k-1)^2\sum_{n=k}^{\infty}(-1)^n q^{n(2k-1)}\\
&= \sum_{n=1}^{\infty}\frac{(2n)^2}{1+q^{2n}}\Big\{(-1)^n q^n-q^{n(1+2n)}\Big\}\\
&+ \sum_{k=1}^{\infty}\frac{(2k-1)^2}{1+q^{2k-1}}q^{k(2k-1)}.
\end{aligned}$$

Their combination leads us to the following:

$$\begin{aligned}
\clubsuit(q) &= \mathrm{Eq(G3.1a)} \;=\; \mathrm{Eq(G3.1b)} + \mathrm{Eq(G3.1c)} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{q^{1+2n}}{1+q^{1+2n}}(1+2n)^2 + 4\sum_{n=1}^{\infty}(-1)^n \frac{n^2q^n}{1+q^{2n}}.
\end{aligned}$$

Replacing q by $-q$, we have finally established the q-series identity:

$$\begin{aligned}
\Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^6 &= \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^6 \;=\; 1+4\clubsuit(-q) \\
&= 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}} - 4\sum_{n=0}^{\infty}(-1)^n\frac{(1+2n)^2q^{1+2n}}{1-q^{1+2n}}.
\end{aligned}$$

Extracting the coefficient of q^n, we get the formula for $r_6(n)$ stated in the Theorem. This completes the solution of representations by six square sums. $\square$

G4. Representations by eight square sums

Following the same procedure to the last section, we can also show the eight square sum theorem. But the proof is much easier this time.

The theorem states that there hold the q-series identity:

$$\Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^8 = \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^8 \;=\; 1+16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-q)^n}$$

and the corresponding formula for the numbers of representations by eight squares

$$r_8(n) \;=\; 16\sum_{d\,|\,n}(-1)^{n+d}\,d^3.$$

The proof is divided into two parts.

G4.1. Putting with $b = c = d = e = -1$ in Bailey's very well-poised non-terminating bilateral series identity, we can write the result as

$$\begin{aligned}\frac{(q;q)_\infty(q/a;q)_\infty(qa;q)_\infty^7}{(qa^2;q)_\infty(-q;q)_\infty^4(-qa;q)_\infty^4} &= 1+\sum_{k=1}^{\infty}\frac{1-q^{2k}a}{1-a}\frac{(-1;q)_k^4}{(-qa;q)_k^4}(qa^2)^k\\ &\quad+\sum_{k=1}^{\infty}\frac{1-q^{-2k}a}{1-a}\frac{(-1;q)_{-k}^4}{(-qa;q)_{-k}^4}(qa^2)^{-k}\\ &= 1+\sum_{k=1}^{\infty}\frac{q^k}{1-a}\left\{\left(a^{2k}-q^{2k}a^{1+2k}\right)\frac{(-1;q)_k^4}{(-qa;q)_k^4}\right.\\ &\qquad\left.+\left(q^{2k}a^{2k}-a^{1+2k}\right)\frac{(-1/a;q)_k^4}{(-q;q)_k^4}\right\}.\end{aligned}$$

Letting $a \to 1$, we can compute, through L'Hôspital's rule, the limit of the summand as follows:

$$\begin{aligned}&16q^k\frac{1-q^{2k}}{(1+q^k)^4}\left\{\frac{(1+2k)q^{2k}-2k}{1-q^{2k}}+4\sum_{i=1}^{k}\frac{q^i}{1+q^i}\right.\\ &\qquad\left.+\frac{(1+2k)-2kq^{2k}}{1-q^{2k}}-4\sum_{j=0}^{k-1}\frac{q^j}{1+q^j}\right\}\\ &=\frac{16q^{2k}}{(1+q^k)^4}\left\{4-q^k-q^{-k}\right\}.\end{aligned}$$

Therefore we have found the expression:

$$\left\{\frac{(q;q)_\infty}{(-q;q)_\infty}\right\}^8 = 1+16\sum_{k=1}^{\infty}\frac{q^{2k}}{(1+q^k)^4}\left\{4-q^k-q^{-k}\right\}.$$

G4.2. Denoting the last sum with respect to k by $\diamondsuit(q)$ and then recalling the binomial expansion

$$\frac{q^{2k}}{(1+q^k)^4} = \sum_{\ell=2}^{\infty}(-1)^\ell\binom{1+\ell}{3}q^{k\ell}$$

we can manipulate $\diamondsuit(q)$ in the following manner:

$$\begin{aligned}
\diamondsuit(q) &= \sum_{k=1}^{\infty}\frac{q^{2k}}{(1+q^k)^4}\Big\{4-q^k-q^{-k}\Big\}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=2}^{\infty}(-1)^{\ell}\binom{1+\ell}{3}\Big\{4-q^k-q^{-k}\Big\}q^{k\ell}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{\ell}\Big\{4\binom{1+\ell}{3}+\binom{\ell}{3}+\binom{2+\ell}{3}\Big\}q^{k\ell}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{\ell}\ell^3q^{k\ell} = \sum_{\ell=1}^{\infty}(-1)^{\ell}\frac{q^{\ell}}{1-q^{\ell}}\ell^3
\end{aligned}$$

where the following binomial sum has been used

$$4\binom{1+\ell}{3}+\binom{\ell}{3}+\binom{2+\ell}{3} = \ell^3.$$

Now replacing q by $-q$, we derive the q-series identity:

$$\begin{aligned}
\Big\{\sum_{n=-\infty}^{\infty}q^{n^2}\Big\}^8 &= \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^8 = 1+16\diamondsuit(-q)\\
&= 1+16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-q)^n}.
\end{aligned}$$

Extracting the coefficient of q^n, we get the formula for $r_8(n)$ stated in the Theorem. This completes the solution of representations by eight square sums. $\square$

G5. Jacobi's identity and q-difference equations

Among q-difference equations, there is a beautiful result due to Jacobi (1829), which will be proved and generalized in this section.

G5.1. Jacobi's q-difference equation. The identity on eight infinite products states that

$$\big(-q;\,q^2\big)_\infty^8-\big(q;\,q^2\big)_\infty^8 = 16q\big(-q^2;\,q^2\big)_\infty^8$$

which has been commented by Jacobi (1829) as "*aequatio identica satis abstrusa*".

Its proof can be fulfilled by means of the Jacobi-triple product identity and Lagrange's four square theorem.

In fact, multiplying both sides by $\left(q^2;\, q^2\right)_\infty^4$ and then noticing that

$$\left[q^2,\, q,\, q;\, q^2\right]_\infty \quad = \quad \sum_{m=-\infty}^{+\infty} (-1)^m\, q^{m^2}$$

we can reformulate the eight product difference equation as follows

$$2q\sum_{m=0}^{\infty} r_4(1+2m)\, q^{2m} \;=\; q\Big\{\sum_{n=-\infty}^{+\infty} q^{n(n+1)}\Big\}^4 \;=\; q\sum_{n=0}^{\infty} s_4(n)\, q^{2n}$$

where $s_4(n)$ is the number of integer solutions of Diophantine equation

$$n \quad = \quad \binom{1+x_1}{2} + \binom{1+x_2}{2} + \binom{1+x_3}{2} + \binom{1+x_4}{2}$$

which counts the number of ways expressing n as sums of four triangles. It is equal to the number of integer solutions of Diophantine equation

$$4+8n \quad = \quad (1+2x_1)^2 + (1+2x_2)^2 + (1+2x_3)^2 + (1+2x_4)^2.$$

The last one is in turn the number of odd integer solutions of Diophantine equation

$$4+8n \quad = \quad y_1^2 + y_2^2 + y_3^2 + y_4^2$$

whose integer solutions enumerated by $r_4(4+8n)$ may be divided into two categories: odd integer solutions counted by $s_4(n)$ and even integer solutions by $r_4(1+2n)$. Therefore we have

$$s_4(n) \quad = \quad r_4(4+8n) - r_4(1+2n) \quad = \quad 2r_4(1+2n)$$

which leads us to Jacobi's q-difference equation.

G5.2. Theorem. Generalizing the Jacobi q-difference equation, we prove, by combining the telescoping method with bailey's bilateral ${}_6\psi_6$-series identity, the following theorem due to Chu (1992).

For five parameters related by multiplicative relation $A^2 = bcde$, there holds

$$\begin{aligned} & \langle A/b;q\rangle_\infty\, \langle A/c;q\rangle_\infty\, \langle A/d;q\rangle_\infty\, \langle A/e;q\rangle_\infty && \text{(G5.1a)} \\ - \quad & \langle b;q\rangle_\infty\, \langle c;q\rangle_\infty\, \langle d;q\rangle_\infty\, \langle e;q\rangle_\infty && \text{(G5.1b)} \\ = \quad & b\,\langle A;q\rangle_\infty\, \langle A/bc;q\rangle_\infty \langle A/bd;q\rangle_\infty\, \langle A/be;q\rangle_\infty && \text{(G5.1c)} \end{aligned}$$

where the q-shifted factorial for $|q| < 1$ is defined by

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1 - xq^n) \quad \text{and} \quad \langle x;q\rangle_\infty \;=\; (x;q)_\infty \times (q/x;q)_\infty.$$

This identity reduces to Jacobi's equation under parameter replacements

$$q \to q^2 :\; A = -q^2 \quad \text{and} \quad b = c = d = e = -q.$$

Proof. Define the factorial fractions by

$$T_k := \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_k.$$

It is trivial to check factorization

$$\begin{aligned} & (1-q^kA/b)(1-q^kA/c)(1-q^kA/d)(1-q^kA/e) \\ - & (1-q^kb)(1-q^kc)(1-q^kd)(1-q^ke) \\ = & bq^k(1-q^{2k}A)(1-A/bc)(1-A/bd)(1-A/be) \end{aligned}$$

which leads us to the following difference relation:

$$\begin{aligned} T_k - T_{k+1} &= \left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q\right]_k \\ &\times \frac{\left\{\begin{matrix} (1-q^kA/b)(1-q^kA/c)(1-q^kA/d)(1-q^kA/e) \\ -(1-q^kb)\,(1-q^kc)\,(1-q^kd)\,(1-q^ke) \end{matrix}\right\}}{(1-A/b)(1-A/c)(1-A/d)(1-A/e)} \\ &= \left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q\right]_k \\ &\times \frac{bq^k(1-q^{2k}A)(1-A/bc)(1-A/bd)(1-A/be)}{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}. \end{aligned}$$

Reformulating the last relation as

$$\begin{aligned} & \frac{1-q^{2k}A}{1-A}\left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q\right]_k q^k \\ = & \left\{T_k - T_{k+1}\right\}\frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \end{aligned}$$

and then applying the telescoping method, we derive the bilateral finite summation formula:

$$\sum_{k=m}^{n-1} \frac{1-q^{2k}A}{1-A}\left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q\right]_k q^k \tag{G5.2a}$$

$$= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \sum_{k=m}^{n-1} \left\{T_k - T_{k+1}\right\} \tag{G5.2b}$$

$$= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \left\{T_m - T_n\right\}. \tag{G5.2c}$$

Recalling the definition of T_k and keeping in mind of $A^2 = bcde$, we have

$$\begin{aligned}
\lim_{n\to+\infty} T_n &= \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_\infty \\
\lim_{m\to-\infty} T_m &= \lim_{m\to+\infty} \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_{-m} \\
&= \lim_{m\to+\infty} \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_m \\
&= \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}$$

Now letting $m \to -\infty$ and $n \to +\infty$ in (G5.2), we get a closed formula for the non-terminating bilateral convergent series:

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} q\sqrt{A}, & -q\sqrt{A}, & b, & c, & d, & e \\ \sqrt{A}, & -\sqrt{A}, & qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q; q\right] \\
= \;& \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \\
\times \;& \left\{ \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_\infty \right. \\
& \left. - \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_\infty \right\}.
\end{aligned}$$

Alternatively, the last bilateral sum can be evaluated by Bailey's ${}_6\psi_6$-series identity with $A^2 = bcde$ as follows:

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} qA^{1/2}, & -qA^{1/2}, & b, & c, & d, & e \\ A^{1/2}, & -A^{1/2}, & qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q; \frac{qA^2}{bcde}\right] \\
= \;& \left[\begin{matrix} qA, q/A, qA/bc, qA/bd, qA/be, qA/cd, qA/ce, qA/de \\ qA/b,\ qA/c,\ qA/d,\ qA/e,\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}$$

Equating the right members of both results, we get the following relation:

$$\begin{aligned}
& b \left[\begin{matrix} A,\ q/A,\ A/bc,\ A/bd,\ A/be,\ qA/cd,\ qA/ce,\ qA/de \\ A/b,\ A/c,\ A/d,\ A/e,\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty \\
= \;& \left[\begin{matrix} qb/A, qc/A, qd/A, qe/A \\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty - \left[\begin{matrix} b,\ c,\ d,\ e \\ A/b, A/c, A/d, A/e \end{matrix} \,\middle|\, q\right]_\infty
\end{aligned}$$

which is equivalent to the q-difference equation

$$\begin{aligned}
& b\ \langle A; q\rangle_\infty\ \langle A/bc; q\rangle_\infty \langle A/bd; q\rangle_\infty\ \langle A/be; q\rangle_\infty \\
&= \langle A/b; q\rangle_\infty\ \langle A/c; q\rangle_\infty\ \langle A/d; q\rangle_\infty\ \langle A/e; q\rangle_\infty \\
&- \langle b; q\rangle_\infty\ \langle c; q\rangle_\infty\ \langle d; q\rangle_\infty\ \langle e; q\rangle_\infty .
\end{aligned}$$

This proves the q-difference equation stated in Theorem G5.2. $\square$

G5.3. A trigonometric identity. The discovery of Theorem G5.2 has been inspired by the following interesting fact. Given five parameters related instead by additive relation $2A = b + c + d + e$, it can be verified that

$$(A-b)(A-c)(A-d)(A-e) \ - \ bcde \ = \ A(A-b-c)(A-b-d)(A-b-e).$$

Surprisingly, it is also true even if we replace each linear factor with its sine function:

$$\begin{aligned} & \sin(A-b)\ \sin(A-c)\ \sin(A-d)\ \sin(A-e)\ -\ \sin b\ \sin c\ \sin d\ \sin e \\ = \ & \sin A\ \sin(A-b-c)\ \sin(A-b-d)\ \sin(A-b-e). \end{aligned}$$

The q-difference equation displayed in Theorem G5.2 may be considered as the q-analogue of this trigonometric identity.

According to the factorial fraction

$$\frac{\langle x; q\rangle_\infty}{(1-q)(q;\ q)_\infty^2} \ = \ \frac{1-x}{1-q}\prod_{n=1}^{\infty}\frac{(1-q^n x)(1-q^n/x)}{(1-q^n)^2}$$

we have the following limit relation

$$\begin{aligned} \lim_{q\to 1}\frac{\langle q^x; q\rangle_\infty}{(1-q)(q;\ q)_\infty^2} \ &= \ \lim_{q\to 1}\frac{1-q^x}{1-q}\prod_{n=1}^{\infty}\frac{(1-q^{n+x})(1-q^{n-x})}{(1-q^n)^2} \\ &= \ x\prod_{n=1}^{\infty}\Big\{1-\frac{x^2}{n^2}\Big\} \ = \ \frac{\sin(\pi x)}{\pi}. \end{aligned}$$

Replacing first b, c, d, e respectively by q^b, q^c, q^d, q^e in the q-difference equation stated in Theorem G5.2, then dividing both sides by $(1-q)^4(q;q)_\infty^8$ and finally letting $q \to 1$, we get the following trigonometric formula:

$$\begin{aligned} & \sin\pi A\ \sin\pi(A-b-c)\ \sin\pi(A-b-d)\ \sin\pi(A-b-e) \\ = \ & \sin\pi(A-b)\ \sin\pi(A-c)\ \sin\pi(A-d)\ \sin\pi(A-e) \\ - \ & \sin\pi b\ \sin\pi c\ \sin\pi d\ \sin\pi e, \quad (b+c+d+e=2A) \end{aligned}$$

which is the equivalent form of the trigonometric identity to be proved. $\square$

CHAPTER H

Congruence Properties of Partition Function

Congruence properties of $p(n)$, the number of partitions of n, were first discovered by Ramanujan on examining the table of the first 200 values of $p(n)$ constructed by MacMahon (1915):

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In general, if β is a prime, then the congruence with some fixed $\gamma \in \mathbb{N}$

$$p(\beta n+\gamma) \equiv_\beta 0 \quad \text{for all} \quad n \in \mathbb{N}$$

is called the Ramanujan congruence modulo β. Ahlgren and Boylan (2003) confirmed recently that the three congruences displayed above are the only Ramanujan ones.

There exist other congruences of partition function, but non Ramanujan one's. For example, the simplest congruence modulo 13 recorded by Atkin and O'Brien (1967) can be reproduced as follows:

$$p(1331 \times 13n + 237) \equiv 0 \pmod{13}.$$

In order to facilitate the demonstration of the three Ramanujan congruences, we will first show the following general congruence relation about the partition function.

Congruence Lemma on Partition Function: *Let β be a prime and γ an integer. Define the $\wp(m)$-sequence by*

$$\wp(m) = \left[q^m\right] \left\{q^{\beta-\gamma}(q;q)_\infty^{\beta-1}\right\}.$$

If all the coefficients $\wp(\beta m)$ for $m \in \mathbb{N}$ are multiples of β, then there holds the corresponding Ramanujan congruence, i.e., $p(\beta n+\gamma)$ are divisible by β for all $n \in \mathbb{N}$.

PROOF. Writing $x \equiv_\beta y$ for congruence relation $x \equiv y \pmod{\beta}$, then we have the binomial congruence

$$\binom{\beta-1+k}{\beta-1} \equiv_\beta \begin{cases} 1, & k \equiv_\beta 0 \\ 0, & k \not\equiv_\beta 0. \end{cases}$$

By means of binomial expansion, we can derive congruence relation

$$\begin{aligned} \frac{1-q^\beta}{(1-q)^\beta} &= (1-q^\beta)\sum_{k=0}^{\infty}\binom{\beta-1+k}{\beta-1}q^k \\ &\equiv_\beta (1-q^\beta)\sum_{k=0}^{\infty} q^{\beta k} = 1. \end{aligned}$$

Therefore we have accordingly the formal power series congruence

$$\begin{aligned} q^{\beta-\gamma}\frac{(q^\beta;\, q^\beta)_\infty}{(q;\, q)_\infty} &= q^{\beta-\gamma}(q;\, q)_\infty^{\beta-1}\frac{(q^\beta;\, q^\beta)_\infty}{(q;\, q)_\infty^\beta} \\ &\equiv q^{\beta-\gamma}\prod_{k=1}^{\infty}(1-q^k)^{\beta-1} \pmod{\beta} \end{aligned}$$

which implies consequently the following congruence relation

$$\wp(\beta m) \equiv \left[q^{\beta m}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\, q^\beta)_\infty}{(q;\, q)_\infty}\right\} \pmod{\beta} \quad \text{for} \quad m \in \mathbb{N}.$$

According to the generating function of partitions and the Cauchy product of formal power series, we get the following relation

$$\begin{aligned} p(\beta n+\gamma) &= \left[q^{\beta(1+n)}\right]\left\{q^{\beta-\gamma}\Big/\prod_{k=1}^{\infty}(1-q^k)\right\} \\ &= \left[q^{\beta(1+n)}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\, q^\beta)_\infty}{(q;\, q)_\infty}\Big/(q^\beta;\, q^\beta)_\infty\right\} \\ &= \sum_{m=0}^{1+n} p(1+n-m)\left[q^{\beta m}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\, q^\beta)_\infty}{(q;\, q)_\infty}\right\} \\ &\equiv \sum_{m=0}^{1+n} p(1+n-m)\,\wp(\beta m) \pmod{\beta}. \end{aligned}$$

Hence $p(\beta n+\gamma)$ is divisible by β as long as all the coefficients $\wp(\beta m)$ for $m \in \mathbb{N}$ are multiples of β. This completes the proof of the congruence lemma. $\square$

By means of this lemma, we will present Ramanujan's original proof for the first two congruences and the proof for the third one due to Winquist (1969). In addition, the corresponding generating functions will be determined for the first two cases.

H1. Proof of $p(5n+4) \equiv 0 \pmod 5$

There holds the Ramanujan congruence modulo 5:

$$p(5n+4) \;\equiv\; 0 \pmod{5}. \tag{H1.1}$$

In view of the congruence lemma on partition function, we should show that the coefficients of q^{5m} in the formal power series expansion of $q(q;\, q)_\infty^4$ are divisible by 5 for all $m \in \mathbb{N}$.

By means of Euler's pentagon number theorem and the Jacobi triple product identity, consider the formal power series expansion

$$\begin{aligned} q(q;\, q)_\infty^4 \;&=\; q\prod_{m=1}^{\infty}(1-q^m)\prod_{n=1}^{\infty}(1-q^n)^3 \\ &=\; \sum_{i=0}^{+\infty}\sum_{j=-\infty}^{+\infty}(-1)^{i+j}\,(1+2i)\,q^{1+j^2+\binom{1+i}{2}+\binom{1+j}{2}}. \end{aligned}$$

In accordance with congruences

$$\binom{k+1}{2} \equiv_5 \begin{cases} 0, & k \equiv_5 0 \\ 1, & k \equiv_5 1 \\ 3, & k \equiv_5 2 \\ 1, & k \equiv_5 3 \\ 0, & k \equiv_5 4 \end{cases}$$

it is not hard to check that the residues of q-exponent in the formal power series just-displayed

$$1+j^2+\binom{1+i}{2}+\binom{1+j}{2}$$

modulo 5 are given by the following table:

$j\backslash i$	0	1	2	3	4
0	1	2	4	2	1
1	3	4	1	4	3
2	3	4	1	4	3
3	1	2	4	2	1
4	2	3	0	3	2

From this table, we see that if the q-exponent $1+j^2+\binom{1+i}{2}+\binom{1+j}{2}$ is a multiple of 5, so is the coefficient $1+2i$, which corresponds to the only case $i \equiv_5 2$ and $j \equiv_5 4$.

We can also verify this fact by reformulating the congruence relation on the q-exponent as

$$\begin{aligned} 0 &\equiv_5 1+j^2+\binom{1+i}{2}+\binom{1+j}{2} \\ &\equiv_5 8\Big\{1+j^2+\binom{1+i}{2}+\binom{1+j}{2}\Big\} \\ &\equiv_5 (1+2i)^2+2(1+j)^2. \end{aligned}$$

This congruence can be reached only when

$$\begin{aligned} (1+2i)^2 \equiv_5 0 &\implies i \equiv_5 2 \\ 2(1+j)^2 \equiv_5 0 &\implies j \equiv_5 4 \end{aligned}$$

because the corresponding residues modulo 5 read respectively as

$$(1+2i)^2 \equiv_5 0,1,4 \quad \text{and} \quad 2(1+j)^2 \equiv_5 0,2,3.$$

Therefore the coefficients of q^{5m} in the formal power series expansion of $q(q;q)_\infty^4$ are divisible by 5. This completes the proof of the Ramanujan congruence (H1.1).

H2. Generating function for $p(5n+4)$

Furthermore, Ramanujan computed explicitly the generating function:

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n \;=\; 5\frac{(q^5;\, q^5)_\infty^5}{(q;\, q)_\infty^6}. \tag{H2.1}$$

About this identity, Hardy wrote that if he were to select one formula from Ramanujan's work for supreme beauty, he would agree with MacMahon in selecting this one.

The proof presented here is essentially due to Ramanujan.

H2.1. Let $p := q^{1/5}$ and ω be a 5-th primitive root of unit $\omega = e^{\frac{2\pi}{5}\sqrt{-1}}$.

Recall the generating function of partitions

$$\frac{q}{(q;\, q)_\infty} \;=\; q\sum_{n=0}^{\infty} p(n)q^n.$$

Replacing q by $q\omega^k$ and then summing the equations with k over $0 \le k \le 4$, we have

$$\sum_{k=0}^{4} \frac{q\omega^k}{\left(q\omega^k; q\omega^k\right)_\infty} \;=\; \sum_{n=0}^{\infty} p(n)q^{n+1} \sum_{k=0}^{4} \omega^{k(n+1)}.$$

It is not hard to verify that

$$\sum_{k=0}^{4} \omega^{k(n+1)} \;=\; \begin{cases} 5, & n+1 \equiv_5 0 \\ 0, & n+1 \not\equiv_5 0 \end{cases} \tag{H2.2}$$

where the last line is justified by the finite geometric series

$$\sum_{k=0}^{4} \omega^{k(n+1)} \;=\; \frac{1-\omega^{5(n+1)}}{1-\omega^{n+1}} \quad \text{provided that } n+1 \not\equiv_5 0.$$

Specifying $n+1$ with $5m+5$, we have

$$\sum_{m=0}^{\infty} p(5m+4)q^{5m+5} \;=\; \frac{1}{5}\sum_{k=0}^{4} \frac{q\omega^k}{\left(q\omega^k; q\omega^k\right)_\infty}.$$

Replacing q by $p := q^{1/5}$, we can reformulate the last equation as

$$\sum_{m=0}^{\infty} p(5m+4)q^{m} \;=\; \frac{1}{5q}\sum_{k=0}^{4} \frac{p\omega^k}{\left(p\omega^k; p\omega^k\right)_\infty}. \tag{H2.3}$$

H2.2. In order to evaluate the sum displayed in (H2.3), we first show that:

$$\frac{p(q^5;q^5)_\infty}{(p;p)_\infty} \;=\; \frac{1}{\lambda/p - 1 - p/\lambda} \tag{H2.4}$$

where λ is an infinite factorial fraction defined by

$$\lambda := \frac{\left[q^2,\, q^3;\, q^5\right]_\infty}{\left[q,\, q^4;\, q^5\right]_\infty}. \tag{H2.5}$$

Recall the Euler pentagon number theorem:

$$(p;p)_\infty \;=\; \sum_{j=-\infty}^{+\infty} (-1)^j p^{\frac{j(3j+1)}{2}}.$$

It is easy to verify that the pentagon numbers admit only three residue classes modulo 5:

$$\frac{j(3j+1)}{2} \equiv_5 \begin{cases} 0, & j = 0, -2 \pmod 5 \\ 1, & j = -1 \pmod 5 \\ 2, & j = 1,\, 2 \pmod 5). \end{cases}$$

We can accordingly write

$$(p;p)_\infty \;=\; A - pB - p^2C. \tag{H2.6}$$

The coefficients A, B and C can be individually determined by means of Jacobi's triple and the quintuple product identities.

A**-Coefficient:** Specifying the summation index j with $5j$ and $-2-5j$, we can compute A, by means of the quintuple product identity as follows:

$$\begin{aligned} A &= \sum_{j=-\infty}^{+\infty}(-1)^j\Big\{p^{\frac{5j(15j+1)}{2}} + p^{\frac{(5j+2)(15j+5)}{2}}\Big\} \\ &= \sum_{j=-\infty}^{+\infty}(-1)^j\big\{1+p^{5+25j}\big\}p^{75\binom{j}{2}+40j} \\ &= \big[p^{25}, -p^5, -p^{20}; p^{25}\big]_\infty\big[p^{35}, p^{15}; p^{50}\big]_\infty \\ &= \big[q^5, -q, -q^4; q^5\big]_\infty\big[q^7, q^3; q^{10}\big]_\infty. \end{aligned}$$

B**-Coefficient:** It can be evaluated through the Jacobi triple product identity as follows:

$$\begin{aligned} B &= p^{-1}\sum_{j=-\infty}^{+\infty}(-1)^jp^{\frac{(5j-1)(15j-2)}{2}} = \sum_{j=-\infty}^{+\infty}(-1)^jp^{75\binom{j}{2}+25j} \\ &= \big[p^{75}, p^{25}, p^{50}; p^{75}\big]_\infty = \big[q^{15}, q^5, q^{10}; q^{15}\big] = \big[q^5; q^5\big]_\infty. \end{aligned}$$

C**-Coefficient:** Similar to the computation of A, we can compute C, by specifying the summation index j with $5j+1$ and $5j+2$, as follows:

$$\begin{aligned} C &= p^{-2}\sum_{j=-\infty}^{+\infty}(-1)^j\Big\{p^{\frac{(5j+1)(15j+4)}{2}} - p^{\frac{(5j+2)(15j+7)}{2}}\Big\} \\ &= \sum_{j=-\infty}^{+\infty}(-1)^jp^{75\binom{j}{2}+55j}\big\{1-p^{5+15j}\big\} \\ &= \big[p^{25}, -p^{10}, -p^{15}; p^{25}\big]_\infty\big[p^{45}, p^5; p^{50}\big]_\infty \\ &= \big[q^5, -q^2, -q^3; q^5\big]_\infty\big[q^9, q; q^{10}\big]_\infty. \end{aligned}$$

In accordance with (H2.6), we find the following relation

$$\frac{(p;p)_\infty}{(q^5;q^5)_\infty} = \frac{A}{B} - p - p^2\frac{C}{B}$$

where the coefficient-fractions can be simplified as follows:

$$\begin{aligned}
\frac{A}{B} &= \big[-q,-q^4;q^5\big]_\infty \times \big[q^7,q^3;q^{10}\big]_\infty \\
&= \frac{\big[q^2,q^3,q^7,q^8;q^{10}\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty} = \frac{\big[q^2,q^3;q^5\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty} \\
\frac{C}{B} &= \big[-q^2,-q^3;q^5\big]_\infty \times \big[q,q^9;q^{10}\big]_\infty \\
&= \frac{\big[q,q^4,q^6,q^9;q^{10}\big]_\infty}{\big[q^2,q^3;q^5\big]_\infty} = \frac{\big[q,\ q^4;q^5\big]_\infty}{\big[q^2,q^3;q^5\big]_\infty}.
\end{aligned}$$

Observing further that

$$\lambda := \frac{A}{B} = \frac{B}{C} = \frac{\big[q^2,q^3;q^5\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty}$$

we can reformulate (H2.6) as the following reduced expression

$$\frac{(p;p)_\infty}{(q^5;q^5)_\infty} = \lambda - p - \frac{p^2}{\lambda}$$

which is equivalent to

$$\frac{p(q^5;q^5)_\infty}{(p;p)_\infty} = \frac{1}{\lambda/p - 1 - p/\lambda}.$$

H2.3. For the sum displayed in (H2.3), we then compute the common denominator:

$$\prod_{k=0}^{4}(p\omega^k;p\omega^k)_\infty = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}. \tag{H2.7}$$

In fact, the general term of the product with index n reads as

$$\prod_{k=0}^{4}\big\{1-(p\omega^k)^n\big\} = \prod_{k=0}^{4}(1-p^n\omega^{kn}) = \begin{cases}(1-p^n)^5, & n \equiv_5 0\\ (1-p^{5n}), & n \not\equiv_5 0.\end{cases}$$

Therefore we have the following simplified product

$$\begin{aligned}
\prod_{k=0}^{4}(p\omega^k;p\omega^k)_\infty &= \prod_{n=1}^{\infty}\prod_{k=0}^{4}(1-p^n\omega^{kn}) = \prod_{\substack{n=1\\ n\not\equiv_5 0}}^{\infty}(1-p^{5n})\prod_{n=1}^{\infty}(1-p^{5n})^5 \\
&= \prod_{\substack{n=1\\ n\not\equiv_5 0}}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^5 = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}, \quad p := q^{1/5}.
\end{aligned}$$

This can be stated equivalently as the product of λ-polynomials:

$$\prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda} = (q^5;q^5)_\infty^5\prod_{k=0}^{4}\frac{p\omega^k}{(p\omega^k;p\omega^k)_\infty} = q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}.$$

H2.4. Performing replacement $p\to p\omega^\ell$ in (H2.4) and then summing both sides over $0\le\ell\le 4$, we have

$$\sum_{\ell=0}^{4}\frac{p\omega^\ell}{(p\omega^\ell;p\omega^\ell)_\infty} = \frac{1}{(q^5;q^5)_\infty}\sum_{\ell=0}^{4}\frac{1}{\lambda/(p\omega^\ell)-1-p\omega^\ell/\lambda}. \tag{H2.8}$$

For the sum on the right hand side, there holds the following closed form:

$$\sum_{\ell=0}^{4}\frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} = \frac{25}{\prod_{k=0}^{4}\left\{\lambda/p\omega^k-1-p\omega^k/\lambda\right\}}. \tag{H2.9}$$

Then the generating function (H2.1) can be derived from (H2.3) consequently as follows:

$$\begin{aligned}\sum_{m=0}^{\infty}p(5m+4)q^m &= \frac{1}{5q}\sum_{k=0}^{4}\frac{p\omega^k}{(p\omega^k;p\omega^k)_\infty}\\
&= \frac{1}{5q(q^5;q^5)_\infty}\sum_{\ell=0}^{4}\frac{1}{\lambda/(p\omega^\ell)-1-p\omega^\ell/\lambda}\\
&= \frac{5}{q(q^5;q^5)_\infty}\prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda}\\
&= \frac{5}{q(q^5;q^5)_\infty}\times\frac{q(q^5;q^5)_\infty^6}{(q;q)_\infty^6} = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.\end{aligned}$$

H2.5. In order to show the algebraic identity (H2.9), we first reformulate it as follows:

$$\begin{aligned}\sum_{\ell=0}^{4}\frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} &= \prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda}\\
&\times\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\left\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\right\}.\end{aligned}$$

If we can show that the last sum on the right hand side equals 25, then the generating function (H2.1) will be confirmed.

Surprisingly enough, it is true that the sum just mentioned is indeed equal to 25:

$$\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\} \;=\; 25. \tag{H2.10}$$

As the Laurent polynomial in p, we can expand the product

$$\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\} \;=\; \sum_{\kappa=-4}^{4} W(\kappa)\, p^{\kappa}$$

where $\{W(\kappa)\}$ are constants independent of p. Keeping in mind that for each ℓ with $0\le\ell\le4$, the residues of $\{k+\ell\}_{k=1}^{4}$ modulo 5 are, in effect, $\{0\le k\le4\}_{k\neq\ell}$, we can accordingly simplify the sum displayed in (H2.10) as follows:

$$\begin{aligned}
&\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\\
=\;&\sum_{\ell=0}^{4}\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\Big|_{p\to p\omega^\ell}\\
=\;&\sum_{\ell=0}^{4}\sum_{\kappa=-4}^{4} W(\kappa)p^{\kappa}\omega^{\kappa\ell} \;=\; \sum_{\kappa=-4}^{4} W(\kappa)\,p^{\kappa}\sum_{\ell=0}^{4}\omega^{\kappa\ell}\\
=\;&5W(0) \;=\; 5[p^0]\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}.
\end{aligned}$$

Recalling two simple facts about ω with $\omega=e^{\frac{2\pi}{5}\sqrt{-1}}$

$$\prod_{k=1}^{4}\omega^k \;=\; +1 \quad\text{and}\quad \sum_{k=1}^{4}\omega^k \;=\; -1$$

we can compute $W(0)$, by matching the powers of p, as follows:

$$\begin{aligned}
W(0) &= [p^0]\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\\
&= (-1)^4-(-1)^2\sum_{i=1}^{4}\omega^{-i}\sum_{j\neq i}\omega^j+\sum_{1\le i<j\le 4}\omega^{-2i-2j}\\
&= 1-\sum_{i=1}^{4}\omega^{-i}\{-1-\omega^i\}+\frac{1}{2}\sum_{i\neq j}\omega^{-2i-2j}\\
&= 1+\sum_{i=1}^{4}\{1+\omega^{-i}\}+\frac{1}{2}\Big\{\Big(\sum_{k=1}^{4}\omega^{-2k}\Big)^2-\sum_{k=1}^{4}\omega^{-4k}\Big\}\\
&= 1+(4-1)+\frac{1}{2}\big\{(-1)^2+1\big\} \;=\; 5.
\end{aligned}$$

H2.6. Partial fraction method for (H2.9)**.** The algebraic identity (H2.9) can also be demonstrated by means of partial fraction method.

Define the quadratic polynomial by

$$\vartheta(\lambda,p) \;=\; \lambda^2-\lambda p-p^2 \;=\; \big\{\lambda-\lambda_1(p)\big\}\times\big\{\lambda-\lambda_2(p)\big\}$$

where two zeros are given explicitly by

$$\lambda_1(p)=\frac{p}{2}(1+\sqrt{5}) \quad\text{and}\quad \lambda_2(p)=\frac{p}{2}(1-\sqrt{5}).$$

Then we have the partial fraction decomposition

$$25q\prod_{k=0}^{4}\frac{\lambda}{\vartheta(\lambda,p\omega^k)} = \frac{25\lambda^5q}{\big\{\lambda^5-\lambda_1^5(p)\big\}\times\big\{\lambda^5-\lambda_2^5(p)\big\}} \tag{H2.11a}$$

$$= \frac{25}{\prod_{k=0}^{4}\Big\{\lambda/p\omega^k-1-p\omega^k/\lambda\Big\}} \tag{H2.11b}$$

$$= \sum_{\ell=0}^{4}\Big\{\frac{u_\ell}{\lambda-\lambda_1(p\omega^\ell)}+\frac{v_\ell}{\lambda-\lambda_2(p\omega^\ell)}\Big\} \tag{H2.11c}$$

where the coefficients u_ℓ and v_ℓ remain to be determined.

By means of the L'Hôspital rule, we can compute the u_ℓ-coefficient:

$$\begin{aligned} u_\ell &= \lim_{\lambda\to\lambda_1(p\omega^\ell)} \frac{25q\lambda^5\{\lambda-\lambda_1(p\omega^\ell)\}}{\{\lambda^5-\lambda_1^5(p)\}\times\{\lambda^5-\lambda_2^5(p)\}} \\ &= \frac{25q\lambda_1^5(p\omega^\ell)}{\lambda_1^5(p)-\lambda_2^5(p)} \lim_{\lambda\to\lambda_1(p\omega^\ell)} \frac{\lambda-\lambda_1(p\omega^\ell)}{\lambda^5-\lambda_1^5(p)} \\ &= \frac{5q\lambda_1(p\omega^\ell)}{\lambda_1^5(p)-\lambda_2^5(p)} = \frac{\lambda_1(p\omega^\ell)}{\sqrt{5}} \end{aligned}$$

where we have simplified the difference

$$\lambda_1^5(p)-\lambda_2^5(p) = 5q\sqrt{5}.$$

Similarly, we can also determine the v_ℓ-coefficient:

$$\begin{aligned} v_\ell &= \lim_{\lambda\to\lambda_2(p\omega^\ell)} \frac{25q\lambda^5\{\lambda-\lambda_2(p\omega^\ell)\}}{\{\lambda^5-\lambda_1^5(p)\}\times\{\lambda^5-\lambda_2^5(p)\}} \\ &= \frac{25q\lambda_2^5(p\omega^\ell)}{\lambda_2^5(p)-\lambda_1^5(p)} \lim_{\lambda\to\lambda_2(p\omega^\ell)} \frac{\lambda-\lambda_2(p\omega^\ell)}{\lambda^5-\lambda_2^5(p)} \\ &= \frac{5q\lambda_2(p\omega^\ell)}{\lambda_2^5(p)-\lambda_1^5(p)} = -\frac{\lambda_2(p\omega^\ell)}{\sqrt{5}}. \end{aligned}$$

Combining two summand terms in (H2.11c) into a single one

$$\begin{aligned} \frac{u_\ell}{\lambda-\lambda_1(p\omega^\ell)} + \frac{v_\ell}{\lambda-\lambda_2(p\omega^\ell)} &= \frac{\lambda}{\sqrt{5}} \frac{\lambda_1(p\omega^\ell)-\lambda_2(p\omega^\ell)}{\vartheta(\lambda,p\omega^\ell)} \\ = \frac{\lambda p\omega^\ell}{\vartheta(\lambda,p\omega^\ell)} &= \frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} \end{aligned}$$

we establish the algebraic identity

$$\sum_{\ell=0}^{4} \frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} = \frac{25}{\prod_{k=0}^4\Big\{\lambda/p\omega^k-1-p\omega^k/\lambda\Big\}}$$

which is exactly (H2.9) as desired.

H2.7. There exists a polynomial expression of the common denominator in terms of λ:

$$\prod_{k=0}^{4}(\lambda/(p\omega^k)-1-p\omega^k/\lambda) = \lambda^5/q-11-q/\lambda^5.$$

In fact, replacing λ/p by y, we can restate the product as

$$\prod_{k=0}^{4}(y/\omega^k-1-\omega^k/y).$$

Noticing that the zeros of the first factor $y-1-1/y$ are solutions of equation $y-1/y=1$ and so solutions of equation

$$(y-1/y)^3 = (y^3-1/y^3) - 3(y-1/y) = 1$$

which is equivalent to

$$y^3 - 1/y^3 = 4.$$

Furthermore, these zeros of $y-1-1/y$ are also solutions of equation

$$(y-1/y)^5 = y^5 - 1/y^5 - 5(y^3-1/y^3) + 10(y-1/y) = 1.$$

The last equation reads in fact as the following simplified form

$$y^5 - 1/y^5 = 11$$

which implies therefore that $y^5 - 11 - 1/y^5$ is a multiple of $y-1-1/y$.

Noting that $y^5 - 11 - 1/y^5$ is invariant under $y \to y/\omega^k$ for $k = 0, 1, 2, 3, 4$, we deduce that it is also a multiple of the product $\prod_{k=0}^{4}(y/\omega^k - 1 - \omega^k/y)$. Hence we have established the following equation

$$\prod_{k=0}^{4}(y/\omega^k - 1 - \omega^k/y) = y^5 - 11 - 1/y^5$$

thanks for the fact that both sides are *monic polynomials* of the same degree.

H3. Proof of $p(7n+5) \equiv 0 \pmod 7$

There holds the Ramanujan congruence modulo 7:

$$p(7n+5) \;\equiv\; 0 \pmod 7. \tag{H3.1}$$

According to the congruence lemma on partition function, we should show that the coefficients of q^{7m} in the formal power series expansion of $q^2(q;\,q)_\infty^6$ are divisible by 7 for all $m \in \mathbb{N}$.

By means of the limiting version of the Jacobi triple product identity, consider the formal power series expansion

$$\begin{aligned} q^2(q;\,q)_\infty^6 &= q^2\prod_{m=1}^{\infty}(1-q^m)^3\prod_{n=1}^{\infty}(1-q^n)^3 \\ &= \sum_{i,j=0}^{+\infty}(-1)^{i+j}\,(1+2i)(1+2j)\,q^{2+\binom{1+i}{2}+\binom{1+j}{2}}. \end{aligned}$$

Observe that the congruence relation on the q-exponent

$$\begin{aligned} 0 &\equiv_7 2+\binom{1+i}{2}+\binom{1+j}{2} \\ &\equiv_7 8\Big\{2+\binom{1+i}{2}+\binom{1+j}{2}\Big\} \\ &\equiv_7 (1+2i)^2+(1+2j)^2 \end{aligned}$$

can be reached only when

$$\begin{aligned} (1+2i)^2 \equiv_7 0 &\implies i \equiv_7 3 \\ (1+2j)^2 \equiv_7 0 &\implies j \equiv_7 3 \end{aligned}$$

because the corresponding residues modulo 7 read as

$$(1+2k)^2 \equiv_7 0,1,2,4.$$

The coefficients of q^{7m} in the formal power series expansion of $q^2(q;q)_\infty^6$ are therefore divisible by 7. This completes the proof of congruence (H3.1).

H4. Generating function for $p(7n+6)$

Ramanujan discovered also explicitly the generating function.

$$\sum_{n=0}^{\infty} p(7n+6)\,q^n = 7\frac{(q^7;\,q^7)_\infty^3}{(q;\,q)_\infty^4} + 49q\frac{(q^7;\,q^7)_\infty^7}{(q;\,q)_\infty^8}. \tag{H4.1}$$

Following the same line to the proof of (H2.1), we present a derivation of this generating function, which is much more difficult.

H4.1. Let $\rho := q^{1/7}$ and ϖ be a 7-th primitive root of unit $\varpi = e^{\frac{2\pi}{7}\sqrt{-1}}$.

Recall the generating function of partitions

$$\frac{q^2}{(q;\,q)_\infty} = q^2\sum_{n=0}^{\infty} p(n)q^n.$$

Replacing q by $q\varpi^k$ and then summing the equations with k over $0 \le k \le 6$, we have

$$\sum_{k=0}^{6} \frac{q^2\varpi^{2k}}{(q\varpi^k;q\varpi^k)_\infty} = \sum_{n=0}^{\infty} p(n)q^{n+2}\sum_{k=0}^{6}\varpi^{k(n+2)}.$$

It is not hard to verify that

$$\sum_{k=0}^{6} \varpi^{k(n+2)} = \begin{cases} 7, & n+2 \equiv_7 0 \\ 0, & n+2 \not\equiv_7 0 \end{cases} \tag{H4.2}$$

where the last line is justified by the finite geometric series

$$\sum_{k=0}^{6} \varpi^{k(n+2)} = \frac{1-\varpi^{7(n+2)}}{1-\varpi^{n+2}} \quad \text{provided that } n+2 \not\equiv_7 0.$$

Specifying $n+2$ with $7m+7$, we have

$$\sum_{m=0}^{\infty} p(7m+5)q^{7m+7} = \frac{1}{7}\sum_{k=0}^{6} \frac{q^2\varpi^{2k}}{(q\varpi^k; q\varpi^k)_\infty}.$$

Replacing q by $\rho := q^{1/7}$, we can reformulate the last equation as

$$\sum_{m=0}^{\infty} p(7m+5)q^{m} = \frac{1}{7q}\sum_{k=0}^{6} \frac{\rho^2\varpi^{2k}}{(\rho\varpi^k; \rho\varpi^k)_\infty}. \tag{H4.3}$$

H4.2. In order to simplify the sum displayed in (H4.3), we show that:

$$\frac{\rho^2(q^7;q^7)_\infty}{(\rho;\rho)_\infty} = \frac{1}{A/\rho^2 - B/\rho - 1 + \rho^3/AB} \tag{H4.4}$$

where A and B are two infinite factorial fractions defined by

$$A := \left[\begin{matrix} q^2,\ q^5 \\ q,\ q^6 \end{matrix} \,\middle|\, q^7\right]_\infty \quad \text{and} \quad B := \left[\begin{matrix} q^3,\ q^4 \\ q^2,\ q^5 \end{matrix} \,\middle|\, q^7\right]_\infty. \tag{H4.5}$$

Recall again the Euler pentagon number theorem:

$$(\rho;\rho)_\infty = \sum_{j=-\infty}^{+\infty} (-1)^j \rho^{\frac{j(3j+1)}{2}}.$$

It is easy to verify that the pentagon numbers admit only four residue classes modulo 7:

$$\frac{j(3j+1)}{2} \equiv_7 \begin{cases} 0, & j = 0, 2 \pmod 7 \\ 1, & j = 3, 6 \pmod 7 \\ 2, & j = 1 \pmod 7 \\ 5, & j = 4, 5 \pmod 7. \end{cases}$$

We can accordingly write

$$(\rho;\rho)_\infty = C_0 - \rho C_1 - \rho^2 C_2 + \rho^5 C_5. \tag{H4.6}$$

The coefficients C_0, C_1, C_2 and C_5 can be individually determined by means of Jacobi's triple and the quintuple product identities.

C_0-Coefficient: Specifying the summation index j with $7n$ and $7n+2$, we can compute C_0, by means of the quintuple product identity as follows:

$$\begin{aligned} C_0 &= \sum_{n=-\infty}^{+\infty} (-1)^n \big\{1+\rho^{7(1+6n)}\big\}\rho^{147\binom{n}{2}+77n} \\ &= \big[\rho^{49}, -\rho^{7}, -\rho^{42}; \rho^{49}\big]_\infty \big[\rho^{63}, \rho^{35}; \rho^{98}\big]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q^2,\ q^5 \\ q,\ q^6 \end{matrix} \,\Big|\, q^7\right]_\infty = A \times (q^7;q^7)_\infty. \end{aligned}$$

C_1-Coefficient: Similar to the computation of C_0, we can compute C_1, by specifying the summation index j with $7n-1$ and $7n+3$, as follows:

$$\begin{aligned} C_1 &= \sum_{n=-\infty}^{+\infty} (-1)^n \big\{1+\rho^{14(1+6n)}\big\}\rho^{147\binom{n}{2}+56n} \\ &= \big[\rho^{49}, -\rho^{14}, -\rho^{35}; \rho^{49}\big]_\infty \big[\rho^{77}, \rho^{21}; \rho^{98}\big]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q^3,\ q^4 \\ q^2,\ q^5 \end{matrix} \,\Big|\, q^7\right]_\infty = B \times (q^7;q^7)_\infty. \end{aligned}$$

C_2-Coefficient: It can be evaluated through the Jacobi triple product identity with $j = 1+7n$ as follows:

$$\begin{aligned} C_2 &= \sum_{n=-\infty}^{+\infty} (-1)^n \rho^{147\binom{n}{2}+98n} \\ &= \big[\rho^{147}, \rho^{49}, \rho^{98}; \rho^{147}\big]_\infty \\ &= \big[q^{21}, q^{7}, q^{14}; q^{21}\big]_\infty = (q^7;q^7)_\infty. \end{aligned}$$

C_5-Coefficient: Similar to the computation of C_0 and C_1, we can evaluate C_5, by specifying the summation index j with $-7n-2$ and $-7n-3$, as follows:

$$\begin{aligned} C_5 &= \sum_{n=-\infty}^{+\infty} (-1)^n \big\{1-\rho^{7(1+3n)}\big\}\rho^{147\binom{n}{2}+112n} \\ &= \big[\rho^{49}, -\rho^{21}, -\rho^{28}; \rho^{49}\big]_\infty \big[\rho^{91}, \rho^{7}; \rho^{98}\big]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q,\ q^6 \\ q^3,\ q^4 \end{matrix} \,\Big|\, q^7\right]_\infty = \frac{(q^7;q^7)_\infty}{AB}. \end{aligned}$$

In accordance with (H4.6), we find the following relation

$$\frac{(\rho;\rho)_\infty}{\rho^2(q^7;q^7)_\infty} = A/\rho^2 - B/\rho - 1 + \rho^3/AB$$

which is equivalent to (H4.4):

$$\frac{\rho^2(q^7;q^7)_\infty}{(\rho;\rho)_\infty} = \frac{1}{A/\rho^2 - B/\rho - 1 + \rho^3/AB}.$$

H4.3. Replacing $\rho \to \rho\varpi^\ell$ in (H4.4) and then summing both sides over $0 \le \ell \le 6$, we can express the generating function defined by (H4.3) as

$$\begin{aligned}\sum_{m=0}^{\infty} p(7m+5)q^m &= \frac{1}{7q}\sum_{\ell=0}^{6}\frac{\rho^2\varpi^{2\ell}}{(\rho\varpi^\ell;\rho\varpi^\ell)_\infty}\\ &= \frac{1}{7q(q^7;q^7)_\infty}\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell} - B/\rho\varpi^\ell - 1 + \rho^3\varpi^{3\ell}/AB}.\end{aligned}$$

Observing that for each ℓ with $0 \le \ell \le 6$, the residues of $\{k+\ell\}_{k=1}^{6}$ modulo 7 are $\{0 \le k \le 6\}_{k\neq\ell}$, we can accordingly reformulate the sum as follows:

$$\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell} - B/\rho\varpi^\ell - 1 + \rho^3\varpi^{3\ell}/AB} \tag{H4.7a}$$

$$= \sum_{\ell=0}^{6}\prod_{\substack{k=0\\k\neq\ell}}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\} \tag{H4.7b}$$

$$\div \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\} \tag{H4.7c}$$

$$= \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^\ell} \tag{H4.7d}$$

$$\div \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\}. \tag{H4.7e}$$

Let "nn" and "dd" stand for the sum and the product displayed in (H4.7d) and (H4.7e) respectively. We shall reduce these algebraic expressions and find a functional equation between them.

H4.4. As the Laurent polynomial in ρ, we can expand the product displayed in (H4.7d) as follows:

$$\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\} = \sum_{\kappa=-12}^{18} U(\kappa)\,\rho^\kappa$$

where $\{U(\kappa)\}$ are constants independent of ρ. The sum displayed in (H4.7d) can be accordingly reduced to

$$\begin{aligned}
\mathrm{nn} \quad &:= \quad \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^\ell} \\
&= \quad \sum_{\ell=0}^{6}\sum_{\kappa=-12}^{18} U(\kappa)\,\rho^\kappa\,\varpi^{\kappa\ell} \;=\; \sum_{\kappa=-12}^{18} U(\kappa)\,\rho^\kappa \sum_{\ell=0}^{6}\varpi^{\kappa\ell} \\
&= \quad 7\Big\{U(0) + qU(7) + q^{-1}U(-7) + q^2U(14)\Big\}.
\end{aligned}$$

Similarly, we expand the denominator "dd" as a Laurent polynomial in ρ:

$$\prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\} \;=\; \sum_{\kappa=-14}^{21} V(\kappa)\,\rho^\kappa.$$

Noting that the product is invariant under replacement $\rho \to \rho\varpi^\ell$ with $\ell \in \mathbb{Z}$, we can reduce the expression to following:

$$\begin{aligned}
\mathrm{dd} := &\prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^k - 1 + \rho^3\varpi^{3k}/AB\Big\} \\
= \; &V(0) + qV(7) + q^{-1}V(-7) + q^2V(14) + q^{-2}V(-14) + q^3V(21).
\end{aligned}$$

Analogously to the reasoning on the determination of the $W(0)$-coefficient in the proof of the generating function (H2.1), one can respectively compute (manually or by computer algebra) the coefficients for numerator

$$U(0) \quad = \quad 8 + 3\frac{B^2}{A} - 4\frac{A}{B^2} \tag{H4.8a}$$

$$U(7) \quad = \quad -\frac{3}{A^2B^3} - \frac{4}{A^3B} \tag{H4.8b}$$

$$U(-7) \quad = \quad AB^5 - \frac{A^4}{B} + 3A^3B + 4A^2B^3 \tag{H4.8c}$$

$$U(14) \quad = \quad \frac{1}{A^5B^4} \tag{H4.8d}$$

and the coefficients for denominator

$$\begin{aligned}
V(0) &= -8+14\frac{A}{B^2} && \text{(H4.9a)}\\
V(7) &= \frac{14}{A^3B} && \text{(H4.9b)}\\
V(-7) &= 7\frac{A^4}{B}-14A^2B^3-7AB^5-B^7 && \text{(H4.9c)}\\
V(14) &= -\frac{7}{A^5B^4} && \text{(H4.9d)}\\
V(-14) &= A^7 && \text{(H4.9e)}\\
V(21) &= \frac{1}{A^7B^7}. && \text{(H4.9f)}
\end{aligned}$$

They lead us to the polynomial expression for numerator

$$\begin{aligned}
\text{nn} &= \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^\ell} && \text{(H4.10a)}\\
&= 7\Big\{8+3\frac{B^2}{A}-4\frac{A}{B^2}-3\frac{q}{A^2B^3}-4\frac{q}{A^3B} && \text{(H4.10b)}\\
&\quad+\frac{AB^5}{q}-\frac{A^4}{Bq}+3\frac{A^3B}{q}+4\frac{A^2B^3}{q}+\frac{q^2}{A^5B^4}\Big\} && \text{(H4.10c)}
\end{aligned}$$

and the polynomial expression for denominator

$$\begin{aligned}
\text{dd} &= \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\} && \text{(H4.11a)}\\
&= -8+14\frac{A}{B^2}+14\frac{q}{A^3B}+7\frac{A^4}{Bq}-14\frac{A^2B^3}{q} && \text{(H4.11b)}\\
&\quad-7\frac{AB^5}{q}-\frac{B^7}{q}-7\frac{q^2}{A^5B^4}+\frac{A^7}{q^2}+\frac{q^3}{A^7B^7}. && \text{(H4.11c)}
\end{aligned}$$

H4.5. In order to simplify the polynomial expressions for numerator "nn" and denominator "dd", we prove the following astonishing algebraic equation:

$$A^3B-A^2B^3 = q. \tag{H4.12}$$

Recalling the definition of A and B in (H4.5), we can restate the equation as

$$\begin{aligned}
&\langle q^2;q^7\rangle_\infty\,\langle q^2;q^7\rangle_\infty\,\langle q^2;q^7\rangle_\infty\,\langle q^4;q^7\rangle_\infty\\
-\;&\langle q;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\\
=\;&q\,\langle q^5;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty
\end{aligned}$$

which follows immediately from the q-difference equation stated in Theorem G5.2 under parameter specification $Q = q^7$, $b = q$, $c = d = e = q^3$ and $A = q^5$.

With the help of algebraic equation $q = A^3B - A^2B^3$, we can simplify further "nn" and "dd" as the following polynomial expressions

$$\mathrm{nn} = \frac{7^2A}{Bq}\Big\{A^2B^2 - \frac{q}{B} + q\frac{B^3}{A^2}\Big\} \tag{H4.13a}$$

$$= 7^3 + \frac{7^2A}{Bq}\Big\{8AB^4 - 5A^2B^2 - A^3 - B^6\Big\} \tag{H4.13b}$$

$$\mathrm{dd} = \frac{A^2}{B^2q^2}\Big\{8AB^4 - 5A^2B^2 - A^3 - B^6\Big\}^2. \tag{H4.13c}$$

H4.6. In order to determine generating function explicitly, we need an alternative expression for denominator "dd" in terms of infinite shifted factorial fraction.

Observing that the general term of the product with index n reads as

$$\prod_{k=0}^{6}\{1-(\rho\varpi^k)^n\} = \prod_{k=0}^{6}(1-\rho^n\varpi^{kn}) = \begin{cases}(1-\rho^n)^7, & n \equiv_7 0\\ (1-\rho^{7n}), & n \not\equiv_7 0\end{cases}$$

we have therefore the following simplified product

$$\begin{aligned}\prod_{k=0}^{6}(\rho\varpi^k;\rho\varpi^k)_\infty &= \prod_{n=1}^{\infty}\prod_{k=0}^{6}\Big\{1-(\rho\varpi^k)^n\Big\}\\ &= \prod_{\substack{n=1\\ n\not\equiv_7 0}}^{\infty}(1-\rho^{7n})\prod_{n=1}^{\infty}(1-\rho^{7n})^7\\ &= \prod_{\substack{n=1\\ n\not\equiv_7 0}}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^7\end{aligned}$$

which can restated as the following identity:

$$\prod_{k=0}^{6}(\rho\varpi^k;\rho\varpi^k)_\infty = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty}. \tag{H4.14}$$

In view of (H4.4), this gives also another expression for the common denominator:

$$\begin{aligned}\frac{1}{\mathrm{dd}} &= \prod_{k=0}^{6}\frac{1}{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB}\\ &= (q^7;q^7)_\infty^7\prod_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = q^2\frac{(q^7;q^7)_\infty^8}{(q;q)_\infty^8}.\end{aligned}$$

H4.7. By comparing (H4.13b) with (H4.13c), we find that

$$\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell}-B/\rho\varpi^\ell-1+\rho^3\varpi^{3\ell}/AB} \tag{H4.15a}$$

$$= \frac{\mathrm{nn}}{\mathrm{dd}} = \frac{7^3}{\mathrm{dd}}+\frac{7^2}{\sqrt{\mathrm{dd}}}. \tag{H4.15b}$$

Substituting the factorial expression for "dd" in the last fraction, we can finally determine the generating function

$$\begin{aligned}\sum_{m=0}^{\infty}p(7m+5)q^m &= \frac{1}{7q}\sum_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = \frac{1}{7q(q^7;q^7)_\infty}\frac{\mathrm{nn}}{\mathrm{dd}}\\ &= \frac{1}{7q(q^7;q^7)_\infty}\left\{\frac{7^3}{\mathrm{dd}}+\frac{7^2}{\sqrt{\mathrm{dd}}}\right\}\\ &= 7\frac{(q^7;\,q^7)_\infty^3}{(q;\,q)_\infty^4}+49q\frac{(q^7;\,q^7)_\infty^7}{(q;\,q)_\infty^8}.\end{aligned}$$

If we combine (H4.13a) with (H4.13c), we would get another expression of the generating function

$$\begin{aligned}\sum_{m=0}^{\infty}p(7m+5)q^m &= \frac{1}{7q}\sum_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = \frac{1}{7q(q^7;q^7)_\infty}\frac{\mathrm{nn}}{\mathrm{dd}}\\ &= 7\frac{(q^7;\,q^7)_\infty^7}{(q;\,q)_\infty^8}\left\{A^3B-\frac{qA}{B^2}+q\frac{B^2}{A}\right\}\end{aligned}$$

where A and B are shifted factorial fractions given by (H4.5).

Naturally, this form is less elegant that stated in (H4.1). However, it confirms again the Ramanujan congruence modulo 7.

H5. Proof of $p(11n+6) \equiv 0 \pmod{11}$

There holds the Ramanujan congruence modulo 11:

$$p(11n+6) \;\equiv\; 0 \pmod{11}. \tag{H5.1}$$

Recalling the congruence lemma on partition function, we should show that the coefficients of q^{11m} in the formal power series expansion of $q^5(q;\,q)_\infty^{10}$ are divisible by 11 for all $m \in \mathbb{N}$. The simplest proof of this congruence is due to Winquist (1969), which is based on the following formal power series expansion formula:

$$6q^5(q;q)_\infty^{10} \;=\; \sum_{i,j}(-1)^{i+j}(3i-3j-1)(3i+3j-2)^3 q^{3\binom{i}{2}+3\binom{j}{2}+j+5}. \tag{H5.2}$$

H5.1. If the q-exponent in the double sum is a multiple of 11, then we have the following congruence relation

$$\begin{aligned} 0 \;&\equiv_{11}\; 5+j+3\binom{i}{2}+3\binom{j}{2} \\ &\equiv_{11}\; 8\Big\{5+j+3\binom{i}{2}+3\binom{j}{2}\Big\} \\ &\equiv_{11}\; (i-6)^2+(j-2)^2. \end{aligned}$$

This can be reached only when

$$\begin{aligned} (i-6)^2 \equiv_{11} 0 \quad &\Longrightarrow \quad i \equiv_{11} 6 \\ (j-2)^2 \equiv_{11} 0 \quad &\Longrightarrow \quad j \equiv_{11} 2 \end{aligned}$$

in view of the following table on the quadratic residues modulo 11:

$k \pmod{11}$	0	±1	±2	±3	±4	±5
$k^2 \pmod{11}$	0	1	4	9	5	3

The coefficients corresponding to $i \equiv_{11} 6$ and $j \equiv_{11} 2$ are divisible by 11^4 because they contain two factors displayed in (H5.2):

$$\begin{aligned} 3i-3j-1 \;&\equiv_{11}\; 18-6-1 \equiv_{11} 0 \\ 3i+3j-2 \;&\equiv_{11}\; 18+6-2 \equiv_{11} 0. \end{aligned}$$

Therefore the coefficients of q^{11m} in the formal power series expansion of $q^5(q;q)_\infty^{10}$ are divisible by 11.

In order to complete the proof of congruence (H5.1), it remains to show the infinite series identity (H5.2).

H5.2. Define the bivariate function $F(x,y)$ by the following product of ten infinite shifted factorials:

$$F(x,y) \;:=\; (q;q)_\infty^2 \langle x;q\rangle_\infty \langle y;q\rangle_\infty \langle xy;q\rangle_\infty \langle x/y;q\rangle_\infty. \tag{H5.3}$$

We can expand it formally as a Laurent series in x

$$F(x,y) \;=\; \sum_{k=-\infty}^{+\infty} \gamma_k(y)\, x^k.$$

It is trivial to check the functional equation

$$F(x,y) \;=\; -x^3\, F(qx,y)$$

which corresponds to the recurrence relation

$$\gamma_{k+3}(y) \;=\; -q^k\, \gamma_k(y).$$

Iterating this relation for k-times, we find that there exist three formal power series $A(y)$, $B(y)$ and $C(y)$ such that there hold

$$\begin{aligned}
\gamma_{3k}(y) &= -q^{3k-3}\,\gamma_{3k-3}(y) \;=\; (-1)^k q^{3\binom{k}{2}}\, A(y)\\
\gamma_{3k+1}(y) &= -q^{3k-2}\,\gamma_{3k-2}(y) \;=\; (-1)^k q^{3\binom{k}{2}+k}\, B(y)\\
\gamma_{3k+2}(y) &= -q^{3k-1}\,\gamma_{3k-1}(y) \;=\; (-1)^k q^{3\binom{k}{2}+2k}\, C(y).
\end{aligned}$$

Therefore $F(x,y)$ can be written as

$$F(x,y) \;=\; A(y) \sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}}\, x^{3k} \tag{H5.4a}$$

$$+\; B(y) \sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}+k}\, x^{3k+1} \tag{H5.4b}$$

$$+\; C(y) \sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}+2k}\, x^{3k+2}. \tag{H5.4c}$$

Again from the definition of $F(x,y)$, it is easy to verify another functional equation

$$F(x,y) \;=\; -x^3\, F(1/x,y)$$

which can be translated into the following

$$\begin{aligned}
F(x,y) &= A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}}\,x^{3-3k}\\
&+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+k}\,x^{2-3k}\\
&+ C(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+2k}\,x^{1-3k}.
\end{aligned}$$

The reversal of the bilateral series just displayed reads as

$$F(x,y) = A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}q^{3\binom{k}{2}}\,x^{3k} \tag{H5.5a}$$

$$+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+2k}\,x^{2+3k} \tag{H5.5b}$$

$$+ C(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+k}\,x^{1+3k}. \tag{H5.5c}$$

Comparing both expansions (H5.4) and (H5.5) of $F(x,y)$, we find that $B(y) = -C(y)$. This allows us to restate $F(x,y)$ as follows:

$$F(x,y) = A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}q^{3\binom{k}{2}}\,x^{3k} \tag{H5.6a}$$

$$+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}\Big\{x^{1+3k}-x^{2-3k}\Big\}q^{3\binom{k}{2}+k} \tag{H5.6b}$$

where the formal power series $A(y)$ and $B(y)$ remain to be determined.

H5.3. By means of the Jacobi triple product identity, the last expansion for $F(x,y)$ can be reformulated as

$$\begin{aligned}
F(x,y) &= A(y)\ \left[q^3,\ x^3,\ q^3/x^3;q^3\right]_\infty\\
&+ x\,B(y)\left[q^3,\ qx^3,\ q^2/x^3;q^3\right]_\infty\\
&- x^2\,B(y)\left[q^3,\ q^2x^3,\ q/x^3;q^3\right]_\infty.
\end{aligned}$$

Putting $x = q^{1/3}$ in the last equation and then recalling the definition of $F(x,y)$, we find that

$$\begin{aligned} A(y) + q^{1/3}\,B(y) &= \frac{F(q^{1/3},y)}{(q;q)_\infty} = \Big[q^{1/3},\ y,\ q^{1/3}/y; q^{1/3}\Big]_\infty \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k\, q^{\frac{1}{3}\binom{k}{2}}\, y^k. \end{aligned}$$

Based on the binomial congruences

$$\binom{k}{2} \equiv_3 \begin{cases} 0, & k \equiv_3 \ \ 0 \\ 0, & k \equiv_3 +1 \\ 1, & k \equiv_3 -1 \end{cases}$$

we can determine $A(y)$ and $B(y)$ respectively as follows:

$$\begin{aligned} A(y) &= \sum_{k=-\infty}^{+\infty} (-1)^k \Big\{ q^{\frac{1}{3}\binom{3k}{2}} y^{3k} - q^{\frac{1}{3}\binom{3k+1}{2}} y^{3k+1} \Big\} \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k \Big\{ y^{3k} - y^{1-3k} \Big\} q^{3\binom{k}{2}+k} \\ B(y) &= -\sum_{k=-\infty}^{+\infty} (-1)^k q^{\frac{1}{3}\binom{3k-1}{2}-\frac{1}{3}} y^{3k-1} \\ &= -\sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}} y^{3k-1}. \end{aligned}$$

We therefore have the following bivariate formal power series expression

$$F(x,y) = \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}} x^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ y^{3j} - y^{1-3j} \Big\} q^{3\binom{j}{2}+j} \qquad \text{(H5.7a)}$$

$$- \frac{x}{y} \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}} y^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ x^{3j} - x^{1-3j} \Big\} q^{3\binom{j}{2}+j}. \qquad \text{(H5.7b)}$$

H5.4. Define further the bivariate function by formal power series

$$G(x,y) = \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}}\, x^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ y^{3j} - y^{1-3j} \Big\} q^{3\binom{j}{2}+j}. \qquad \text{(H5.8)}$$

Then $F(x,y)$ can be expressed as a skew-symmetric function of x and y:

$$y\,F(x,y) = y\,G(x,y) - x\,G(y,x). \qquad \text{(H5.9)}$$

Recalling the definition of $F(x,y)$, we have

$$\lim_{y\to x} \frac{yF(x,y)}{y-x} = (q;q)_\infty^4 \langle x;q\rangle_\infty^2 \langle x^2;q\rangle_\infty.$$

In view of the symmetric property, we write

$$yG(x,y) - xG(y,x) = \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\left\{\begin{matrix} x^{3i}(y^{1+3j} - y^{2-3j}) \\ -y^{3i}(x^{1+3j} - x^{2-3j}) \end{matrix}\right\}$$

which permits us to compute the corresponding limit:

$$\begin{aligned}
\lim_{y\to x}\frac{yF(x,y)}{y-x} &= \lim_{y\to x}\frac{yG(x,y)-xG(y,x)}{y-x} \\
&= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\lim_{y\to x}\frac{\left\{\begin{matrix} x^{3i}(y^{1+3j} - y^{2-3j}) \\ -y^{3i}(x^{1+3j} - x^{2-3j}) \end{matrix}\right\}}{y-x} \\
&= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\left\{\begin{matrix} (1+3j-3i)\,x^{3i+3j} \\ +(3i+3j-2)x^{1+3i-3j} \end{matrix}\right\} \\
&= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}(1+3j-3i)\left\{x^{3i+3j}-x^{4-3i-3j}\right\}
\end{aligned}$$

where the last line follows from the index involution $i \to 1-i$ on double sums.

Therefore we have established the following expansion formula:

$$(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty = \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j} \tag{H5.10a}$$

$$\times\quad (1+3j-3i)\left\{x^{3i+3j} - x^{4-3i-3j}\right\}. \tag{H5.10b}$$

Multiplying across by x^{-2}, we can rewrite (H5.10) as

$$\begin{aligned}
\frac{(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty}{x^2} &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j} \\
&\times\quad (1+3j-3i)\left\{x^{3i+3j-2} - x^{2-3i-3j}\right\}.
\end{aligned}$$

Applying the derivative operator $\frac{x\partial}{\partial x}$ for three times at $x=1$, we find that

$$6(q;\,q)_\infty^{10} = \sum_{i,j}(-1)^{i+j}(3i-3j-1)(3i+3j-2)^3\,q^{3\binom{i}{2}+3\binom{j}{2}+j}.$$

This is exactly the formal power series expansion (H5.2), which has played the key role in the proof of congruence (H5.1).

H5.5. The crucial identity (H5.10) due to Winquist (1969) can alternatively be proved by means of the quintuple product identity.

$$\begin{aligned}(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\\ &\times \quad (1+3j-3i)\Big\{x^{3i+3j}-x^{4-3i-3j}\Big\}.\end{aligned}$$

The strategy is to simplify the double sum on the right hand side and then reduce it to the product form on the left hand side.

Performing the replacement on summation indices:

$$\left.\begin{aligned} j+i=m\\ j-i=n\end{aligned}\right\} \quad \rightleftharpoons \quad \left.\begin{aligned} i=\frac{m-n}{2}\\ j=\frac{m+n}{2}\end{aligned}\right\} \quad m\equiv_2 n$$

then we can reformulate the double sum as

$$\sum_{m\equiv_2 n}(-1)^m q^{3\binom{(m+n)/2}{2}+3\binom{(m-n)/2}{2}+\frac{m+n}{2}}(1+3n)\Big\{x^{3m}-x^{4-3m}\Big\} \tag{H5.11}$$

where the double sum runs over $-\infty < m,\ n < +\infty$ with m and n having the same parity.

H5.6. Recall the quintuple product identities

$$\begin{aligned}[q,\,z,\,q/z;q]_\infty\left[qz^2,\,q/z^2;q^2\right]_\infty &= \sum_{k=-\infty}^{+\infty}\Big\{1-z^{1+6k}\Big\}\,q^{3\binom{k}{2}}\,\left(q^2/z^3\right)^k\\ &= \sum_{k=-\infty}^{+\infty}\Big\{1-(q/z^2)^{1+3k}\Big\}q^{3\binom{k}{2}}\left(qz^3\right)^k\end{aligned}$$

and their limiting forms:

$$\begin{aligned}\sum_{k=-\infty}^{+\infty}(1+6k)\,q^{3\binom{k}{2}+2k} &= [q,\,q,\,q;q]_\infty\left[q,\,q;q^2\right]_\infty\\ \sum_{k=-\infty}^{+\infty}(1+3k)q^{3\binom{k}{2}+\frac{5}{2}k} &= \left[q,\,q^{1/2},q^{1/2};q\right]_\infty\left[q^2,q^2;q^2\right]_\infty.\end{aligned}$$

We can evaluate the double sum (H5.11) with both m and n being even as

$$\begin{aligned}
&\sum_{m,\,n} q^{3\binom{m+n}{2}+3\binom{m-n}{2}+m+n}(1+6n)\Big\{x^{6m}-x^{4-6m}\Big\}\\
=\;& \sum_{m} q^{6\binom{m}{2}+m}\Big\{x^{6m}-x^{4-6m}\Big\}\sum_{n}\big(1+6n\big)\,q^{6\binom{n}{2}+4n}\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\sum_{m} q^{6\binom{m}{2}}\Big\{1-x^{4(1+3m)}\Big\}\big(q^5/x^6\big)^m\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\left[q^2,qx^2,q/x^2;q^2\right]_\infty\left[q^4/x^4,x^4;q^4\right]_\infty.
\end{aligned}$$

The double sum (H5.11) with both m and n being odd can be reduced similarly to the product:

$$\begin{aligned}
&\sum_{m,\,n} q^{3\binom{m+n+1}{2}+3\binom{m-n}{2}+m+n+1}(4+6n)\Big\{x^{1-6m}-x^{3+6m}\Big\}\\
=\;& \sum_{m} q^{6\binom{m}{2}+4m}\Big\{x^{3+6m}-x^{1-6m}\Big\}\sum_{n}\big(2+6n\big)\,q^{6\binom{n}{2}+5n}\\
=\;& -2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\sum_{m} q^{6\binom{m}{2}}\Big\{1-x^{2(1+6m)}\Big\}\big(q^4/x^6\big)^m\\
=\;& -2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\left[q^2,x^2,q^2/x^2;q^2\right]_\infty\left[q^2x^4,q^2/x^4;q^4\right]_\infty.
\end{aligned}$$

Their sum leads the identity (H5.10) equivalently to the following equation:

$$\begin{aligned}
&(q;q)^4_\infty\langle x;q\rangle^2_\infty\langle x^2;q\rangle_\infty\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\left[q^2,qx^2,q/x^2;q^2\right]_\infty\left[q^4/x^4,x^4;q^4\right]_\infty\\
-\;& 2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\left[q^2,x^2,q^2/x^2;q^2\right]_\infty\left[q^2x^4,q^2/x^4;q^4\right]_\infty.
\end{aligned}$$

We can reduce it by canceling the common factors to the following equivalent q-difference equation:

$$[q,\ x,\ q/x;q]^2_\infty=(q^2;q^2)^2_\infty\left\{\begin{matrix}\left[-q,\ -q,\ -x^2,\ -q^2/x^2;\ q^2\right]_\infty\\ -2x\left[-q^2,-q^2,-qx^2,-q/x^2;q^2\right]_\infty\end{matrix}\right\}\tag{H5.12}$$

whose terms can be reorganized, for convenience, as follows:

$$\begin{aligned}
&\left[-q,-q,-x^2,-q^2/x^2;q^2\right]_\infty-(q;q^2)^2_\infty\left[x,\ q/x;q\right]^2_\infty\\
=\;& x\left[-1,-q^2,-qx^2,-q/x^2;q^2\right]_\infty.
\end{aligned}$$

Rewriting the last identity as

$$\begin{aligned}
&\langle x\sqrt{-1};q\rangle_\infty\,\langle -x\sqrt{-1};q\rangle_\infty\,\langle q^{1/2}\sqrt{-1};q\rangle_\infty\,\langle q^{1/2}\sqrt{-1};q\rangle_\infty && \text{(H5.13a)}\\
-\;&\langle x;q\rangle_\infty\,\langle x;q\rangle_\infty\,\langle q^{1/2};q\rangle_\infty\,\langle -q^{1/2};q\rangle_\infty && \text{(H5.13b)}\\
=\;& x\,\langle\sqrt{-1};q\rangle_\infty\,\langle-\sqrt{-1};q\rangle_\infty\,\langle q^{1/2}x\sqrt{-1};q\rangle_\infty\,\langle q^{1/2}\sqrt{-1}/x;q\rangle_\infty && \text{(H5.13c)}
\end{aligned}$$

we can see without difficulty that it is the special case $b=c=x$, $d=q^{1/2}$, $e=-q^{1/2}$ and $A=q^{1/2}x\sqrt{-1}$ of the identity stated in Theorem G5.2.

This completes the proof of (H5.10). □

H5.7. The identity (H5.12) can also be proved directly.

In fact, by means of the Jacobi triple product identity, its right hand side can be expanded as

$$\text{RHS(H5.12)} = \sum_{i,\,j} q^{i^2+j^2}\Big\{q^{-j}x^{2j} - q^{-i}x^{1+2j}\Big\}.$$

Interchanging two summation indices i and j for the first part and then letting $k := j-i$, we can manipulate the last double sum as follows:

$$\begin{aligned}
\text{RHS(H5.12)} &= \sum_{i,\,j} q^{i^2+j^2-i}\Big\{x^{2i} - x^{1+2j}\Big\}\\
&= \sum_{k} q^{k^2}\Big\{1 - x^{1+2k}\Big\}\sum_{i} q^{4\binom{i}{2}+(1+2k)i}x^{2i}\\
&= \sum_{k} q^{k^2}\Big\{1 - x^{1+2k}\Big\}\left[q^4, -q^{1+2k}x^2, -q^{3-2k}/x^2; q^4\right]_\infty.
\end{aligned}$$

The last triple product can be restated as

$$\begin{aligned}
&\left[q^4, -q^{1+2k}x^2, -q^{3-2k}/x^2; q^4\right]_\infty\\
=\ &\begin{cases} x^{-2\ell}q^{-4\binom{\ell}{2}-\ell}\left[q^4, -qx^2, -q^3/x^2; q^4\right]_\infty, & k = 2\ell;\\ x^{-2\ell}q^{-4\binom{\ell}{2}-3\ell}\left[q^4, -q^3x^2, -q/x^2; q^4\right]_\infty, & k = 2\ell+1.\end{cases}
\end{aligned}$$

Now reformulating the k-sum according to the parity of k, we can express it as a combination of two infinite series:

$$\begin{aligned}
\text{RHS(H5.12)} &= \left[q^4, -qx^2, -q^3/x^2; q^4\right]_\infty \sum_{\ell} q^{4\binom{\ell}{2}+3\ell}x^{-2\ell}\Big\{1 - x^{1+4\ell}\Big\}\\
&+ q\left[q^4, -q^3x^2, -q/x^2; q^4\right]_\infty \sum_{\ell} q^{4\binom{\ell}{2}+5\ell}x^{-2\ell}\Big\{1 - x^{3+4\ell}\Big\}.
\end{aligned}$$

By feeding back the parity of k, we can evaluate the first ℓ-sum as follows:

$$\begin{aligned}
&\sum_{\ell} q^{4\binom{\ell}{2}+3\ell}x^{-2\ell}\Big\{1 - x^{1+4\ell}\Big\}\\
=\ &\sum_{\ell}\Big\{q^{2\ell^2-\ell}x^{2\ell} - q^{2\ell^2+\ell}x^{1+2\ell}\Big\}\\
=\ &\sum_{k}(-1)^k\, q^{\binom{k}{2}}\, x^k = [q, x, q/x; q]_\infty.
\end{aligned}$$

The second ℓ-sum can be reduced similarly as follows:

$$\begin{aligned}
&\sum_{\ell} q^{4\binom{\ell}{2}+5\ell} x^{-2\ell}\Big\{1 - x^{3+4\ell}\Big\}\\
=\ &\sum_{\ell}\Big\{q^{2\ell^2-3\ell}x^{2\ell} - q^{2\ell^2+3\ell}x^{3+2\ell}\Big\}\\
=\ &\sum_{k}(-1)^k\, q^{\binom{k}{2}}\,\big(x/q\big)^k \;=\; -x/q\,[q,x,q/x;q]_\infty\,.
\end{aligned}$$

Combining these expressions, we arrive at the final assault

$$\begin{aligned}
\mathrm{RHS(H5.12)} &= [q,x,q/x;q]_\infty \begin{Bmatrix} \big[q^4,-qx^2,-q^3/x^2;q^4\big]_\infty \\ -x\,\big[q^4,-q^3x^2,-q/x^2;q^4\big]_\infty \end{Bmatrix}\\
&= [q,x,q/x;q]_\infty \sum_{\ell}\Big\{q^{2\ell^2-\ell}x^{2\ell} - q^{2\ell^2+\ell}x^{1+2\ell}\Big\}\\
&= [q,x,q/x;q]_\infty \sum_{k}(-1)^k\, q^{\binom{k}{2}}\, x^k\\
&= [q,x,q/x;q]_\infty^2 \;=\; \mathrm{LHS(H5.12)}.
\end{aligned}$$

This completes the proof of (H5.12).

Appendix: Tannery's Limiting Theorem

In this monograph, we have frequently refered to the Tannery theorem. This theorem deals with the limiting process on infinite series, which can be reproduced as follows.

For a given infinite series $\{v_k(n)\}_{k\geq 0}$, suppose that the series satisfies the following conditions:

- For any fixed k, there holds $\lim\limits_{n\to\infty} v_k(n) = w_k$;
- For any $k \in \mathbb{N}_0$, we have $|v_k(n)| \leq M_k$ with M_k being independent of n and the series $\sum\limits_{k=0}^{\infty} M_k$ is convergent.

Then we have the following limit relation:

$$\lim_{n\to\infty} \sum_{k=0}^{m(n)} v_k(n) \;=\; \sum_{k=0}^{\infty} w_k \;=\; W$$

where $m(n)$ is an increasing integer valued function which tends steadily to infinity as n does.

Proof. For any given $\varepsilon > 0$, first choose a number $\ell = \ell(\varepsilon)$ such that $\sum\limits_{k=\ell}^{\infty} M_k < \varepsilon$ and then let n be taken large enough to make $m(n) > \ell$. This leads us consequently to the following inequality: $\Big|\sum\limits_{k=\ell}^{m(n)} v_k\Big| \leq \sum\limits_{k=\ell}^{m(n)} M_k < \varepsilon.$

Noting also that $\Big|\sum_{k=\ell}^{\infty} w_k\Big| \le \sum_{k=\ell}^{\infty} M_k < \varepsilon$, we can estimate the difference

$$\begin{aligned}\Big|\sum_{k=0}^{m(n)} v_k(n) - W\Big| \quad &\le \quad \Big|\sum_{k=\ell}^{m(n)} v_k(n)\Big| + \Big|\sum_{k=\ell}^{\infty} w_k\Big| + \Big|\sum_{k=0}^{\ell-1} \big\{v_k(n) - w_k\big\}\Big| \\ &< \quad 2\varepsilon + \Big|\sum_{k=0}^{\ell-1} \big\{v_k(n) - w_k\big\}\Big|.\end{aligned}$$

Remember that so far n has only been restricted by the condition $m(n) > \ell$. Since ℓ is independent of n, we can allow n to tend to infinity and obtain

$$\lim_{n\to\infty} \sum_{k=0}^{\ell-1} \big\{v_k(n) - w_k\big\} \;=\; 0 \quad \text{for} \quad \lim_{n\to\infty} v_k(n) = w_k \quad \text{with } k \text{ being fixed.}$$

Hence we have found that for any $\varepsilon > 0$, there holds

$$\lim_{n\to\infty} \Big|\sum_{k=0}^{m(n)} v_k(n) - W\Big| < 2\varepsilon$$

which implies the limit relation:

$$\lim_{n\to\infty} \sum_{k=0}^{m(n)} v_k(n) \;=\; W \;=\; \sum_{k=0}^{\infty} w_k$$

as anticipated in the Tannery theorem. $\square$

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