Chapter VI

Maximal subgroups of the finite classical groups

Here the main references are [1], [2] and [15].

1 Some preliminary facts

(1.1) Definition *Let $1 \neq G$ be a group. A subgroup M of G is said to be* maximal *if $M \neq G$ and there exists no subgroup H such that $M < H < G$.*

If G is finite, by order reasons every subgroup $H \neq G$ is contained in a maximal subgroup. If M is maximal in G, then also every conjugate gMg^{-1} of M in G is maximal. Indeed

$$gMg^{-1} < K < G \implies M < g^{-1}Kg < G.$$

For this reason the maximal subgroups are studied up to conjugation.

(1.2) Lemma *Let $G = G'$ and let M be a maximal subgroup of G. Then:*

(1) *M contains the center Z of G;*

(2) *$\frac{M}{Z}$ is maximal in $\frac{G}{Z}$;*

(3) *the preimage in G of every maximal subgroup of $\frac{G}{Z}$ is maximal in G.*

Proof

(1) Suppose $Z \not\leq M$. Then $M < ZM$ gives $ZM = G$, by the maximality of M. Hence M is normal in G and the factor group $\frac{G}{M}$ is abelian. In fact:

$$\frac{G}{M} = \frac{ZM}{M} \cong \frac{Z}{M \cap Z}.$$

It follows $G' \leq M$, a contradiction, as we are assuming $G' = G$.
Points (2) and (3) follow from the fact that the subgroups of $\frac{G}{Z}$ are those of the form $\frac{K}{Z}$, where K is a subgroup of G which contains Z. ■

(1.3) Lemma *If* $Z(G) = \{1\}$ *then* G *is isomorphic to a subgroup of* $\mathrm{Aut}(G)$.

Proof For every $g \in G$ the map $\gamma : G \to G$ defined by $x \mapsto gxg^{-1}$ is an automorphism of G (called *inner*). Consider the homomorphism $\varphi : G \to \mathrm{Aut}(G)$ defined by: $g \mapsto \gamma$. $\mathrm{Ker}\ \varphi = Z(G)$. Thus, under our assumption, $G \cong \varphi(G) \leq \mathrm{Aut}(G)$. ■

2 Aschbacher's Theorem

Let $\overline{G}_0$ be one of the following groups, with the further assumption that it is simple:

$$\mathrm{PSL}_n(q),\ \mathrm{PSU}_n(q^2),\ \mathrm{PSp}_{2m}(q),\ P\Omega^{\pm}_{2m}(q),\ P\Omega_{2m+1}(q).$$

Suppose that $\overline{G}$ is a group such that $\overline{G}_0 \lhd \overline{G} \leq \mathrm{Aut}(\overline{G}_0)$. By the subgroup structure theorem due to Aschbacher, every maximal subgroup $\overline{H}$ of $\overline{G}$, not containing $\overline{G}_0$, belongs to a class in the table below:

Rough description of the classes of maximal subgroups

$\mathcal{C}_1$	Stabilizers of subspaces	
$\mathcal{C}_2$	Stabilizers of decompositions $V = \oplus_{i=1}^{t} V_i$,	$\dim V_i = m$
$\mathcal{C}_3$	Stabilizers of prime degree extension fields of $\mathbb{F}_q$	
$\mathcal{C}_4$	Stabilizers of tensor decompositions $V = V_1 \otimes V_2$	
$\mathcal{C}_5$	Stabilizers of prime index subfields of $\mathbb{F}_q$	
$\mathcal{C}_6$	Normalisers of symplectic $-$ type $r -$ groups, $(r, q) = 1$	
$\mathcal{C}_7$	Stabilizers of decompositions $\otimes_{i=1}^{t} V_i$,	$\dim V_i = m$
$\mathcal{C}_8$	Classical subgroups	
$\mathcal{S}$	Almost simple absolutely irreducible subgroups	
$\mathcal{N}$	Novelty subgroups	

The 8 classes $\mathcal{C}_i = \mathcal{C}_i(\overline{G})$ consist of "natural" subgroups of $\overline{G}$, which can be described in geometric terms. Class $\mathcal{N}$ exists only for $\overline{G}_0 = P\Omega_8^{\pm}(p^a)$ or $\overline{G}_0 = \mathrm{PSp}_{2m}(2^a)'$ (see [4]). We will describe the structure of the groups in some of these classes in the case:

$$\overline{G} = \overline{G}_0 = \mathrm{PSL}_n(q).$$

It is easier to describe the linear preimages of such groups. To this purpose we set $V = \mathbb{F}^n$, with canonical basis $\{e_1, \ldots, e_n\}$, and $G = \mathrm{SL}_n(q)$.

3 The reducible subgroups $\mathcal{C}_1$

If W is a subspace of V, then its *stabilizer* $G_W := \{g \in G \mid gW = W\}$ is a subgroup of G. If W' is a subspace of V and $\dim W = \dim W'$, there exists $g \in G$ such that $gW = W'$. It follows that $G_{W'} = gG_Wg^{-1}$. So, if W is a subspace of dimension m, up to conjugation we may suppose:

$$W = \langle e_1, \ldots, e_m \rangle, \quad G_W = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid \det(C) = \det(A)^{-1} \right\}.$$

To see its structure we factorize G_W as follows:

$$G_W = U\ C_{q-1}\ (\mathrm{SL}_m(q) \times \mathrm{SL}_{n-m}(q)) \tag{3.1}$$

where

$$U = \left\{ \begin{pmatrix} I_m & B \\ 0 & I_{n-m} \end{pmatrix} \mid B \in \mathrm{Mat}_{m,\, n-m}(q) \right\} \cong (\mathbb{F}_q, +)^{m(n-m)}$$

$U \lhd G_W$,

$$C_{q-1} = \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & I_{m-1} & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & I_{n-m-1} \end{pmatrix} \mid \alpha \in \mathbb{F}_q^* \right\} \cong (\mathbb{F}_q^*, \cdot)$$

cyclic, and

$$\mathrm{SL}_m(q) \times \mathrm{SL}_{n-m}(q) = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid X \in \mathrm{SL}_m(q), Y \in \mathrm{SL}_{n-m}(q) \right\}.$$

Actually we may suppose $m \leq \frac{n}{2}$ since, considering the transpose of G_W, namely

$$G_W^T = \left\{ \begin{pmatrix} A & 0 \\ B^T & C \end{pmatrix} \mid \det(C) = \det(A)^{-1} \right\}$$

we obtain the stabilizer of a subspace of dimension $n-m \geq \frac{n}{2}$, namely of:

$$\langle e_{m+1}, \ldots, e_n \rangle .$$

(3.2) Definition *The groups in class $\mathcal{C}_1$ are called* parabolic subgroups.

They are the only subgroups in the classes $\mathcal{C}_i$, $1 \leq i \leq 8$, which contain a Sylow p-subgroup of $\mathrm{SL}_n(q)$, $q = p^a$. When W is chosen as above, the Sylow p-subgroup consists of the upper unitriangular matrices, namely:

$$\begin{pmatrix} 1 & * & \ldots & * \\ 0 & 1 & \ldots & * \\ & & \ldots & * \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

4 The imprimitive subgroups $\mathcal{C}_2$

Let $n = mt$, $1 \leq m < n$ and consider a decomposition $\mathcal{D}$ of V as a direct sum

$$V = V_1 \oplus \cdots \oplus V_t$$

of t subspaces V_i, all of the same dimension m.

(4.1) Definition *The stabilizer $N_{\mathrm{GL}_n(q)}(\mathcal{D})$ of the above decomposition is the subgroup of G which permutes the spaces V_i among themselves, i.e.,*

$$N_{\mathrm{GL}_n(q)}(\mathcal{D}) := \{ g \in G \mid gV_i = V_j,\ 1 \leq i,j \leq t \} .$$

We study first the structure of $N_{\mathrm{GL}_n(q)}(\mathcal{D})$. Up to conjugation we may assume:

$$V_1 = \langle e_1, \ldots, e_m \rangle ,\ \ldots,\ \ V_t = \left\langle e_{(t-1)m+1}, \ldots, e_n \right\rangle .$$

For each $g \in N_{\mathrm{GL}_n(q)}(\mathcal{D})$, let φ_g be the permutation induced by g on the set $\{V_1, \ldots, V_t\}$. The map

$$\begin{array}{rccc} \varphi : & N_{\mathrm{GL}_n(q)}(\mathcal{D}) & \rightarrow & \mathrm{Sym}(t) \\ & g & \mapsto & \varphi_g \end{array}$$

is a homomorphism and

$$\mathrm{Ker}\,\varphi \;=\; \bigcap_{i=1}^{t} G_{V_i} = \left\{ \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ \ldots & \ldots & \ldots & \\ & & & A_t \end{pmatrix} \mid A_i \in \mathrm{GL}_m(q) \right\} \;\cong\; \mathrm{GL}_m(q)^t.$$

Denote by H the subgroup of $\mathrm{GL}_t(q)$ consisting of all permutation matrices.

Then the group:

$$\widehat{H} := H \otimes I_m = \{h \otimes I_m \mid h \in H\} \ \leq \ \mathrm{GL}_n(q)$$

permutes the V_i-s in all possible ways. Hence $\widehat{H} \leq N_{\mathrm{GL}_n(q)}(\mathcal{D})$ and

$$\varphi(\widehat{H}) = \mathrm{Sym}(t).$$

It follows:

$$N_{\mathrm{GL}_n(q)}(\mathcal{D}) = (\mathrm{Ker}\,\varphi) \ \varphi\left(\widehat{H}\right) \ \cong \ \mathrm{GL}_m(q)^t\,\mathrm{Sym}(t) \ = \ \mathrm{GL}_m(q) \wr \mathrm{Sym}(t).$$

Finally we have to determine $N_G(\mathcal{D}) = N_{\mathrm{GL}_n(q)}(\mathcal{D}) \cap \mathrm{SL}_n(q)$. To this purpose, let

$$\sigma = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I_{n-2} \end{pmatrix}.$$

Then $\langle \sigma, \mathrm{Alt}(t)\rangle$ is a subgroup of $N_G(\mathcal{D})$ which maps onto $\mathrm{Sym}(t)$. It follows that

$$N_G(\mathcal{D}) = (\mathrm{Ker}\varphi \cap \mathrm{SL}_n(q))\, \langle \sigma, \mathrm{Alt}(t)\rangle\,.$$

Note that $\mathrm{Ker}\,\varphi \ \cap \mathrm{SL}_n(q)$ can be factorized as the product of the group:

$$\left\{\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ \cdots & \cdots & \cdots & \\ & & & B_t \end{pmatrix} \mid B_i \in \mathrm{SL}_m(q)\right\} \ \cong \ \mathrm{SL}_m(q)^t$$

and the group

$$\left\{\begin{pmatrix} \mathrm{diag}(\alpha_1,\dots,1) & & & \\ & \mathrm{diag}(\alpha_2,\dots,1) & & \\ & & \cdots & \\ & & & \mathrm{diag}\left((\prod_{i=1}^{t-1}\alpha_i)^{-1},\dots,1\right) \end{pmatrix} \mid \alpha_i \in \mathbb{F}_q^*\right\}$$

is isomorphic to $\left(C_{q-1}\right)^{t-1}$. Thus:

$$\frac{N_G(\mathcal{D})}{\mathrm{SL}_m(q)^t \ \left(C_{q-1}\right)^{t-1}} \ \cong \ \mathrm{Sym}(t).$$

Equivalently:

$$N_G(\mathcal{D}) \ = \ \mathrm{SL}_m(q)^t \ \left(C_{q-1}\right)^{t-1}.\mathrm{Sym}(t) \quad (\mathrm{non-split\ extension}).$$

(4.2) Remark *For $m = 1$, the subgroup $N_{\mathrm{GL}_n(q)}(\mathcal{D})$ coincides with the standard monomial subgroup.*

5 The irreducible subgroups $\mathcal{C}_3$

(5.1) Lemma *Let $\mathbb{K}$ be a subfield of the field $\mathbb{F}$. Two matrices $A, B \in \mathrm{Mat}_n(\mathbb{K})$ are conjugate under $\mathrm{GL}_n(\mathbb{K})$ if and only if they are conjugate under $\mathrm{GL}_n(\mathbb{F})$.*

Proof The rational canonical forms C_A e C_B of A and B respectively lie in $\mathrm{Mat}_n(\mathbb{K})$. If A, B are conjugate under $\mathrm{GL}_n(\mathbb{F})$, we have $C_A = C_B$. Hence A and B are conjugate also under $\mathrm{GL}_n(\mathbb{K})$, having the same rational canonical form. The converse is obvious. ■

(5.2) Lemma $\mathrm{Mat}_n(q)$ *contains a self-centralizing subalgebra $R \cong \mathbb{F}_{q^n}$. Moreover*

$$\frac{N_{\mathrm{GL}_n(q)}(R)}{C_{\mathrm{GL}_n(q)}(R)} \cong \mathrm{Gal}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong C_n \text{ (cyclic group of order } n).$$

Proof Let $p(t)$ be an irreducible polynomial of degree n in $\mathbb{F}_q[t]$. Denoting by A its companion matrix, we obtain the subring:

$$\mathbb{F}_q[A] = \mathbb{F}_q I_n + \mathbb{F}_q A + \cdots + \mathbb{F}_q A^{n-1} \cong \frac{\mathbb{F}_q[t]}{\langle p(t) \rangle} \cong \mathbb{F}_{q^n}.$$

Since $\mathbb{F}_q^n$ is an irreducible A-module, the centralizer C of A in $\mathrm{Mat}_n(q)$ is a field. The multiplicative group $C \setminus \{0\}$ is generated by a matrix $B \in \mathrm{Mat}_n(q)$. Since the minimal polynomial of B has degree $\leq n$, the dimension of C over $\mathbb{F}_q$ does not exceed n. We conclude that $C = \mathbb{F}_q[A]$. Thus we take $R = \mathbb{F}_q[A]$.

The Jordan form of A in $\mathrm{Mat}_n(q^n)$ is $J_A = \mathrm{diag}\left(\epsilon, \epsilon^q, \ldots, \epsilon^{q^{n-1}}\right)$ where ϵ is a root of $p(t)$ in $\mathbb{F}_{q^n}$. It follows that J_A is conjugate to $(J_A)^q$ in $\mathrm{GL}_n(q^n)$. By the previous Lemma, there exists $g \in \mathrm{GL}_n(q)$ such that $g^{-1}Ag = A^q$. Clearly g normalizes R. Moreover the automorphism $\gamma : R \to R$ such that $X \mapsto g^{-1}Xg$ for all $X \in R$, has order n. Hence it generates the Galois group $\mathrm{Gal}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$.

Finally, let y be an element of the normalizer of R in $\mathrm{GL}_n(q)$. The map $\nu : R \to R$ such that $X \to y^{-1}Xy$ for all $X \in R$, is a field automorphism. The scalar matrices, which form the subfield of R of order q, are fixed by ν. We conclude that $\nu \in \mathrm{Gal}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. ■

The subgroups of class $\mathcal{C}_3$ are $N(R) \cap \mathrm{SL}_n(q)$, where $N(R)$ is defined as in the previous Lemma.

6 Groups in class $\mathcal{S}$

They arise from absolutely irreducible representations of simple groups. We give only some examples.

6.1 The Suzuki groups $Sz(q)$ in $\mathrm{Sp}_4(q)$

The Suzuki groups ${}^2B_2(q) = Sz(q)$ are simple groups of order $q^2(q-1)(q^2+1)$, with $q = 2^{2r+1}$, $r \geq 1$. They were discovered by M.Suzuki in 1960. $Sz(q)$ was originally defined as the subgroup of $\mathrm{SL}_4(2^{2r+1})$ generated by:

$$T := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

and by the groups:

$$Q := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha^r & 1 & 0 & 0 \\ \beta & \alpha & 1 & 0 \\ \alpha^{2r+1} + \alpha^r\beta + \beta^{2r} & \alpha^{r+1} + \beta & \alpha^r & 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{F}_q \right\}. \tag{6.2}$$

T and Q fix the symplectic form T. Hence $Sz(q)$ is a subgroup of $\mathrm{Sp}_4(q)$, with respect to T. For $q \geq 8$ it is a maximal subgroup.

6.2 Representations of $\mathrm{SL}_2(\mathbb{F})$

Let $\mathbb{F}$ be a field of characteristic $p \geq 0$ and V be the vector space of homogeneous polynomials in two variables x, y, of degree $d-1$, over $\mathbb{F}$. Every matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{F})$$

acts in a natural way on the basis $\mathcal{B} = \left\{x^{d-1}, x^{d-2}y, \ldots, y^{d-1}\right\}$ of V, via:

$$x^i y^j \mapsto (a_{11}x + a_{21}y)^i \, (a_{12}x + a_{22}y)^j \, .$$

Call $\alpha : V \to V$ the extension by linearity of this action. The homomorphism

$$h_d : \mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_d(\mathbb{F}) \tag{6.3}$$

such that each $A \in \mathrm{SL}_2(\mathbb{F})$ maps to the matrix of α with respect to $\mathcal{B}$, is a representation of degree d of $\mathrm{SL}_2(\mathbb{F})$. This representation is absolutely irreducible whenever $0 < d \leq p$ (see also [3]). When d is even and $\mathbb{F} = \mathbb{F}_q$, with q appropriate, it gives rise to maximal subgroups of $\mathrm{Sp}_d(q)$.

(6.4) Example *For $d = 4$, the homomorphism $h_4 : \mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_4(\mathbb{F})$ acts as:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{pmatrix}. \tag{6.5}$$

7 Exercises

(7.1) Exercise *Let W and W' be subspaces of $\mathbb{F}^n$. Show that there exists $g \in \mathrm{SL}_n(\mathbb{F})$ such that $gW = W'$ if and only if they have the same dimension.*

(7.2) Exercise *In $\mathrm{Mat}_3(7)$ find a field of order 7^3, its centralizer and its normalizer.*

(7.3) Exercise *Show that the representation (6.5) fixes a symplectic form.*

(7.4) Exercise *Write explicitly an absolutely irreducible representation of $\mathrm{SL}_2(7)$ of degree 6, fixing a symplectic form.*