CHAPTER 3: SOME PRELIMINARY RESULTS AND THE MAIN LEMMA

Having prepared the way in the preceding chapter, we now undertake a formal proof of Theorem 1.9. As a starting point, it seems convenient to bring together various notations and definitions already met on the preceding pages.

3.1. In the following, $\mathbb{R}^n$ will denote Euclidean n-dimensional space over the real numbers $\mathbb{R}$, endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$; n is an integer not less than 2. Points in $\mathbb{R}^n$ will be denoted by x,y,z; measurable sets by E,F,G; compact sets by K; open sets by A and Ω; open balls by B. When we want to specify the center x and the radius t of B, then we write $B_{x,t}$. Projection of points or sets in $\mathbb{R}^n$ onto the first n-1 variables will always be denoted by a "prime", such as x',A',B', and so on. Hence, in particular, we have $x = (x',x_n)$ and $B' = \{y'\in\mathbb{R}^{n-1} : |y'-x'| < t\}$ if $B = B_{x,t}$. The symbol "0" however, will denote the origin of both $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ (and, of course, the real number "zero"): which one of them, will be clear from the context. We shall also abbreviate B_t and B'_t for $B_{o,t}$ and $B'_{o,t}$ respectively.

Whener $F,G \subset \mathbb{R}^n$, the notation $F \subset\subset G$ means that the closure of F is a compact subset of G, while $F \Delta G$ denotes the symmetric difference $(F \cup G) \setminus (F \cap G)$. The characteristic function of a set $E \subset \mathbb{R}^n$ will be denoted by ϕ_E. Convergence of a sequence $\{E_h\}$ to E in Ω always means the $L^1_{loc}(\Omega)$-convergence of the corresponding characteristic functions, i.e.:

$$(3.1)\quad E_h \to E \text{ locally in } \Omega \text{ iff } \int_A |\phi_{E_h}(x)-\phi_E(x)|\,dx \to 0 \quad \forall A\subset\subset\Omega .$$

We saythat E is a *Caccioppoli set* iff the distributional gradient $D\phi_E = (D_1\phi_E,\dots,D_n\phi_E)$ of ϕ_E is a Radon vector measure with locally finite total variation $|D\phi_E|$:

$$|D\phi_E|(A) < +\infty \qquad \forall A \subset\subset \mathbb{R}^n .$$

We have of course

$$|D\phi_E|(\Omega) = \sup \{\int_E \operatorname{div}\Phi(x)dx \ :\Phi \in C_o^1(\Omega;\mathbb{R}^n), |\Phi| \le 1\}$$

for every open set Ω of $\mathbb{R}^n$; according to the Gauss-Green theorem, we thus get

$$|D\phi_E|(\Omega) = H_{n-1}(\partial E \cap \Omega)$$

whenever $\partial E \cap \Omega$ is sufficiently smooth. Here, for every real $s \geq 0$, H_s denotes the s-dimensional Hausdorff measure on $\mathbb{R}^n$. We also set (see [12], 2.10.2):

$$\omega_s = \Gamma^s(\tfrac{1}{2})/\ \Gamma(s/2+1)$$

When k is a positive integer, ω_k yields precisely the k-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^k$.

The relevant facts about Caccioppoli sets can be found in [19] and in the recent book [27]. For our purposes, it suffices to recall that every Caccioppoli set E possesses, at $|D\phi_E|$ - almost all points $x \in \partial E$, an unit inner normal vector $\nu_E(x)$, defined through the following relation:

(3.2) $$\nu_E(x) = \lim_{t \to 0^+} \nu(E, B_{x,t})$$

where for short:

(3.3) $$\nu(E,G) = \frac{D\phi_E(G)}{|D\phi_E|(G)}$$

whenever $G \subset\subset \mathbb{R}^n$. The collection of points x where such a limit exists and has unit length, is commonly known as the *reduced boundary* of E, denoted by ∂^*E. See also (2.27). We remark explicitely, that when speaking of a Caccioppoli set E, we let ∂E denote the boundary of E in the measure-theoretical sense, i.e.

(3.4) $$x \in \partial E \quad \text{iff} \quad 0 < \text{meas}\,(E \cap B_{x,t}) < \text{meas}\,(B_{x,t}) \quad \forall t>0.$$

Whenever $x \in \partial^*E$, we have (see [19], Theorem 3.8, or [27], 2.3(23")):

(3.5) $$\lim_{t\to 0^+} t^{1-n}|D\phi_E|(B_{x,t}) = \omega_{n-1} \ .$$

We use vector addition and multiplication to define translations and homothetic transformations in $\mathbb{R}^n$. Thus:

$$E + x_o = \{x : x-x_o \in E\} \quad \text{and} \quad tE = \{x : t^{-1}x \in E\} \ ,$$

for $E \subset \mathbb{R}^n$, $x_o \in \mathbb{R}^n$ and $t>0$.

Clearly, whenever E is a Caccioppoli set, then so are $E+x_o$ and tE, and for every $G \subset \mathbb{R}^n$ it holds:

(3.6) $$|D\phi_{E+x_o}|(G+x_o) = |D\phi_E|(G), \ |D\phi_{tE}|(tG) = t^{n-1}|D\phi_E|(G).$$

Analogous relations hold for the measures $D_i\phi_E$, $i=1,\dots,n$, as well as for the following non-negative measures:

(3.7) $$\mu_E = |D\phi_E| - D_n\phi_E$$

$$(3.8) \qquad \omega_E \equiv |D\phi_E| - \left(\sum_{i=1}^{n} (D_i\phi_E)^2\right)^{\frac{1}{2}}$$

i.e. $\omega_E(G) = |D\phi_E|(G) - |D\phi_E(G)|$, $\forall G \subset\subset \mathbb{R}^n$. We shall preferably write $\omega(E,G)$ instead of $\omega_E(G)$. All these measures are obvioulsy invariant under orthogonal transformations.

We recall that the quantity $t^{1-n}\,\omega(E,B_{x,t})$ is usually called the *excess*, see section 2.7. Finally, we recall the definition of the functional ψ (see 1.13):

$$(3.9) \qquad \psi(E,A) = |D\phi_E|(A) - \inf\{|D\phi_F|(A) : F \Delta E \subset\subset A\} \ , \ A \subset\subset \mathbb{R}^n$$

which is also invariant under translations and orthogonal transformations, while clearly

$$(3.10) \qquad \psi(tE,tA) = t^{n-1}\,\psi(E,A) \qquad \forall t > 0$$

we have in addition (see [37]):

$$(3.11) \qquad A_1 \subset A_2 \subset\subset \mathbb{R}^n \Longrightarrow \psi(E,A_1) \le \psi(E,A_2) \qquad \forall E$$

$$(3.12) \qquad E_h \to E \quad \text{locally in } A \Longrightarrow \psi(E,A) \le \liminf_{h\to+\infty} \psi(E_h,A)$$

$$(3.13) \qquad \left.\begin{array}{l} E_h \to E \quad \text{locally in } A \\ \psi(E_h,A) \to \psi(E,A) \end{array}\right\} \Longrightarrow |D\phi_{E_h}|(A_1) \to |D\phi_E|(A_1),$$

for every $A_1 \subset\subset A$ such that A_1 is open and $|D\phi_E|\,(\partial A_1) = 0$.
For, assuming $A_1 \subset A_2$ we get

$$|D\phi_E|(A_1) - \inf\{|D\phi_F|(A_1) : F \Delta E \subset\subset A_1\} = |D\phi_E|(A_2) - \inf\{|D\phi_F|(A_2) : F \Delta E \subset\subset A_1\}$$

which proves (3.11).

As for (3.12), if $E_h \to E$ in A and F is such that $F \Delta E \subset\subset A$, then (reasoning possibly on a subsequence of $\{E_h\}$), we can pick an open subset A_2 of A, with lipschitz boundary, satisfying:

(3.14) $F\Delta E \subset\subset A_2 \subset\subset A$, $|D\phi_{E_h}|(\partial A_2)=|D\phi_E|(\partial A_2)=0 \ \forall h$, $\int_{\partial A_2}|\phi_{E_h}-\phi_E|dH_{n-1}\to 0$.

Setting $F_h = (E_h \setminus A_2) \cup (F \cap A_2)$ we find

$$|D\phi_{F_h}|(A) = |D\phi_{E_h}|(A \setminus A_2) + \int_{\partial A_2} |\phi_{E_h}-\phi_E|dH_{n-1} + |D\phi_F|(A_2)$$

from which, observing that $F_h \Delta E_h \subset\subset A$, we get

(3.15) $\psi(E_h,A) \geq |D\phi_{E_h}|(A)-|D\phi_{F_h}|(A)=|D\phi_{E_h}|(A_2)-|D\phi_F|(A_2)-\int_{\partial A_2} |\phi_{E_h}-\phi_E|dH_{n-1}$

By letting $h \to +\infty$ we then find for every F such that $F \Delta E \subset\subset A$:

$$\liminf_{h\to+\infty} \psi(E_h,A) \geq |D\phi_E|(A_2)-|D\phi_F|(A_2) = |D\phi_E|(A)-|D\phi_F|(A)$$

(recall (3.14) and the lower semicontinuity of $|D\phi.|(A)$ with respect to the local convergence in A), and from this (3.12) follows at once.

Finally, assume that $E_h \to E$ in A and that $\psi(E_h,A) \to \psi(E,A)$, and fix $A_1 \subset\subset A$ such that A_1 is open and $|D\phi_E|(\partial A_1) = 0$. Then choose F, A_2, and F_h as above, with in addition $A_1 \subset\subset A_2$. By (3.15) we get

$$\psi(E,A) = \lim_{h\to+\infty} \psi(E_h,A) \geq \limsup_{h\to+\infty} |D\phi_{E_h}|(A_2)-|D\phi_F|(A_2)$$

$$\geq \limsup_{h\to+\infty} |D\phi_{E_h}|(A_1)+\liminf_{h\to+\infty} |D\phi_{E_h}|(A_2\setminus\bar{A}_1)-|D\phi_F|(A_2)$$

$$\geq \limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) + |D\phi_E|(A_2 \setminus \bar{A}_1) - |D\phi_F|(A_2)$$

$$= \limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) - |D\phi_E|(\bar{A}_1) + |D\phi_E|(A) - |D\phi_F|(A)$$

which holds $\forall F : F \Delta E \subset\subset A$. When combined with (3.9), this gives

$$\limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) \leq |D\phi_E|(\bar{A}_1) \equiv |D\phi_E|(A_1)$$

by our assumptions. Since

$$|D\phi_E|(A_1) \leq \liminf_{h \to +\infty} |D\phi_{E_h}|(A_1)$$

by semicontinuity, we obtain eventually (3.13).

We now establish some helpful inequalities, involving ψ and ω. See also Prop. 2.8.

3.2. *Lemma*.

If E_1, E_2 are Caccioppoli sets in $\mathbb{R}^n$, and B is an n-ball, then

(3.16) $$\psi(E_1,B)-\psi(E_2,B) \leq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

(3.17) $$\psi(E_1,B)-\psi(E_2,B) \leq \omega(E_1,B)-\omega(E_2,B) + 2\int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

Proof. First we remark that for any Caccioppoli set $F \subset \mathbb{R}^n$, the term

$$\int_{\partial B} \phi_F \, dH_{n-1}$$

denotes the integral of the *inner trace* of F over ∂B (see e.g. [19], Chapter 2). Given such an F, we set for $B = B_{x,t}$ and $\tau \in (0,1)$:

$$F_\tau = (F \cap B_{x,\tau t}) \cup (E_2 \setminus B_{x,\tau t})$$

so that:

$$|D\phi_{F_\tau}|(B_{x,t}) \leq |D\phi_F|(B_{x,t}) - (|D\phi_F| - |D\phi_{E_2}|)(B_{x,t} \setminus B_{x,\tau t}) +$$

$$+ \int_{\partial B_{x,\tau t}} |\phi_{E_2} - \phi_F| \, dH_{n-1} .$$

Assuming $F \Delta E_1 \subset\subset B_{x,t}$, we get easily:

$$|D\phi_{E_1}|(B) - |D\phi_F|(B) \leq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \psi(E_2,B) -$$

$$- (|D\phi_F| - |D\phi_{E_2}|)(B_{x,t} \setminus B_{x,\tau t}) + \int_{\partial B_{x,\tau t}} |\phi_{E_2} - \phi_{E_1}| dH_{n-1}$$

Hence, letting $\tau \to 1$ and taking the supremum over such F's, we find

(3.18) $$\psi(E_1,B) \leq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \psi(E_2,B) + \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

which is exactly (3.16).

Now, since

(3.19) $$D\phi_F(B) = \int_{B_{x,t}} d\, D\phi_F = \int_{\partial B_{x,t}} t^{-1} \phi_F(y)(y-x) dH_{n-1}(y)$$

for every Caccioppoli set $F \subset \mathbb{R}^n$, we have

(3.20) $$|D\phi_{E_1}(B)| - |D\phi_{E_2}(B)| \leq \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

Adding (3.18) and (3.20), and rearranging, we get (3.17).

3.3. The following inequality was proved in Section 2.5 and 2.6 (see especially (2.16), (2.22) and (2.23)):

$$\omega(E,Q_s) \leq 2\psi(E,Q_t)+2(1-p^2)^{-1}\left[(s/t)^{n+1}+p^2\right] \cdot\omega(E,Q_t) \tag{3.21}$$

It holds $\forall s,t : 0<s<t<T$, under the following assumptions:[16]

$$E = \{x : |x'| < T,\ x_n > u(x')\} \tag{3.22}$$

$$Q_r = \{x : |x'| < r,\ |x_n - u(o)| < r\} \tag{3.23}$$

where $u \in C^1(B'_T)$ is such that

$$p \equiv \sup\{|Du(x')| : |x'| < T\} < 1 . \tag{3.24}$$

We conclude this section by recalling two further relations, which are proved e.g. in [27], 2.5.4 (1) and 2.5.1, respectively:

$$|\nu(E,G_1) - \nu(E,G_2)| \leq 2\left[\frac{\omega(E,G_2)}{|D\phi_E|(G_1)}\right]^{1/2} \tag{3.25}$$

(which holds for every Caccioppoli set $E \subset \mathbb{R}^n$, and every $G_1 \subset G_2 \subset\subset \mathbb{R}^n$ such that $|D\phi_E|(G_1) > 0$. See (2.33) and (3.3))

$$\left[\int_{\partial B_{x,1}} |\phi_E(x+s(y-x)) - \phi_E(x+t(y-x))|\, dH_{n-1}(y)\right]^2 \leq$$

(3

$$\leq 2\left[t^{1-n}|D\phi_E|(B_{x,t}) - s^{1-n}|D\phi_E|(B_{x,s}) + (n-1)\int_s^t r^{-n}|D\phi_E|(B_{x,r})\,dr\right]$$

$$\cdot\left[t^{1-n}|D\phi_E|(B_{x,t})-s^{1-n}|D\phi_E|(B_{x,t})+(n-1)\int_s^t r^{-n}\psi(E,B_{x,r})dr\right],$$

which is valid for every Caccioppoli set $E \subset \mathbb{R}^n$, every point $x \in \mathbb{R}^n$, and every $s,t : 0 < s < t$. This last inequality will be used in the next section to establish some fundamental area and volume density ratio bounds for (a special class of) almost minimal boundaries.

3.4. *Proposition.*

Suppose we are given a Caccioppoli set $E \subset \mathbb{R}^n$ and a non-negative function $\alpha(t)$, defined on $(0,1)$ and satisfying

$$\int_0^1 t^{-1}\alpha(t)dt < +\infty. \tag{3.27}$$

If for some point x and some radius $t \in (0,1)$ it holds

$$\psi(E,B_{x,t}) \leq \alpha(t)\cdot t^{n-1} \tag{3.28}$$

then for the same x,t we also have: [17]

$$t^{1-n}|D\phi_E|(B_{x,t}) \leq \alpha(t) + n\omega_n/2 \tag{3.29}$$

If (3.28) holds for every $t \in (0,T_o)$, with T_o fixed in $(0,1)$, then

(3.30) $t^{1-n}|D\phi_E|(B_{x,t})+(n-1)\cdot\int_o^t r^{-1}\alpha(r)dr$ is a non-decreasing function on $(0,T_o)$

Finally, assuming that (3.28) holds for every $x \in B_{x_o,T_o}$ and every $t \in (0,T_o)$ then we have:

(3.31) $t^{1-n}|D\phi_E|(B_{x,t}) \geq \omega_{n-1} - (n-1)\int_0^t r^{-1}\alpha(r)dr$

(3.32) $t^{-n}\cdot \min\{\text{meas}(E\cap B_{x,t}), \text{meas}(B_{x,t}\setminus E\} \geq \omega_{n-1}/n - \int_0^t r^{-1}\alpha(r)dr$

both $\forall x \in \partial E\cap B_{x_o,T_o}$, $\forall t\in(0,T_o)$. In this case moreover assuming that $\alpha(t)$ is non-decreasing and infinitesimal at 0, if we set

(3.33) $$E_h = t_h^{-1}(E-x)$$

for $x \in \partial E\cap B_{x_o,T_o}$ and $t_h \downarrow 0$, then a subsequence of $\{E_h\}$ will converge to a minimal cone $C \subset \mathbb{R}^n$, with $0 \in \partial C$.

Remark.

Evidently, when E has almost minimal boundary in Ω, with $x_o \in \Omega$ and $\alpha(t)$ satisfying (3.27) (see Def. 1.5 and 1.13), then a convenient T_o can be found so that (3.28) holds $\forall x \in B_{x_o,T_o}$ and $\forall t \in (0,T_o)$. Accordingly, (3.29)-(3.32) all hold $\forall x \in \partial E\cap B_{x_o,T_o}$, $\forall t \in (0,T_o)$. Also notice that (3.27) is weaker than (α_4) of section 1.11.

Proof. (see section 2 of [37] for the special case $\alpha(t) = ct^{2\alpha}$).
From (3.28) and (3.9) we get $\forall\tau \in (0,1)$:

$$\alpha(t)\cdot t^{n-1} \geq |D\phi_E|(B_{x,t}) - \min\{|D\phi_{E\cup B_{x,\tau t}}|(B_{x,t}), |D\phi_{E\setminus B_{x,\tau t}}|(B_{x,t})\}$$

(3.34) $$\geq |D\phi_E|(B_{x,t}) - |D\phi_E|(B_{x,t}\setminus B_{x,\tau t}) - \min\{\int_{\partial B_{x,\tau t}}\phi_E dH_{n-1}, \int_{\partial B_{x,\tau t}}(1-\phi_E)dH_{n-1}\}$$

$$\geq |D\phi_E|(B_{x,\tau t}) - \frac{1}{2}n\omega_n(\tau t)^{n-1}$$

from wich we get (3.29), by letting $\tau \to 1$.

(3.30) follows easily from (3.26), (3.27) and (3.28). If $x \in \partial^*E \cap B_{x_o,T_o}$ then (3.5),(3.27) and (3.30) imply (3.31). For a generic point x in ∂E, (3.31) follows by approximation, since $\partial E = \overline{\partial^* E}$.

Now, arguing as in (3.34) and using (3.31) we find $\forall t \in (0,T_o)$:

$$\omega_{n-1}-(n-1)\int_0^t r^{-1}\alpha(r)dr \le \alpha(t)+t^{1-n}\cdot\min\{\int_{\partial B_{x,t}}\Phi_E dH_{n-1},\int_{\partial B_{x,t}}(1-\phi_E)dH_{n-1}\}$$

On rearranging and integrating between 0 and t we obtain:

$$\min\{\text{meas}(E\cap B_{x,t}),\ \text{meas}(B_{x,t}\setminus E)\}\ge\int_0^t[\omega_{n-1}-(n-1)\int_0^s r^{-1}\alpha(r)dr-\alpha(s)]s^{n-1}ds$$

$$=\omega_{n-1}t^n/n+(1/n-1)t^n\int_0^t r^{-1}\alpha(r)dr-(1/n)\int_0^t r^{n-1}\alpha(r)dr$$

$$\ge[\omega_{n-1}/n-\int_0^t r^{-1}\alpha(r)dr]\cdot t^n$$

which proves (3.32).

Finally, for E_h as in (3.33), $r > o$, and h sufficiently large (so that $rt_h < T_o$) we have, in view of (3.6),(3.29) and the new assumptions on $\alpha(t)$:

$$(3.35)\quad |D\phi_{E_h}|(B_r)=t_h^{1-n}|D\phi_E|(B_{x,rt_h})\le r^{n-1}(\alpha(T_o)+n\omega_n/2)$$

Hence, a subsequence of E_h (not relabeled) will converge to some limit set C, locally in $\mathbb{R}^n$. On the other hand, we have (see (3.10)):

$$\psi(E_h,B_r) = t_h^{1-n}\ \psi(E,B_{x,rt_h}) \leq \alpha(rt_h)\cdot r^{n-1} \tag{3.36}$$

by (3.28), so that

$$\psi(C,B_r) = 0 \qquad \forall r > 0 \tag{3.37}$$

in view of (3.12), that is, C has minimal boundary in $\mathbb{R}^n$ (Def. 1.2).

From (3.26); (3.37) and (3.13) we deduce

$$|D\phi_{E_h}|\ (B_r) \to |D\phi_C|(B_r) \qquad \text{for a.e. } r > 0$$

or (see (3.35)):

$$(rt_h)^{1-n}\ |D\phi_E|(B_{x,rt_h}) \to r^{1-n}|D\phi_C|(B_r) \quad \text{for a.e. } r>0, \tag{3.38}$$

as $h \to +\infty$.

Setting

$$b = \lim_{t\to 0^+} [t^{1-n}\ |D\phi_E|(B_{x,t})] \tag{3.39}$$

((3.30) shows that the limit in question exists, while (3.29) and (3.31) give upper and lower bounds for b), we conclude that

$$r^{1-n}|D\phi_C|(B_r) = b \in [\omega_{n-1},\ n\omega_n/2\ ] \quad \text{for a.e. } r > 0. \tag{3.40}$$

Substitution of (3.37) and (3.40) into (3.26) then yields

$$\int_{\partial B_1} |\phi_C(sy) - \phi_C(ty)|\ dH_{n-1}(y) = 0$$

for almost every $t > 0$, and almost every $s \in (0,t)$, thus proving that C is (equivalent to) a minimal cone, with $0 \in \partial C$ (see (3.4) and (3.40)).

3.5. The main result of the present chapter is the following Lemma 3.6, which extends De Giorgi's Lemma of section 2.12.

Its proof will be achieved by comparing the given set E with level sets L of a suitable mollification of ϕ_E - much as in the original paper of De Giorgi [8], using however a more direct argument. The results contained in sections 3.2 and 3.4 then show that the comparison surface ∂L is appropriately "close" to ∂E. The area excess of ∂L being nicely controlled (section 3.3), this yields the desired estimation of the excess of ∂E.

A few properties of mollifiers are now in order. We introduce the following "tent function"

(3.41) $$\eta(x) = c(n)\cdot\max\{1-|x|,0\}, \quad \text{with} \quad c(n) = (n+1)/\omega_n$$

first considered by E. Giusti [19], chapter 7. Clearly, η is a non-negative, symmetric, Lipschitz-continuous mollifier, whose integral is 1 and whose support coincides with the unit ball in $\mathbb{R}^n$. We set as usual, for $\varepsilon > 0$ and $g \in L^1_{loc}(\mathbb{R}^n)$:

(3.42) $$\begin{cases} \eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x) \\ g_\varepsilon(x) = (g*\eta_\varepsilon)(x) = \int \eta_\varepsilon(x-y)g(y)\,dy \end{cases}.$$

Then, whewever F is a Caccioppoli set in $\mathbb{R}^n$, $\varepsilon > 0$, and $f_\varepsilon = \phi_F * \eta_\varepsilon$, we have

(3.43) $$f_\varepsilon \text{ is of class } C^1$$

(3.44) $$\int_{B_t} |f_\varepsilon - \phi_F|\,dx \leq \varepsilon\cdot|D\phi_F|(B_{t+\varepsilon}) \qquad \forall t > 0$$

$$(3.45)\qquad \int_{B_t} |Df_\varepsilon|\,dx \leqq |D\phi_F|(B_{t+\varepsilon}) \qquad \forall\, t > 0$$

(3.46) if $0 < t < 1/n$ and $n^2t^2 < f_\varepsilon(x) < 1 - n^2t^2$, then $\mathrm{dist}(x,\partial F)<(1-t)\varepsilon$.

See [19], Lemma 7.1 and 7.2, for the simple proof.

From now on we suppose that $\alpha(t)$ satisfies (α_1)-(α_4) of section 1.11. We also introduce the notation $\beta \preccurlyeq \alpha$ to indicate a non-decreasing function β, defined on $(0,1)$, and satisfying $0 \leqq \beta(t) \leqq \alpha(t)$ $\forall t \in (0,1)$.

We are now is a position to state and prove the following result (compare with Lemma 2.12).

3.6. Main Lemma.

For any $n \geqq 2$, *any* α *as in* 1.11, *and any* $\tau \in (0,2^{-4})$, *there exists a constant* $\sigma^* = \sigma^*(n,\alpha,\tau) \in (0,1)$, *such that whenever* $F \subset \mathbb{R}^n$, $\sigma \in (0,\sigma^*]$, *and* $\beta \preccurlyeq \alpha$ *satisfy the following hypotheses:*

(H_1) $\qquad \psi(F,B_{x,t}) \leqq \beta(t)\cdot t^{n-1} \qquad \forall x \in B_1,\ \forall t \in (0,1)$

(H_2) $\qquad \omega(F,B_1) \leqq \sigma$

then:

$$(3.47)\qquad \omega(F,B_\tau) \leqq c_1\beta(1) + c_2\sigma\tau^{n+1}$$

where c_1, c_2 are positive constants, depending only on the dimension n.

Proof. Without loss of generality, we can assume that $|D\phi_F(B_1)| = D_n\phi_F(B_1)$, so that (see (3.7)) : $\omega(F,B_1) = \mu_F(B_1)$. We split the proof into three steps.

Step 1. Given n and α as above, we prove first the existence of a constant $\sigma^{\#} \in (0,1)$ and of a function $g : (0,\sigma^{\#}] \to (0,1)$, with $g(\sigma) = o(1)$ ($\sigma^{\#}$ and g depending on n and α), such that *whenever* $F \subset \mathbb{R}^n$ *and* $\sigma \in (0,\sigma^{\#}]$ *satisfy:*

$$(h_1) \qquad \psi(F,B_{x,t}) \leq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_1,\ \forall t \in (0,1)$$

$$(h_2) \qquad \omega(F,B_1) = \mu_F(B_1) \leq \sigma$$

then

$$(3.48) \qquad \frac{D_n f(x)}{|Df(x)|} > 1-g(\sigma) \quad \forall x \in B_1 : |x| < 1-2\sigma^{1/2(n-1)} \text{ and } n^2\sigma^2 < f(x) < 1-n^2\sigma^2 ,$$

where

$$(3.49) \qquad f = \phi_F * \eta_\varepsilon , \qquad \varepsilon = \sigma^4 .$$

We observe that when $\sigma \geq 4^{1-n}$, the set of points in (3.48) is empty, so that there is nothing to prove in this case. Thus, we assume $\sigma < 4^{1-n}$. In addition, we observe that

$$(3.50) \qquad D_n f(x) = \int \eta_\varepsilon(x-y)\, d\, D_n\phi_F(y) \text{ and } |Df(x)| \leq \int \eta_\varepsilon(x-y)\, d\, |D\phi_F|(y)$$

since, by definition, $f(x) = \int \eta_\varepsilon(x-y)\phi_F(y)\,dy$. Hence, (3.48) will be proved if we can show that

$$(3.51) \qquad \int \eta_\varepsilon(x-y)\, d\mu_F(y) < g(\sigma) \int \eta_\varepsilon(x-y)\, d\, |D\phi_F|(y)$$

for any x satisfying

(3.52) $$|x| < 1-2\sigma^{1/2(n-1)}\ , \ n^2\sigma^2 < f(x) < 1 - n^2\sigma^2 .$$

To this aim, we define for x as in (3.52) and $\sigma \in (0,4^{1-n})$:

$$\gamma = 1 - \sigma^{n+1} \quad \text{and} \quad G = B_{x,\varepsilon} - B_{x,\gamma\varepsilon}$$

We observe that, from (3.41),(3.42):

(3.53) $$\int_G \eta_\varepsilon(x-y)d\mu_F(y) \leq c(n)\varepsilon^{-n}\sigma^{n+1}\mu_F(G) \leq 2c(n)\varepsilon^{-1}\sigma^{n+1}(\alpha(1)+n\omega_n/2)$$

by virtue of (h_1), (3.29), and the monotonicity of α ((α_1) of 1.11); on the other hand, from (3.46) and the assumption $\sigma < 4^{1-n}$ we conclude, that for every x as in (3.52) it is possible to find $z \in \partial E$ such that $|x-z| < (1-\sigma)\varepsilon$. Therefore:

$$\int \eta_\varepsilon(x-y)d\,|D\phi_F|(y) \geq c(n)\varepsilon^{-n}(\sigma/2)|D\phi_F|(B_{x,(1-\sigma/2)\varepsilon})$$

(3.54) $$\geq c(n)\varepsilon^{-n}(\sigma/2)\ |D\phi_F|(B_{z,\sigma\varepsilon/2})$$

$$\geq c(n)\varepsilon^{-1}(\sigma/2)^n[\omega_{n-1}-(n-1)\int_0^{\sigma\varepsilon/2} t^{-1}\alpha(t)dt]$$

by (3.31). In view of 1.11, (α_4), we can certainly choose $\sigma^{\#}\in(0,1)$ such that

(3.55) $$\int_0^{\sigma^{\#}} t^{-1}\alpha(t)dt \leq \omega_{n-1}/2(n-1)$$

Hence, from (3.54) we derive

(3.56) $$\int \eta_\varepsilon(x-y)d|D\phi_F|(y) \geq c(n)\omega_{n-1}\, 2^{-n-1}\varepsilon^{-1}\sigma^n > 0 .$$

Combining (3.53) and (3.56) we obtain:

(3.57) $$\int_G \eta_\varepsilon(x-y)d\mu_F(y) \leqq 2^{n+2}\omega_{n-1}^{-1}\, g_1(\sigma) \int \eta_\varepsilon(x-y)d|D\phi_F|(y)$$

where

(3.58) $$g_1(\sigma) = (\alpha(1) + n\omega_n/2)\sigma .$$

Now, put

$$\delta = \sigma^{n+1}/2 \quad \text{and} \quad D = \partial F \cap B_{x,\gamma\varepsilon}$$

Due to the boundedness of D, we can find a finite number of points in D, wich we call $z_1,\ldots,z_h$, with the property that:

(3.59) $$B_{z_i,\delta\varepsilon} \cap B_{z_j,\delta\varepsilon} = \emptyset \text{ if } i \neq j, \text{ and } D \subset \bigcup_{i=1}^{h} B_{z_i,2\delta\varepsilon}.$$

We write $B_{i,t}$ for $B_{z_i,t}$ $(i=1,\ldots,h)$, and observe that $B_{i,2\delta\varepsilon} \subset B_{x,\varepsilon}$ whence

(3.60) $$\int_{B_{x,\gamma\varepsilon}} \eta_\varepsilon(x-y)d\mu_F(y) \leqq \sum_{i=1}^{h} \int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)d\mu_F(y)$$

(3.61) $$\int_{B_{x,\varepsilon}} \eta_\varepsilon(x-y)d|D\phi_F|(y) \geqq \sum_{i=1}^{h} \int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)d|D\phi_F|(y)$$

For every $i = 1,\ldots,h$, we find

(3.62) $$\int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)d\mu_F(y) \leqq c(n)\,\varepsilon^{-n}(1+2\delta - |x-z_i|/\varepsilon)\cdot\mu_F(B_{i,2\delta\varepsilon})$$

$$(3.63)\quad \int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y) \geq c(n)\varepsilon^{-n}(1-\delta-|x-z_i|/\varepsilon)\cdot|D\phi_F|(B_{i,\delta\varepsilon})$$

$$\geq c(n)2^{-1}\omega_{n-1}\delta^{n-1}\varepsilon^{-1}(1-\delta-|x-z_i|/\varepsilon)$$

(here, we used again (3.31), and the assumtion (3.55)). Since $|x-z_i| < (1-\sigma^{n+1})\varepsilon$, we have $1-\delta-|x-z_i|/\varepsilon > \sigma^{n+1} - \delta = \sigma^{n+1}/2 > 0$
Hence, taking the quotient of (3.62) over (3.63), we find

$$(3.64)\quad \Big(\int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)\,d\mu_F(y)\Big)\cdot\Big(\int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y)\Big)^{-1} \leq 2^{n+2}\omega_{n-1}^{-1}(2\delta\varepsilon)^{1-n}\mu_F(B_{i,2\delta\varepsilon})$$

which holds for each $i = 1,\dots,h$.

Now we put $s = 2\delta\varepsilon = \sigma^{n+5}$ (see (3.49), $t = \sigma^{1/2(n-1)}$, and use definition (3.7) together with (3.30), to deduce that:

$$s^{1-n}\mu_F(B_{i,s}) = s^{1-n}|D\phi_F|(B_{i,s}) - s^{1-n}\,D_n\phi_F(B_{i,s})$$

$$(3.65)\quad \leq t^{1-n}|D\phi_F|(B_{i,t})+(n-1)\int_s^t r^{-1}\alpha(r)dr-s^{1-n}\,D_n\phi_F(B_{i,s})$$

$$\leq t^{1-n}\,\mu_F(D_{i,t})+(n-1)\int_0^t r^{-1}\alpha(r)dr+\big[t^{1-n}D_n\phi_F(B_{i,t})-s^{1-n}D_n\phi_F(B_{i,s})\big]$$

We have $B_{i,t} \subset B_1$, hence the first term in the right-hand side of the last inequality is not larger than $\sigma^{1/2}$, by (h_2) and our assumptions. On the account of (3.19),(3.26) the term in square brakets is easily estimated by

$$2^{1/2}\Big[t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s})+(n-1)\cdot\int_s^t r^{-n}|D\phi_F|(B_{i,r})dr\Big]^{1/}$$

$$\Big[t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s})+(n-1)\int_s^t r^{-n}\psi(F,B_{i,r})dr\Big]^{1/2}$$

As before,

$$t^{1-n}|D\phi_F|(B_{i,t}) = t^{1-n}\mu_F(B_{i,t})+t^{1-n}\ D_n\phi_F(B_{i,t}) \leqq \sigma^{1/2}+\omega_{n-1}$$

as a consequence of (3.19). Therefore, from (h_1) and (3.31) we get

$$t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s}) \leqq \sigma^{1/2}+(n-1)\int_o^s r^{-1}\alpha(r)dr.$$

Similarly, from (h_1) and (3.29) we get

$$\int_s^t r^{-n}|D\phi_F|(B_{i,r})dr \leqq \int_s^t r^{-1}(\alpha(r)+n\omega_n/2)dr=2^{-1}n\omega_n\lg(t/s)+\int_s^t r^{-1}\alpha(r)dr,$$

$$\int_s^t r^{-n}\psi(F,B_{i,r})dr \leqq \int_s^t r^{-1}\alpha(r)dr.$$

Collecting terms and going back to (3.65) we find, for each $i=1,\ldots,h$:

$$s^{1-n}\ \mu_F(B_{i,s}) < \sigma^{1/2}+(n-1)\int_o^t r^{-1}\alpha(r)dr +$$

(3.66) $$+\ 2\left[\sigma^{1/2}+(n-1)\int_0^t r^{-1}\alpha(r)dr+2^{-1}(n-1)n\omega_n\ \lg(t/s)\right]^{1/2}$$

$$\cdot\left[\sigma^{1/2}+(n-1)\int_0^t r^{-1}\alpha(r)dr\ \right]^{1/2}.$$

Recalling that $s=\sigma^{n+5}$, $t=\sigma^{1/2(n-1)}$, we derive from (3.66),(3.64), (3.60) and (3.61)

(3.67) $$\int_{B_{x,\gamma\varepsilon}} \eta_\varepsilon(x-y)d\mu_F(y) < 2^{n+2}\omega_{n-1}^{-1}\ g_2(\sigma)\int_{B_{x,\varepsilon}} \eta_\varepsilon(x-y)d|D\phi_F|(y)$$

where

(3.68) $$g_2(\sigma)=\sigma^{1/2}+(n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr+2\cdot\left[\sigma^{1/2}+(n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr\right]^{1/2}\cdot$$

$$\cdot\left[\sigma^{1/2}+(n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr-2^{-1}(n-1)n\omega_n(2n^2+8n-11)\cdot\lg\sigma^{1/2(n-1)}\ \right]^{1/2}$$

On adding (3.57) and (3.67), we get (3.51) with

$$g(\sigma)=2^{n+2}\ \omega_{n-1}^{-1}\left[g_1(\sigma)+g_2(\sigma)\right].$$

g_1 and g_2 given by (3.58) and (3.68). In order to assure that g is infinitesimal at 0, the only point to check is the following:

(3.69) $$\lim_{\sigma\to 0^+}(-\lg\sigma^{1/2(n-1)})\cdot\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr=0$$

Now, the monotonicity of α implies that

$$(-\lg\sigma^{1/2(n-1)}) \cdot \int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr \leqq \left(\int_{\sigma^{1/2(n-1)}}^1 r^{-1}dr\right)\cdot\alpha^{1/2}(\sigma^{1/2(n-1)})\cdot \int_0^{\sigma^{1/2(n-1}} r^{-1}\alpha^{1/2}(r)dr$$

$$\leqq \left(\int_{\sigma^{1/2(n-1)}}^1 r^{-1}\alpha^{1/2}(r)dr\right) . \int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha^{1/2}(r)dr$$

and (3.69) follows from 1.11, (α_4).

We can then choose $\sigma^{\#} \in (0,1)$ such that (3.55) holds and, in addition, such that

$$g(\sigma) < 1 \qquad \forall\ \sigma \leq \sigma^{\#}$$

From (3.50), (3.51) we deduce (3.48), thus concluding the proof of the first step.

Step 2. According to Step 1, assumtions (h_1) and (h_2) imply

(3.70) $$\frac{D_n f(x)}{|Df(x)|} > 1 - g(\sigma) > 0$$

for every x as in (3.52), provided $\sigma \in (0,\sigma^{\#}]$. At this point, we can start on the study of the level sets of the function f, defined by (3.49). To this end, we also assume

(3.71) $$\sigma \leqq 2^{8(1-n)}$$

so that in particular $7/8 \leqq 1-2\sigma^{1/2(n-1)}$ and $1-2n^2\sigma^2 > 3/4$. For $\lambda \in]0,1[$, we define

(3.72) $$L_\lambda = \{x : f(x) \geqq \lambda\}$$

and observe that, according to (3.70), for every $\lambda \in (n^2\sigma^2,\ 1-n^2\sigma^2)$,

$$\partial L_\lambda \cap B_{7/8} = \{x \in B_{7/8} : f(x) = \lambda\}$$

is the graph, over a certain open set $A'_\lambda \subset \mathbb{R}^{n-1}$, of a certain function $u_\lambda \in C^1(A'_\lambda)$. Denoting by $\nu_\lambda(x)$ the unit inner normal to L_λ at $x \in \partial L_\lambda$, we have also:

$$\nu_\lambda(x) = -\frac{Df(x)}{|Df(x)|} = (1+|Du_\lambda(x')|^2)^{-1/2}\cdot(-Du_\lambda(x'),1)$$

for every $x \in \partial L_\lambda \cap B_{7/8}$, i.e. $x = (x', u_\lambda(x'))$, with $x' \in A'_\lambda$. As a consequence, (3.70) yields $p_\lambda^2 \leqq g(\sigma)(2-g(\sigma))(1-g(\sigma))^{-2}$, $\forall \lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$, where:

(3.73) $$p_\lambda \equiv \sup\{|Du_\lambda(x')| : x' \in A'_\lambda\}$$

In particular, we get

(3.74) $$p_\lambda \leqq t \quad \text{whenever} \quad g(\sigma) \leqq 1-(1+t^2)^{-1/2}$$

On the other hand, it is not difficult to show, that if for every such λ ∂L_λ passes "sufficiently close to the origin", while being "flat enough", then each domain A'_λ contains an $(n-1)$-dimensional ball of fixed radius. For example, let us suppose that for a fixed $\lambda \in (n^2\sigma^2,\ 1-n^2\sigma^2)$ it holds:

(3.75) $$\partial L_\lambda \cap B_{1/8} \neq \emptyset$$

We already know that $A'_\lambda \subset B'_{7/8}$ and $|u_\lambda(x')| < 7/8$ $\forall x' \in A'_\lambda$

Moreover, if σ is chosen in such a way that

(3.76) $$g(\sigma) \leqq 1-4 \bullet 17^{-1/2} = .0299$$

(recall that g is infinitesimal at 0), then (3.74) yields

(3.77) $$p_\lambda \leqq 1/4.$$

Now, according to (3.75), we pick $z \in \partial L_\lambda \cap B_{1/8}$, i.e. $z = (z', u_\lambda(z'))$ with $z' \in A'_\lambda$, $|z'| < 1/8$, and $|u_\lambda(z')| < 1/8$. If x' is any other point in A'_λ, then

(3.78) $$|u_\lambda(x')| \leqq |u_\lambda(z')| + p_\lambda(|x'| + |z'|) < 3/8$$

while, if $x' \in \partial A'_\lambda$, then $(x', u_\lambda(x')) \in \partial B_{7/8}$, hence (3.78) yields

$$|x'| \geqq 7/8 - |u_\lambda(x')| > 1/2$$

and we conclude immediately that $B'_{1/2} \subset A'_\lambda$.

Thus, see (3.71) and (3.76), if σ satisfies

(3.79) $$\sigma \leqq \sigma^{\#}, \sigma \leqq 2^{8(1-n)}, \quad \text{and} \quad g(\sigma) \leqq 1-4\cdot 17^{-1/2}$$

with $\sigma^{\#}$ and g as in Step 1, and if in addition $\partial L_\lambda \cap B_{1/8} \neq \emptyset$ for a certain $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$, then (see also (3.78)):

(3.80) $$B'_{1/2} \subset A'_\lambda \subset B'_{7/8} \quad \text{and} \quad \partial L_\lambda \cap \tilde{Q}_{1/2} = \partial L_\lambda \cap (B'_{1/2} \times \mathbb{R}) \cap B_{7/8} =$$

$$= \{x : x' \in B'_{1/2}, x_n = u_\lambda(x')\}$$

where $\tilde{Q}_{1/2}$ denotes the cylinder of radius 1/2 around the origin, i.e.

$$\tilde{Q}_{1/2} = B'_{1/2} \times (-1/2, 1/2) \subset B_{7/8}.$$

In the same hypotheses, from (3.78) we get also $|u_\lambda(0)| \leqq 5/32$, so that $|u_\lambda(0)| + r < 1/2$ whenever $r < 1/4$. Thus, setting

(3.81) $$Q_{\lambda,r} = \{x : |x'| < r, \ |x_n - u_\lambda(0)| < r\}$$

(compare with (3.23)), we obtain from (3.80) and (3.77):

(3.82) $$\partial L_\lambda \cap B_r \subset \partial L_\lambda \cap [B'_r \times (-1/2,1/2)] = \{x : x' \in B'_r, x_n = u_\lambda(x')\} = \partial L_\lambda \cap Q_{\lambda,1}$$

for every $r \in (0,1/4)$. Finally, we can easily check that
$Q_{\lambda,r} \subset B_{3r} \quad \forall r \in (1/8,1/4)$.

Step 3. We are now ready to conclude the proof of the Main Lemma. As in Step 2, we denote by σ a positive number satisfying (3.79), by λ a number in the interval $[0,1]$, and by L_λ the corresponding level est of the function $f = \phi_F * \eta_\varepsilon$, with $\varepsilon = \sigma^4$.

According to the preceding assumptions (see the implications following (3.71), we have in particular:

(3.83) $$1 - 2n^2\sigma^2 > 3/4 .$$

Furthermore, it is easy to check that

(3.84) $$\int_0^1 d\lambda \int_B |\phi_{L_\lambda} - \phi_F| dx = \int_B |f - \phi_F| dx, \quad \int_0^1 d\lambda \int_{\partial B} |\phi_{L_\lambda} - \phi_F| dH_{n-1} = \int_{\partial B} |f - \phi_F| dH_{n-1}$$

(here, only the fact that f lies between 0 and 1 really matters). Finally, we recall the following "coarea formula":

(3.85) $$\int_0^1 |D\phi_{L_\lambda}|(B) d\lambda = \int_B |Df(x)| dx$$

(see [19], theorem 1.23, or [28], theorem 1.6).

From (3.17) we get for all $t < 1$ and almost all $\lambda \in [0,1]$:

$$\omega(F,B_t) \leqq \psi(F,B_t) + \omega(L_\lambda,B_t) + 2\int_{\partial B_t} |\phi_{L_\lambda} - \phi_F| dH_{n-1}$$

which, integrated over $(\tau, 2\tau)$, yields:

$$\omega(F,B_\tau) \leqq \beta(1) + \omega(L_\lambda,B_{2\tau})+2\tau^{-1}\int_{B_{1/8}} |\phi_{L_\lambda}-\phi_F|\,dx, \tag{3.86}$$

because of the monotonicity of ω and ψ, our hypothesis (H_1), and the fact we are assuming $\tau < 2^{-4}$. We now suppose that $\partial L_\lambda \cap B_{2\tau} \neq \emptyset$, for every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$: otherwise, we would have $\omega(L_\lambda,B_{2\tau}) = 0$ for some of such λ's, and the proof of the Lemma would obviously be easier. We are then precisely in the situation discussed in Step 2 (see 3.75)). Hence, according to (3.77), (3.80), and (3.82), we derive from (3.21):

$$\begin{aligned}\omega(L_\lambda,B_{2\tau}) \leqq \omega(L_\lambda,Q_{\lambda,2\tau}) &\leqq 2\psi(L_\lambda,Q_{\lambda,t}) + \\ &+ 2(1-p_\lambda^2)^{-1}[(2\tau/t)^{n+1}+p_\lambda^2]\omega(L_\lambda,Q_{\lambda,t})\end{aligned} \tag{3.87}$$

for every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$ and every $t \in (2\tau, 1/4)$.

Recalling (see the last assertion of Step 2) that $Q_{\lambda,t} \subset B_{3t}$ $\forall t \in (1/8,1/4)$, we get from (3.86),(3.87):

$$\begin{aligned}\omega(F,B_\tau) &\leqq \beta(1)+2\psi(L_\lambda,B_{3t})+2(1-p_\lambda^2)^{-1}[(2\tau/t)^{n+1}+p_\lambda^2]\omega(L_\lambda,B_{3t}) + \\ &+ 2\tau^{-1}\int_{B_{1/8}} |\phi_{L_\lambda}-\phi_F|\,dx\end{aligned} \tag{3.88}$$

which holds for any $t \in (1/8,1/4)$ and any $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$.

To focus on the real substance of the proof, it seems now convenient to adopt the following

Convention. Throughout be rest of the present section, c will denote constants not necessarily the same at any occurence. Similarly, c(n) will denote a *generic* positive constant, depending only on n.

We remark that all these constants (in particular, $c_1(n), c_2(n)$ in (3.97)) are easily computable.

We use again (3.17) to estimate $\omega(L_\lambda, B_{3t})$ in (3.88), thus getting for t, λ as before:

$$\omega(F,B_\tau) \leqq \beta(1) + c\psi(L_\lambda, B_{3t}) + c\,[(2\tau/t)^{n+1} + p_\lambda^2]\,\omega(F,B_{3t}) +$$

(3.89)

$$+ c \int_{\partial B_{3t}} |\phi_{L\lambda} - \phi_F^-|\, dH_{n-1} + 2\tau^{-1} \int_{B_{1/8}} |\phi_{L_\lambda} - \phi_F|\, dx$$

since $p_\lambda \leqq 1/4$ (see (3.77)).

Next, in addition to (3.79), we assume that σ also sat... ...s:

$$(3.90) \qquad g(\sigma) \leqq 1 - [1 + (16\tau)^{n+1}]^{-1/2}$$

From (3.74) we obtain $p_\lambda^2 \leqq (16\tau)^{n+1}$, and thus the third term in the right-hand side of (3.89) can be estimated by $c(n)\,\sigma\tau^{n+1}$, in view of (H_2).

To estimate the second term in the right-hand side of (3.89) we use instead (3.16), which yields, in view of (H_1):

$$\psi(L_\lambda, B_t) \leqq \beta(1) + |D\phi_{L_\lambda}|(B_t) - |D\phi_F|(B_t) + \int_{\partial B_t} |\phi_{L_\lambda} - \phi_F|\, dH_{n-1}$$

Going back to (3.89) we find

$$\omega(F,B_\tau) \leqq c\beta(1) + c(n)\sigma\tau^{n+1} + c\,[|D\phi_{L_\lambda}|(B_t) - |D\phi_F|(B_t)] +$$

$$+ c\int_{\partial B_t} |\phi_{L_\lambda} - \phi_F|\, dH_{n-1} + 2\tau^{-1} \int_{B_{1/8}} |\phi_{L\lambda} - \phi_F|\, dx$$

for every $t \in (3/8,3/4)$ and every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$. By integrating in λ we get ((3.83), (3.84), and (3.85)):

$$\omega(F,B_\tau) \leqq c\,\beta(1)+c(n)\sigma\tau^{n+1}+c\Big[\int_{B_t} |Df(x)|dx-|D\phi_F|(B_t)\Big]+c(n)\sigma^2 \int_{B_{3/4}} |Df(x)|dx +$$

$$+ c \int_{\partial B_t} |f-\phi_F|dH_{n-1} + (c/\tau) \int_{B_{1/8}} |f-\phi_F|dx$$

for every $t \in (3/8,3/4)$. Finally, by integrating in t we obtain:

$$\omega(F,B_\tau) \leqq c\beta(1)+c(n)\sigma\tau^{n+1}+c\int_{3/8}^{3/4} dt\Big(\int_{B_t} |Df(x)|dx-|D\phi_F|(B_t)\Big) +$$

(3.91)

$$+ c(n)\sigma^2 \int_{B_{3/4}} |Df(x)|dx + c \int_{B_{3/4}} |f-\phi_F|dx+(c/\tau) \int_{B_{1/8}} |f-\phi_F|dx$$

Now, (3.16) implies that for all $t < 1$ and almost all $\lambda \in [0,1]$:

$$|D\phi_F|(B_t) \leqq |D\phi_{L_\lambda}|(B_t)+\psi(F,B_t) +\int_{\partial B_t} |\phi_{L_\lambda}-\phi_F|dH_{n-1}$$

from which, integrating first in $\lambda \in [0,1]$, and then in $t \in (0,3/8)$, we find, on the account of (3.84), (3.85) and (H_1):

(3.92) $$\int_0^{3/8} (|D\phi_F|(B_t)- \int_{B_t} |Df(x)|dx)dt \leqq (3/8)\beta(1)+ \int_{B_{3/8}} |f-\phi_F|dx$$

Moreover, setting $h(x) = \max\{3/4-|x|,0\}$, we find easily that

(3.93) $$\int h(x)|Df(x)|dx = \int_0^{3/4} dt \int_{B_t} |Df(x)|dx$$

$$\int h(x)d|D\phi_F|(x) = \int_0^{3/4} |D\phi_F|(E_t)dt$$

since the level sets $\{x : h(x) > t\}$ of h are empty whenever $t \geqq 3/4$, and coincide with $B_{3/4-t}$ whenever $0 \leqq t < 3/4$. We notice that h is Lipschitz-continuous, with Lipschitz constant 1 , so that $|h*\eta_\varepsilon - h| \leqq \sigma^4$. Therefore, recalling (3.50), we find:

(3.94) $\int h(x)|Df(x)|dx \leqq \int (h*\eta_\varepsilon)(y)d|D\phi_F|(y) \leqq \sigma^4|D\phi_F|(B_1)+\int h(y)d|D\phi_F|(y)$

In conclusion, from (3.92),(3.93), and (3.94), we get:

$$\int_{3/8}^{3/4} dt\left(\int_{B_t} |Df(x)|dx - |D\phi_F|(B_t)\right) \leqq (3/8)\beta(1)+\sigma^4|D\phi_F|(B_1)+\int_{B_{3/8}} |f-\phi_F|dx$$

which, combined with (3.91) and (3.44), (3.45), yields:

(3.95) $\omega(F,B_\tau) \leqq c\beta(1)+c(n)\sigma\tau^{n+1}+c\sigma^2|D\phi_F|(B_1)\cdot[\sigma^2+c(n)+\sigma^2/\tau]$.

By (H_1) and (3.29) we have $|D\phi_F|(B_1) \leqq \beta(1)+n\omega_n/2$. Hence, assuming that

(3.96) $$\sigma \leqq \tau^{n+1}$$

we get from (3.95):

(3.97) $$\omega(F,B_\tau) \leqq c_1(n)\beta(1) + c_2(n)\sigma\tau^{n+1}$$

as required. Lemma 3.6 is then completely proved, provided we choose $\sigma^* \in (0,1)$ such that each $\sigma \leqq \sigma^*$ satisfies (3.79),(3.90), and (3.96).