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REGULARITY RESULTS
FOR
ALMOST MINIMAL ORIENTED HYPERSURFACES IN $\mathbb{R}^n$

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Preface. - *This work is intended as an introduction to the regularity theory of oriented boundaries in* $\mathbb{R}^n$ *which are almost minimal for the area functional. It is based partly on an earlier manuscript which contained the proof of the main theorem presented below, and partly on lecture notes for a course by the author at the University of Lecce.*

The reader is presumed to have some knowledge of the basic facts concerning Caccioppoli sets: sections 2.1 to 2.4 of the book of Massari and Miranda (see [27] of the bibliography at the end of the volume) will serve the scope.

With the exception of a few "classical" inequalities, which proofs can also be found in [27], the exposition is essentially self-contained.

The first half of the work, Chapters 1 and 2, is introductory. We begin by recalling the (by now classical) Regularity Theorem of minimal boundaries, in the framework of Caccioppoli - De Giorgi Miranda's theory. Almost minimal boundaries are then defined, and a corresponding Regularity Theorem is stated. The remainder of Chapter 1 illustrates these concepts and results with several examples and applications.

A specialized version of the Theorem is derived in Chapter 2. The proof utilizes important ideas of Campanato, originally introduced in the context of elliptic equations. The role of the "area excess" as a regularity parameter is then emphasized, and De Giorgi's Lemma - the key result of the theory - is finally presented, in its original form.

An extended version of this Lemma is obtained in Chapter 3. While results of this type are usually proved "by contradiction" (see however the recent paper of Schoen and Simon [34]), we have been able to modify the original argument of De Giorgi, to get a simpler direct proof of the Lemma. Moreover this way the constants involved can be explicitly computed.

The proof of the Regularity Theorem is then completed in Chapter 4. Some notes and a bibliography conclude the work. Connections with related papers, notably with the important works of Almgren, Bombieri, and Schoen-Simon, are indicated, particulary at the beginning of Chapter 1 and at the and of Chapter 2.

I am indebted to many collegues and friends for a number of stimulating discussions during the preparation of this work. I am particularly grateful to U.Massari and M.Miranda for valuable comments and suggestions, and to E.H.A. Gonzalez for his kind hospitality at the University of Lecce. Without their constant interest and encouragement, these notes could hardly have appeared in the present form.

Finally, I would like to thank Ms. Anna Palma for her excellent typing of the manuscript.

CHAPTER 1: ALMOST MINIMAL BOUNDARIES

We begin with an informal presentation of the material to be discussed in the sequel. While doing this, free use will be made of concepts and elementary results that will be discussed more deeply later on (especially in Chapter 3).

1.1. We fix an open set Ω in $\mathbb{R}^n$, $n \geq 2$, and consider sets E,F,... whose boundaries $\partial E, \partial F, \ldots$ have locally finite "surface area" in Ω:

$$|D\phi_E|(A) < +\infty \qquad \forall A \subset\subset \Omega .$$

The quantity $|D\phi_E|(A)$ may be thought of as the area (in some generalized sense) of $\partial E \cap A$, where A is an arbitrary open an bounded set, strictly contained in Ω; indeed, it coincides with $H_{n-1}(\partial E \cap A)$ whenever ∂E is, locally within Ω, a smooth hypersurface (section 3.1).

A basic definition and a corresponding fundamental result are now in order.

1.2. Definition of minimal boundaries.

We say that the boundary of E is *minimal* in Ω iff

$$(1.1) \qquad |D\phi_E|(A) \leq |D\phi_F|(A) \qquad \forall A \subset\subset \Omega,\ \forall F: F \Delta E \subset\subset A$$

.e., iff any local variation of E in Ω increases surface area (Fig.1).

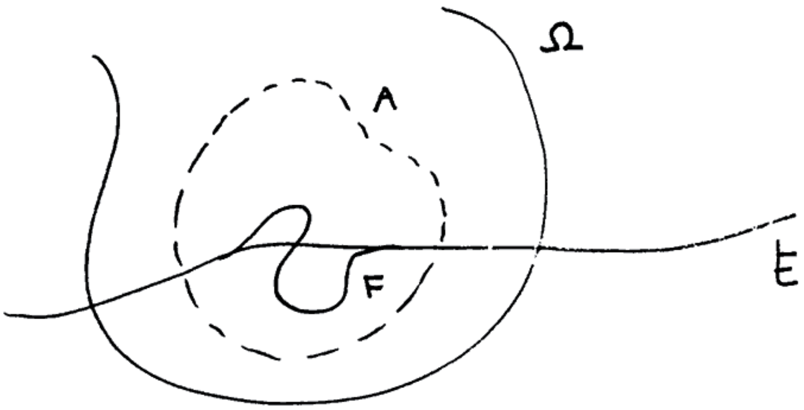


FIGURE 1

1.3. Regularity of minimal boundaries.

Let E have minimal boundary in $\Omega \subset \mathbb{R}^n$. Then

(r_1) $\partial^* E \cap \Omega$ is an analytic hypersurface

(r_2) $H_{n-1}[(\partial E \setminus \partial^* E) \cap \Omega] = 0$.

Furthermore, assuming that $\{E_h\}$ be a sequence of sets with minimal boundary in Ω, and that $\{x_h\}$ be a corresponding sequence of points, satisfying:

$$E_h \to E_\infty \ , \quad x_h \in \partial E_h ; \quad x_h \to x_\infty \in \Omega$$

then

(r_3) $x_\infty \in \partial E_\infty$.

If in addition $x_\infty \in \partial^* E_\infty$, then

(r_4) $x_h \in \partial^* E_h$ for any large h, and $\nu_{E_h}(x_h) \to \nu_{E_\infty}(x_\infty)$.

We recall that $\partial^* E$ denotes the "reduced boundary" of E, i.e. the collection of those points $x \in \partial E$ where an approximate inner unit normal vector $\nu_E(x)$ exists, and that the convergence $E_h \to E_\infty$ is to be intended in the L^1_{loc} - sense on Ω. See section 3.1 again.

1.4. Conclusion (r_1) above, which undoubtedly contains the essence of the Regularity Theorem, was proved by E. de Giorgi in 1960-61 (see [8] and [9]), and then rederived together with (r_2) in 1965 by M.Miranda (see [28])[1]. Two years after, Miranda proved (r_3) and (r_4) as well, see [29]. Thus, in 1967 the Regularity Theorem for minimal boundaries (in the form appearing above) was completely demonstrated. In the setting of Caccioppoli sets, i.e. sets with finite surface area, Theorem 1.3 may well be called the basic regularity result.

In the meantime, various different settings were proposed, in which the classical questions related to Plateau's problem (the problem of finding a surface of least area among those surfaces which span a given curve) could receive a satisfactory answer. We mention the work of Reinfenberg [32,33], Féderer-Fleming [14], Fleming [15], Almgren [3], Allard [1], etc. A considerable effort was directed toward a complete understanding of the structure of the singularities of minimal boundaries: the work Fleming [15], Triscari [38], Almgren [2] and Simons [35] culminated in the celebrated proof -first given by Bombieri, De Giorgi and Giusti in 1969 (see [6] and also [26])- of the minimality of the cone

$$C=\{x\in\mathbb{R}^8 : x_1^2 + \dots + x_4^2 < x_5^2 + \dots + x_8^2\}$$

(*Simons' cone*), which is singular at the origin. As a consequence, the best possible estimate of the Hausdorff dimension of the singular set $\partial E \setminus \partial^* E$ could be obtained by H.Federer [13], thus improving (r_2) above.

More general variational problems of "least area" type have since been considered, especially those concerning surfaces of prescribed mean curvature, possibly with obstacles or subject to given constraints. In this respect, the work of Almgren [4] is really impressive, for both the deepness and the generality of the results obtained. Working with different methods, E. Bombieri [5] and R. Schoen - L. Simon [34] developed quite recently a simplified version of (part of) Almgren's Regularity Theory.[2]

Restricting our attention to the theory of Caccioppoli sets in $\mathbb{R}^n$, we should mention the important contribution of Miranda [30] and Massari [23, 24], on the obstacle problem for minimal boundaries and, respectively, on the regularity of boundaries of prescribed mean curvature. These two problems will be properly discussed later on in this chapter.

Urged by the consideration of these and other particular cases, one is naturally led to the search of a class of "almost minimal boundaries", for which a Regularity Theorem like Theorem 1.3 could be proved. In this respect, the following definition seems quite natural:

1.5. <u>Definition of almost minimal boundaries</u>[3].

The boundary of E is said to be *almost minimal* in $\Omega \subset \mathbb{R}^n$ iff for every $A \subset\subset \Omega$ there exist $T \in (0, \mathrm{dist}(A, \partial\Omega))$ and $\alpha : (0,T) \to [0, +\infty)$, with α non-decreasing and [4] $\alpha(t) = o(1)$, such that

$$(1.2) \qquad |D\phi_E|(B_{x,t}) \leq |D\phi_F|(B_{x,t}) + \alpha(t) \cdot t^{n-1}$$

for every $x \in A$, every $t \in (0,T)$, and every $F : F \Delta E \subset\subset B_{x,t}$

As usual, $B_{x,t}$ denotes the open n-ball with centre x and radius t. See Fig. 2.

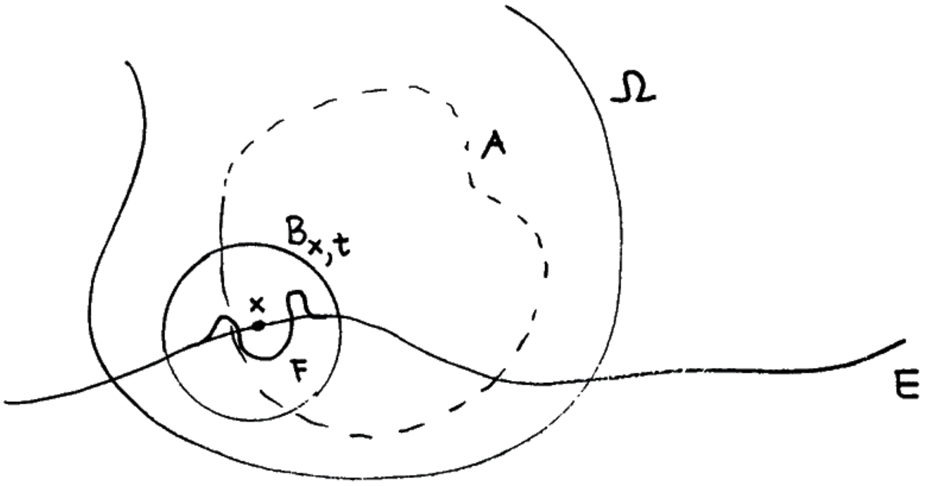


FIGURE 2.

1.6. The assumptions on $\alpha(t)$ are in some sense the minimal ones we can make if we want to prove regularity. Indeed, let us consider a smooth hypersurface S in $\mathbb{R}^n$, and let us choose $R > 0$ and the reference system so that

$$S \cap Q_R = \text{graph of } u \text{ over } B'_R$$

with $u \in C^1(B'_R)$, $u(0)=0$, $Du(0)=0$. Here, Q_R denotes the "vertical" cylinder

$$Q_R = \{x = (x', x_n) \in \mathbb{R}^n : |x'| < R, \ |x_n| < R\}$$

and B'_R its projection on the space of the first (n-1) variables:

$$B'_R = \{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \ |x'| < R\} .$$

See Fig. 3.

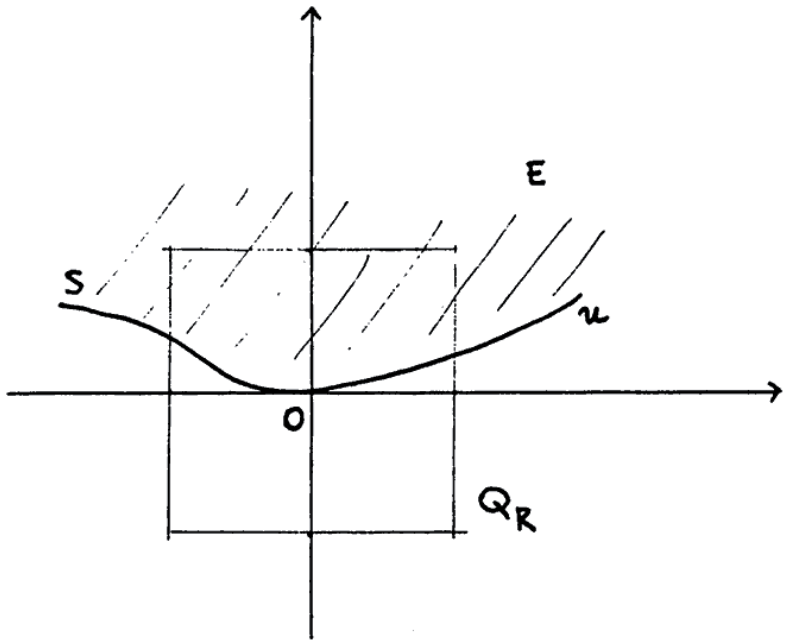


FIGURE 3.

Now, given an (n-1)-ball $B'_t \subset B'_R$ (with centre O), and a function $v \in C^1(B'_R)$ such that the support of u-v lies in B'_t, we have:

$$\int_{B'_t} (1+|Du|^2)^{\frac{1}{2}} dx' - \int_{B'_t} (1+|Dv|^2)^{\frac{1}{2}} dx' \leq$$

$$\leq \int_{B'_t} [(1+|Du|^2)^{\frac{1}{2}} - 1] dx' \leq \frac{1}{2} \int_{B'_t} |Du|^2 dx' \leq \alpha(t) \cdot t^{n-1}$$

provided we take:

$$\alpha(t) \equiv \frac{1}{2} \omega_{n-1} \cdot \sup_{B'_t} |Du(x')|^2 .$$

From this one easily concludes that (1.2) holds for E = epi(u), with a function $\alpha(t)$ of the type described in Definition 1.5. Consequently, any set with smooth (C^1) boundary has almost minimal boundary, in the sense of Def. 1.5.

1.7. On the other hand, we see that the cone

$$E = \{x = (x_1, x_2) : |x_1| < |x_2|\} \subset \mathbb{R}^2$$

whose boundary has a singularity at 0, satisfies

$$|D\phi_E|(B_t) \leq |D\phi_F|(B_t) + \alpha(t)\cdot t \quad \forall F : F \Delta E \subset\subset B_t$$

the *best* choice fot $\alpha(t)$ being the *constant* $2(2-\sqrt{2})$ (see Fig. 4). This also shows the "necessity" of the assumptions on $\alpha(t)$.

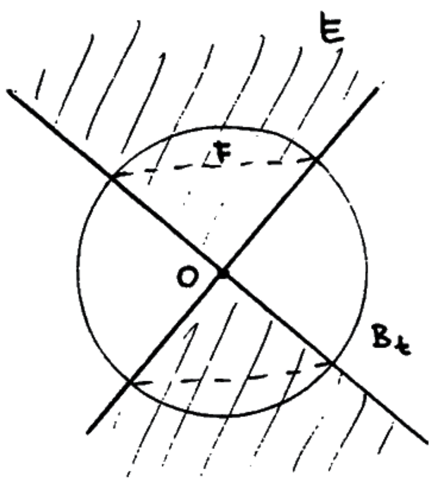


FIGURE 4.

.8. A second important class of almost minimal boundaries is
stituted by surfaces of prescribed mean curvature which satisfies
suitable integrability condition. To be specific, let us consider
ven function $H \in L^1_{loc}(\Omega)$, together with a *local minimizer* E of
the following functional[5]

$$(1.3) \qquad \mathscr{F}_H(F,A) = |D\phi_F|(A) + \int_{A\cap F} H(x)dx$$

which is defined for any Caccioppoli set F and any $A \subset\subset \Omega$. That is, suppose that

$$(1.4) \qquad \mathscr{F}_H(E,A) \leq \mathscr{F}_H(F,A) \qquad \forall A \subset\subset \Omega\ , \quad \forall F : F \Delta E \subset\subset A.$$

Whenever $\partial E \cap A$ is locally of class C^2, and H is a continuous function on Ω , we see immediately that the mean curvature of ∂E coincides, at any point $x \in \partial E \cap \Omega$, with $H(x)$ (just compute the first variation of $\mathscr{F}_H$!). Motivated by the preceding observation, we call the local minimizers of $\mathscr{F}_H$: "sets of generalized mean curvature H in Ω".

Now, for any ball $B_t \subset\subset \Omega$, we have from (1.3),(1.4):

$$(1.5) \qquad |D\phi_E|(B_t) \leq |D\phi_F|(B_t) + \int_{B_t} |H(x)|dx$$

Assuming $H \in L^n_{loc}(\Omega)$, we find (according to Hölder's inequality) that (1.2) holds true, with (essentially)

$$\alpha(t) = \omega_n^{1-1/n} \left(\int_{B_t} |H|^n\, dx\right)^{1/n}$$

More precisely, we observe that if E is a local minimizer of $\mathscr{F}_H$ in Ω, $A \subset\subset \Omega$, and $T \in (0,\mathrm{dist}(A,\partial\Omega))$, then $\int_{B_{x,t}} |H|^n\, dy$ is, for any $t \in (0,T)$, a continuous function of $x \in \bar{A}$, which will then achieve its maximum value at some point $x_t \in \bar{A}$. Setting

$$\alpha(t) = \omega_n^{1-1/n} \cdot \left(\int_{B_{x_t,t}} |H|^n dx \right)^{1/n} \tag{1.6}$$

we get (1.2), with a function $\alpha(t)$ depending on A, non-decreasing on (0,T) and infinitesimal at 0. We shall return to this problem later on, in section 1.14.

We are now in a position to state our main result (compare with Theorem 1.3).

1.9. Regularity Theorem for almost minimal boundaries.

Let E have almost minimal boundary in $\Omega \subset \mathbb{R}^n$, in the sense of Def. 1.5. Assume in addition that $\alpha(t)$ (the function appearing in (1.2)) be such that

$$t^{-1} \cdot \alpha(t) \quad \text{is non-increasing on} \quad (0,T)$$

and

$$\int_0^T t^{-1} \cdot \alpha^{1/2}(t)\, dt < +\infty.$$

Then

(R_1) $\partial^* E \cap \Omega$ is a C^1 hypersurface

(R_2) $H_s[(\partial E \setminus \partial^* E) \cap \Omega] = 0 \qquad \forall s > n - 8.$

Furthermore, assuming that $\{E_h\}$ be a sequence of sets with *uniformly* almost minimal boundaries in Ω (i.e., such that (1.2) holds for every E_h, with T and $\alpha(t)$ independent of h), and that $\{x_h\}$ be a corresponding sequence of points, satisfying

$$E_h \to E_\infty \ , \ x_h \in \partial E_h, \quad x_h \to x_\infty \in \Omega$$

then

$$(R_3) \qquad\qquad x_\infty \in \partial E_\infty \ .$$

If in addition $x_\infty \in \partial^* E_\infty$, then

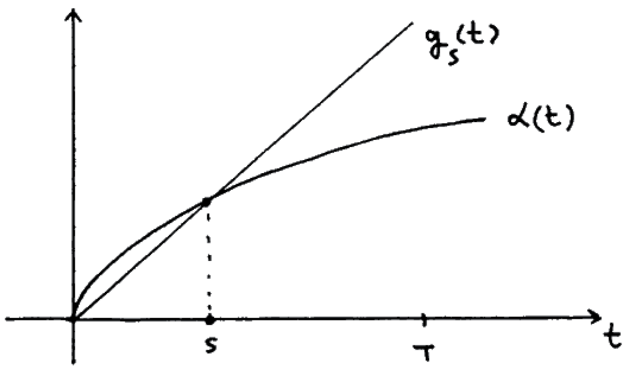
$$(R_4) \qquad x_h \in \partial^* E_h \ \text{ for any large } h, \text{ and } \ \nu_{E_h}(x_h) \to \nu_{E_\infty}(x_\infty).$$

1.10. Regarding (R_1), we have specifically the following estimate of the oscillation of the unit normals to ∂E :

$$(1.7) \qquad |\nu_E(x) - \nu_E(y)| \le c_1 \int_0^{|x-y|} t^{-1} \alpha^{1/2}(t)dt + c_2|x-y|^{1/2}$$

which holds for every $x,y \in \partial^* E$ such that $|x-y|$ is sufficiently small. Thus, we see that the integrability of $t^{-1} \alpha^{1/2}(t)$ is an essential ingredient of the Regularity Theorem.

The other hypotesis of the Theorem, namely, the monotonicity of $t^{-1}\alpha(t)$, is more a convenience than a necessity, and it is assumed only with the aim of simplifying calculations. At any rate, we see that when α is a (non decreasing, infinitesimal at 0, and) concave function of $t \in (0,T)$, then it certainly satisfies that condition. See Fig. 5.



$\alpha(t) \geq g_s(t)$

$= (t/s)\cdot\alpha(s)$

$\forall t \in (0,s)$.

FIGURE 5.

The proof of Theorem 1.9 occupies the second part of the present work (Chapters 3 and 4). Before starting with the formal demonstration, it seems appropriate to illustrate with examples the applicability of the Theorem itself, and to discuss in some details the method of the proof.

For convenience ot the reader, we list the assumptions on $\alpha(t)$ under which Theorem 1.9 will be proved.

1.11. Complete set of hypotheses on $\alpha(t)$.

(α_1) $\alpha : (0,1) \to \mathbb{R}$ is non-decreasing and bounded

(α_2) $\alpha(t) = o(1)$

(α_3) $t^{-1}\alpha(t)$ is non-increasing on $(0,1)$

(α_4) $\int_0^1 t^{-1}\,\alpha^{1/2}(t)dt < +\infty$.

1.12. Now we consider some explicit examples.

The simplest choice is perhaps $\alpha(t) = c.t^{2\alpha}$; conditions above are then all satisfied, for any $c \geq 0$ and any $\alpha \in (0,\frac{1}{2}]$.

Moreover, in this case[6] (1.7) becomes

$$(1.8) \qquad |\nu_E(x) - \nu_E(y)| \leq \text{const} \cdot |x-y|^\alpha$$

which amounts to saying that $\partial^* E \in C^{1,\alpha}$. In a sense, this is an optimal result, since the converse is also true (and easy to prove, see Example 1.14 (v) below); that is, if $E = \text{epi}(u)$ with $u \in C^{1,\alpha}$, then ∂E is almost minimal, with $\alpha(t) \leq \text{const}.t^{2\alpha}$.

For $\varepsilon > 0$, the function

$$\alpha(t) = c\ [\lg(e/t)]^{-2(1+\varepsilon)}$$

(truncated at a suitable level, in order to save concavity), also satisfies $(\alpha_1 - \alpha_4)$. On the contrary, it does not satisfy (α_4) when $\varepsilon = 0$.

A similar behaviour is exhibited by the function

$$\alpha(t) = c\ [\lg(e/t)]^{-2} \cdot [\lg(e\ \lg(e/t))]^{-2(1+\varepsilon)} .$$

1.13. Before giving examples of almost minimal boundaries, we introduce the following functional

$$(1.9) \qquad \psi(E,A) = |D\phi_E|(A) - \inf\{|D\phi_F|(A) : F \Delta E \subset\subset A\}$$

which is defined for every Caccioppoli set E and every $A \subset\subset \mathbb{R}^n$. With the aid of ψ, the definition of almost minimality can be restated in a more compact form, just by replacing (1.2) of Def. 1.5 by

$$(1.10) \qquad \psi(E,B_{x,t}) \leq \alpha(t) \cdot t^{n-1} \qquad \forall x \in A\ ,\ \forall t \in (0,T).$$

The quantity $t^{1-n}\,\psi(E,B_{x,t})$ may be called "Deviation from minimality of E in $B_{x,t}$", abbreviated: Dev(E,x,t). Thus, almost minimal boundaries are those boundaries whose deviation from minimality is controlled from above, in any ball $B_{x,t}$, by a suitable function of the radius t, which is non-decreasing in t and infinitesimal at 0.

Simple properties of the functional ψ are stated and proved in Chapter 3, see (3.10)-(3.13). For the present usage, we anticipate that ψ is non-decreasing with respect to the second variable, i.e. $\psi(E,A_1) \leq \psi(E,A_2)$ whenever $A_1 \subset A_2$.

1.14. Examples of almost minimal boundaries.

(i) *Minimal boundaries* (Def. 1.2) are evidently almost minimal, with $\alpha(t) \equiv 0$ (and $\psi \equiv 0$).

(ii) *Boundaries with prescribed generalized mean curvature.*

We saw in the preceding pages that any local minimizer of the functional $\mathcal{F}_H$ (defined by (1.3)) in Ω, corresponding to a mean curvature function $H \in L^n_{loc}(\Omega)$, has almost minimal boundary in Ω, since it verifies (1.2) (or equivalently (1.10)), with a function $\alpha(t)$ - given by (1.6) - satisfying (α_1) and (α_2) of Hypotheses 1.11. It may happen however, that the remaining assumptions (α_3),(α_4) are not satisfied. We see this with the aid of the following example.

Consider, for $n = 2$, the function

$$H(x) = H(x_1,x_2) = -Mr^{-1}(\lg(e/r))^{-5/2} \tag{1.11}$$

where $r = |x| \in (0,1]$ and $M > 1$. For every $t \in (0,1]$ we have

$$\int_{B_t} |H|^2\, dx = (\pi/2)M^2(\lg(e/t))^{-4}$$

so that $H \in L^2(B_1)$. However, with $\alpha(t) = (\int_{B_t} |H|^2 dx)^{1/2}$, see (1.6), we find

$$\int_0^1 t^{-1} \alpha^{1/2}(t)\, dt = +\infty$$

thus violating condition (α_4).

Nevertheless, putting

$$H(x) = 0 \quad \text{for} \quad r = |x| \in (1,2] \tag{1.12}$$

and taking into account the symmetry of the problem, we realize immediately that $E = B_1$ is local minimizer of $\mathscr{F}_H$ in $\Omega = B_2$, at least when M is large enough. Indeed, if

$$M > \left[\int_0^1 (\lg(e/s))^{-5/2}\, ds\right]^{-1}$$

then B_1 is the *unique* solution to the problem [5]

$$|D\phi_E|(\bar{B}_2) + \int_E H(x)dx \to \min, \quad \text{with} \quad E \subset B_2.$$

Therefore, the choice (1.11),(1.12) for H provides no counterexample to regularity! Actually, the question whether a regularity theorem holds for boundaries of prescribed mean curvature $H \in L^n_{loc}(\Omega)$ has not been settled in full generality. We shall return to this question in a moment, after discussing the general case $H \in L^p_{loc}(\Omega)$, with either $1 \leq p < n$ or $n < p \leq +\infty$.

In the former instance, simple examples show that singularities may appear, even in low dimension; while, on the contrary, the conclusions ot Theorem 1.9 hold in the latter case.

We see this as follows: first, notice that the lipschitz function $w(x) = |x|$, $x \in \mathbb{R}^n$, $n \geq 2$, is a *weak solution* of the non-homogeneous minimal surface equation

$$\text{(1.13)} \qquad \operatorname{Div} \frac{Dw(x)}{(1+|Dw(x)|^2)^{\frac{1}{2}}} = h(x)$$

corresponding to $h(x) = (n-1)/\sqrt{2}|x|$ - a function which belongs to the Lebesgue class $L^{\tau n}_{loc}(\mathbb{R}^n)$ for every $\tau < 1$. By this we mean, as usual, that

$$\int < \frac{Dw}{(1+|Dw|^2)^{\frac{1}{2}}}, D\phi> dx = - \int h\phi \, dx \qquad \forall \phi \in C^1_o(\mathbb{R}^n)$$

(w is of course a classical C^2 solution of (1.13) in $\mathbb{R}^n \setminus \{0\}$). Consequently, the mean curvature of the graph of w (an n-dimensional cartesian surface in $\mathbb{R}^{n+1}$) is summable to any power less than n (notice that when n=1, the corresponding mean curvature is given by $\sqrt{2}$ times the Dirac mass at 0).

Next, put $E = \text{epi}(u) \subset \mathbb{R}^n$, with $u(x') = |x'|$, $x' \in \mathbb{R}^{n-1}$, $n \geq 2$, and consider a solution G of the following problem [5]

$$|D\phi_G|(\bar{B}_1) + \int_{B_1 \cap G} H(x)dx \to \min, \quad \text{with } G \setminus \bar{B}_1 = E \setminus \bar{B}_1$$

where $H(x) = -c/|x|$ if $x \in E$, $= 0$ otherwise (c being a large positive constant). Plainly, $G \cap B_1 \subset E \cap B_1$, since E is convex and H vanishes outside E. Furthermore, $0 \in \partial G$, suice from

$$|D\phi_{E_r}| + \int_{E_r} H dx < 0 \qquad (E_r \equiv E \cap B_{o,r})$$

which holds $\forall r \in (0,1)$ if c is large enough, we derive

$$|D\phi_{G \cup E_r}|(\bar{B}_1) + \int_{(G \cup E_r) \cap B_1} H dx < |D\phi_G|(\bar{B}_1) + \int_{G \cap B_1} H dx$$

whenever $G \cap E_{2r} = \emptyset$ (see Fig. 6).

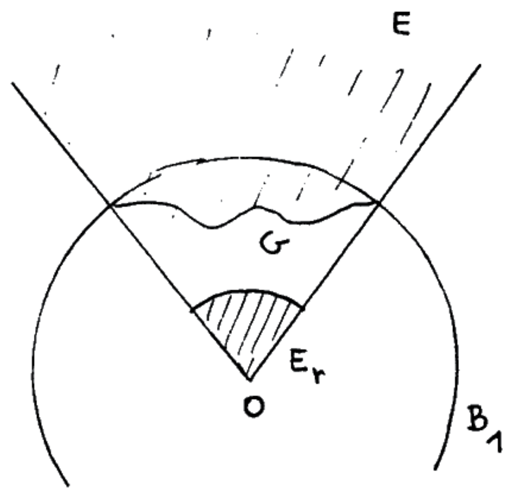


FIGURE 6.

In conclusion, G is a local minimizer of $\mathscr{F}_H$ in B_1, with $H \in L^{\tau n}(B$ $\forall \tau < 1$, and 0 is a singular point of ∂G, thus showing that Theorem 1.9 (particularly, conclusion (R_2)) does not hold for boundaries of prescribed mean curvature $H \in L^p_{loc}(\Omega)$, with $p < n$.

On the other hand, if E is a local minimizer of $\mathscr{F}_H$ in Ω, with $H \in L^p_{loc}(\Omega)$ and $p > n$, and if $A \subset\subset \Omega$, $T < \text{dist}(A, \partial\Omega)$, $x \in A$, $t \in (0, T$ then (1.5), (1.10) and Hölder inequality yield:

$$\psi(E,B_{x,t}) \leq \int_{B_{x,t}} |H|\,dy \leq \omega_n^{1-1/p} \, \|H\|_{L^p(B_{x,t})} \; t^{n-n/p}$$

$$\leq \text{const}(n, \|H\|_{L^p(A_T)}) \cdot t^{1-n/p} \cdot t^{n-1}$$

$$\equiv \alpha(t) \cdot t^{n-1}$$

where $A_T = \{x : \text{dist}(x,A) < T\} \subset\subset \Omega$. Since $1-n/p \in (0,1)$ in this case, we know (recall 1.12) that $\alpha(t) = c \cdot t^{1-n/p}$ satisfies $(\alpha_1 \div \alpha_4)$, so that the Regularity Theorem 1.9 applies to this case.

We can also consider mean curvature functions H belonging to more general function spaces. Let us introduce e.g. the Morrey space $L^{p,\lambda}_{loc}(\Omega)$ $(p \geq 1, \lambda \geq 0)$:

$$u \in L^{p,\lambda}_{loc}(\Omega) \text{ iff } u \in L^p_{loc}(\Omega) \text{ and } \forall A \subset\subset \Omega :$$

(1.14)
$$\sup_{\substack{x \in A \\ 0<t<\text{diam}A}} \left(t^{-\lambda} \int_{A \cap B_{x,t}} |u|^p \, dy\right) < +\infty$$

Some elementary properties of Morrey spaces can be found in [20], Chapter 4.

It should be clear by the foregoing considererations that any local minimizer of $\mathscr{F}_H$ in Ω , with $H \in L^{1,n-1+\alpha}_{loc}(\Omega)$ and $\alpha > 0$, satisfies (1.10) with $\alpha(t) = \text{const}.t^{\alpha}$, so that Theorem 1.9 applies equally well to the present situation. Notice that $L^p_{loc} \subset L^{1,n-1+(p-n)/p}$, by Hölder inequality, so that the case $H \in L^p_{loc}(\Omega)$ with $p > n$ appears as a particular instance in this general picture.

As previously seen, things are not so clear in the borderline case $H \in L^n_{loc}(\Omega)$: in particular, we do not know yet whether conclusions (R_1) and (R_2) of Theorem 1.9 extend to local minimizers of $\mathscr{F}_H$, when $H \in L^n_{loc}(\Omega)$. The following example may shed some light on the question.

Let $E \subset \mathbb{R}^n$, $n \geq 3$, be the epigraph of a radial function $u = u(r)$, where $r = |x'|$ and $x' \in \mathbb{R}^{n-1}$. Assume that $u \in C^1(0,+\infty)$, with derivative $u' > 0$, and that

$$u(r) \to 0 \ , \quad u'(r) \to M \in [0,+\infty] \tag{1.15}$$

as $r \to o^+$. See Fig. 7.

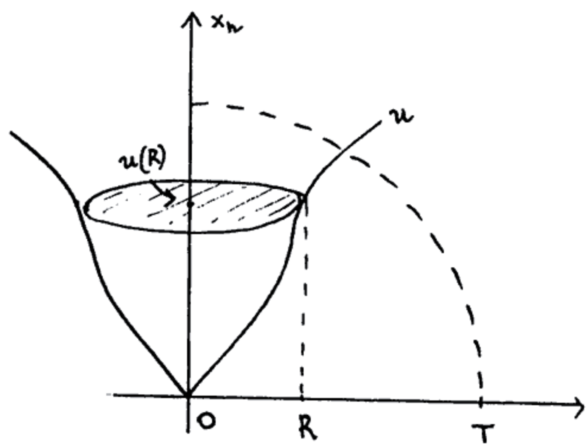


FIGURE 7.

If E is a local minimizer of $\mathscr{F}_H$ in $\mathbb{R}^n$, with $H \in L^n_{loc}(\mathbb{R}^n)$, and if E_R denotes the set

$$E_R = E \cap \{x : x_n > u(R)\}$$

then $\forall R > 0$ and $\forall T > (R^2+u^2(R))^{\frac{1}{2}}$ we find (see (1.3) and Fig. 7):

$$\begin{aligned} 0 &\le \mathscr{F}_H(E_R,B_T) - \mathscr{F}_H(E,B_T) \\ &= \omega_{n-1} R^{n-1} - (n-1)\omega_{n-1} \int_0^R r^{n-2}(1+u'^2(r))^{\frac{1}{2}} dr - \int_{E\setminus E_R} H(x)dx \end{aligned} \tag{1.16}$$

On using succesively Hölder inequality and the isoperimetric inequality we get

$$\int_{E\setminus E_R} |H|dx \le n^{-1}\omega_n^{-1/n} \|H\|_{L^n(E\setminus E_R)} |D\phi_{E\setminus E_R}| \equiv g_n(R)|D\phi_{E\setminus E_R}| \tag{1.17}$$

which, combined with (1.16), yields

$$(n-1)[1-g_n(R)] \int_0^R r^{n-2}(1+u'^2)^{\frac{1}{2}} dr \le [1+g_n(R)]\cdot R^{n-1}. \tag{1.18}$$

Were M in (1.15) positive, we would deduce from (1.18)

$$[1-g_n(R)]\cdot(1+\varepsilon^2)^{\frac{1}{2}} \le 1+g_n(R) \qquad \forall\varepsilon \in (0,M),\ \forall R < R_\varepsilon,$$

a contradiction, since $g_n(R) \to 0$ when $R \to 0$ (see (1.17)).

Therefore, $M = 0$, thus showing that ∂E is everywhere smooth.

(iii) Minimal boundaries with a volume constraint.

It is a well-known fact that among the sets having a given measure v in $\mathbb{R}^n$, the n-ball B_R of radius $R = (v/\omega_n)^{1/n}$ is the one which minimizes surface area. A variety of (less trivial) examples of the same type are usually encountered in Capillarity Theory. For instance, one can think of a liquid drop of given mass and resting on a given surface (as shown in Fig. 8), as a local minimizer of an "energy functional" (whose analytic expression is, roughly speaking, the sum of "surface terms" plus "curvature terms", corresponding respectively to the surface forces - like surface tension - and body forces - like gravity - acting on the drop) in a certain class of admissible configurations, all with the same fixed mass (see e.g. [11]).

FIGURE 8

Let us now introduce the following abstract definition:

E *has minimal boundary in* Ω *with a volume constraint iff*

(1.19) $$|D\phi_E|(A) \leq |D\phi_F|(A) \quad \forall A \subset\subset \Omega,\ \forall F : F \Delta E \subset\subset A \ \textit{and}\ |F\cap A| = |E\cap A|$$

where $|G|$ denotes the Lebesgue measure of $G \subset \mathbb{R}^n$. The preceding definition extends to "curvature functionals" like (1.3) in the obvious way.

We proved in [21] that such an E satisfies (1.10) with $\alpha(t) = ct$: Theor. 1.9 then yields the regularity of ∂^*E, together with the usual estimate of the Hausdorff dimension of $\partial E \setminus \partial^* E$.

Actually, the main body of [21] was devoted to the proof of the fact, that whenever (1.19) holds (with, of course, $|E \cap A| > 0$ and $|A \setminus E| > 0$), then two balls B_1, B_2 of arbitrarily small radius r can be found, such that $B_1 \subset E \cap A$ and $B_2 \subset A \setminus E$.

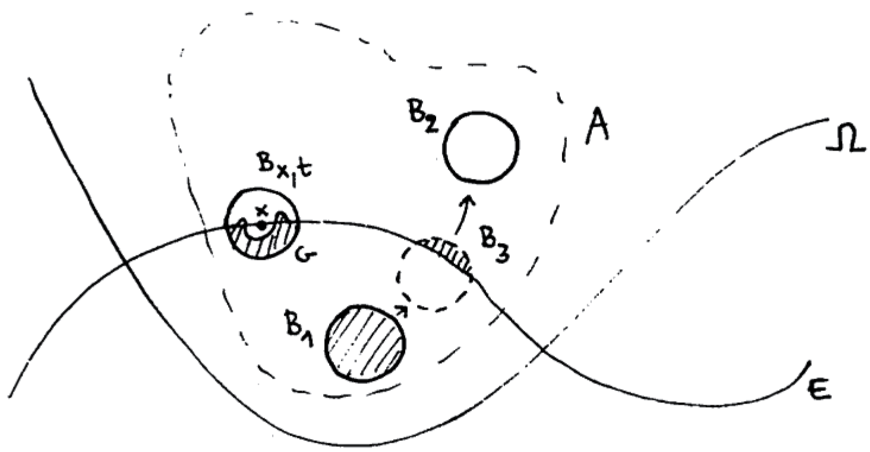


FIGURE 9.

Assuming this, and having fixed $x \in \partial E \cap A$, $t \in (0,r)$, and G such that

$$(1.20) \qquad B_{x,t} \subset\subset A \setminus (\overline{B_1 \cup B_2}), \quad G \,\Delta\, E \subset\subset B_{x,t}, |G \cap B_{x,t}| < |E \cap B_{x,t}$$

we see that we can move B_1 toward B_2 (in a continuous fashion, and taking care of remaining strictly within $A \setminus \bar{B}_{x,t}$; see Fig. 9), until a new position, denoted by B_3, is reached, such that

$$|F \cap A| = |E \cap A|, \text{ where } F \equiv (G \cap B_{x,t}) \cup (E \setminus B_{x,t}) \cup B_3.$$

From (1.19) we then derive essentially (see [21], prop. 1, for the precise calculations):

$$|D\phi_E|(B_{x,t}) \leq |D\phi_G|(B_{x,t}) + \int_{\partial B_3} \phi_{E\cup B_3}\, dH_{n-1} - |D\phi_E|(B_3)$$

(1.21) $$\leq |D\phi_G|(B_{x,t}) + (n/r)|B_3 \setminus E|$$

(see section 1.15) to follow for the proof of this last inequality). As the case when $|G\cap B_{x,t}| \geq |E\cap B_{x,t}|$ can be treated similarly

(just by interchanging the role of B_1, B_2), we see that (1.21) holds $\forall G : G\Delta E \subset\subset B_{x,t}$. In addition, the way B_3 was chosen shows that $|B_3 \setminus E| \leq \omega_n t^n$, which, combined with (1.21), yields

$$\psi(E, B_{x,t}) \leq (n/r)\omega_n\, t^n$$

as claimed.

(iv) Minimal boundaries with obstacles.

A second constraint we can impose on our solution is that it has to avoid some "obstacle". Stated more precisely:

E has minimal boundary in Ω with respect to the obstacle L iff

(1.22) $$E \supset L\cap\Omega \quad \textit{and} \quad |D\phi_E|(A) \leq |D\phi_F|(A)$$

$$\forall A \subset\subset \Omega,\ \forall F : F\Delta E \subset\subset A \quad \textit{and} \quad F\supset L\cap A.$$

Assuming this, and having fixed $A \subset\subset \Omega$ and $F : F\Delta E \subset\subset A$, we get

$$|D\phi_E|(A) \leq |D\phi_{F\cup L}|(A)$$

since $F \cup L$ is an admissible variation of E (i.e. $F \cup L \supset L \cap A$; see Fig. 10).

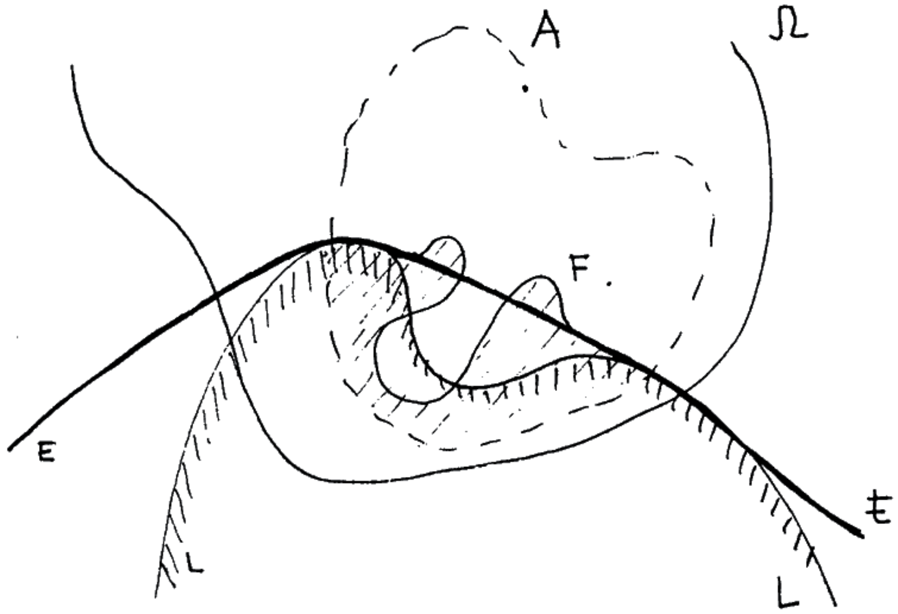


FIGURE 10.

By virtue of the inequality

$$|D\phi_{E_1 \cup E_2}|(A) + |D\phi_{E_1 \cap E_2}|(A) \leq |D\phi_{E_1}|(A) + |D\phi_{E_2}|(A)$$

(see [27], 2.1.2 (10)), we then find

$$|D\phi_E|(A) - |D\phi_F|(A) \leq |D\phi_L|(A) - |D\phi_{F \cap L}|(A)$$

$$\leq \psi_o(L,A) \tag{1.23}$$

where (see (1.9)):

$$\psi_o(L,A) = |D\phi_L|(A) - \inf\{|D\phi_G|(A) : G \Delta L \subset\subset A, G \subset L\} \leq \psi(L,A). \tag{1.24}$$

On taking the supremum of the left-hand side of (1.23), as F varies *freely* among the local variations of E in A, we get in conclusion:

$$\psi(E,A) \leq \psi_o(L,A) \leq \psi(L,A). \tag{1.25}$$

Thus, we realize that in any ball $B \subset\subset \Omega$ the deviation from minimality of E (a solution of the least area problem with obstacle L) is controlled by the deviation from minimality of the obstacle itself! (see 1.13).

Therefore, whenever ∂L is almost minimal in Ω, the same is true for ∂E. In fact, in view of example (v) immediately following, by using (1.25), the Regularity Theorem 1.9, and (essentially) the fact that a set with minimal boundary in $\mathbb{R}^n$, which in addition contains a half space, is ifself a half space (see [30], Theor. 1), we can prove that if ∂L is of class $C^{1,\alpha}$ in Ω $(0 < \alpha < 1)$, then ∂E is likewise of class $C^{1,\alpha}$ in a neighbourhood of ∂L. We refer to section 3 of [37] for a deeper analysis of the regularity of minimal boundaries with obstacles.

We remark that the result just quoted holds for $\alpha = 0$ as well (see [30], Theor.2). The proof in this case requires special attention, and in fact that result *cannot* be deduced directly from (1.25) and Theor. 1.9 alone (the reason being that the deviation from minimality of a set with C^1 boundary cannot be controlled, in general, by a "good" function $\alpha(t)$ - in particular, one satisfying (α_4) of Hypotheses 1.11; see example (v) below). The case $\alpha = 1$ also requires a special analysis, see [7]. Finally, we remark that the regularity result just quoted does *not* generally hold, for obstacles with lipschitz boundary (in constrast to what happens in the "cartesian case",see e.g. [10,16,18]).

To see this, merely consider the lipschitz cone

$$L = \{x \in \mathbb{R}^8 : x_8 > (x_1^2 + \ldots + x_7^2)^{\frac{1}{2}}\}$$

which is contained in Simons'cone C (see 1.4): although C has minimal boundary in $\mathbb{R}^n$ (and thus also with respect to L), ∂C is not lipschitz (not even a local graph near $0 \in \partial L \cap \partial C$.

(v) Smooth hypersurfaces

We saw in section 1.6 that whenever ∂E is of class C^1, then ∂E is in particular an almost minimal boundary. Generally however, the function $\alpha(t)$ which controls the deviation from minimality of E does not satisfy the integrability hypothesis (α_4) of 1.11. Here we have a simple example of such a situation.

Consider the function $u : (-1,1) \to (0,1)$ defined by

$$\begin{cases} u(t) = \int_0^t [\lg(e/s)]^{-1}\, ds & \text{when } t \in (0,1) \\ u(0) = 0, \quad u(t) = u(-t) \end{cases}$$

and put $E = \text{epi}(u) \subset \mathbb{R}^2$, so that ∂E is of class C^1 in the open square $\Omega = (-1,1)^2$.

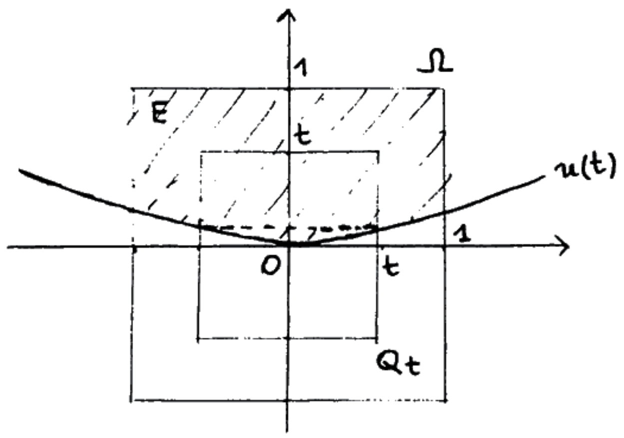


FIGURE 11.

Setting $Q_t = (-t,t)^2$, we find (see Fig. 11):

$$\psi(E,Q_t) = 2 \int_0^t \{[1+(\lg(e/s))^{-2}]^{\frac{1}{2}} - 1\}\, ds$$

that is [4], $\psi(E,Q_t) \sim t[\lg(e/t)]^{-2}$. Consequently, the function controlling the deviation from minimality of E is essentially

$$\alpha(t) = [\lg(e/t)]^{-2}$$

which does not satisfy (α_4) (recall 1.12).

Things are better in the case when ∂E is of class $C^{1,\alpha}$, $0<\alpha<1$, i.e. when (locally) $\partial E = \mathrm{epi}(u)$, with $u \in C^1(\mathbb{R}^{n-1})$ and

(1.26) $$|Du(y') - Du(z')| \leq L\, |y'-z'|^{\alpha}.$$

In this case, arguing as in 1.6, we find

$$\int_{B'_t} (1+|Du|^2)^{\frac{1}{2}}\, dx' - \int_{B'_t} (1+|Dv|^2)^{\frac{1}{2}}\, dx' \leq \frac{1}{2} \int_{B'_t} |Du|^2 dx'$$

$$\leq \frac{1}{2}\, \omega_{n-1}\, L^2 t^{n-1} \cdot t^{2\alpha}$$

(recall we are assuming $Du(o) = 0$), so that [7]

$$\psi(E,B_{x,t}) \leq c.t^{n-1+2\alpha} \qquad \text{whenever} \qquad \partial E \in C^{1,\alpha}.$$

As the preceding example (iv) indicates, this fact is of considerable importance in connection with the obstacle problem for minimal boundaries. Also recall the remark following (1.8).

We conclude the present section with the short proof of an inequality, which find application in several cases (see [36], section 1), and which was used in Example 1.14 (iii) above.

1.15. An isoperimetric inequality.[8]

Given a ball B_R of radius R in $\mathbb{R}^n$ and a subset L of B_R, there holds:

$$\int_{\partial B_R} \phi_L \, dH_{n-1} \leq |D\phi_L|(B_R) + (n/R)|L| \tag{1.27}$$

Equality holds in (1.27) iff either $L = \emptyset$ or $L = B_R$.

Proof. Clearly, (1.27) is homogeneous in R, hence it suffices to prove it when R=1. Assuming this, we apply the Gauss-Green Theorem to the vector field $\Phi(x) = \phi_L(x)\cdot x$, $x \in \mathbb{R}^n$, thus obtaining

$$\begin{aligned}\int_{\partial B_1} \phi_L(x)dH_{n-1}(x) &= n \int_{B_1} \phi_L(x)dx + \int_{B_1} \langle x, D\phi_L(x)\rangle \\ &\leq n\ |L| + |D\phi_L|(B_1)\end{aligned} \tag{1.28}$$

which is (1.27) for R = 1. Recalling that (see [27], 2.3):

$$D\phi_L = \nu_L \, H_{n-1}\big|_{\partial^* L} \, , |D\phi_L| = H_{n-1}\big|_{\partial^* L} \tag{1.29}$$

we conclude that equality holds in (1.28) only if

$$\langle x, \nu_L(x)\rangle = 1 \qquad H_{n-1} \text{ - a.e. on } \partial^* L \cap B_1$$

which is possible only when

$$|x| = 1 \qquad H_{n-1} \text{ - a.e. on } \partial^* L \cap B_1$$

i.e., only when $H_{n-1}(\partial^* L \cap B_1) = 0$. This last assertion implies tha

L is equivalent either to the empty set or to B_1 itself.

The converse being obvious, we are done. From (1.27), observing that $L \subset B_R$ implies $|L| \leq \omega_n R^n$, we deduce

$$(1.30) \qquad \int_{\partial B_R} \phi_L \, dH_{n-1} \leq |D\phi_L|(B_R) + n\omega_n^{1/n} |L|^{(n-1)/n} \qquad \forall L \subset B_R$$

still with equality iff either $L = \emptyset$ or $L = B_R$. We see in addition that the only bounded sets $\Omega \subset \mathbb{R}^n$ for which the inequality

$$(1.31) \qquad \int_{\partial\Omega} \phi_L \, dH_{n-1} \leq |D\phi_L|(\Omega) + n\omega_n^{1/n} |L|^{(n-1)/n}$$

holds, for every choice of $L \subset \Omega$, are exactly the n-balls. To be convinced, put $L = \Omega$ in (1.31) and recall the isoperimetric inequality:

$$(1.32) \qquad |D\phi_E| \geq n\omega_n^{1/n} |E|^{(n-1)/n}$$

which is valid for every bounded $E \subset \mathbb{R}^n$, with equality iff E is an n-ball. See [27], 2.2.2(2).

CHAPTER 2: THE AREA EXCESS AND DE GIORGI'S LEMMA

In this section we shall be primarily concerned with some key ideas underlying the proof of Theorem 1.9. Techniques and concepts relevant to that proof will be introduced in a rather "natural" way, by working out an explicit example in Regularity Theory.

2.1. As we showed in the preceding chapter, in the case when $\alpha(t)$ - the function controlling the deviation from minimality - is of the following type:

$$\alpha(t) = ct^{2\alpha} \quad , \quad 0 < \alpha < 1$$

then we have an "optimal regularity result", in the sense that [9]

$$\partial E \in C^{1,\alpha} \Rightarrow Dev(E,x,t) \leq ct^{2\alpha}$$

(2.1)

$$Dev(E,x,t) \leq ct^{2\alpha} \Rightarrow \partial^* E \in C^{1,\alpha}$$

See 1.12 and 1.14 (v). The appearance of the *reduced* boundary $\partial^* E$ in the last implication is unavoidable, on the account of the existence of minimal cones with singularities. In the special case when ∂E is already known to be of class C^1, we have then clearly a perfectly symmetric situation:

(2.2) if $\partial E \in C^1$, then $\partial E \in C^{1,\alpha} \Longleftrightarrow Dev(E,x,t) \leq ct^{2\alpha}$

It seems convenient to give the simple (relative to that of Theor. 1.9) proof of this fact, one reason being that while doing this we will quickly meet a certain regularity parameter, which will play a basic role in the subsequent sections.

To begin with, we introduce a new class of function spaces, including both the Morrey spaces $L^{p,\lambda}(\Omega)$ (see (1.14)) and the Hölder spaces

$C^{o,\alpha}(\Omega)$.

2.2. Definition of Campanato spaces.

Given: Ω open and bounded in $\mathbb{R}^n$, $p \geq 1$, $\lambda \geq 0$;

we say that

$$u \in \mathscr{L}^{p,\lambda}(\Omega) \text{ iff } u \in L^p(\Omega) \text{ and } \sup_{\substack{x\in\Omega \\ o<t<\operatorname{diam}\Omega}} \left(t^{-\lambda} \int_{\Omega \cap B_{x,t}} |u-u_{x,t}|^p \, dy\right) < +\infty$$

where $u_{x,t}$ (also denoted $\{u\}_{x,t}$ is the average of u on $B_{x,t}$

$$u_{x,t} = \{u\}_{x,t} = |B_{x,t}|^{-1} \int_{B_{x,t}} u(y)dy.$$

A basic fact about Campanato spaces is that $\mathscr{L}^{p,\lambda}$ is isomorphic to $C^{o,(\lambda-n)/p}$, *provided* $\lambda \in (n,n+p]$ *and* $\partial\Omega$ satisfies a suitable regularity condition (e.g. $\partial\Omega$ is locally lipschitz). See [20], Chapter 4, Theor. 1.6.

2.3. For convenience of the reader, we now recall an elementary property of averages:

if $A \subset\subset \mathbb{R}^n$, $u \in L^2(A)$, and $u_A = |A|^{-1} \int_A u dx$, then

$$(2.3) \qquad \int_A |u-u_A|^2 dx = \int_A (|u|^2 - |u_A|^2) dx \leq \int_A |u-\lambda|^2 dx \qquad \forall \lambda \in \mathbb{R}$$

along with some simple facts about harmonic functions:

if $B = B_{x,R} \subset \mathbb{R}^n$, $u \in C^1(\bar{B})$, and v is the harmonic function associated with u on B, i.e. satisfying [10]

$$(2.4)\qquad \begin{cases} \Delta v = v_{x_i x_i} = 0 & \text{in } B \\ v = u & \text{on } \partial B \end{cases}$$

then

$$(2.5)\qquad \int_B \langle Du, Dv\rangle dy = \int_B |Dv|^2\, dy \leq \int_B |Du|^2\, dy$$

$$(2.6)\qquad \int_B |Du-Dv|^2 dy = \int_B (|Du|^2 - |Dv|^2)dy$$

$$(2.7)\qquad \{Du\}_{x,R} = \{Dv\}_{x,r} \qquad \forall r \in (0,R]$$

$$(2.8)\qquad r^{-(n+2)} \int_{B_{x,r}} |Dv - \{Dv\}_{x,r}|^2\, dy \quad \text{is a non-decreasing}$$

function of $r \in (0,R)$.

Assertions (2.5) to (2.7) are easy consequences of the Gauss-Green Theorem. As for (2.8), observe that any weak solution w of a homogeneous elliptic partial differential equation with constant coefficients:

$$a_{ij}\, w_{x_i x_j} = 0$$

satisfies

$$\int_{B_s} |w - \{w\}_s|^2 \leq c_1 (s/t)^{n+2} \int_{B_t} |w-\{w\}_t|^2$$

for a suitable constant c_1 (depending on the ellipticity constant and on n), and for every s,t : $0 < s < t$; see [20], Chapter 4, Lemma 2.2.

The fact that $c_1 = 1$ when w is harmonic requires additional care: its proof may be based upon a classical result about the uniform approximation of harmonic functions by means of homogeneous harmonic polynomials (as in [8]; see e.g. [27], 2.5.2, prop. 1).

Finally, we list two elementary algebraic inequalities

$$(2.9)\quad a^2-b^2 \le 2(1+b^2)^{\frac{1}{2}} \cdot \left[(1+a^2)^{\frac{1}{2}} - (1+b^2)^{\frac{1}{2}}\right] + (a^2-b^2)^2/4$$

$$(2.10)\quad a^2-b^2 \le 2(1+a^2)^{\frac{1}{2}} \cdot \left[(1+a^2)^{\frac{1}{2}} - (1+b^2)^{\frac{1}{2}}\right]$$

both valid $\forall a,b \in \mathbb{R}$ (the proof is a straightforward calculation), together with the following result (see [17], Lemma 2.2):

2.4. A useful Lemma.

For any choice of the constants a,α,β with $a>0$, $\alpha>\beta>0$, it is possible to find two new constants $\varepsilon=\varepsilon(a,\alpha,\beta) > 0$ and $c=c(a,\alpha,\beta)>0$ such that, whenever $\omega : (0,T) \to (0,+\infty)$ is a non-decreasing function, satisfying

$$(2.11)\qquad \omega(s) \le a\left[(s/t)^\alpha + \varepsilon\right]\cdot \omega(t) + bt^\beta$$

for some $T > 0$ and some $b \ge 0$, and for every s,t : $0<s<t<T$, then it holds:

$$(2.12)\qquad \omega(s) \le c\left[(s/t)^\beta\, \omega(t) + bs^\beta\right]$$

still for every s,t : $0 < s < t < T$.

The proof of Lemma 2.4 goes as follows: fix $\gamma \in (\beta,\alpha)$ and $\tau \in (0,1)$

so that $2a\tau^{\alpha} < \tau^{\gamma}$, and the define

$$\varepsilon = \tau^{\alpha} \quad , \; c^{-1} = \tau^{\beta}(\tau^{\beta}-\tau^{\gamma}).$$

Given $s,t : 0 < s < t < T$, consider $t' = t$, $s' = \tau t$, and apply (2.11) to obtain

$$\omega(\tau t) \leq a(\tau^{\alpha}+\varepsilon)\, \omega(t) + bt^{\beta} \leq \tau^{\gamma}\omega(t) + bt^{\beta}$$

in view of our initial assumptions. By induction:

$$\omega(\tau^{k+1}t) \leq \tau^{(k+1)\gamma}\cdot\omega(t) + bt^{\beta}\, \tau^{k\beta}\cdot \sum_{j=o}^{k} \tau^{j(\gamma-\beta)} \qquad \forall\, k \geq o$$

whence

$$(2.13) \qquad \omega(\tau^{k}t) \leq \tau^{k\beta}(\tau^{\beta}-\tau^{\gamma})^{-1}\cdot(\omega(t)+bt^{\beta}) \qquad \forall\, k \geq 0.$$

Since $0 < s < t$, there will exist a unique $k \geq 0$ s.t. $\tau^{k+1}t \leq s < \tau^{k}t$, so that $\tau^{k} \leq \tau^{-1}\cdot(s/t)$. In conclusion, we get

$$\omega(s) \leq \omega(\tau^{k}t) \leq c \;\left[(s/t)^{\beta}\omega(t)+bs^{\beta}\right]$$

by (2.13), the monotonicity of ω, and the choice of c.

2.5. At this point, we dispose of all the ingredients needed for the proof of (2.2). Notice that the validity of the implication $\Longrightarrow$ in (2.2) has already been shown in Example 1.11 (v), hence we concentrate on the reverse implication.

To be specific, let us consider a function u of class C^1 in some $(n-1)$-ball $B'_{2T} = \{x' \in \mathbb{R}^{n-1} : |x'| < 2T\}$, and let us assume that

$$(2.14) \qquad p \equiv \sup\,\{|Du(x')| \,:\, x' \in B'_{2T}\} < 1 \;.$$

We fix $x' \in B'_T$ and $s,t : 0 < s < t < T$, and denote by Q_r the cylinder

$$Q_r = \{y = (y', y_n) \in \mathbb{R}^n : |y'-x'| < r, \ |y_n - u(x')| < r\},$$

by E the epigraph of u over B'_{2T}, and by v the harmonic function associated with u on $B'_{x',t}$ (see (2.4)).

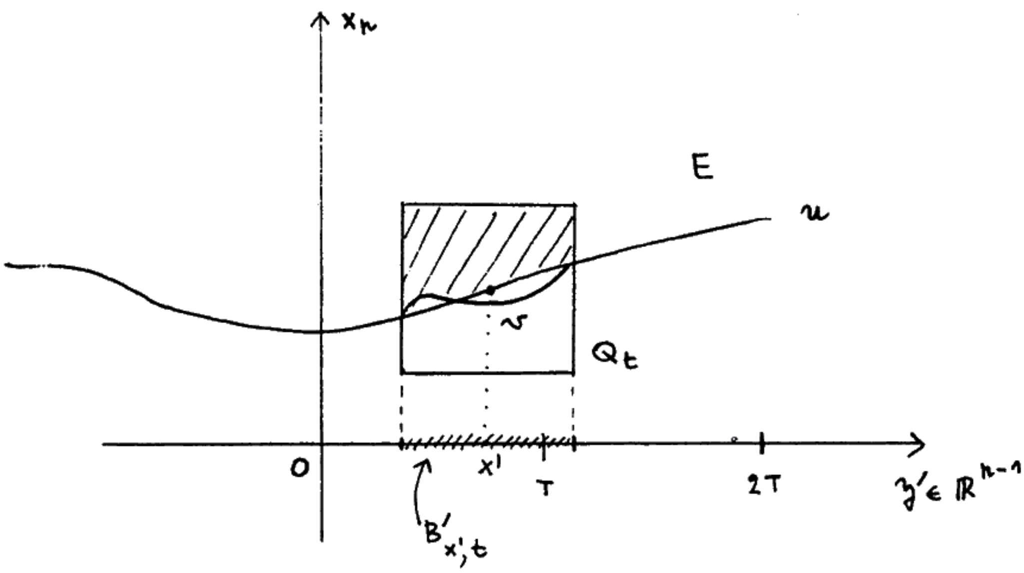


FIGURE 12.

By using successively (2.3),(2.6),(2.7),(2.8),(2.9),(2.10),(2.3), (2.5) and (2.7), we find [11]

$$\int_s |Du - \{Du\}_s|^2 \le \int_s |Du - \{Du\}_t|^2 \le 2 \int_s |Du - Dv|^2 + 2 \int_s |Dv - \{Du\}_t|^2$$

$$\le 2 \int_t (|Du|^2 - |Dv|^2 + 2 \int_s |Dv - \{Dv\}_s|^2$$

$$\leq 2 \int_t (|Du|^2 - |\{Du\}_t|^2) + 2 \int_t (|\{Du\}_t|^2 - |Dv|^2) +$$

$$+ 2(s/t)^{n+1} \cdot \int_t |Dv - \{Dv\}_t|^2$$

$$\leq 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \int_t [(1+|Du|^2)^{\frac{1}{2}} - (1+|\{Du\}_t|^2)^{\frac{1}{2}}] +$$

$$+ \frac{1}{2} \int_t (|Du|^2 - |\{Du\}_t|^2)^2 + 2(s/t)^{n-1} \cdot \int_t (|Dv|^2 - |\{Dv\}_t|^2) +$$

$$+ 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \cdot \int_t [(1+|\{Du\}_t|^2)^{\frac{1}{2}} - (1+|Dv|^2)^{\frac{1}{2}}]$$

$$\leq 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \cdot \int_t [(1+|Du|^2)^{\frac{1}{2}} - (1+|Dv|^2)^{\frac{1}{2}}] +$$

$$+ 2p^2 \int_t |Du - \{Du\}_t|^2 + 2(s/t)^{n+1} \int_t (|Du|^2 - |\{Du\}_t|^2)$$

since:

$$(2.15) \quad (|Du|^2 - |\{Du\}_t|^2)^2 = \langle Du + \{Du\}_t, Du - \{Du\}_t \rangle^2 \leq 4p^2 |Du - \{Du\}_t|^2$$

by Cauchy-Schwarz inequality and (2.14).

In conclusion, we have in view of (2.14), (1.9), and (2.3):

$$(2.16) \quad \int_{B'_{x',s}} |Du - \{Du\}_s|^2 \leq 4(1+p^2)^{\frac{1}{2}} \psi(E,Q_t) + 2[(s/t)^{n+1} + p^2] \int_{B'_{x',t}} |Du - \{Du\}_t|^2$$

Now, if

$$(2.17) \quad \psi(E,Q_t) \leq ct^{n-1+2\alpha} \quad , \quad 0 < \alpha < 1$$

then, setting

$$(2.18) \qquad \omega(r) = \int_{B'_{x',r}} |Du-\{Du\}_r|^2$$

we get from (2.14),(2.16),(2.17):

$$(2.19) \qquad \omega(s) \leq 4\cdot 2^{\frac{1}{2}}\, c t^{n-1+2\alpha} + 2\left[(s/t)^{n+1}+p^2\right]\cdot \omega(t) \quad \forall s,t: 0<s<t<T$$

and thus also

$$(2.20) \qquad \omega(s) \leq \text{const.}\, s^{n-1+2\alpha} \qquad \forall s \in (0,T)$$

by virtue of Lemma 2.4, *provided* p *is sufficiently small.*

Consequelty, if

(i) (2.17) holds uniformly, for every cylinder Q_t with center at points $(x',u(x'))$ and radius t, such that $|x'| < T$ and $t \in (0,T)$;

(ii) p is sufficiently small, depending on α (see (2.19) and Lemma 2.4);

then

$$(2.21) \qquad \int_{B'_{x',t}} |Du-\{Du\}_t|^2 \leq \text{const.}\, t^{n-1+2\alpha} \qquad \forall x': |x'| < T,\ \forall t \in (0,T).$$

In view of the isomorphism between Campanato and Hölder spaces (particularly [12], between $\mathcal{L}^{2,n-1+2\alpha}$ and $C^{0,\alpha}$, see 2.2), we get in conclusion that $u \in C^{1,\alpha}(B'_{T/2})$.

2.6. Conditions (i) and (ii) above are clearly satisfied in the case under consideration. Indeed, whenever $E \subset \mathbb{R}^n$ has, in some open set Ω , a locally smooth (of class C^1) boundary ∂E, which in addition is almost minimal in Ω (Def. 1.5), with $\alpha(t) = ct^{2\alpha}$ and $0<\alpha<1$, then we can always arrange things so that (i) and (ii) above - with p defined by (2.14), and with u giving a local parametrization of (a piece of) $\partial E \cap \Omega$, see 1.6 and Fig. 3 - are satisfied. The preceding discussion then shows that ∂E if of class $C^{1,\alpha}$ in Ω , thus concluding the proof of (2.2).

The key role of the quantity $\int_{B'_r} |Du-\{Du\}_r|^2$ as a regularity parameter has also been stressed by the preceding discussion, see (2.18) - (2.21). Now, as the calculations above show, we have [13]

$$2(1+p^2)^{\frac{1}{2}} \int_{B'_r} [(1+|Du|^2)^{\frac{1}{2}}-(1+|\{Du\}_r|^2)^{\frac{1}{2}}] \leq \int_{B'_r} (|Du|^2-|\{Du\}_r|^2) \leq$$

$$\leq 2(1-p^2)^{-1}(1+p^2)^{\frac{1}{2}} \int_{B'_r} [(1+|Du|^2)^{\frac{1}{2}}-(1+|\{Du\}_r|^2)^{\frac{1}{2}}] \tag{2.22}$$

whenever $p < 1$. The integral in the left-hand side of (2.22) can be rewritten in terms of E (recall that $E = epi(u)$, with $u \in C^1$ and $p<1$), because of the following relations (see [19], 3.4 and 4.10):

$$\begin{aligned} D_i\phi_E(B'_r \times \mathbb{R}) &= D_i\phi_E(Q_r) = \int_{B'_r} D_i u(y')dy' \qquad i = 1,\dots,n-1 \\ D_n\phi_E(B'_r \times \mathbb{R}) &= D_n\phi_E(Q_r) = H_{n-1}(B'_r) \end{aligned} \tag{2.23}$$

which imply that

$$|D\phi_E(Q_r)| = \int_{B'_r} (1+|\{Du\}_r|^2)^{\frac{1}{2}} dy'$$

while clearly

$$|D\phi_E|(Q_r) = \int_{B'_r} (1+|Du|^2)^{\frac{1}{2}} dy'$$

It is then apparent that the quantity

$$|D\phi_E|(Q_r) - |D\phi_E(Q_r)|$$

also represents a fundamental regularity parameter. This justifies the following definition.

2.7. Definition of the Excess.

For every $A \subset\subset \mathbb{R}^n$ and every Caccioppoli set $E \subset \mathbb{R}^n$ we put

$$\omega(E,A) = |D\phi_E|(A) - |D\phi_E(A)| . \tag{2.24}$$

The quantity $t^{1-n} \cdot \omega(E,B_{x,t})$ is usually known as the "area excess of E in $B_{x,t}$", denoted by Exc(E,x,t). Compare with (1.9), and the definition following (1.10).

Just as ψ was an "index of minimality", so is ω an "index of flatness": for, it is clear that if ∂E is *flat* near one of its points (so that we can assume that $\partial E \cap B_T = \{x \in B_T : x_n = 0\}$), then (see (2.23) and Fig. 13):

$$\omega(E,B_T) = |D\phi_E|(B_T) - D_n\phi_E(B_T) = 0$$

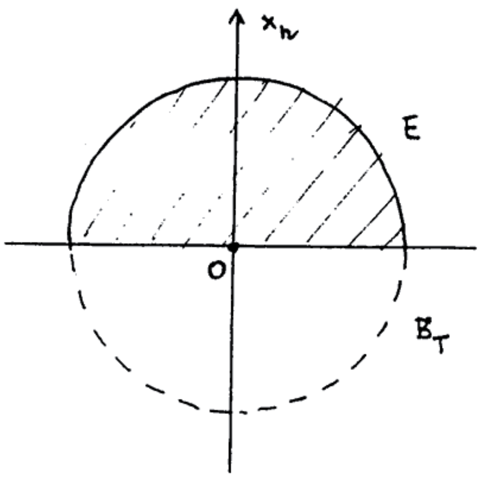


FIGURE 13.

Reciprocally, if $0 \in \partial E$ and $\omega(E,B_T) = 0$, then, on choosing the reference system so that

$$D_i\phi_E(B_T) = 0 \quad \text{when} \quad i=1,\dots,n-1; \quad D_n\phi_E(B_T) \geq 0$$

we get (see (1.29)):

$$0 < \int_{\partial^* E \cap B_T} dH_{n-1} = |D\phi_E|(B_T) = D_n\phi_E(B_T) = \int_{\partial^* E \cap B_T} \nu_E^{(n)} \, dH_{n-1}$$

Consequently

$$\nu_E^{(n)} = 1 \qquad H_{n-1}\text{-a.e. on} \quad \partial^* E \cap B_T$$

which implies, in view of known results (see e.g. [19], Theor. 4.8), that

$$\partial E \cap B_T = \{x \in B_T : x_n = 0\}.$$

Here we have a few illustrative examples:

(i) for the cone $E = \{|x_1| < |x_2|\} \subset \mathbb{R}^2$ (see 1.7) one has

$$\begin{cases} Dev(E,0,t) = 2(2-\sqrt{2}) \\ Exc(E,0,t) = 4 \end{cases}$$

(ii) for Simons'cone $C = \{x_1^2+\ldots+x_4^2 < x_5^2+\ldots+x_8^2\} \subset \mathbb{R}^8$ (see 1.4) one has instead

$$\begin{cases} Dev(C,0,t) = 0 \\ Exc(C,0,t) = const.>0 \end{cases}$$

(iii) for the epigraph $E=\{x_2 > |x_1|^{1+\alpha}\} \subset \mathbb{R}^2$, with $0\leq\alpha\leq 1$ (see 1.14 (v)) one has finally[4]

$$Dev(E,0,t) = Exc(E,0,t) \sim c_\alpha t^{2\alpha}.$$

The following proposition shows that some of the features exhibited by the preceding examples are of a general nature:

2.8. *Proposition.*

For every Caccioppoli set $E \subset \mathbb{R}^n$ we have

$$(2.25) \qquad 0 \leq Dev(E,x,t) \leq Exc(E,x,t) \leq t^{1-n}\, |D\phi_E|(B_{x,t}) \quad \forall x\in\mathbb{R}^n, \forall t>0.$$

Furthermore:

$$(2.26) \qquad Exc(E,x,t) = o(1) \qquad \forall x \in \partial^*E.$$

Proof. Let $B = B_{x,t}$ be an arbitrary n-ball, and $F : F \Delta E \subset\subset B$. Then

$$\phi_E(B) = \int_{\partial E} \phi_E(y)(y-x)t^{-1}\, dH_{n-1}(y) = \int_{\partial B} \phi_F(y)(y-x)t^{-1} dH_{n-1}(y) = D\phi_F(B)$$

whence

$$|D\phi_E|(B) - |D\phi_E\ (B)| = |D\phi_E|(B) - |D\phi_F(B)| \geq |D\phi_E|(B) - |D\phi_F|(B)$$

and (2.25) follows at once.

Now, recall that $x \in \partial^* E$ iff

$$(\nu_1) \qquad |D\phi_E|(B_{x,t}) > 0 \qquad \forall t > 0$$

$$(2.27) \quad (\nu_2) \qquad \lim_{t \to o^+} \frac{D\phi_E(B_{x,t})}{|D\phi_E|(B_{x,t})} \triangleq \nu_E(x) \quad \text{exists, and}$$

$$(\nu_3) \qquad |\nu_E(x)| = 1$$

when this is the case, one has moreover (see (3.5)):

$$(2.28) \qquad |D\phi_E|(B_{x,t}) \sim \omega_{n-1}\, t^{n-1}$$

Conclusion (2.26) is then clear, since

$$(2.29) \qquad \mathrm{Exc}(E,x,t) = t^{1-n}\, |D\phi_E|(B_{x,t}) \left[1 - \frac{|D\phi_E\ (B_{x,t})|}{|D\phi_E|\ (B_{x,t})} \right]$$

2.9. We have just seen that $x \in \partial^* E$ implies $\mathrm{Exc}(E,x,t) \to 0$ as

$t \to 0$. When is the converse true? (i.e., under what additional assumptions does the infinitesimal character of the excess at a given boundary point imply the existence of the "normal" ν_E at that point?). This is a crucial point of our program. We begin our analysis by considering a simple counterexample.

Let $E = \{x_3 > r^{\frac{1}{2}}\} \subset \mathbb{R}^3$, with $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ (Fig. 14)

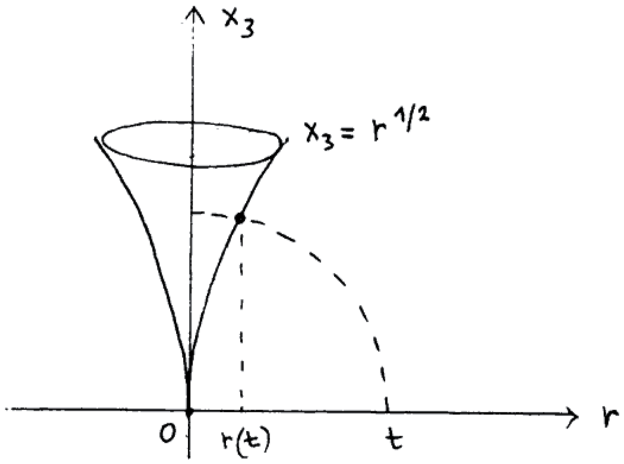


FIGURE 14.

Then $0 \in \partial E$, and

$$\begin{cases} |D\phi_E|(B_t) = 2\pi \int_0^{r(t)} r(1+1/4r)^{\frac{1}{2}} dr \\ D_1\phi_E(B_t) = D_2\phi_E(B_t) = 0 \ , \quad D_3\phi_E(B_t) = \pi r^2(t) \end{cases}$$

with $r(t) = \frac{1}{2}\left[(1+4t^2)^{\frac{1}{2}}-1\right]$. We immediately check that

$$(2.30) \quad \mathrm{Exc}(E,0,t) = t^{-2}\left[2\pi \int_0^{r(t)} r(1+1/4r)^{\frac{1}{2}} dr - \pi r^2(t)\right] \to 0 \quad \text{as} \quad t \to 0,$$

and that there exists

$$(2.31)\qquad \nu_E(0) = \lim_{t\to o^+} \frac{D\phi_E(B_t)}{|D\phi_E|(B_t)} = 0 \in \mathbb{R}^3$$

However, $|\nu_E(o)| = 0$ implies that $0 \notin \partial^*E$ (recall (2.27)). It follows from (2.29), (2.30) and (2.31) that

$$\lim_{t\to o} t^{-2} |D\phi_E|(B_t) = 0 ,$$

a fact that could also be checked directly.

2.10. Now, let us suppose that $x \in \partial E$ and that (contrary to what happens in the preceding example) there holds

$$(2.32)\qquad |D\phi_E|(B_{x,t}) \geq c_1 t^{n-1} \qquad \forall t \in (0,T)$$

with $c_1 > 0$. We anticipate (see prop. 3.4) that every set with almost minimal boundary does satisfy (2.32).

It follows from (2.29), (2.32) that *if* $Exc(E,x,t) = o(1)$ *and* $\nu_E(x)$ *exists, then it has unit length,* and consequently $x \in \partial^*E$. In order to be sure of the existence of $\nu_E(x)$, we employ the following inequality

$$(2.33)\qquad \left| \frac{D\phi_E(G_1)}{|D\phi_E|(G_1)} - \frac{D\phi_E(G_2)}{|D\phi_E|(G_2)} \right| \leq 2 \left[\frac{\omega(E,G_2)}{|D\phi_E|(G_1)} \right]^{1/2}$$

which holds $\forall G_1 \subset G_2 \subset \mathbb{R}^n$ with $|D\phi_E|(G_1) > 0$ (see [27],2.5.4 (1)).

From (2.32) and (2.33) we deduce

$$(2.34)\qquad \left| \frac{D\phi_E(B_{x,s})}{|D\phi_E|(B_{x,s})} - \frac{D\phi_E(B_{x,t})}{|D\phi_E|(B_{x,t})} \right| \leq 2\, c_1^{-1/2}\, (t/s)^{(n-1)/2}\ \mathrm{Exc}(E,x,t)^{\frac{1}{2}}$$

for every $s,t : 0<s<t<T$.

Now consider the abstract situation in which a given function

$$\nu : (0,T) \to \dot{\bar{B}}_1 \subset \mathbb{R}^n \quad \text{satisfies}$$

$$(2.35)\qquad |\nu(s) - \nu(t)| \leq (t/s)^{(n-1)/2}\cdot g(t) \qquad \forall s,t: 0<s<t<T,$$

with $g(t) = o(1)$. Observe that (2.34) is a special case of (2.35). A simple calculation shows that the function

$$\nu(t) = (\text{sen lg lg}(e/t),\ \cos \text{lg lg}\,(e/t)),\quad 0<t<1$$

satisfies (2.35) with $T = 1$, $n=2$, and $g(t) = \sqrt{2}/\lg(e/t) = o(1)$. Nevertheless, $\alpha(t)$ has no limit as $t \to o$.

Condition (2.35) implies the existence of that limit, *provided* we have a reasonable "quantitative" hypothesis[14], regarding the convergence of $g(t)$ to 0. This is the case for instance when $g(t) \leq ct^{\alpha}$, $\alpha > o$; indeed, given $t\in(0,T)$ and $\tau \in(0,1)$, for every $h,k \geq 1$ we find, on the account of (2.35):

$$|\nu(\tau^{h+k}t) - \nu(\tau^h t)| \leq \sum_{i=o}^{k=1} |\nu(\tau^{h+i+1}t) - \nu(\tau^{h+i}t)|$$

$$(2.36)\qquad \leq \tau^{(1-n)/2} \sum_{i=o}^{k-1} g(\tau^{h+i}t)$$

$$\leq c t^{\alpha} \tau^{(1-n)/2} \tau^{h\alpha} \sum_{i=0}^{\infty} \tau^{\alpha i}$$

$$\leq \text{const}\,(c,\tau,n,\alpha)\cdot t^{\alpha} \tau^{h\alpha}$$

which shows that $\{\nu(\tau^h t)\}_h$ is a Cauchy sequence in $\mathbb{R}^n$, for every $t \in (0,T)$ and every $\tau \in (0,1)$. Put

$$\nu_o = \lim_{h\to+\infty} \nu(2^{-(h+1)}T) \tag{2.37}$$

and observe that $\forall t \in (0,T/2)$ there exists (and is unique) an integer $h = h(t) \geq 1$ such that

$$2^{-(h+1)}T \leq t < 2^{-h}T \tag{2.38}$$

with in addition

$$\lim_{t\to o^+} h(t) = +\infty . \tag{2.39}$$

Consequently

$$|\nu(t)-\nu_o| \leq |\nu(2^{-(h+1)}T)-\nu_o| + |\nu(2^{-(h+1)}T) - \nu(t)|$$

$$\leq | \quad \text{"} \quad\quad \text{"} \quad | + c\cdot 2^{(n-1)/2}\cdot t^{\alpha}$$

by virtue of (2.35) and (2.38). From (2.37), (2.39) we then find

$$\nu_o = \lim_{t\to o} \nu(t)$$

thus proving our assertion.

2.11. We deduce from the foregoing considerations that when the set E, the point $x \in \partial E$, and the radius $T > 0$ are such that:

$$|D\phi_E|(B_{x,t}) \geq c_1 t^{n-1} \qquad \forall t \in (0,T), \text{ with } c_1 > 0, \text{ and}$$

$$\mathrm{Exc}(E,x,t) \to 0 \quad \text{as} \quad t \to 0, \text{ in a certain "controlled may"}$$

(e.g., as $t^{2\alpha}$), then $x \in \partial^* E$.

It is not difficult to show that the first condition is satisfied, whenever ∂E is almost minimal. The point is that almost minimality implies the second condition as well, *at least when the excess, corresponding to the initial radius* T, *is conveniently small.* [15]

This fundamental result was originally proved by E.De Giorgi for minimal boundaries,in the form of the following lemma.

2.12. *Lemma* (De Giorgi [8,9])

For every $n \geq 2$ there exists a constant $\sigma = \sigma(n) > 0$ such that whenever the set $E \subset \mathbb{R}^n$, the point $x \in \partial E$, and the radius $t > 0$ satisfy

$$\begin{cases} \psi(E,B_{x,2t}) = 0 \\ \mathrm{Exc}(E,x,2t) \leq \sigma \end{cases}$$

then:

$$\mathrm{Exc}(E,x,t) \leq \sigma/2.$$

The iterative character of this result is apparent: a repeated application of the lemma yields the right estimation of the excess, which in addition turns out to be uniform in a neighbourhood of the given point. One derives from this the regularity of the set of boundary points, where the initial value of the excess is bounded by σ

A lemma of this sort is at the root of the various Regularity Theorems which extended De Giorgi's work: see [28,23,19,27], where the proof of such a result is obtained "by contradiction", as it was the case for the proof of Lemma 2.12 in De Giorgi's paper [8].

Moreover, a similar result is among the main tools in the Regularity Theory for almost minimal currents (and varifolds): see [4,5], where the proof is still obtained "by contradiction", and [34], where a more direct proof is developed.

It will be our aim in the next chapter to give a direct proof of a variation of Lemma 2.12, which will prove particularly useful for the demonstration of Theorem 1.9.

CHAPTER 3: SOME PRELIMINARY RESULTS AND THE MAIN LEMMA

Having prepared the way in the preceding chapter, we now undertake a formal proof of Theorem 1.9. As a starting point, it seems convenient to bring together various notations and definitions already met on the preceding pages.

3.1. In the following, $\mathbb{R}^n$ will denote Euclidean n-dimensional space over the real numbers $\mathbb{R}$, endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$; n is an integer not less than 2. Points in $\mathbb{R}^n$ will be denoted by x,y,z; measurable sets by E,F,G; compact sets by K; open sets by A and Ω; open balls by B. When we want to specify the center x and the radius t of B, then we write $B_{x,t}$. Projection of points or sets in $\mathbb{R}^n$ onto the first n-1 variables will always be denoted by a "prime", such as x',A',B', and so on. Hence, in particular, we have $x = (x',x_n)$ and $B' = \{y' \in \mathbb{R}^{n-1} : |y'-x'| < t\}$ if $B = B_{x,t}$. The symbol "0" however, will denote the origin of both $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ (and, of course, the real number "zero"): which one of them, will be clear from the context. We shall also abbreviate B_t and B'_t for $B_{o,t}$ and $B'_{o,t}$ respectively.

Whener $F,G \subset \mathbb{R}^n$, the notation $F \subset\subset G$ means that the closure of F is a compact subset of G, while $F \Delta G$ denotes the symmetric difference $(F \cup G) \setminus (F \cap G)$. The characteristic function of a set $E \subset \mathbb{R}^n$ will be denoted by ϕ_E. Convergence of a sequence $\{E_h\}$ to E in Ω always means the $L^1_{loc}(\Omega)$-convergence of the corresponding characteristic functions, i.e.:

$$(3.1) \quad E_h \to E \text{ locally in } \Omega \text{ iff } \int_A |\phi_{E_h}(x) - \phi_E(x)|\, dx \to 0 \quad \forall A \subset\subset \Omega .$$

We say that E is a *Caccioppoli set* iff the distributional gradient $D\phi_E = (D_1\phi_E,\dots,D_n\phi_E)$ of ϕ_E is a Radon vector measure with locally finite total variation $|D\phi_E|$:

$$|D\phi_E|(A) < +\infty \qquad \forall A \subset\subset \mathbb{R}^n .$$

We have of course

$$|D\phi_E|(\Omega) = \sup \{ \int_E \operatorname{div}\Phi(x)dx : \Phi \in C_0^1(\Omega;\mathbb{R}^n), |\Phi| \le 1\}$$

for every open set Ω of $\mathbb{R}^n$; according to the Gauss-Green theorem, we thus get

$$|D\phi_E|(\Omega) = H_{n-1}(\partial E \cap \Omega)$$

whenever $\partial E \cap \Omega$ is sufficiently smooth. Here, for every real $s \ge 0$, H_s denotes the s-dimensional Hausdorff measure on $\mathbb{R}^n$. We also set (see [12], 2.10.2):

$$\omega_s = \Gamma^s(\tfrac{1}{2})/\ \Gamma(s/2+1)$$

When k is a positive integer, ω_k yields precisely the k-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^k$.

The relevant facts about Caccioppoli sets can be found in [19] and in the recent book [27]. For our purposes, it suffices to recall that every Caccioppoli set E possesses, at $|D\phi_E|$ - almost all points $x \in \partial E$, an unit inner normal vector $\nu_E(x)$, defined through the following relation:

(3.2)
$$\nu_E(x) = \lim_{t\to 0^+} \nu(E, B_{x,t})$$

where for short:

(3.3) $$\nu(E,G) = \frac{D\phi_E(G)}{|D\phi_E|(G)}$$

whenever $G \subset\subset \mathbb{R}^n$. The collection of points x where such a limit exists and has unit length, is commonly known as the *reduced boundary* of E, denoted by ∂^*E. See also (2.27). We remark explicitely, that when speaking of a Caccioppoli set E, we let ∂E denote the boundary of E in the measure-theoretical sense, i.e.

(3.4) $$x \in \partial E \quad \text{iff} \quad 0 < \text{meas}\,(E \cap B_{x,t}) < \text{meas}\,(B_{x,t}) \quad \forall t>0.$$

Whenever $x \in \partial^*E$, we have (see [19], Theorem 3.8, or [27], 2.3(23")):

(3.5) $$\lim_{t\to 0^+} t^{1-n}|D\phi_E|(B_{x,t}) = \omega_{n-1}\ .$$

We use vector addition and multiplication to define translations and homothetic transformations in $\mathbb{R}^n$. Thus:

$$E + x_o = \{x : x - x_o \in E\} \quad \text{and} \quad tE = \{x : t^{-1}x \in E\}\ ,$$

for $E \subset \mathbb{R}^n$, $x_o \in \mathbb{R}^n$ and $t>0$.

Clearly, whenever E is a Caccioppoli set, then so are $E+x_o$ and tE, and for every $G \subset \mathbb{R}^n$ it holds:

(3.6) $$|D\phi_{E+x_o}|(G+x_o) = |D\phi_E|(G),\ |D\phi_{tE}|(tG) = t^{n-1}|D\phi_E|(G).$$

Analogous relations hold for the measures $D_i\phi_E$, $i=1,\dots,n$, as well as for the following non-negative measures:

(3.7) $$\mu_E = |D\phi_E| - D_n\phi_E$$

(3.8) $$\omega_E \equiv |D\phi_E| - \left(\sum_{i=1}^{n} (D_i\phi_E)^2\right)^{\frac{1}{2}}$$

i.e. $\omega_E(G) = |D\phi_E|(G) - |D\phi_E(G)|$, $\forall G \subset\subset \mathbb{R}^n$. We shall preferably write $\omega(E,G)$ instead of $\omega_E(G)$. All these measures are obvioulsy invariant under orthogonal transformations.

We recall that the quantity $t^{1-n}\,\omega(E,B_{x,t})$ is usually called the *excess*, see section 2.7. Finally, we recall the definition of the functional ψ (see 1.13):

(3.9) $$\psi(E,A) = |D\phi_E|(A) - \inf\{|D\phi_F|(A) : F \Delta E \subset\subset A\}, \quad A \subset\subset \mathbb{R}^n$$

which is also invariant under translations and orthogonal transformations, while clearly

(3.10) $$\psi(tE,tA) = t^{n-1}\,\psi(E,A) \qquad \forall t > 0$$

we have in addition (see [37]):

(3.11) $$A_1 \subset A_2 \subset\subset \mathbb{R}^n \Longrightarrow \psi(E,A_1) \leq \psi(E,A_2) \qquad \forall E$$

(3.12) $$E_h \to E \text{ locally in } A \Longrightarrow \psi(E,A) \leq \liminf_{h\to+\infty} \psi(E_h,A)$$

(3.13) $$\left.\begin{array}{l} E_h \to E \text{ locally in } A \\ \psi(E_h,A) \to \psi(E,A) \end{array}\right\} \Longrightarrow |D\phi_{E_h}|(A_1) \to |D\phi_E|(A_1),$$

for every $A_1 \subset\subset A$ such that A_1 is open and $|D\phi_E|(\partial A_1) = 0$.

For, assuming $A_1 \subset A_2$ we get

$$|D\phi_E|(A_1) - \inf\{|D\phi_F|(A_1) : F\Delta E \subset\subset A_1\} = |D\phi_E|(A_2) - \inf\{|D\phi_F|(A_2) : F\Delta E \subset\subset A_1\}$$

which proves (3.11).

As for (3.12), if $E_h \to E$ in A and F is such that $F \Delta E \subset\subset A$, then (reasoning possibly on a subsequence of $\{E_h\}$), we can pick an open subset A_2 of A, with lipschitz boundary, satisfying:

(3.14) $F \Delta E \subset\subset A_2 \subset\subset A$, $|D\phi_{E_h}|(\partial A_2) = |D\phi_E|(\partial A_2) = 0 \ \forall h$, $\int_{\partial A_2} |\phi_{E_h} - \phi_E| \, dH_{n-1} \to 0$.

Setting $F_h = (E_h \setminus A_2) \cup (F \cap A_2)$ we find

$$|D\phi_{F_h}|(A) = |D\phi_{E_h}|(A \setminus A_2) + \int_{\partial A_2} |\phi_{E_h} - \phi_E| \, dH_{n-1} + |D\phi_F|(A_2)$$

from which, observing that $F_h \Delta E_h \subset\subset A$, we get

(3.15) $\psi(E_h, A) \geq |D\phi_{E_h}|(A) - |D\phi_{F_h}|(A) = |D\phi_{E_h}|(A_2) - |D\phi_F|(A_2) - \int_{\partial A_2} |\phi_{E_h} - \phi_E| \, dH_{n-1}$

By letting $h \to +\infty$ we then find for every F such that $F \Delta E \subset\subset A$:

$$\liminf_{h \to +\infty} \psi(E_h, A) \geq |D\phi_E|(A_2) - |D\phi_F|(A_2) = |D\phi_E|(A) - |D\phi_F|(A)$$

(recall (3.14) and the lower semicontinuity of $|D\phi_\cdot|(A)$ with respect to the local convergence in A), and from this (3.12) follows at once.

Finally, assume that $E_h \to E$ in A and that $\psi(E_h, A) \to \psi(E, A)$, and fix $A_1 \subset\subset A$ such that A_1 is open and $|D\phi_E|(\partial A_1) = 0$. Then choose F, A_2, and F_h as above, with in addition $A_1 \subset\subset A_2$. By (3.15) we get

$$\psi(E,A) = \lim_{h \to +\infty} \psi(E_h, A) \geq \limsup_{h \to +\infty} |D\phi_{E_h}|(A_2) - |D\phi_F|(A_2)$$

$$\geq \limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) + \liminf_{h \to +\infty} |D\phi_{E_h}|(A_2 \setminus \bar{A}_1) - |D\phi_F|(A_2)$$

$$\geq \limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) + |D\phi_E|(A_2 \setminus \bar{A}_1) - |D\phi_F|(A_2)$$

$$= \limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) - |D\phi_E|(\bar{A}_1) + |D\phi_E|(A) - |D\phi_F|(A)$$

which holds $\forall F : F \triangle E \subset\subset A$. When combined with (3.9), this gives

$$\limsup_{h \to +\infty} |D\phi_{E_h}|(A_1) \leq |D\phi_E|(\bar{A}_1) \equiv |D\phi_E|(A_1)$$

by our assumptions. Since

$$|D\phi_E|(A_1) \leq \liminf_{h \to +\infty} |D\phi_{E_h}|(A_1)$$

by semicontinuity, we obtain eventually (3.13).

We now establish some helpful inequalities, involving ψ and ω See also Prop. 2.8.

3.2. *Lemma*.

If E_1, E_2 are Caccioppoli sets in $\mathbb{R}^n$, and B is an n-ball, then

$$\psi(E_1,B)-\psi(E_2,B) \leq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1} \tag{3.16}$$

$$\psi(E_1,B)-\psi(E_2,B) \leq \omega(E_1,B)-\omega(E_2,B) + 2\int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1} \tag{3.17}$$

Proof. First we remark that for any Caccioppoli set $F \subset \mathbb{R}^n$, the term

$$\int_{\partial B} \phi_F \, dH_{n-1}$$

denotes the integral of the *inner trace* of F over ∂B (see e.g. [19], Chapter 2). Given such an F, we set for $B = B_{x,t}$ and $\tau \in (0,1)$:

$$F_\tau = (F \cap B_{x,\tau t}) \cup (E_2 \setminus B_{x,\tau t})$$

so that:

$$|D\phi_{F_\tau}|(B_{x,t}) \leq |D\phi_F|(B_{x,t}) - (|D\phi_F| - |D\phi_{E_2}|)(B_{x,t} \setminus B_{x,\tau t}) +$$

$$+ \int_{\partial B_{x,\tau t}} |\phi_{E_2} - \phi_F| \, dH_{n-1} .$$

Assuming $F \Delta E_1 \subset\subset B_{x,t}$, we get easily:

$$|D\phi_{E_1}|(B) - |D\phi_F|(B) \leqq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \psi(E_2,B) -$$

$$- (|D\phi_F| - |D\phi_{E_2}|)(B_{x,t} \setminus B_{x,\tau t}) + \int_{\partial B_{x,\tau t}} |\phi_{E_2} - \phi_{E_1}| dH_{n-1}$$

Hence, letting $\tau \to 1$ and taking the supremum over such F's, we find

(3.18) $$\psi(E_1,B) \leqq |D\phi_{E_1}|(B) - |D\phi_{E_2}|(B) + \psi(E_2,B) + \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

which is exactly (3.16).

Now, since

(3.19) $$D\phi_F(B) = \int_{B_{x,t}} d\, D\phi_F = \int_{\partial B_{x,t}} t^{-1} \phi_F(y)(y-x) dH_{n-1}(y)$$

for every Caccioppoli set $F \subset \mathbb{R}^n$, we have

(3.20) $$|D\phi_{E_1}(B)| - |D\phi_{E_2}(B)| \leqq \int_{\partial B} |\phi_{E_1} - \phi_{E_2}| dH_{n-1}$$

Adding (3.18) and (3.20), and rearranging, we get (3.17).

3.3. The following inequality was proved in Section 2.5 and 2.6 (see especially (2.16), (2.22) and (2.23)):

(3.21) $$\omega(E,Q_s) \leq 2\psi(E,Q_t)+2(1-p^2)^{-1}[(s/t)^{n+1}+p^2] \cdot \omega(E,Q_t)$$

It holds $\forall s,t : 0<s<t<T$, under the following assumptions:[16]

(3.22) $$E = \{x : |x'| < T,\ x_n > u(x')\}$$

(3.23) $$Q_r = \{x : |x'| < r,\ |x_n - u(o)| < r\}$$

where $u \in C^1(B'_T)$ is such that

(3.24) $$p \equiv \sup\{|Du(x')| : |x'| < T\} < 1 .$$

We conclude this section by recalling two further relations, which are proved e.g. in [27], 2.5.4 (1) and 2.5.1, respectively:

(3.25) $$|\nu(E,G_1) - \nu(E,G_2)| \leq 2\left[\frac{\omega(E,G_2)}{|D\phi_E|(G_1)}\right]^{1/2}$$

(which holds for every Caccioppoli set $E \subset \mathbb{R}^n$, and every $G_1 \subset G_2 \subset\subset \mathbb{R}^n$ such that $|D\phi_E|(G_1) > 0$. See (2.33) and (3.3))

(3 $$\left[\int_{\partial B_{x,1}} |\phi_E(x+s(y-x)) - \phi_E(x+t(y-x))|\,dH_{n-1}(y)\right]^2 \leq$$

$$\leq 2[t^{1-n}\,|D\phi_E|(B_{x,t}) - s^{1-n}|D\phi_E|(B_{x,s}) + (n-1)\int_s^t r^{-n}|D\phi_E|(B_{x,r})\,dr]$$

$$\cdot\left[t^{1-n}|D\phi_E|(B_{x,t})-s^{1-n}|D\phi_E|(B_{x,t})+(n-1)\int_s^t r^{-n}\psi(E,B_{x,r})dr\right],$$

which is valid for every Caccioppoli set $E \subset \mathbb{R}^n$, every point $x \in \mathbb{R}^n$, and every $s,t : 0 < s < t$. This last inequality will be used in the next section to establish some fundamental area and volume density ratio bounds for (a special class of) almost minimal boundaries.

3.4. *Proposition.*

Suppose we are given a Caccioppoli set $E \subset \mathbb{R}^n$ and a non-negative function $\alpha(t)$, defined on $(0,1)$ and satisfying

$$\int_0^1 t^{-1}\alpha(t)dt < +\infty. \tag{3.27}$$

If for some point x and some radius $t \in (0,1)$ it holds

$$\psi(E,B_{x,t}) \leq \alpha(t)\cdot t^{n-1} \tag{3.28}$$

then for the same x,t we also have: [17]

$$t^{1-n}|D\phi_E|(B_{x,t}) \leq \alpha(t) + n\omega_n/2 \tag{3.29}$$

If (3.28) holds for every $t \in (0,T_o)$, with T_o fixed in $(0,1)$, then

$$t^{1-n}|D\phi_E|(B_{x,t})+(n-1)\cdot\int_o^t r^{-1}\alpha(r)dr \text{ is a non-decreasing function on } (0,T_o) \tag{3.30}$$

Finally, assuming that (3.28) holds for every $x \in B_{x_o,T_o}$ and every $t \in (0,T_o)$ then we have:

(3.31) $t^{1-n}|D\phi_E|(B_{x,t}) \geqq \omega_{n-1} - (n-1)\int_0^t r^{-1}\alpha(r)dr$

(3.32) $t^{-n}\cdot \min\{\text{meas}(E\cap B_{x,t}), \text{meas}(B_{x,t}\setminus E\} \geqq \omega_{n-1}/n - \int_0^t r^{-1}\alpha(r)dr$

both $\forall x \in \partial E\cap B_{x_o,T_o}$, $\forall t\in(0,T_o)$. In this case moreover assuming that $\alpha(t)$ is non-decreasing and infinitesimal at 0, if we set

(3.33) $$E_h = t_h^{-1}(E-x)$$

for $x\in \partial E\cap B_{x_o,T_o}$ and $t_h \downarrow 0$, then a subsequence of $\{E_h\}$ will converge to a minimal cone $C\subset \mathbb{R}^n$, with $0\in\partial C$.

Remark.

Evidently, when E has almost minimal boundary in Ω, with $x_o\in\Omega$ and $\alpha(t)$ satisfying (3.27) (see Def. 1.5 and 1.13), then a convenient T_o can be found so that (3.28) holds $\forall x\in B_{x_o,T_o}$ and $\forall t\in(0,T_o)$. Accordingly, (3.29)-(3.32) all hold $\forall x\in\partial E\cap B_{x_o,T_o}$, $\forall t\in(0,T_o)$. Also notice that (3.27) is weaker than (α_4) of section 1.11.

Proof. (see section 2 of [37] for the special case $\alpha(t) = ct^{2\alpha}$).
From (3.28) and (3.9) we get $\forall\tau\in(0,1)$:

$$\alpha(t)\cdot t^{n-1} \geq |D\phi_E|(B_{x,t}) - \min\{|D\phi_{E\cup B_{x,\tau t}}|(B_{x,t}), |D\phi_{E\setminus B_{x,\tau t}}|(B_{x,t})\}$$

(3.34) $$\geq |D\phi_E|(B_{x,t}) - |D\phi_E|(B_{x,t}\setminus B_{x,\tau t}) - \min\{\int_{\partial B_{x,\tau t}}\phi_E\, dH_{n-1}, \int_{\partial B_{x,\tau t}}(1-\phi_E)dH_{n-1}\}$$

$$\geq |D\phi_E|(B_{x,\tau t}) - \frac{1}{2}n\omega_n(\tau t)^{n-1}$$

from wich we get (3.29), by letting $\tau \to 1$.

(3.30) follows easily from (3.26), (3.27) and (3.28). If $x \in \partial^* E \cap B_{x_o,T_o}$ then (3.5),(3.27) and (3.30) imply (3.31). For a generic point x in ∂E, (3.31) follows by approximation, since $\partial E = \overline{\partial^* E}$.

Now, arguing as in (3.34) and using (3.31) we find $\forall t \in (0,T_o)$:

$$\omega_{n-1} - (n-1) \int_0^t r^{-1}\alpha(r)dr \le \alpha(t) + t^{1-n} \cdot \min\{\int_{\partial B_{x,t}} \phi_E dH_{n-1}, \int_{\partial B_{x,t}} (1-\phi_E) dH_{n-1}\}$$

On rearranging and integrating between 0 and t we obtain:

$$\min\{\text{meas}(E \cap B_{x,t}), \text{meas}(B_{x,t} \setminus E)\} \ge \int_0^t [\omega_{n-1} - (n-1)\int_0^s r^{-1}\alpha(r)dr - \alpha(s)] s^{n-1} ds$$

$$= \omega_{n-1} t^n/n + (1/n-1)t^n \int_0^t r^{-1}\alpha(r)dr - (1/n)\int_0^t r^{n-1}\alpha(r)dr$$

$$\ge [\omega_{n-1}/n - \int_0^t r^{-1}\alpha(r)dr] \cdot t^n$$

which proves (3.32).

Finally, for E_h as in (3.33), $r > 0$, and h sufficiently large (so that $rt_h < T_o$) we have, in view of (3.6),(3.29) and the new assumptions on $\alpha(t)$:

$$(3.35) \quad |D\phi_{E_h}|(B_r) = t_h^{1-n} |D\phi_E|(B_{x,rt_h}) \le r^{n-1}(\alpha(T_o) + n\omega_n/2)$$

Hence, a subsequence of E_h (not relabeled) will converge to some limit set C, locally in $\mathbb{R}^n$. On the other hand, we have (see (3.10)):

$$(3.36) \qquad \psi(E_h, B_r) = t_h^{1-n}\, \psi(E, B_{x,rt_h}) \le \alpha(rt_h) \cdot r^{n-1}$$

by (3.28), so that

$$(3.37) \qquad \psi(C, B_r) = 0 \qquad \forall r > 0$$

in view of (3.12), that is, C has minimal boundary in $\mathbb{R}^n$ (Def. 1.2).

From (3.26), (3.37) and (3.13) we deduce

$$|D\phi_{E_h}|(B_r) \to |D\phi_C|(B_r) \qquad \text{for a.e. } r > 0$$

or (see (3.35)):

$$(3.38) \qquad (rt_h)^{1-n}\, |D\phi_E|(B_{x,rt_h}) \to r^{1-n} |D\phi_C|(B_r) \quad \text{for a.e. } r>0,$$

as $h \to +\infty$.

Setting

$$(3.39) \qquad b = \lim_{t \to o^+} \left[t^{1-n}\, |D\phi_E|(B_{x,t}) \right]$$

((3.30) shows that the limit in question exists, while (3.29) and (3.31) give upper and lower bounds for b), we conclude that

$$(3.40) \qquad r^{1-n} |D\phi_C|(B_r) = b \in [\omega_{n-1}, n\omega_n/2\,] \quad \text{for a.e. } r > 0.$$

Substitution of (3.37) and (3.40) into (3.26) then yields

$$\int_{\partial B_1} |\phi_C(sy) - \phi_C(ty)|\, dH_{n-1}(y) = 0$$

for almost every $t > 0$, and almost every $s \in (0,t)$, thus proving that C is (equivalent to) a minimal cone, with $0 \in \partial C$ (see (3.4) and (3.40)).

3.5. The main result of the present chapter is the following Lemma 3.6, which extends De Giorgi's Lemma of section 2.12.

Its proof will be achieved by comparing the given set E with level sets L of a suitable mollification of ϕ_E - much as in the original paper of De Giorgi [8], using however a more direct argument. The results contained in sections 3.2 and 3.4 then show that the comparison surface ∂L is appropriately "close" to ∂E. The area excess of ∂L being nicely controlled (section 3.3), this yields the desired estimation of the excess of ∂E.

A few properties of mollifiers are now in order. We introduce the following "tent function"

(3.41) $$\eta(x) = c(n)\cdot\max\{1-|x|,0\}, \quad \text{with} \quad c(n) = (n+1)/\omega_n$$

first considered by E. Giusti [19], chapter 7. Clearly, η is a non-negative, symmetric, Lipschitz-continuous mollifier, whose integral is 1 and whose support coincides with the unit ball in $\mathbb{R}^n$. We set as usual, for $\varepsilon > 0$ and $g \in L^1_{loc}(\mathbb{R}^n)$:

(3.42) $$\begin{cases} \eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x) \\ g_\varepsilon(x) = (g*\eta_\varepsilon)(x) = \int \eta_\varepsilon(x-y)g(y)dy \end{cases} .$$

Then, whewever F is a Caccioppoli set in $\mathbb{R}^n$, $\varepsilon > 0$, and $f_\varepsilon = \phi_F * \eta_\varepsilon$, we have

(3.43) $$f_\varepsilon \text{ is of class } C^1$$

(3.44) $$\int_{B_t} |f_\varepsilon - \phi_F| \, dx \le \varepsilon \cdot |D\phi_F|(B_{t+\varepsilon}) \qquad \forall t > 0$$

(3.45) $$\int_{B_t} |Df_\varepsilon|\,dx \leq |D\phi_F|(B_{t+\varepsilon}) \qquad \forall\ t > 0$$

(3.46) if $0 < t < 1/n$ and $n^2t^2 < f_\varepsilon(x) < 1 - n^2t^2$, then $\mathrm{dist}(x,\partial F)<(1-t)\varepsilon$.

See [19], Lemma 7.1 and 7.2, for the simple proof.

From now on we suppose that $\alpha(t)$ satisfies (α_1)-(α_4) of section 1.11. We also introduce the notation $\beta \preccurlyeq \alpha$ to indicate a non-decreasing function β, defined on $(0,1)$, and satisfying $0 \leq \beta(t) \leq \alpha(t)$ $\forall t\in(0,1)$.

We are now is a position to state and prove the following result (compare with Lemma 2.12).

3.6. Main Lemma.

For any $n \geq 2$, *any* α *as in* 1.11, *and any* $\tau \in (0,2^{-4})$, *there exists a constant* $\sigma^* = \sigma^*(n,\alpha,\tau) \in (0,1)$, *such that whenever* $F \subset \mathbb{R}^n$, $\sigma \in (0,\sigma^*]$, *and* $\beta \preccurlyeq \alpha$ *satisfy the following hypotheses:*

(H_1) $$\psi(F,B_{x,t}) \leq \beta(t)\cdot t^{n-1} \qquad \forall x \in B_1,\ \forall t \in (0,1)$$

(H_2) $$\omega(F,B_1) \leq \sigma$$

then:

(3.47) $$\omega(F,B_\tau) \leq c_1\beta(1) + c_2\sigma\tau^{n+1}$$

where c_1,c_2 are positive constants, depending only on the dimension n.

Proof. Without loss of generality, we can assume that $|D\phi_F(B_1)| = D_n\phi_F(B_1)$, so that (see (3.7)) : $\omega(F,B_1) = \mu_F(B_1)$. We split the proof into three steps.

Step 1. Given n and α as above, we prove first the existence of a constant $\sigma^{\#} \in (0,1)$ and of a function $g : (0,\sigma^{\#}] \to (0,1)$, with $g(\sigma) = o(1)$ ($\sigma^{\#}$ and g depending on n and α), such that *whenever* $F \subset \mathbb{R}^n$ *and* $\sigma \in (0,\sigma^{\#}]$ *satisfy:*

$$(h_1) \qquad \psi(F,B_{x,t}) \leqq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_1,\ \forall t \in (0,1)$$

$$(h_2) \qquad \omega(F,B_1) = \mu_F(B_1) \leqq \sigma$$

then

$$(3.48) \qquad \frac{D_n f(x)}{|Df(x)|} > 1-g(\sigma) \quad \forall x\in B_1 : |x| < 1-2\sigma^{1/2(n-1)} \ \textit{and}\ n^2\sigma^2 < f(x) < 1-n^2\sigma^2 ,$$

where

$$(3.49) \qquad f = \phi_F * \eta_\varepsilon \ , \qquad \varepsilon = \sigma^4 .$$

We observe that when $\sigma \geqq 4^{1-n}$, the set of points in (3.48) is empty, so that there is nothing to prove in this case. Thus, we assume $\sigma<4^{1-n}$. In addition, we observe that

$$(3.50) \qquad D_n f(x) = \int \eta_\varepsilon(x-y)\,d\,D_n\phi_F(y) \ \text{and}\ |Df(x)| \leqq \int \eta_\varepsilon(x-y)\,d\,|D\phi_F|(y)$$

since, by definition, $f(x) = \int \eta_\varepsilon(x-y)\phi_F(y)dy$. Hence, (3.48) will be proved if we can show that

$$(3.51) \qquad \int \eta_\varepsilon(x-y)d\mu_F(y) < g(\sigma) \int \eta_\varepsilon(x-y)\,d\,|D\phi_F|(y)$$

for any x satisfying

(3.52) $$|x| < 1-2\sigma^{1/2(n-1)} \quad , \; n^2\sigma^2 < f(x) < 1 - n^2\sigma^2 .$$

To this aim, we define for x as in (3.52) and $\sigma \in (0,4^{1-n})$:

$$\gamma = 1 - \sigma^{n+1} \quad \text{and} \quad G = B_{x,\varepsilon} - B_{x,\gamma\varepsilon}$$

We observe that, from (3.41),(3.42):

(3.53) $$\int_G \eta_\varepsilon(x-y)d\mu_F(y) \leq c(n)\varepsilon^{-n}\sigma^{n+1}\mu_F(G) \leq 2c(n)\varepsilon^{-1}\sigma^{n+1}(\alpha(1)+n\omega_n/2)$$

by virtue of (h_1), (3.29), and the monotonicity of α ((α_1) of 1.11); on the other hand, from (3.46) and the assumption $\sigma < 4^{1-n}$ we conclude, that for every x as in (3.52) it is possible to find $z \in \partial E$ such that $|x-z| < (1-\sigma)\varepsilon$. Therefore:

$$\int \eta_\varepsilon(x-y)d\,|D\phi_F|(y) \geq c(n)\varepsilon^{-n}(\sigma/2)|D\phi_F|(B_{x,(1-\sigma/2)\varepsilon})$$

(3.54) $$\geq c(n)\varepsilon^{-n}(\sigma/2)\;|D\phi_F|(B_{z,\sigma\varepsilon/2})$$

$$\geq c(n)\varepsilon^{-1}(\sigma/2)^n[\omega_{n-1}-(n-1)\int_0^{\sigma\varepsilon/2} t^{-1}\alpha(t)dt]$$

by (3.31). In view of 1.11, (α_4), we can certainly choose $\sigma^{\#} \in (0,1)$ such that

(3.55) $$\int_0^{\sigma\#} t^{-1}\alpha(t)dt \leq \omega_{n-1}/2(n-1)$$

Hence, from (3.54) we derive

(3.56) $$\int \eta_\varepsilon(x-y)d|D\phi_F|(y) \geq c(n)\omega_{n-1}\,2^{-n-1}\varepsilon^{-1}\sigma^n > 0 .$$

Combining (3.53) and (3.56) we obtain:

(3.57)
$$\int_G \eta_\varepsilon(x-y)\,d\mu_F(y) \leqq 2^{n+2}\omega_{n-1}^{-1}\, g_1(\sigma) \int \eta_\varepsilon(x-y)\,d|D\phi_F|(y)$$

where

(3.58)
$$g_1(\sigma) = (\alpha(1) + n\omega_n/2)\sigma .$$

Now, put

$$\delta = \sigma^{n+1}/2 \quad \text{and} \quad D = \partial F \cap B_{x,\gamma\varepsilon}$$

Due to the boundedness of D, we can find a finite number of points in D, wich we call $z_1,\ldots,z_h$, with the property that:

(3.59)
$$B_{z_i,\delta\varepsilon} \cap B_{z_j,\delta\varepsilon} = \emptyset \text{ if } i \neq j, \text{ and } D \subset \bigcup_{i=1}^{h} B_{z_i,2\delta\varepsilon}.$$

We write $B_{i,t}$ for $B_{z_i,t}$ $(i=1,\ldots,h)$, and observe that $B_{i,2\delta\varepsilon} \subset B_{x,\varepsilon}$ whence

(3.60)
$$\int_{B_{x,\gamma\varepsilon}} \eta_\varepsilon(x-y)\,d\mu_F(y) \leqq \sum_{i=1}^{h} \int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)\,d\mu_F(y)$$

(3.61)
$$\int_{B_{x,\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y) \geqq \sum_{i=1}^{h} \int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y)$$

For every $i = 1,\ldots,h$, we find

(3.62)
$$\int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)\,d\mu_F(y) \leqq c(n)\,\varepsilon^{-n}(1+2\delta - |x-z_i|/\varepsilon)\cdot\mu_F(B_{i,2\delta\varepsilon})$$

$$(3.63)\quad \int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y) \geqq c(n)\varepsilon^{-n}(1-\delta-|x-z_i|/\varepsilon)\cdot|D\phi_F|(B_{i,\delta\varepsilon})$$

$$\geqq c(n)2^{-1}\omega_{n-1}\delta^{n-1}\varepsilon^{-1}(1-\delta-|x-z_i|/\varepsilon)$$

(here, we used again (3.31), and the assumtion (3.55)). Since $|x-z_i| < (1-\sigma^{n+1})\varepsilon$, we have $1-\delta-|x-z_i|/\varepsilon > \sigma^{n+1} - \delta = \sigma^{n+1}/2 > 0$
Hence, taking the quotient of (3.62) over (3.63), we find

$$(3.64)\quad \Big(\int_{B_{i,2\delta\varepsilon}} \eta_\varepsilon(x-y)\,d\mu_F(y)\Big)\cdot\Big(\int_{B_{i,\delta\varepsilon}} \eta_\varepsilon(x-y)\,d|D\phi_F|(y)\Big)^{-1} \leqq 2^{n+2}\omega_{n-1}^{-1}(2\delta\varepsilon)^{1-n}\mu_F(B_{i,2\delta\varepsilon})$$

which holds for each $i = 1,\dots,h$.

Now we put $s = 2\delta\varepsilon = \sigma^{n+5}$ (see (3.49), $t = \sigma^{1/2(n-1)}$, and use definition (3.7) together with (3.30), to deduce that:

$$s^{1-n}\mu_F(B_{i,s}) = s^{1-n}|D\phi_F|(B_{i,s}) - s^{1-n}\,D_n\phi_F(B_{i,s})$$

$$(3.65)\quad \leqq t^{1-n}|D\phi_F|(B_{i,t})+(n-1)\int_s^t r^{-1}\alpha(r)dr-s^{1-n}\,D_n\phi_F(B_{i,s})$$

$$\leqq t^{1-n}\,\mu_F(D_{i,t})+(n-1)\int_0^t r^{-1}\alpha(r)dr+\big[t^{1-n}D_n\phi_F(B_{i,t})-s^{1-n}D_n\phi_F(B_{i,s})\big]$$

We have $B_{i,t} \subset B_1$, hence the first term in the right-hand side of the last inequality is not larger than $\sigma^{1/2}$, by (h_2) and our assumptions. On the account of (3.19),(3.26) the term in square brakets is easily estimated by

$$2^{1/2}\left[t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s})+(n-1)\cdot\int_s^t r^{-n}|D\phi_F|(B_{i,r})dr\right]^{1/}$$

$$\left[t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s})+(n-1)\int_s^t r^{-n}\psi(F,B_{i,r})dr\right]^{1/2}$$

As before,

$$t^{1-n}|D\phi_F|(B_{i,t}) = t^{1-n}\mu_F(B_{i,t})+t^{1-n}\ D_n\phi_F(B_{i,t}) \leqq \sigma^{1/2}+\omega_{n-1}$$

as a consequence of (3.19). Therefore, from (h_1) and (3.31) we get

$$t^{1-n}|D\phi_F|(B_{i,t})-s^{1-n}|D\phi_F|(B_{i,s}) \leqq \sigma^{1/2}+(n-1)\int_o^s r^{-1}\alpha(r)dr.$$

Similarly, from (h_1) and (3.29) we get

$$\int_s^t r^{-n}|D\phi_F|(B_{i,r})dr \leqq \int_s^t r^{-1}(\alpha(r)+n\omega_n/2)dr=2^{-1}n\omega_n \lg(t/s)+\int_s^t r^{-1}\alpha(r)dr\ ,$$

$$\int_s^t r^{-n}\psi(F,B_{i,r})dr \leqq \int_s^t r^{-1}\alpha(r)dr.$$

Collecting terms and going back to (3.65) we find, for each $i=1,\dots,h$:

$$s^{1-n}\ \mu_F(B_{i,s}) < \sigma^{1/2}+(n-1)\int_o^t r^{-1}\alpha(r)dr +$$

$$(3.66) \qquad + 2\left[\sigma^{1/2} + (n-1)\int_0^t r^{-1}\alpha(r)dr + 2^{-1}(n-1)n\omega_n \lg(t/s)\right]^{1/2}$$

$$\cdot\left[\sigma^{1/2} + (n-1)\int_0^t r^{-1}\alpha(r)dr\right]^{1/2} .$$

Recalling that $s = \sigma^{n+5}$, $t = \sigma^{1/2(n-1)}$, we derive from (3.66),(3.64), (3.60) and (3.61)

$$(3.67) \qquad \int_{B_{x,\gamma\varepsilon}} \eta_\varepsilon(x-y)d\mu_F(y) < 2^{n+2}\omega_{n-1}^{-1} g_2(\sigma) \int_{B_{x,\varepsilon}} \eta_\varepsilon(x-y)d|D\phi_F|(y)$$

where

$$g_2(\sigma) = \sigma^{1/2} + (n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr + 2\cdot\left[\sigma^{1/2} + (n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr\right]^{1/2} \cdot$$

$$(3.68) \qquad \cdot\left[\sigma^{1/2} + (n-1)\int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr - 2^{-1}(n-1)n\omega_n(2n^2+8n-11)\cdot\lg\sigma^{1/2(n-1)}\right]^{1/2}$$

On adding (3.57) and (3.67), we get (3.51) with

$$g(\sigma) = 2^{n+2}\ \omega_{n-1}^{-1}\left[g_1(\sigma) + g_2(\sigma)\right] .$$

g_1 and g_2 given by (3.58) and (3.68). In order to assure that g is infinitesimal at 0, the only point to check is the following:

$$(3.69) \qquad \lim_{\sigma\to 0^+} (-\lg \sigma^{1/2(n-1)}) \cdot \int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr = 0$$

Now, the monotonicity of α implies that

$$(-\lg\sigma^{1/2(n-1)}) \cdot \int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha(r)dr \leqq \left(\int_{\sigma^{1/2(n-1)}}^1 r^{-1}dr\right)\cdot\alpha^{1/2}(\sigma^{1/2(n-1)})\cdot \int_0^{\sigma^{1/2(n-1}} r^{-1}\alpha^{1/2}(r)dr$$

$$\leqq \left(\int_{\sigma^{1/2(n-1)}}^1 r^{-1}\alpha^{1/2}(r)dr\right). \int_0^{\sigma^{1/2(n-1)}} r^{-1}\alpha^{1/2}(r)dr$$

and (3.69) follows from 1.11, (α_4).

We can then choose $\sigma^{\#} \in (0,1)$ such that (3.55) holds and, in addi tion, such that

$$g(\sigma) < 1 \qquad \forall\ \sigma \leq \sigma^{\#}$$

From (3.50), (3.51) we deduce (3.48), thus concluding the proof of the first step.

Step 2. According to Step 1, assumtions (h_1) and (h_2) imply

(3.70) $$\frac{D_n f(x)}{|Df(x)|} > 1 - g(\sigma) > 0$$

for every x as in (3.52), provided $\sigma \in (0,\sigma^{\#}]$. At this point, we can start on the study of the level sets of the function f, defined by (3.49). To this end, we also assume

(3.71) $$\sigma \leqq 2^{8(1-n)}$$

so that in particular $7/8 \leqq 1-2\sigma^{1/2(n-1)}$ and $1-2n^2\sigma^2 > 3/4$. For $\lambda \in]0,1[$, we define

(3.72) $$L_\lambda = \{x : f(x) \geqq \lambda\}$$

and observe that, according to (3.70), for every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$,

$$\partial L_\lambda \cap B_{7/8} = \{x \in B_{7/8} : f(x) = \lambda\}$$

is the graph, over a certain open set $A'_\lambda \subset \mathbb{R}^{n-1}$, of a certain function $u_\lambda \in C^1(A'_\lambda)$. Denoting by $\nu_\lambda(x)$ the unit inner normal to L_λ at $x \in \partial L_\lambda$, we have also:

$$\nu_\lambda(x) = -\frac{Df(x)}{|Df(x)|} = (1+|Du_\lambda(x')|^2)^{-1/2} \cdot (-Du_\lambda(x'),1)$$

for every $x \in \partial L_\lambda \cap B_{7/8}$, i.e. $x = (x', u_\lambda(x'))$, with $x' \in A'_\lambda$. As a consequence, (3.70) yields $p_\lambda^2 \leqq g(\sigma)(2-g(\sigma))(1-g(\sigma))^{-2}$, $\forall \lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$, where:

(3.73) $$p_\lambda \equiv \sup\{|Du_\lambda(x')| : x' \in A'_\lambda\}$$

In particular, we get

(3.74) $$p_\lambda \leqq t \quad \text{whenever} \quad g(\sigma) \leqq 1-(1+t^2)^{-1/2}$$

On the other hand, it is not difficult to show, that if for every such λ ∂L_λ passes "sufficiently close to the origin", while being "flat enough", then each domain A'_λ contains an (n-1)-dimensional ball of fixed radius. For example, let us suppose that for a fixed $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$ it holds:

(3.75) $$\partial L_\lambda \cap B_{1/8} \neq \emptyset$$

We already know that $A'_\lambda \subset B'_{7/8}$ and $|u_\lambda(x')| < 7/8$ $\forall x' \in A'_\lambda$

Moreover, if σ is chosen in such a way that

(3.76) $$g(\sigma) \leqq 1-4 \cdot 17^{-1/2} = .0299$$

(recall that g is infinitesimal at 0), then (3.74) yields

(3.77) $$p_\lambda \leq 1/4.$$

Now, according to (3.75), we pick $z \in \partial L_\lambda \cap B_{1/8}$, i.e. $z = (z', u_\lambda(z'))$ with $z' \in A'_\lambda$, $|z'| < 1/8$, and $|u_\lambda(z')| < 1/8$. If x' is any other point in A'_λ, then

(3.78) $$|u_\lambda(x')| \leq |u_\lambda(z')| + p_\lambda(|x'| + |z'|) < 3/8$$

while, if $x' \in \partial A'_\lambda$, then $(x', u_\lambda(x')) \in \partial B_{7/8}$, hence (3.78) yields

$$|x'| \geq 7/8 - |u_\lambda(x')| > 1/2$$

and we conclude immediately that $B'_{1/2} \subset A'_\lambda$.

Thus, see (3.71) and (3.76), if σ satisfies

(3.79) $$\sigma \leq \sigma^{\#}, \sigma \leq 2^{8(1-n)}, \quad \text{and} \quad g(\sigma) \leq 1-4\cdot 17^{-1/2}$$

with $\sigma^{\#}$ and g as in Step 1, and if in addition $\partial L_\lambda \cap B_{1/8} \neq \emptyset$ for a certain $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$, then (see also (3.78)):

(3.80) $$B'_{1/2} \subset A'_\lambda \subset B'_{7/8} \quad \text{and} \quad \partial L_\lambda \cap \tilde{Q}_{1/2} = \partial L_\lambda \cap (B'_{1/2} \times \mathbb{R}) \cap B_{7/8} =$$

$$= \{x : x' \in B'_{1/2}, x_n = u_\lambda(x')\}$$

where $\tilde{Q}_{1/2}$ denotes the cylinder of radius 1/2 around the origin, i.e.

$$\tilde{Q}_{1/2} = B'_{1/2} \times (-1/2, 1/2) \subset B_{7/8}.$$

In the same hypotheses, from (3.78) we get also $|u_\lambda(0)| \leq 5/32$, so that $|u_\lambda(0)| + r < 1/2$ whenever $r < 1/4$. Thus, setting

(3.81) $$Q_{\lambda,r} = \{x : |x'| < r, \ |x_n - u_\lambda(0)| < r\}$$

(compare with (3.23)), we obtain from (3.80) and (3.77):

(3.82) $$\partial L_\lambda \cap B_r \subset \partial L_\lambda \cap [B'_r \times (-1/2,1/2)] = \{x : x' \in B'_r, x_n = u_\lambda(x')\} = \partial L_\lambda \cap Q_{\lambda,r}$$

for every $r \in (0,1/4)$. Finally, we can easily check that
$Q_{\lambda,r} \subset B_{3r} \quad \forall r \in (1/8,1/4)$.

Step 3. We are now ready to conclude the proof of the Main Lemma. As in Step 2, we denote by σ a positive number satisfying (3.79), by λ a number in the interval $[0,1]$, and by L_λ the corresponding level est of the function $f = \phi_F * \eta_\varepsilon$, with $\varepsilon = \sigma^4$.

According to the preceding assumptions (see the implications following (3.71), we have in particular:

(3.83) $$1 - 2n^2\sigma^2 > 3/4 .$$

Furthermore, it is easy to check that

(3.84) $$\int_0^1 d\lambda \int_B |\phi_{L_\lambda} - \phi_F| dx = \int_B |f-\phi_F| dx, \quad \int_0^1 d\lambda \int_{\partial B} |\phi_{L_\lambda} - \phi_F| dH_{n-1} = \int_{\partial B} |f-\phi_F| dH_{n-1}$$

(here, only the fact that f lies between 0 and 1 really matters).
Finally, we recall the following "coarea formula":

(3.85) $$\int_0^1 |D\phi_{L_\lambda}|(B) d\lambda = \int_B |Df(x)| dx$$

(see [19], theorem 1.23, or [28], theorem 1.6).

From (3.17) we get for all $t < 1$ and almost all $\lambda \in [0,1]$:

$$\omega(F,B_t) \leqq \psi(F,B_t) + \omega(L_\lambda,B_t) + 2\int_{\partial B_t} |\phi_{L_\lambda} - \phi_F| dH_{n-1}$$

which, integrated over $(\tau,2\tau)$, yields:

$$(3.86) \qquad \omega(F,B_\tau) \leqq \beta(1) + \omega(L_\lambda,B_{2\tau})+2\tau^{-1}\int_{B_{1/8}} |\phi_{L_\lambda}-\phi_F|\,dx,$$

because of the monotonicity of ω and ψ, our hypothesis (H_1), and the fact we are assuming $\tau < 2^{-4}$. We now suppose that $\partial L_\lambda \cap B_{2\tau} \neq \emptyset$, for every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$: otherwise, we would have $\omega(L_\lambda, B_{2\tau}) = 0$ for some of such λ's, and the proof of the Lemma would obviously be easier. We are then precisely in the situation discussed in Step 2 (see 3.75)). Hence, according to (3.77), (3.80), and (3.82), we derive from (3.21):

$$\omega(L_\lambda,B_{2\tau}) \leqq \omega(L_\lambda,Q_{\lambda,2\tau}) \leqq 2\psi(L_\lambda,Q_{\lambda,t}) +$$

$$(3.87) \qquad + 2(1-p_\lambda^2)^{-1}[(2\tau/t)^{n+1}+p_\lambda^2]\,\omega(L_\lambda,Q_{\lambda,t})$$

for every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$ and every $t \in (2\tau, 1/4)$.

Recalling (see the last assertion of Step 2) that $Q_{\lambda,t} \subset B_{3t}$ $\forall t \in (1/8, 1/4)$, we get from (3.86),(3.87):

$$\omega(F,B_\tau) \leqq \beta(1)+2\psi(L_\lambda,B_{3t})+2(1-p_\lambda^2)^{-1}[(2\tau/t)^{n+1}+p_\lambda^2]\,\omega(L_\lambda,B_{3t}) +$$

$$(3.88) \qquad + 2\tau^{-1}\int_{B_{1/8}} |\phi_{L_\lambda}-\phi_F|\,dx$$

which holds for any $t \in (1/8,1/4)$ and any $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$.

To focus on the real substance of the proof, it seems now convenient to adopt the following

Convention. Throughout be rest of the present section, c will denote constants not necessarily the same at any occurence. Similarly, c(n) will denote a *generic* positive constant, depending only on n.

We remark that all these constants (in particular, $c_1(n), c_2(n)$ in (3.97)) are easily computable.

We use again (3.17) to estimate $\omega(L_\lambda, B_{3t})$ in (3.88), thus getting for t, λ as before:

$$\omega(F,B_\tau) \leqq \beta(1) + c\psi(L_\lambda, B_{3t}) + c[(2\tau/t)^{n+1} + p_\lambda^2]\omega(F,B_{3t}) +$$

(3.89)

$$+ c \int_{\partial B_{3t}} |\phi_{L\lambda} - \phi_F^-| dH_{n-1} + 2\tau^{-1} \int_{B_{1/8}} |\phi_{L_\lambda} - \phi_F| dx$$

since $p_\lambda \leqq 1/4$ (see (3.77)).

Next, in addition to (3.79), we assume that σ also sat... ...s:

$$g(\sigma) \leqq 1 - [1+(16\tau)^{n+1}]^{-1/2} \tag{3.90}$$

From (3.74) we obtain $p_\lambda^2 \leqq (16\tau)^{n+1}$, and thus the third term in the right-hand side of (3.89) can be estimated by $c(n)\,\sigma\tau^{n+1}$, in view of (H_2).

To estimate the second term in the right-hand side of (3.89) we use instead (3.16), which yields, in view of (H_1):

$$\psi(L_\lambda, B_t) \leqq \beta(1) + |D\phi_{L_\lambda}|(B_t) - |D\phi_F|(B_t) + \int_{\partial B_t} |\phi_{L_\lambda} - \phi_F| dH_{n-1}$$

Going back to (3.89) we find

$$\omega(F,B_\tau) \leqq c\beta(1) + c(n)\sigma\tau^{n+1} + c[|D\phi_{L_\lambda}|(B_t) - |D\phi_F|(B_t)] +$$

$$+ c\int_{\partial B_t} |\phi_{L_\lambda} - \phi_F| dH_{n-1} + 2\tau^{-1} \int_{B_{1/8}} |\phi_{L\lambda} - \phi_F| \, dx$$

for every $t \in (3/8,3/4)$ and every $\lambda \in (n^2\sigma^2, 1-n^2\sigma^2)$. By integrating in λ we get ((3.83), (3.84), and (3.85)):

$$\omega(F,B_\tau) \leq c\beta(1)+c(n)\sigma\tau^{n+1}+c\Big[\int_{B_t} |Df(x)|dx-|D\phi_F|(B_t)\Big]+c(n)\sigma^2 \int_{B_{3/4}} |Df(x)|dx +$$

$$+ c \int_{\partial B_t} |f-\phi_F|dH_{n-1} + (c/\tau)\int_{B_{1/8}} |f-\phi_F|dx$$

for every $t \in (3/8,3/4)$. Finally, by integrating in t we obtain:

$$\omega(F,B_\tau) \leq c\beta(1)+c(n)\sigma\tau^{n+1}+c\int_{3/8}^{3/4} dt\Big(\int_{B_t} |Df(x)|dx-|D\phi_F|(B_t)\Big) +$$

(3.91)

$$+ c(n)\sigma^2 \int_{B_{3/4}} |Df(x)|dx + c\int_{B_{3/4}} |f-\phi_F|dx+(c/\tau)\int_{B_{1/8}} |f-\phi_F|dx$$

Now, (3.16) implies that for all $t < 1$ and almost all $\lambda \in [0,1]$:

$$|D\phi_F|(B_t) \leq |D\phi_{L_\lambda}|(B_t)+\psi(F,B_t) + \int_{\partial B_t} |\phi_{L_\lambda}-\phi_F|dH_{n-1}$$

from which, integrating first in $\lambda \in [0,1]$, and then in $t \in (0,3/8)$, we find, on the account of (3.84), (3.85) and (H_1):

$$(3.92) \quad \int_0^{3/8} \Big(|D\phi_F|(B_t)-\int_{B_t} |Df(x)|dx\Big)dt \leq (3/8)\beta(1)+\int_{B_{3/8}} |f-\phi_F|dx$$

Moreover, setting $h(x) = \max\{3/4-|x|,0\}$, we find easily that

$$(3.93) \quad \int h(x)|Df(x)|dx = \int_0^{3/4} dt \int_{B_t} |Df(x)|dx$$

$$\int h(x)d|D\phi_F|(x) = \int_0^{3/4} |D\phi_F|(B_t)dt$$

since the level sets $\{x : h(x) > t\}$ of h are empty whenever $t \geq 3/4$, and coincide with $B_{3/4-t}$ whenever $0 \leq t < 3/4$. We notice that h is Lipschitz-continuous, with Lipschitz constant 1, so that $|h*\eta_\varepsilon - h| \leq \sigma^4$. Therefore, recalling (3.50), we find:

$$(3.94)\quad \int h(x)|Df(x)|dx \leq \int (h*\eta_\varepsilon)(y)d|D\phi_F|(y) \leq \sigma^4|D\phi_F|(B_1)+\int h(y)d|D\phi_F|(y)$$

In conclusion, from (3.92),(3.93), and (3.94), we get:

$$\int_{3/8}^{3/4} dt\left(\int_{B_t} |Df(x)|dx - |D\phi_F|(B_t)\right) \leq (3/8)\beta(1)+\sigma^4|D\phi_F|(B_1)+\int_{B_{3/8}} |f-\phi_F|dx$$

which, combined with (3.91) and (3.44), (3.45), yields:

$$(3.95)\quad \omega(F,B_\tau) \leq c\beta(1)+c(n)\sigma\tau^{n+1}+c\sigma^2|D\phi_F|(B_1)\cdot[\sigma^2+c(n)+\sigma^2/\tau].$$

By (H_1) and (3.29) we have $|D\phi_F|(B_1) \leq \beta(1)+n\omega_n/2$. Hence, assuming that

$$(3.96)\qquad \sigma \leq \tau^{n+1}$$

we get from (3.95):

$$(3.97)\qquad \omega(F,B_\tau) \leq c_1(n)\beta(1) + c_2(n)\sigma\tau^{n+1}$$

as required. Lemma 3.6 is then completely proved, provided we choose $\sigma^* \in (0,1)$ such that each $\sigma \leq \sigma^*$ satisfies (3.79),(3.90), and (3.96).

CHAPTER 4: PROOF OF THE REGULARITY THEOREM

We are now in a position to complete the proof of Theorem 1.9.

We split the demonstration into three steps. First we treat the regularity of the reduced boundary of a set with almost minimal boundary, then we consider sequences of sets with uniformly almost minimal boundaries, and finally we discuss the Hausdorff dimension of the singular points.

A general remark is in order: since the conclusions of Theor. 1.9 are of local character, it is clear that, given a set E with almost minimal boundary in Ω , we can restrict our analysis to a (sufficiently small) neighbourhood of on arbitrary point of Ω (actually, the only interesting case is when that point is in $\partial E \cap \Omega$). Our main assumption will then be

$$\psi(E,B_{x_o,t_o}) \leq \alpha(t)\cdot t^{n-1} \quad \forall x \in B_{x_o,T_o} \ , \ \forall t\in(0,T_o)$$

with $\alpha(t)$ as in section 1.11. See also the remark in section 3.4.

Step 1. Given $n \geq 2$, α as in 1.11, and τ satisfying $0<\tau<\min\{2^{-4},1/2c_2\}$ where c_2 is the constant appearing in (3.47), we indicate by $\sigma^* \in (0,1)$ the constant whose existence is granted by the Main Lemma 3.6.

Let now $E \subset \mathbb{R}^n$, $x_o \in \partial E$, $R_o \in (0,1)$, and $\sigma_o \in (0,\sigma^*]$ be such that:

(4.1) $$\int_0^{R_o} t^{-1}\alpha(t)dt \leqq \omega_{n-1}/2(n-1)$$

(4.2) $$\alpha(R_o) \leqq \sigma_o \tau^n/4 c_1$$

(4.3) $$\psi(E,B_{x,t}) \leqq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_{x_o,R_o} \text{ and } \forall t\in(0,R_o)$$

(4.4) $$\omega(E,B_{x_o,R_o}) \leqq \sigma_o R_o^{n-1}$$

(Roughly speaking, we are assuming that the excess is small, on a (small) initial ball in which ∂E is almost minimal. Applying the Main Lemma iteratively, we first show that for every integer $h \geq 0$ it holds:

(4.5) $$\omega(E,B_{x_o,R_h}) \leqq \sigma_h R_h^{n-1}$$

where:

$$R_h = \tau^h R_o$$

(4.6) $$\sigma_h = c_3 \sum_{i=1}^{h} c_4^{i-1} \alpha(R_{h-i}) + c_4^h \sigma_o$$

$$c_3 = c_1 \tau^{1-n}, \quad c_4 = c_2 \tau^2$$

and c_1, c_2 are as in (3.47).

In fact, (4.5) reduces simply to (4.4) when $h = 0$. Assuming that (4.5) holds for a certain $h \geqq 0$, and setting

$$F_h = R_h^{-1}(E-x_o), \quad \beta_h(t) = \alpha(R_h t) \quad \text{for} \quad t \in (0,1]$$

we find from (4.3) and (4.5);

$$\psi(F_h,B_{x,t}) \leqq \beta_h(t) \cdot t^{n-1} \qquad \forall x \in B_1, \ \forall t \in (0,1)$$

$$\omega(F_h,B_1) \leqq \sigma_h$$

Clearly, $\beta_h \prec \alpha$ (section 3.5), while $\sigma_h \leqq \sigma_o \leqq \sigma^*$ $\forall h$: for, if $h \geqq 0$, then

$$\tau^i \alpha(R_{h-i}) \leqq \alpha(R_h) \qquad \forall i = 0,\dots,h$$

since $t^{-1}\alpha(t)$ is non-increasing on $(0,1)$ (recall (α_3) of 1.11); hence, from (4.6) we obtain

$$(4.7)\qquad \sigma_h \leqq c_3\tau^{-1}\alpha(R_h)\sum_{i=1}^{h}(c_2\tau)^{i-1}+c_4^h\sigma_o \leqq 2c_1\tau^{-n}+\tau^h\sigma_o \leqq \sigma_o \quad \forall h \geqq 0,$$

according to (4.2), and our initial assumption $\tau < 1/2c_2$. We are then precisely in the situation covered by the Main Lemma 3.6, from which we derive

$$\omega(F_h,B_\tau) \leqq c_1\beta_h(1)+c_2\sigma_h\tau^{n+1} = (c_3\alpha(R_h)+c_4\sigma_h)\tau^{n-1} = \sigma_{h+1}\tau^{n-1}$$

according to (4.6). In conclusion, we find

$$\omega(E,B_{x_o,R_{n+1}}) \leqq \sigma_{h+1}\, R_{h+1}^{n-1}$$

which is exactly (4.5), with h+1 in place of h.

Next, we show that in the hypotheses (4.1)-(4.4), $x_o \in \partial^*E$. To this aim, we observe that from (3.25) and for every $h,k \geqq 0$:

$$|\nu(E,B_{x_o,R_{h+k}}) - \nu(E,B_{x_o,R_h})| \leqq \sum_{i=o}^{k-1}|\nu(E,B_{x_o,R_{h+i+1}})-\nu(E,B_{x_o,R_{h+i}})|$$

$$(4.8)\qquad \leqq 2\sum_{i=o}^{k-1}\left[\frac{\omega(E,B_{x_o,R_{h+i}})}{|D\phi_E|\ (B_{x_o,R_{h+i+1}})}\right]^{1/2}$$

$$\leqq 2^{3/2}(\omega_{n-1}\tau^{n-1})^{-1/2}\cdot\sum_{i=o}^{k-1}\sigma_{h+i}^{1/2}$$

by virtue of (4.5),(3.31),and (4.1). See section 2.10.

According to (4.7), we have:

$$\sigma_{h+i} \leqq 2c_1 \tau^{-n}\alpha(R_{h+i}) + \tau^{h+i}\sigma_o$$

whence

$$\sum_{i=o}^{k-1} \sigma_{h+i}^{1/2} \leqq (2c_1\tau^{-n})^{\frac{1}{2}} \cdot \sum_{i=o}^{k-1} \alpha^{1/2}(R_{h+i}) + (\tau^h\sigma_o)^{\frac{1}{2}} \cdot \sum_{i=o}^{k-1} \tau^{i/2}$$

(4.9)
$$\leqq (2c_1\tau^{-n})^{\frac{1}{2}} \cdot (1-\tau)^{-1} \cdot \int_{R_{h+k}}^{R_h} t^{-1}\alpha^{1/2}(t)dt + 2\tau^{h/2}$$

since $t^{-1}\alpha^{1/2}(t)$ is also non-increasing, by (α_3) of 1.11. By the same reason, we have also:

(4.10) $$\int_o^{R_h} t^{-1}\alpha^{1/2}(t)dt \leqq \tau^{-k}\int_o^{R_{h+k}} t^{-1}\alpha^{1/2}(t)dt \qquad \forall h,k \geqq 0.$$

Thus, substitution of (4.9) into (4.8) yields, for every $h,k \geqq 0$:

(4.11)
$$|\nu(E,B_{x_o,R_{h+k}}) - \nu(E,B_{x_o,R_h})| \leqq 4(c_1/\omega_{n-1})^{\frac{1}{2}} \cdot \tau^{1/2-n} \cdot (1-\tau)^{-1} \cdot \int_o^{R_h} t^{-1}\alpha^{1/2}(t)dt + 2^{5/2}(\omega_{n-1}\tau^{n-1})^{-\frac{1}{2}} \cdot \tau^{h/2}$$

which shows that $\{\nu(E,B_{x_o,R_h})\}$ is a Cauchy sequence. Calling ν its limit, we find

$$0 \leqq 1 - |\nu| = \lim_{h\to+\infty} \frac{\omega(E, B_{x_o, R_h})}{|D\phi_E|(B_{x_o, R_h})} \leqq \lim_{h\to+\infty} (2\omega_{n-1}^{-1}\,\sigma_h) = 0$$

by (4.5),(3.31),(4.1) and (4.7).

Now, let $t \in (0, R_o)$, and call $h = h(t)$ the unique, non-negative integer, for wich

$$R_{h+1} \leqq t < R_h$$

Arguing as above (see in particular (4.8), (4.9),(4.10), and (4.11)), we find

(4.12)

$$|\nu - \nu(E, B_{x_o, t})| \leqq |\nu - \nu(E, B_{x_o, R_h})| + 2\left[\frac{\omega(E, B_{x_o, R_h})}{|D\phi_E|(B_{x_o, R_{h+1}})}\right]^{1/2}$$

$$\leqq 4(c_1/\omega_{n-1})^{\frac{1}{2}}.(2-\tau).\tau^{1/2-n}.(1-\tau)^{-1}.\int_0^{R_h} r^{-1}\alpha^{1/2}(r)dr + 3\cdot 2^{3/2}(\tau^{n-1})^{-\frac{1}{2}}\tau^{h/2}$$

$$\leqq c_5 \int_0^{R_{h+1}} r^{-1}\alpha^{1/2}(r)dr + c_6\,\tau^{(h+1)/2}$$

$$\leqq c_5 \int_0^{t} r^{-1}\alpha^{1/2}(r)dr + c_6(t/R_o)^{\frac{1}{2}},$$

where c_5, c_6 depend only on n and τ.

In conclusion, see (3.2), we have $\nu = \nu_E(x_o)$, i.e. $x_o \in \partial^* E$ as claimed. Similarly, in the same hypotheses (4.1)-(4.4) we can prove that $\partial E = \partial^* E$ in a neighborhood of x_o.

For, let $N \geqq 1$ be such that $\sigma_N \leqq \sigma_o \tau^{n-1}$ (see (4.7)), and set

$$\delta = (1-\tau)\tau^N R_o < R_o$$

Then, for every $x \in B_{x_o,\delta}$ we have $B_{x,\tau^{N+1}R_o} \subset B_{x_o,\tau^N R_o}$, whence:

$$(\tau^{N+1}R_o)^{1-n} \cdot \omega(E,B_{x,\tau^{N+1}R_o}) \leqq (\tau^N R_o)^{1-n} \cdot \tau^{1-n} \cdot \omega(E,B_{x_o,\tau^N R_o}) \leqq \tau^{1-n}\sigma_N \leqq \sigma_o$$

by virtue of (4.5). Accordingly, we are again in the situation considered at the very beginning of Step 1, i.e. (4.1)-(4.4) all hol with x_o and R_o replaced by any $x \in B_{x_o,\delta} \cap \partial E$ and, respective $R = \tau^{N+1} R_o < R_o$. It follows from the preceding discussion that $x \in \partial^* E$, for any such x. Moreover, see (4.12), for every $x \in \partial E \cap B_{x_o,\delta}$ and every $t \in (0,R)$, we have:

$$(4.13) \qquad |\nu_E(x) - \nu(E,B_{x,t})| \leqq c_5 \int_0^t r^{-1}\alpha^{1/2}(r)dr + c_6(t/R)^{1/2}$$

Using (4.13), we can easily show that ν_E varies smoothly on ∂E near x_o. To this aim, we put $\delta_1 = \tau^2 R/2 < \delta/2$ and, given $x,y \in \partial E \cap B_{x_o,\delta_1}$ with $x \neq y$, we denote by h the unique, positive integer for which

$$(4.14) \qquad \tau^{h+2} R \leqq |x-y| < \tau^{h+1} R.$$

Then we define $s = (1-\tau)\cdot\tau^h R$, $t = \tau^h R$, so that $B_{x,s} \subset B_{y,t}$. It follows from (3.25) that

$$|\nu(E,B_{x,s}) - \nu(E,B_{y;t})| \leq 2 \left[\frac{\omega(E,B_{y,t})}{|D\phi_E|(B_{x,s})}\right]^{1/2}$$

Hence, repeating the preceding argument, and using (4.13), we get

$$|\nu_E(x) - \nu_E(y)| \leq c_7 \int_o^t r^{-1}\alpha^{1/2}(r)dr + c_8(t/R)^{1/2}$$

where, as usual, c_7 and c_8 depend only on n and τ. Finally, recalling that $t = \tau^h R$, we find from (4.10) and (4.14):

$$|\nu_E(x) - \nu_E(y)| \leq c_7\tau^{-2} \int_o^{|x-y|} r^{-1}\alpha^{1/2}(r)dr + c_8\tau^{-1}(|x-y|/R)^{\frac{1}{2}}$$

wich proves the continuity of the normal vector ν_E on $\partial E \cap B_{x_o,\delta_1}$. In particular, when $\alpha(t) \leq \text{const.} \cdot t^\alpha$ for $\alpha \in (0,1)$, we obtain that ν_E is of class $C^{o,\alpha/2}$ (see also section 1.12).

To conclude with the first part of the Regularity Theorem, we have only to show that in the case when ∂E is almost minimal in Ω and $x_o \in \partial^* E \cap \Omega$, then it is possible to pick R_o and σ_o such that (4.1)-(4.4) all hold. This is certainly true, because of almost minimality (see sections 1.5, 1.11, and 1.13), and since $t^{1-n}\omega(E,B_{x,t})$ tends to zero as $t \to o^+$, whenever $x \in \partial^* E$ (recall (2.26)).

Step 2. Now, given α as in 1.11, $T_o \in (0,1)$, and $x_o \in \mathbb{R}^n$, we suppose that

$$(4.15) \qquad \psi(E_h,B_{x,t}) \leq \alpha(t)\cdot t^{n-1} \quad \forall x \in B_{x_o,T_o},\ \forall t \in (0,T_o),\ \forall h \geq 1$$

Moreover, we assume $E_h \to E_\infty$ on $B_{x_o,2T_o}$. If $x_h \in \partial E_h$ and $x_h \to x_\infty \in B_{x_o,T_o}$, then clearly $B_{x_h,r} \cap E_h \to B_{x_\infty,r} \cap E_\infty$ $\forall r \in (0,d)$, with $d = T_o - |x_o - x_\infty|$. Furthermore, $B_{x_h,r} \subset B_{x_\infty,d}$ whenever $r < d/2$ and h is large enough. From (3.32) we get immediately $x_\infty \in \partial E_\infty$, as required. Next, we assume $x_\infty \in \partial^* E_\infty$, and fix τ and σ^* as in Step 1. Reasoning possibly on subsequences of $\{E_h\}$, we can choose $r \in (0,d)$ and $h_o \geqq 1$ such that $\forall h \geqq h_o$:

$$\int_0^r t^{-1}\alpha(t)dt \leqq \omega_{n-1}/2(n-1)$$

$$\alpha(r) \leqq \sigma^* \tau^n / 4c_1$$

$$r^{1-n}\omega(E_\infty, B_{x_\infty,r}) \leqq 2^{-n-1}\sigma^* \tag{4.16}$$

$$|x_h - x_\infty| < r/2$$

$$r^{1-n} \cdot \int_{\partial B_{x_\infty,r}} |\phi_{E_h} - \phi_{E_\infty}| dH_{n-1} \leqq 2^{-n-2} \sigma^*$$

As a consequence of the almost minimality of ∂E_h, we derive from (4.16) and (3.17)

$$r^{1-n}\omega(E_h, B_{x_\infty,r}) \leqq 2^{1-n}\sigma^* \quad \text{and} \quad B_{x_h,r/2} \subset B_{x_\infty,r} \quad \forall h \geqq h_o.$$

Hence:

$$\omega(E_h, B_{x_h,r/2}) \leqq \sigma^* \cdot (r/2)^{n-1}$$

by virtue of the monotonicity of ω . Thus, for every $h \geqq h_o$, we see that $E_h, x_h, r/2$, and σ^* are precisely in the situation already discussed at the beginning of Step 1: we get, in particular, $x_h \in \partial^* E_h \quad \forall h \geqq h_o$, while (see (4.13)):

$$(4.17)\quad |\nu_{E_h}(x_h)-\nu(E_h,B_{x_h,t})| \leqq c_5 \int_0^t s^{-1}\alpha^{1/2}(s)ds+c_6(2t/r)^{\frac{1}{2}} \quad \forall h \geqq h_o,\ \forall t\in(0,r/2)$$

Similarly, observing that E_∞ is also almost minimal (because of (4.15) and (3.12)), we obtain

$$(4.18)\quad |\nu_{E_\infty}(x_\infty)-\nu(E_\infty,B_{x_\infty,t})| \leqq c_5 \int_0^t s^{-1}\alpha^{1/2}(s)ds+c_6(2t/r)^{\frac{1}{2}} \quad \forall t\in(0,r/2).$$

Moreover, it si not difficult to show that

$$(4.19)\quad \limsup_{h\to+\infty} |\nu(E_h,B_{x_h,t})-\nu(E_\infty,B_{x_\infty,t})| \leqq c_9\alpha(t) \text{ for a.e. } t \in (0,r/2).$$

This follows e.g. by inserting

$$\frac{D\phi_{E_h}(B_{x_\infty,t})}{|D\phi_{E_h}|(B_{x_h,t})}\ ,\quad \nu(E_h,B_{x_\infty,t})\ ,\quad \text{and}\quad \frac{D\phi_{E_\infty}(B_{x_\infty,t})}{|D\phi_{E_h}|(B_{x_\infty,t})}$$

as intermediate points between $\nu(E_h,B_{x_h,t})$ and $\nu(E_\infty,B_{x_\infty,t})$, and then by using (3.19),(3.16) and almost minimality to estimate the four partial distances.

Combining (4.17),(4.18), and (4.19) we get immediately the convergence of $\nu_{E_h}(x_h)$ toward $\nu_{E_\infty}(x_\infty)$.

As a by-product of the preceding discussion, we obtain that whenever the open se A contains the singular points of ∂E_∞, then it also contains the singular points of ∂E_h, for h large enough. More precisely, denoting by Σ_h the singular set $\partial E_h \setminus \partial^* E_h$, from the assumption

$$\Sigma_\infty \cap K \subset A$$

(K compact $\subset B_{x_o, T_o}$), we derive immediately that

$$\Sigma_h \cap K \subset A$$

for every sufficiently large h. This, in turn, implies that

$$H_s^\infty(\Sigma_\infty \cap K) \geqq \limsup_{h \to +\infty} H_s^\infty(\Sigma_h \cap K) \tag{4.20}$$

where, for every real $s \geqq 0$ and every $X \subset \mathbb{R}^n$ we de

$$H_s^\infty(X) = \omega_s 2^{-s} \inf\{\sum_{i=1}^\infty (\operatorname{diam} A_i)^s : A_i \text{ open}, X \subset \bigcup_{i=1}^\infty A_i\}$$

(see [13], p. 767, and [27], 2.6.4).

We end this part by recalling two general facts concerning H_s^∞ (see [12], 2.10.2 and 2.10.19 (2), and [27], 2.6.4):

$$H_s^\infty(X) = 0 \qquad \text{if and only if} \qquad H_s(X) = 0 \tag{4.21}$$

$$\limsup_{t \to 0^+} \omega_s^{-1} t^{-s} H_s^\infty(X \cap B_{x,t}) \geqq 2^{-s} \quad \text{for } H_s\text{-a.e. } x \in X. \tag{4.22}$$

Step 3. To conclude the proof of the Regularity Theorem, we have only to show that $H_s(\Sigma_E \cap \Omega) = 0$, whenever E has almost minimal boundary in $\Omega \subset \mathbb{R}^n$ and $s > n - 8$, with:

$$\Sigma_E = \partial E \setminus \partial^* E.$$

This follows easily by "blowing-up" at singular points (see the final part of Prop. 3.4), and then by using known results concerning the existence and non-existence of singular minimal cones in $\mathbb{R}^n$, for which we refer the reader to [27], sections 2.6 and 2.7.

By (4.22), assuming that E satisfies:

(4.23) $$\psi(E,B_{x,t}) \leq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_{x_o,T_o}\ ,\ \forall t \in (0,T_o)$$

and that

(4.24) $$H_s(\Sigma_E \cap B_{x_o,T_o}) > 0,$$

we can choose $x \in \Sigma_E \cap B_{x_o,T_o}$ and a sequence $\{t_h\}$, satisfying

(4.25) $$t_n \downarrow 0 \quad \text{and} \quad H_s^\infty(\Sigma_E \cap B_{x,t_h}) \geqq \omega_s 2^{-s-1} t_h^s \qquad \forall h.$$

Setting $E_h = t_h^{-1}(E-x)$, and passing to a subsequence if necessary, we find (in view of Prop. 3.4) that $\{E_h\}$ converges to a minimal cone $C_o \subset \mathbb{R}^n$, for which

(4.26) $$H_s^\infty(\Sigma_{C_o} \cap B_1) > 0\ ,$$

by virtue of (4.20) and (4.25). This way, starting from a set $E \subset \mathbb{R}^n$ with almost minimal boundary (see (4.23)) and satisfying (4.24), we obtain a *minimal* cone C_o with the same property, namely:

(4.27) $$C_o \subset \mathbb{R}^n \quad \text{and} \quad H_s(\Sigma_{C_o} \cap B_1) > 0$$

(see 4.26) and (4.21)). Now, it is well known that minimal cones in $\mathbb{R}^n$ have smooth boundary up to dimension 7 (included). Therefore, if (4.24) holds for a certain $s \geqq 0$, then necessarily $n \geqq 8$.

On the account of Simon's cone $C \subset \mathbb{R}^8$ (see 1.4), we see that (4.27) may really hold, when $n = 8$ and $s = 0$.

On the other hand, if (4.27) holds with $s > 0$, then we can repeat the above procedure, blowing-up ∂C_o near a singular point different from the vertex, thus getting a minimal *cylinder* $Q = C_1 \times \mathbb{R}$, with the property that $H_s(\Sigma_Q) > 0$. In such a case however, the transversal section C_1 of Q would likewise be a minimal cone in $\mathbb{R}^{n-1}$, with in addition:

$$H_{s-1}(\Sigma_{C_1}) > 0 \ .$$

An easy induction then shows, that if (4.24) holds with $s \geqq m$ (m a non-negative integer), then there exists a minimal cone $C_m \subset \mathbb{R}^{n-m}$, satisfying

$$H_{s-m}(\Sigma_{C_m}) > 0.$$

From the preceding discussion, we see that (4.24) implies $s \leqq n-8$. In view of the preceding considerations, this concludes the proof of the Regularity Theorem.

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