1 Introduction

The results achieved in recent years by the theory of product differentiation may well explain its increasing relevance in the analysis of industrial organization and in the study of the sources of market power. The theory rests on the idea that in the presence of a differentiated demand, the strategic interaction among firms develops along two lines: the prices charged and the characteristics chosen by a firm and its competitors. One of the most investigated topics in this field is the analysis of locational equilibria in a horizontally differentiated market. The horizontal Hotelling model has been widely used in order to discuss problems related to the spatial price competition, the optimal product attributes, the optimal plant location, etc., and has found applications in the spatial economics literature, as well as in trade and banking theory. These models primarily focus on the existence of a Principle of Maximal or Minimum Differentiation (Economides 1986). This existence problem amounts to asking whether the interplay between the structure of consumers' preferences for the differentiated product and the optimal strategic behaviour of firms results into too little or too much product diversity.

Following Hotelling (1929), D'Aspremont, Gabszewicz and Thisse (1979) established a Principle of Maximal Differentiation, by assuming that the intensity of consumer preferences for their ideal product may be reformulated in a locational setup in terms of quadratic transportation costs: in a duopoly market, the firms try to set up apart from each other - differentiate at most their product - in order to relax price competition. This finding sharply contrasts with the acclaimed Principle of Minimum Differentiation of the original Hotelling model where firms, in the presence of linear transportation costs, choose to cluster in the product space. Examples of the tendency for competitors to reduce differences in distance or in the product characteristics space can be easily found in the real world. Conversely, examples of maximal differentiation can be identified in a truly locational perspective - e.g. the attitude for shopping centers and supermarkets to locate outside the urban center - but it is much more difficult to observe maximal differentiation in the characteristics space. The existence of a greater or a lower differentiation clearly depends on the interplay of a price competition effect and a demand effect: when the latter prevails, the firms' strategies are found to exhibit a strong tendency towards agglomeration in the middle; by contrast, when the demand effect is outweighed by the price competition effect, moving away is the optimal behaviour. Recently, this debate has been extended to cover situations with 'low' demand (Hinloopen and van Marrewijk 1999, Chirco, Lambertini and Zagonari, 2000) and to multi-dimensional models (Caplin and Nalebuff 1986, Neven and Thisse 1990, Tabuchi 1994).

One common property of traditional locational models is the assumption that consumers are uniformly distributed over the characteristics space; with a few exceptions, the situations in which the consumers' preferences are concentrated on a subsection of the available varieties have been neglected. In these cases one would expect that competitors produce fairly similar types of products, in order to better match the tastes of the relatively largest share of consumers (Beath and

Katsoulacos 1991). The problem of the optimal prices and locations has been explicitly solved by Tabuchi and Thisse (1995) with a triangular and symmetric distribution. They show that, given that distribution, any symmetric location around the middle cannot be an equilibrium. Indeed, two asymmetric equilibria arise, characterized by strong product differentiation between the firms, with one of them locating outside the support of the customer distribution.Their results, however, heavily depend on the non differentiability of the consumers density function, which generates a discontinuity of the reaction functions in correspondence of any symmetric location.

In this paper, we aim at extending Tabuchi and Thisse analysis in two directions. We offer a simple parametrization of the degree of consumers' concentration around the middle - which include the uniform and the triangular distribution as limit cases. This allows us to solve the price-location problem as a function of the degree of consumers concentration. Within this setup, we are able to show that a symmetric equilibrium exists, provided the density is differentiable at the center of its support. Moreover, we are able to give some theoretical support to the idea that a higher concentration of consumers around the center induces firms to reduce the optimal product differentiation. Finally, we find that the asymmetric equilibria identified by Tabuchi and Thisse may arise for a lower degree of consumers concentration than that implied by the triangular distribution and that these asymmetric equilibria may coexist with a symmetric one.

The paper is organized as follows. In section 2 we describe the basic model and discuss the simple parametrization of consumers' concentration adopted in the sequel. The explicit solution of the price-location problem is presented in section 3. Some comments and concluding remarks are provided in Section 4.

2 The model

Let us consider a market for a horizontally differentiated product, where the population of consumers is normalized to 1. Consumers, indexed with x, are distributed over the interval $[0,1]$, according to a density $f(x,w)$, where the parameter w that can be viewed as a concentration index of the consumers' tastes. More precisely, the density $f(x,w)$ is characterized as follows:

$$\begin{array}{lll} & f(x,1)=1, & \text{for } x\in[0,1] \\ & f(x,0)=2-2\left|2x-1\right|, & \text{for } x\in[0,1] \\ \text{for } 0<w<1 & f(x,w)=\dfrac{4}{1-w^2}x & \text{for } x<\dfrac{1-w}{2}, \\ & f(x,w)=\dfrac{2}{1+w} & \text{for } x\in\left[\dfrac{1-w}{2},\dfrac{1+w}{2}\right], \\ & f(x,w)=\dfrac{4}{1-w^2}(1-x) & \text{for } x<\dfrac{1+w}{2} \end{array}$$

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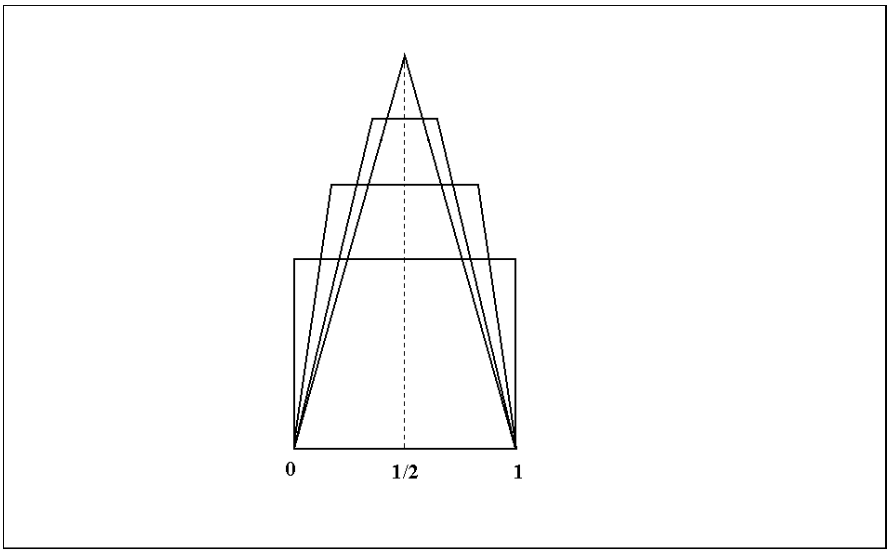

Figure 1: The density function for different values of the concentration parameter

As shown in Figure 1, $f(x,w)$ is symmetric around $x = 1/2$; for $w = 1$ it describes a uniform distribution while, as w decreases it concentrates towards the middle becoming trapezoidal and collapsing to a triangle for $w = 0$. Roughly speaking, our density is a trapezoid, with longest base equal to 1, shortest base equal to w and altitudo equal to $\frac{2}{1+w}$. For a given $w \in (0,1]$, we define 'central interval' the interval $x \in \left[\frac{1-w}{2}, \frac{1+w}{2}\right]$, while we call 'left external' and 'right external' interval respectively the intervals $x \in \left[0, \frac{1-w}{2}\right)$ and $x \in . \left(\frac{1+w}{2}, 1\right]$.

In this framework we consider a duopoly model in which both firms, firm 1 and firm 2, produce a differentiated product at a constant and equal to zero marginal cost. The location x chosen by each firm represents the good it decides to produce: the ideal consumer's product may match with the product offered, otherwise consumers choose to buy a "less than ideal" product paying a transportation cost that we consider quadratic in distance. Each consumer takes at most one unit of the product, so that total demand for the good offered by the firm located in x is given by the number of customers it patronizes. In the sequel we shall assume full market coverage.

Let us denote with a the distance of firm 1 from the origin, while b is the distance of firm 2. In order to exclude the possibility of leapfrogging by either firms we assume $a < b$ - where $a \in (-\infty, \infty)$ and $b \in (-\infty, \infty)$ - and marginal consumer lying between the two firms. As is well known, the price-location problem is a two-stage game in which at the first stage the firms choose their location and at the second stage choose their prices. The game is simultaneous.

The optimal firms' behaviour obviously differs according to the value of w. The results in terms of optimal locations are well known in the literature when $w = 1$ and when $w = 0$: in the unconstrained Hotelling game with a uniform distribution of consumers the firms maximize profits by locating at $-1/4$ e $5/4$ (Lambertini, 1994); moreover, Tabuchi and Thisse (1995) demonstrate that with a triangular distribution two asymmetric equilibria arise, $\left(-\sqrt{6}/9, 5\sqrt{6}/18\right)$ and $\left(1 - 5\sqrt{6}/18, 1 + \sqrt{6}/9\right)$. The following analysis will focus on the price-location equilibria for intermediate values of the parameter w, i.e. when the density becomes trapezoidal.

3 Consumer concentration and equilibrium prices and locations

We look for a subgame perfect equilibrium through backward induction, solving first for the prices and then for the locations as a function of the exogenous parameter w and the optimal prices determined in the first stage. Notice that if firm 1 and 2 set a price respectively equal to p_1 and p_2 being located respectively in a and b, the above hypotheses on transportation costs, unit demand and full market coverage imply that the marginal consumer's location is

$$z = \left(\frac{1}{2}\left[\frac{p_2 - p_1}{b - a} + b - a\right] + a\right) \tag{1}$$

Clearly, given the shape of our density, the firms' reaction functions in both stages of the game will be different according to the fact that the firms know that their behaviour implies that the marginal consumer lies in the 'central interval' or in the two external intervals, i.e. $z \in \left[\frac{1-w}{2}, \frac{1+w}{2}\right]$ - central interval - or $z \in \left[0, \frac{1-w}{2}\right)$ and $z \in .\left(\frac{1+w}{2}, 1\right]$ - external interval. We solve the model under both conjectures and verify under which conditions one or more equilibria exist in which conjectures are fulfilled. Notice that, given the simmetry of the density, the possible existence of a subgame perfect equilibrium such that $z \in \left[0, \frac{1-w}{2}\right)$ implies the existence of a specular equilibrium, with the marginal consumer lying in a specular position within the interval $\left(\frac{1+w}{2}, 1\right]$. This allows to restrict the analysis to one external area only.

3.1 The marginal consumer lies in the central interval

Given the hypothesis of unit consumers' demand and given our normalization, the market demand for each good corresponds to its market share. Therefore, consumer lying between the two firms. As is well known, the price-location problem is a two-stage game in which at the first stage the firms choose their location and at the second stage choose their prices. The game is simultaneous.

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3.1 The marginal consumer lies in the central interval

Given the hypothesis of unit consumers' demand and given our normalization, the market demand for each good corresponds to its market share. Therefore, the demand for the two firms are respectively:

$$\begin{aligned} q_1 &= F(z, w) \\ q_2 &= 1 - F(z, w) \end{aligned}$$

where F is the cumulative function of f. As long as $z \in \left[\frac{1-w}{2}, \frac{1+w}{2}\right]$,

$$q_1(z, w) = \frac{2z - \frac{1}{2}(1+w)}{1+w}$$

so that, by substituting (1), we get:

$$q_1 = \left(\frac{\frac{p_2 - p_1}{b-a} + b + a + \frac{1}{2}(w-1)}{1+w} \right) \tag{2}$$

The demand accruing to the firm 2 will be:

$$q_2 = \left[1 - \left(\frac{\frac{p_2 - p_1}{b-a} + b + a + \frac{1}{2}(w-1)}{1+w} \right) \right] \tag{3}$$

Since there are no production costs, the profit functions of the two firms are:

$$\pi_1 = \left[p_1 \left(\frac{\frac{p_2 - p_1}{b-a} + b + a + \frac{1}{2}(w-1)}{1+w} \right) \right] \tag{4}$$

$$\pi_2 = \left[p_2 \left(1 - \frac{\frac{p_2 - p_1}{b-a} + b + a + \frac{1}{2}(w-1)}{1+w} \right) \right] \tag{5}$$

3.1.1 The price stage

By differentiating the firms' profit functions and solving the first order condition with respect to prices, we find the following reaction functions:

$$\begin{aligned} p_1 &= \frac{1}{2}\left[p_2 + b^2 - a^2 - \frac{1}{2}(a + b + w(b-a)) \right] \\ p_2 &= \frac{1}{2}\left[p_1 - b^2 + a^2 - \frac{1}{2}(3(a+b) + w(b-a)) \right] \end{aligned}$$

The Nash equilibrium in prices is therefore:

$$p_1 = \frac{1}{3}\left(b^2 - a^2\right) + \frac{1}{6}(b-a) + \frac{1}{2}w(b-a) \tag{6}$$

$$p_2 = \frac{1}{3}\left(a^2 - b^2\right) + \frac{5}{6}(b-a) + \frac{1}{2}w(b-a) \tag{7}$$

3.1.2 The location stage

Substituting the optimal prices in (4) and (5) , profits are expressed as a function of locations and w:

$$\begin{aligned}
\pi_1^* &= \frac{1}{6}\left[\frac{1}{3}\left(b^2-a^2\right)+\frac{1}{6}(b-a)+\frac{1}{2}w(b-a)\right]\frac{2(b+a)+1+3w}{1+w} \\
\pi_2^* &= \frac{1}{6}\left[\frac{5}{6}(b-a)+\frac{1}{2}w(b-a)-\frac{1}{3}\left(b^2-a^2\right)\right]\frac{5+3w-2(a+b)}{1+w}
\end{aligned}$$

The first and second order condition for profit maximization are satisfied for

$$a=-\frac{1}{6}-\frac{1}{2}w+\frac{1}{3}b \tag{8}$$

$$b=\frac{1}{3}a+\frac{5}{6}+\frac{1}{2}w \tag{9}$$

The solution of the system (8) and (9) gives the optimal symmetric locations $a^*=\frac{1}{8}-\frac{3}{8}w$ e $b^*=\frac{7}{8}+\frac{3}{8}w$. If firms locate in a^* and b^*, their optimal prices are $p_1^*=p_2^*=\frac{3}{8}+\frac{3}{4}w+\frac{3}{8}w^2$ and the indifferent consumer is located in $\frac{1}{2}$: the conjecture that the indifferent consumer lies in the central area is fulfilled. We can therefore establish the following proposition:

Proposition 1 *For all values of $w\in(0,1]$ there exist a subgame perfect symmetric Nash equilibrium in prices and locations.*

Notice that the optimal locations coincide with those identified in Lambertini (1994), $a=-\frac{1}{4}$ e $b=\frac{5}{4}$, when $w=1$. The optimal prices are increasing in w: a higher degree of concentration around the midlle (lower w) induces firms to move inwards in order to match the tastes of a growing share of consumers: the more concentrated is the consumer distribution, the less the firms differentiate their products.. This reduced differentiation strenghten price competition. The overall equilibrium shows clearly a dominance of the demand effect: the advantage of acquiring the consumers in the central area dominates the advantage of softening competition through a large product differentiation.

3.2 The marginal consumer lies in one of the external intervals

Now we want to verify whether there exist subgame perfect equilibria, such that the marginal consumer falls in the left external interval $\left[0,\frac{1-w}{2}\right]$. In this interval the density function's slope is $\frac{4}{1-w^2}$. Hence, as $z\in\left[0,\frac{1-w}{2}\right]$, the demand for firm 1 is:

$$q_1(z,w)=\frac{z^2}{\frac{1}{2}(1+w)(1-w)}$$

Substituting (1) in the above expression we obtain the following demand functions in terms of the locations and prices:

$$
\begin{aligned}
q_1 &= \left(\frac{\left(\frac{1}{2} \left[\frac{p_2 - p_1}{b-a} + b - a \right] + a \right)^2}{\frac{1}{2}(1+w)(1-w)} \right) \\
q_2 &= \left[1 - \frac{\left(\frac{1}{2} \left[\frac{p_2 - p_1}{b-a} + b - a \right] + a \right)^2}{\frac{1}{2}(1+w)(1-w)} \right]
\end{aligned}
$$

The profit functions are:

$$
\begin{aligned}
\pi_1 &= p_1 \left(\frac{\left(\frac{1}{2} \left[\frac{p_2 - p_1}{b-a} + b - a \right] + a \right)^2}{(1+w)\left(\frac{1}{2} - \frac{1}{2}w\right)} \right) \\
\pi_2 &= p_2 \left[1 - \frac{\left(\frac{1}{2} \left[\frac{p_2 - p_1}{b-a} + b - a \right] + a \right)^2}{(1+w)\left(\frac{1}{2} - \frac{1}{2}w\right)} \right]
\end{aligned}
$$

3.2.1 The price stage

The first and second order conditions for profit maximization with respect to firm 1's price are satisfied by the following reaction function:[1]

$$p_1 = \frac{1}{3}\left(p_2 + b^2 - a^2\right) \tag{10}$$

As far as firm 2 is concerned, the first and second order conditions are satisfied by the reaction function:[2]

$$p_2 = -\frac{2}{3}b^2 + \frac{2}{3}a^2 + \frac{2}{3}p_1 + \frac{1}{3}\sqrt{\left[p_1 + (a^2 - b^2)\right]^2 + \left[6\left(b^2 + a^2\right) - 12ba\right]\left(1 - w^2\right)} \tag{11}$$

[1] The first order condition is satisfied also by $p_1 = \left(p_2 + b^2 - a^2\right)$. However, at this solution the second order condition for a maximum is not satisfied for $w < 1$.

[2] Again, we have two solutions satisfying the FOC. The other solution

$$p_2^2 = -\frac{2}{3}b^2 + \frac{2}{3}a^2 + \frac{2}{3}p_1 - \frac{1}{3}\sqrt{\left[p_1 + (a^2 - b^2)\right]^2 + \left[6\left(b^2 + a^2\right) - 12ba\right]\left(1 - w^2\right)}$$

does not satisfies the second order condition.

The solution of the system (10) and (11) gives the following Nash equilibrium in prices:

$$
\begin{aligned}
p_1 &= \rho\,(b-a) \\
p_2 &= 3\rho b - 3\rho a - b^2 + a^2
\end{aligned}
$$

where ρ is a root of the polynomial $8x^2 - 2\,(b+a)\,x + \left(w^2-1\right)$.

The existence of two solutions demonstrates that the reaction functions intersect twice. Since the two roots of the polynomial are

$$
\begin{aligned}
x_1 &= -\frac{1}{8}\,(a+b) + \frac{1}{8}\sqrt{(a+b)^2 + 8\,(1-w^2)} \\
x_2 &= -\frac{1}{8}\,(a+b) - \frac{1}{8}\sqrt{(a+b)^2 + 8\,(1-w^2)}
\end{aligned}
$$

we may establish that these intersections occur at the following two price couples:

$$
\begin{aligned}
p_1 &= -\frac{1}{8}\,(a-b)\left(a+b-\sqrt{(a+b)^2 + 8\,(1-w^2)}\right) && (12) \\
p_2 &= -\frac{3}{8}\left(a+b-\sqrt{(a+b)^2 + 8\,(1-w^2)}\right)(a-b) + a^2 - b^2 && (13)
\end{aligned}
$$

using solution x_1, or

$$
\begin{aligned}
p_1 &= -\frac{1}{8}\,(a-b)\left(a+b+\sqrt{(a+b)^2 + 8\,(1-w^2)}\right) && (14) \\
p_2 &= -\frac{3}{8}\,(a-b)\left(a+b+\sqrt{(a+b)^2 + 8\,(1-w^2)}\right) + a^2 - b^2 && (15)
\end{aligned}
$$

using solution x_2. It must be noticed, however, that only (14) and (15) entail positive prices at equilibrium for both firms. Therefore this is the only economically meaningful economic solution to the price game.

3.2.2 The location stage

The profit functions calculated at the optimal prices are:

$$
\pi_1 = \frac{1}{32}\,(a-b)\,\frac{\left(a+b+\sqrt{(a+b)^2 + 8\,(1-w^2)}\right)^3}{(1+w)\,(-1+w)}
$$

$$
\pi_2 = \left[-\tfrac{3}{8}\left(a+b+\sqrt{(a+b)^2 + 8\,(1-w^2)}\right)(a-b) + a^2 - b^2\right]
$$
$$
\left[1 - \frac{\left(\frac{1}{2}\left[\frac{\left[-\frac{3}{8}\left(a+b+\sqrt{(a+b)^2+8(1-w^2)}\right)(a-b)+a^2-b^2\right]+\left[\frac{1}{8}(a-b)\left(a+b+\sqrt{(a+b)^2+8(1-w^2)}\right)\right]}{b-a}+b-a\right]+a\right)^2}{\frac{1}{2}(1-w^2)}\right]
$$

By differentiating firm 1's profits with respect to its location, we get the following first order condition:

$$\frac{1}{32}\left(a+b+\sqrt{(a+b)^2+8\left(1-w^2\right)}\right)^3 \frac{\sqrt{(a+b)^2+8\left(1-w^2\right)}+3\left(a-b\right)}{\left(w^2-1\right)\sqrt{(a+b)^2+8\left(1-w^2\right)}}=0$$

which gives the optimal location:[3]

$$a=\frac{5}{4}b-\frac{1}{4}\sqrt{9b^2+16\left(1-w^2\right)} \tag{16}$$

If we now differentiate firm 2's profits with respect to its location, and substitute the reaction function (16), tedious calculations (see the Appendix) show that we can identify the following acceptable solution:

$$b=\frac{5}{18}\sqrt{6\left(1-w^2\right)} \tag{17}$$

Using (17) into (16) we have

$$a=-\frac{1}{9}\sqrt{6\left(1-w^2\right)} \tag{18}$$

Therefore, equations (17) and (18) give the optimal locations as a function of w, under the conjecture that the marginal consumer lies in the interval $\left[0,\dfrac{1-w}{2}\right)$.

We now have to verify whether there is a range of w such that this conjecture is actually fulfilled. We first notice that when $w=0$ - i.e. when the density describing the consumers' preferences is a symmetric triangle - the equations (17) and (18) collapse to $a=-\frac{1}{9}\sqrt{6}$ and $b=\frac{5}{18}\sqrt{6}$, that correspond exactly to Tabuchi and Thisse's solutions. In general, when evaluated at the optimal locations (17) and (18), the price equations (14) and (15) become respectively:

$$\begin{aligned} p_1 &= \frac{7}{18}\left(1-w^2\right) && (19)\\ p_2 &= \frac{7}{9}\left(1-w^2\right) && (20)\end{aligned}$$

By substituting in (17)-(20) into (1), we find the marginal consumer's location as a function of w:

$$z=\frac{1}{2}\frac{\left(1-w^2\right)}{\sqrt{6\left(1-w^2\right)}}+\frac{1}{12}\sqrt{6\left(1-w^2\right)}$$

[3]The above FOC has two solutions. The other is $a=\frac{5}{4}b+\frac{1}{4}\sqrt{9b^2+16\left(1-w^2\right)}$, which does not satisfy the condition $a<b$.

This allows us to establish that the firms' conjectures generating the asymmetric equilibrium (17)-(20) are fulfilled if

$$\frac{1}{2}\frac{\left(1-w^2\right)}{\sqrt{6\left(1-w^2\right)}}+\frac{1}{12}\sqrt{6\left(1-w^2\right)}<\frac{1-w}{2}$$

i.e., if $w<\frac{1}{5}$. By a similar reasoning, it can be proved that, under the same condition on w, a specular asymmetric equilibrium exists, with the marginal consumer lying in the right external interval, with firms located respectively at $a=1-5\sqrt{6\left(1-w^2\right)}/18$ and $b=\sqrt{6\left(1-w^2\right)}/9$. We can therefore establish the following proposition:

Proposition 2 *For $0<w<\frac{1}{5}$ there exist three subgame perfect Nash equilibria in prices and locations, a symmetric equilibrium and two asymmetric ones.*

Notice that in the asymmetric equilibria one firm locates outside the market area, while the other locates in the external interval opposite to that in which lies the marginal consumer. As w increases in the admissible range $\left[0,\frac{1}{5}\right)$, both firms move inwards. Given w, the firm locating within the market area may charge higher prices and enjoy higher profits.

It may be interesting to ask what happens when $w=\frac{1}{5}$. In this case, the asymmetric equilibria defined above make the marginal consumer fall in $\frac{2}{5}$, or specularly in $\frac{3}{5}$, i.e. in correspondence of the hedges of the density function. This is a situation similar to that Tabuchi and Thisse describe with respect to a possible symmetric equilibrium: since the density is not differentiable, the reaction functions are indeed discontinuous.

Let us assume that the solution (17)-(20) holds for $w=\frac{1}{5}$. Then the following results would apply:

$$\begin{aligned} a &= -\frac{4}{15},\ b=\frac{2}{3} \\ p_1 &= \frac{28}{75},\ p_2=\frac{56}{75},\ z=\frac{2}{5} \\ \pi_1 &= \frac{28}{225},\ \pi_2=\frac{112}{225} \end{aligned}$$

In order to ensure that it is indeed an equilibrium, we must exclude the profitability of unilateral deviations from the candidate equilibrium location, in correspondence of the admissible prices for such a location. Let us define the alternative location for firm 1:

$$a'=-\frac{4}{15}+\epsilon$$

If firm 1 locates in a', while firm 2 locates in $\frac{2}{3}$, the marginal consumer lies in the central interval, and the price rules (6)-(7) apply, so that

$$\begin{aligned} p_1 &= \frac{268}{675}-\frac{4}{15}\epsilon-\frac{1}{3}\left(-\frac{4}{15}+\epsilon\right)^2 \\ p_2 &= \frac{488}{675}-\frac{14}{15}\epsilon+\frac{1}{3}\left(-\frac{4}{15}+\epsilon\right)^2 \end{aligned}$$

When evaluated at these prices, and at $a' = -\frac{4}{15} + \epsilon$, and $b = \frac{2}{3}$, the profits of firm 1 turn out to be

$$\frac{5}{6}\left(\frac{268}{675} - \frac{4}{15}\epsilon - \frac{1}{3}\left(-\frac{4}{15} + \epsilon\right)^2\right)\left(\frac{\frac{44}{135} - \frac{2}{3}\epsilon + \frac{2}{3}\left(-\frac{4}{15} + \epsilon\right)^2}{\frac{14}{15} - \epsilon} + \epsilon\right) > \frac{28}{225}$$

for arbitrarily small positive values of ϵ. This is enough to prove that, for $w = \frac{1}{5}$, the solutions (17)-(20) are not subgame perfect equilibria and allows us to establish that for $w = 1/5$ there exists only a subgame perfect symmetric Nash equilibrium in prices and locations, defined by equations (6)-(9).

4 Remarks and conclusions

In this paper we have analysed the effects of the consumers' concentration towards the middle of the space of product characteristics, in a a model of horizontal differentiation with quadratic transportation costs. The consumers' density is assumed to be symmetric and trapezoidal; if the size of the market is normalized to 1, this allows to consider the lenght of the shortest base as a mean preserving spread of consumers' preferences. Clearly, the traditional uniform distribution and a symmetric triangular distribution can be nested into this setup as limit cases.

We have proved that as far as the shortest base is positive - i.e. the distribution is differentiable at 1/2 - a symmetric subgame perfect Nash equilibrium exists in the two stage price-location game. The result we achieve is rather intuitive: starting from the optimal solution obtained under the standard uniform distribution, as preferences become more concentrated around the middle, both firms move inwards and reduce the degree of product differentiation. This clearly reinforces price competition and results in lower equilibrium prices. This result is consistent with a more general intuition that homogeneity of consumers might have important implications in terms of reducing the firms'market power (Benassi, Chirco, and Scrimitore, 2002).

Moreover, our discussion shows that the asymmetric equilibria identified by Tabuchi and Thisse may coexist with the above symmetric equilibrium. For a relevant range of values of our mean preserving spread parameter - when preferences become sufficiently concentrated - two asymmetric subgame perfect equilibria appear, with one firm producing a relatively 'average' product, and the other firm choosing to locate outside the characteristics space. Once one firm decides to produce a product which meets the taste of the large share of consumers located around the middle, the other firm finds it optimal to avoid a destructive price competition by choosing a product with 'extreme' and 'out of market' characteristics. However, this peculiar location choice requires that a low price is charged, in order to capture at least the consumers located at the nearest tail of the distribution. This solution is such that as w increases within its admissible range - the distribution becomes more dispersed - both

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While the simple setup discussed in this paper allows for an explicit general solution which covers the situations previously discussed in the literature, it is nevertheless clear that the relation between any concentration index of the consumers' preferences and the properties of equilibria should be framed in a more general setting, independently of the possibility of defining analytical solutions. This is an important issue of the research agenda on product differentiation.

Appendix

By solving with respect to b the first order condition for firm 2's profit maximization at the location stage, we obtain the following critical values

- $b = -a + \frac{3}{2}\sqrt{2\left(1-w^2\right)}$
- $b = -a - \frac{3}{2}\sqrt{2\left(1-w^2\right)}$
- $b = \rho$, where ρ is a root of the polynomial:

$$\begin{aligned} &4x^4 + 6x^3 a + \left(-30w^2 + 30\right)x^2 + \\ &+ \left(-2a^3 + 21aw^2 - 21a\right)x - 2a^2 + 2a^2w^2 - 18 + 36w^2 - 18w^4 \end{aligned} \tag{A1}$$

Let us consider these solutions.

- Consider first the solution $b = -a + \frac{3}{2}\sqrt{2\left(1-w^2\right)}$. Given the reaction function of firm 1, we have to solve the following system in order to discuss the candidate optimal locations of firm 1 and firm 2:

$$\begin{aligned} a &= \frac{5}{4}b - \frac{1}{4}\sqrt{\left(9b^2 + 16\left(1-w^2\right)\right)} \\ b &= -a + \frac{3}{2}\sqrt{2\left(1-w^2\right)} \end{aligned}$$

  Again we have two solutions: the couple of locations $a_1 = \frac{7}{6}\sqrt{2\left(1-w^2\right)}$ and $b_1 = \frac{1}{3}\sqrt{2\left(1-w^2\right)}$, and the couple $a_2 = \frac{1}{3}\sqrt{2\left(1-w^2\right)}$, and $b_2 = \frac{7}{6}\sqrt{2\left(1-w^2\right)}$. The first couple implies a value for b lower than a. This solution is therefore unacceptable. However, if eqts (14) and (15) were evaluated at the second couple, the marginal consumer would lie at $\frac{3}{4}\sqrt{2\left(1-w^2\right)}$. Since the disequation $\frac{3}{4}\sqrt{2\left(1-w^2\right)} > \dfrac{1-w}{2}$ is always satisfied, at this solution the firms' conjectures would not be fulfilled.

- It is easy to check that, if $b = -\left(a + \frac{3}{2}\sqrt{2\left(1-w^2\right)}\right)$, the values of b that solve the system of the two reaction functions are both smaller than a. This contradicts the assumption $a < b$.

We now consider the polynomial (A1). Its solutions obtained by substituting firm 1's optimal reply are:

$$\begin{aligned} x &= \sqrt{2\left(w^2-1\right)} \\ x &= -\frac{5}{18}\sqrt{6\left(1-w^2\right)} \\ x &= \frac{5}{18}\sqrt{6\left(1-w^2\right)} \end{aligned}$$

- We can immediately rule out the complex solution $x = \sqrt{2\left(w^2 - 1\right)}$.

- If $b = -\frac{5}{18}\sqrt{6\left(1 - w^2\right)}$, the optimal location of firm 1 is $a = -\frac{29}{36}\sqrt{6\left(1 - w^2\right)}$. In this case $a < b$, but both optimal solutions are negative and this contrasts again with the conjectures about the location of the marginal consumer.

- The last solution is indeed the only acceptable one. If $b = \frac{5}{18}\sqrt{6\left(1 - w^2\right)}$, then $a = -\frac{1}{9}\sqrt{6\left(1 - w^2\right)}$. Using these optimal locations into (14) and (15) we may verify that the marginal consumers is in the left external interval for $w < \frac{1}{5}$. Notice that this solution collapses to that obtained by Tabuchi and Thisse by setting $w = 0$.

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