3 Product and process innovations

All firms producing consumption goods start their production after buying a patent of price v_i. New patents for new varieties are produced in the R&D sector, and their production is described by the following function

$$\dot{n}_i = \frac{1}{a} n_i L_R \qquad (12)$$

where a is an inverse measure of labor productivity in the innovative sector.

Expression (12) shows that the number of new patents produced in the R&D sector is proportional to the units of workers employed in the same sector and to the number of the already existing varieties, whose production process is characterized by the smallest value of γ, that is γ_i. Therefore, we share the assumption in Grossman and Helpman (1991) that nonrivalry of ideas in the innovative sector gives rise to increasing returns. We could have chosen a different functional form for (12) which would have avoided the scale effect in Grossman and Helpman (1991). However, we chose this specification because we would like to show that we can identify a potential way through which the presence of the scale effect can be mitigated, given that, as we show later, its consequences are attenuated when process innovations may take place.

Moreover, we assume that γ values are decreasing in m, with

$$\gamma_1 > \gamma_2 > ... > \gamma_m > .... > \gamma_{i-1} > \gamma_i \qquad (13)$$

As a consequence of (12) and (13), when innovations take place, that is when $\dot{n}_i > 0$, new varieties are produced with the most efficient production process, characterized by the smallest available value of γ, that is γ_i.

We know from (12) that labor demand in the innovative sector is

$$L_R = \frac{\dot{n}_i}{n_i} a \qquad (14)$$

Following Grossman and Helpman (1991), we assume that stock market value of a patent, v_i, is at any time equal to the present discounted value of the stream of all following profits. Hence, given the interest rate r on a safe asset, we can write that

$$v_i = \int_0^\infty e^{-rt} \pi_i dt \tag{15}$$

The innovative sector is perfectly competitive and the level of employment of workers in the R&D sector is such that it maximizes profits

$$\pi_R = v_i \dot{n}_i - wL_R \tag{16}$$

From the first order condition on the previous maximization problem, we obtain the nominal wage as a function of the price of any new patent v_i and the number of varieties of type i, that is

$$w = \frac{v_i n_i}{a} \tag{17}$$

At this stage, we need to specify how γ evolves over time. We assume that the value of γ evolves along a learning curve and we think that it is more likely that researchers obtain a further improvement in production processes associated to new patents, which reduces the smallest value of γ, when the number of patents associated to the existing more efficient technology, n_i, becomes sufficiently large and when the potential demand dimension is large, that is when L is large. The reason for the first effect is that knowledge accumulates over time and, consequently, it makes further improvements possible. The rationale for the second effect, which will be discussed later, is that when demand is larger, researchers' efforts are increased and productivity improvements take place sooner. Moreover, in this case, researchers know that there is a competition effect generated by the entrance of a higher number of firms due to the fact that, as we will show later, larger values of L, other things equal, tend to increase the rate of growth of varieties, $\dot{n}_i/n_i$, and consequently n_i, lowering expected rewards by researchers. In fact, expression (8) shows that expected profits on varieties of type i are lower when n_i is higher. However, given that (8) also shows that profits for varieties i are higher, the lower γ_i, researchers increase their efforts to find improvements in the available production technology to avoid the larger reduction in profits when L is larger and to try to improve the profitability of new varieties because, once a reduction in γ_i

occurs, there is a gap in the present value of the flow of all future operating profits between old firms and new type firms

$$\int_{t_i}^{\infty} e^{-rt} \left(\pi_i(t) - \pi_m(t)\right) dt = (1-\alpha) \int_{t_i}^{\infty} e^{-rt} \frac{1}{\sum_{j=1}^{i} n_j(t) \left(\gamma_j(t)\right)^{1-\sigma}} \left(\frac{1}{\gamma_i^{\sigma-1}} - \frac{1}{\gamma_m^{\sigma-1}} \right) dt > 0 \qquad (18)$$

We observe, in passing, that the gap described in previous expression decreases as t increases.

At this point we notice that workers and firms engaged in the R&D sector have incentives in pursuing process innovations, because these innovations allow them to increase the purchasing power of their wages in terms of new, more productive goods. In fact, as (15) and (18) show, process innovations increase patents' prices paid by firms in the final sector, given that they raise profits realizable by these firms. The increase in patents' prices is, in turn, accompanied by an increase in the wage w of workers employed in the R&D sector, because, in the framework we use, their wage is related to the value of their marginal product of the R&D sector, which depends on patents' prices (17). Then, (6) shows that higher wages results in higher purchasing power in terms of consumption goods, only when process innovations take place, because γ decreases and w/p_i increases. Thus, researchers, who are also consumers, have an incentive to obtain process innovations. For this reason, we think that it is likely to assume that researchers make deliberate efforts not only to produce new patents, but also to have more productive processes and we assume that process innovations take place in the R&D sector provided that a sufficient level of knowledge is accumulated.

Moreover, we can also assume that decreases in γ are more frequent, or larger, with larger population because the larger demand could allow researchers to exploit increasing returns to scale or because workers know, as we show later, that a larger population is associated with a larger number of researchers in equilibrium, and, therefore, with a larger number of goods on which process innovations could increase their purchasing power.

Therefore, following previous reasoning, we may think that the evolution of γ_i depends on n_i

and L with

$$\dot{\gamma}_i = f(n_i, L) \tag{19}$$

To give an example, and we wont need to assume it in the following of the paper, a simple specification for (19), could be the following

$$\dot{\gamma}_i = \begin{cases} c_i < 0 & \text{if } n_i L = \chi_i \\ 0 & \text{if } n_i L < \chi_i \end{cases} \tag{20}$$

where χ_i is a threshold value, which once reached allows us to represent the development of new and more productive varieties. The threshold is not a constant, given that different stages of development may require a different number of patents or different sizes of the population and demand to induce further process improvements. Moreover, c_i expresses the size of the process innovation, whose value is not constant, given that process innovations are certainly not at all equal in their effects and that they may have different impacts on various stages of the growth process.

In more details, once γ decreases, from that point in time onward, index i will represent the new more productive varieties. In particular, to make clear the use of our notation, we note that varieties of type i can also be named with the last integer number of the series for m, which we call h with $m = 1, 2, ..., (h = i)$. Once there is an improvement along the learning curve, described for instance by (19), the series continues in the following way: $m = 1, 2, ..., h, (h + 1 = i)$. Moreover, we notice that in the moment of the change in γ, (12) can be written as follows

$$n_{h+1} = \dot{n}_{h+1} = \frac{1}{a} n_h L_R \tag{21}$$

According to (20), the evolution of γ is related to the "adjusted" size of the market, $n_i L$.

In summary, and more generally with (19), for a given size of the market L, more firms operating in the economy along the frontier (higher n_i) increase the accumulated knowledge which allows researchers in the R&D sector to be able to find the way to introduce further improvements in productivity of firms associated with new patents. But these improvements are more likely to occur when larger dimensions of the market, L, push researchers to increase their efforts in searching for process innovations; firstly, to avoid the decrease in the value of new patents which otherwise would be generated by smaller profits due to the higher competition and, secondly, to increase their purchasing power on a larger number of new more productive varieties.

We know that the no arbitrage condition between patents and a safe asset implies that the following Fisher equation must be satisfied for every value of m

$$\frac{\pi_m}{v_m} + \frac{\dot{v}_m}{v_m} = r \tag{22}$$

We recall that while for $m \neq i$ innovation does not introduce any new varieties, these are developed for the $i-th$ group of firms.

4 Moving equilibrium

In this section we describe the properties of the equilibrium of the model, which will be characterized as a *moving equilibrium*, given that we assume that the number of firms is the slow variable of the economy, while all other variables are the fast variables.[5]

In particular, we know from expression (11) that in equilibrium the labor market is clearing. From (10), (6) and (17) we obtain that employment in the final sector is

$$L_C = \frac{\alpha}{w} = \frac{\alpha a}{v_i n_i} \tag{23}$$

Thus, in any periods between the two subsequent reduction in γ_i, the market clearing condition (11) can be rewritten as

$$L = \frac{\dot{n}_i}{n_i} a + \frac{\alpha a}{v_i n_i}$$

Let us denote with V_i the inverse of the value of the aggregate existing stock of patents of firms of type i, $V_i = \frac{1}{v_i n_i}$. Then, from the previous equation, we derive the growth rate of firms of type

[5] See Schlicht (1985, 1997).