Appendix

In this appendix we prove that, in the case of the lognormal distribution, $\lim_{y\to\infty}\eta_\theta = -\infty$.[12] Writing out the whole expression, we get

$$\eta_\theta(y,\theta) = \frac{1}{8}\sqrt{2}\frac{\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)}{\ln^3\theta}\left\{\frac{4\left(\ln^{\frac{1}{2}}\theta\right)\pi\ln^2 y-4\left(\ln^{\frac{1}{2}}\theta\right)\pi\left(\ln^2 y\right)\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}+\right.$$

$$-\frac{\left(\ln^{\frac{5}{2}}\theta\right)\pi-\left(\ln^{\frac{5}{2}}\theta\right)\pi\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}-\frac{4\left(\ln^{\frac{3}{2}}\theta\right)\pi-4\left(\ln^{\frac{3}{2}}\theta\right)\pi\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}$$

$$\left.+\frac{2\sqrt{2}\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)\left(\ln^2\theta\right)\sqrt{\pi}-4\sqrt{2}\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)(\ln\theta)\sqrt{\pi}\ln y}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}\right\}$$

This function can be written as

$$\frac{\sqrt{2}}{8}\frac{e^{-z^2}}{\alpha^6}\left[\frac{4\alpha\pi(1-\Phi(z))\ln^2 y-\alpha^5\pi(1-\Phi(z))-4\alpha^3\pi(1-\Phi(z))+2\alpha^2\sqrt{2\pi}e^{-z^2}\left(\alpha^2-2\ln y\right)}{\theta\pi^{\frac{3}{2}}(\Phi(z)-1)^2}\right]$$

where $z=\frac{\sqrt{2}}{4}\frac{2\ln y+\ln\theta}{\alpha}$ and $\alpha=\sqrt{\ln\theta}$, so that $\ln y=\sqrt{2}\alpha z-\frac{\alpha^2}{2}$. Substituting for the latter, we get

$$\frac{\sqrt{2}}{8}\frac{e^{-z^2}}{\alpha^6}\left[\frac{\alpha\pi(1-\Phi(z))\left(8\alpha^2z^2-4\sqrt{2}\alpha^3z-4\alpha^2\right)+4\sqrt{2}\alpha^3e^{-z^2}\left(\alpha-\sqrt{2}z\right)}{\theta\pi^{\frac{3}{2}}(\Phi(z)-1)^2}\right]$$

which is in terms of z, and collapses to

$$\frac{\sqrt{2}}{2}\frac{e^{-z^2}}{\alpha^3}\left[\frac{\pi\left(1-\Phi\left(z\right)\right)\left(2z^2-\sqrt{2}\alpha z-1\right)-\sqrt{2\pi}e^{-z^2}\left(\sqrt{2}z-\alpha\right)}{\theta\pi^{\frac{3}{2}}\left(\Phi\left(z\right)-1\right)^2}\right]$$

$$=\frac{e^{-z^2}\sqrt{2\pi}}{2\theta\alpha^3\pi^{\frac{3}{2}}}\left[\frac{\sqrt{\pi}\left(1-\Phi\right)\left(2z^2-\sqrt{2}\alpha z-1\right)-e^{-z^2}\left(2z-\sqrt{2}\alpha\right)}{\left(1-\Phi(z)\right)^2}\right] \quad \text{(a.1)}$$

[12] We owe very useful suggestions about this proof to profs A.Leaci and D.Scolozzi.

To ease notation, let $k = \dfrac{\sqrt{2\pi}}{2\theta\alpha^3\pi^{\frac{3}{2}}}$, a constant. Then (a.1) can be written as

$$ke^{-z^2}\left[\frac{\left(2z-\sqrt{2}\alpha\right)\left(\sqrt{\pi}\left(1-\Phi(z)\right)z-e^{-z^2}\right)-\sqrt{\pi}\left(1-\Phi(z)\right)}{\left[1-\Phi(z)\right]^2}\right]$$

$$=\frac{ke^{-z^2}}{1-\Phi(z)}\left\{\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}-\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}-\sqrt{\pi}\right\} \quad \text{(a.2)}$$

Thus finding the $\lim_{y\to\infty}\eta_\theta = -\infty$ amounts to studying the behaviour of (a.2) as z tends to infinity. We now consider the curly brackets of (a.2), to calculate

$$\lim_{z\to\infty}\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))} \quad \text{and} \quad \lim_{z\to\infty}\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}$$

As to the former, we can use Hôpital's rule to obtain

$$\lim_{z\to\infty}\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}=2\lim_{z\to\infty}\frac{\dfrac{2\sqrt{\pi}\left(1-\Phi(z)\right)z}{e^{-z^2}}-1}{-\dfrac{2}{\sqrt{\pi}}}=-\sqrt{\pi} \quad \text{(a.3)}$$

where we use the fact that, by Hôpital's rule,

$$\lim_{z\to+\infty}\frac{\left(1-\Phi(z)\right)z}{e^{-z^2}}=\frac{1}{\sqrt{\pi}} \quad \text{(a.4)}$$

As to the latter, notice that

$$\lim_{z\to\infty}\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}=\lim_{z\to\infty}\left\{\frac{\sqrt{2}\alpha}{z}\frac{z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}\right\}=0 \quad \text{(a.5)}$$

where we use (a.3).

Finally, the term outside the curly brackets tends to infinity. To see this, notice that

$$\lim_{z\to\infty}\frac{ke^{-z^2}}{1-\Phi(z)}=k\lim_{z\to\infty}\frac{e^{-z^2}}{z(1-\Phi(z))}\cdot z=\infty$$

using (a.4). Putting togethere these limits, $\lim_{y\to\infty}\eta_\theta=-\infty$.

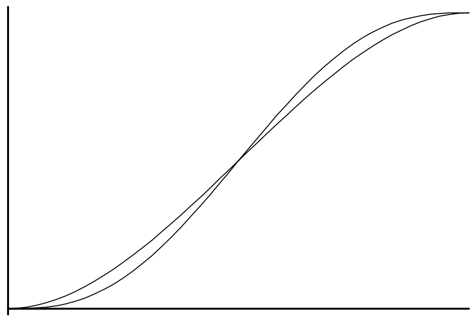

Fig. 1a. Single crossing of distributions

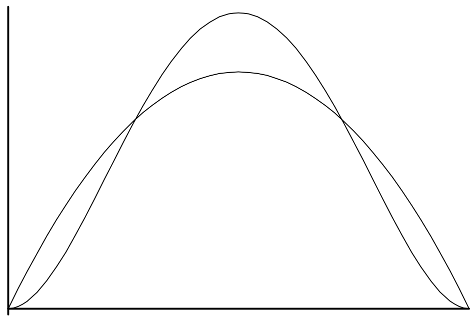

Fig. 1b. Double-crossing of densities

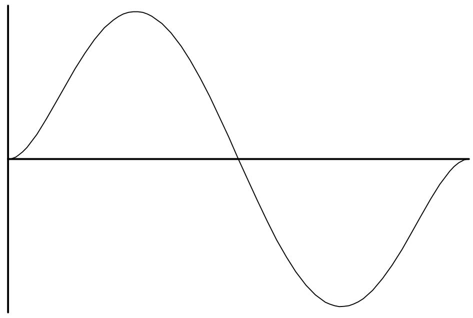

Fig. 1c. The function $F_\theta\left(y,\theta\right)$

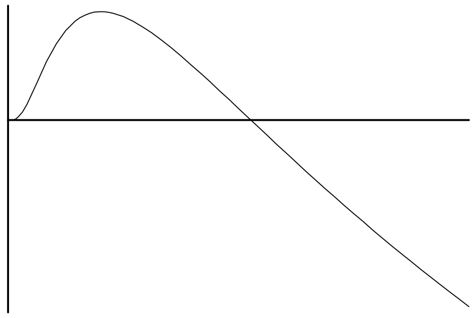

Fig. 2. The function $\eta_\theta(p,\theta)$