2 Personal income distribution and market demand

We model income distribution as a continuous differentiable unimodal density function $f(y, \theta)$, defined over some positive interval (y_m, y_M), $0 \leq y_m < y_M \leq \infty$. The parameter θ is a mean preserving spread. As is well known, in probability theory this is a measure of the degree of riskiness of a distribution. The reason why θ can be fruitfully applied to model income distributions, is that *via* changes in θ one can study the effects of changes income dispersion, as distinct from changes in aggregate (average) income – loosely speaking, an increase in θ shifts income frequencies towards the tails of the density function, while a decrease in θ raises the frequency of central income values.Using a mean preserving spread as defined here, amounts to ranking equal-mean distributions by second-order stochastic dominance. In the literature on income distribution, it is well known that such ranking is equivalent to Lorenz dominance: θ is thus a proper inequality index satisfying the Pigou-Dalton's "principle of transfers" (Atkinson, 1970).[2]

Formally, letting $h \in (y_m, y_M)$ denote the modal income, and letting subscripts denote partial derivatives, the following holds:

$$\left.\begin{array}{l} f_y(h, \theta) = 0 \\ f_y(y, \theta) > 0 \text{ for } y < h \\ f_y(y, \theta) < 0 \text{ for } y > h \end{array}\right\} \tag{1}$$

$$\left.\begin{array}{ll} \displaystyle\int_{y_m}^{y} F_\theta(x, \theta)dx \geq 0, & y < y_M \\ \displaystyle\int_{y_m}^{y_M} F_\theta(x, \theta)dx = 0 & \end{array}\right\} \tag{2}$$

where $F(y, \theta) = \int_{y_m}^{y} f(x, \theta)dx$ is the cumulative distribution function.

We specialize our model by imposing some regularity conditions on $F(\cdot, \theta)$. First, we assume that the mean preserving spread is of the simple type (Rothschild and Stiglitz, 1971), i.e. the crossing of distributions implied by (2) takes place only once. Secondly, we assume that the shift of the frequencies towards the tails associated to an increase in θ is such that the old and new density functions intersect only twice.

[2] Accordingly, an increase in θ shifts unambiguously down the Lorenz curve. The link between inequality orderings and stochastic dominance has been recently sudied, e.g., by Formby *et al.* (1999).

To see the implications of these restrictions, consider the function F_θ. Both assumptions are captured by this taking a shape like that exhibited in Fig. 1c: single crossing of the distributions (Fig. 1a) implies that F_θ crosses zero in the interior of (y_m, y_M) only once; double crossing of the densities (Fig. 1b) implies that this function has only one maximum and one minimum over (y_m, y_M). It should be stressed that this behaviour is shared by many commonly used distributions subject to mean preserving shocks.[3]

Figure 1 about here

This simple figure brings out a very general property of the effects of changes in θ under our assumptions. Four intervals can be identified according as F_θ and f_θ have the same or the opposite sign. Indeed, for any $y \in (y_m, y_A]$, $F_\theta > 0$ and $f_\theta \geq 0$ (with equality only for $y = y_A$); for any $y \in (y_A, y_B]$, $F_\theta \geq 0$ (with equality only for $y = y_B$) and $f_\theta < 0$; for any $y \in (y_B, y_C]$, $F_\theta < 0$ and $f_\theta \leq 0$ (with equality only for $y = y_C$); for any $y > y_C$, $F_\theta \leq 0$ (with equality only for $y = y_M$) and $f_\theta > 0$ (clearly, this holds in the limit if $y_M = \infty$). For ease of notation, we label intervals as $A = (y_m, y_A]$, $B = (y_A, y_B]$, $C = (y_B, y_C]$, $D = (y_C, y_M)$. Of course, the boundary values of these intervals in the interior of (y_m, y_M) are functions of θ, i.e. $y_i = y_i(\theta)$, $i = A, B, C$.

Income distribution is immediately connected to market demand, whenever each consumer chooses discretely between buying or not buying one unit of the commodity, according as the quoted price is lower or higher than his reservation price – the distribution of reservation prices across consumers can reasonably be thought of as mirroring somehow that of the consumers' incomes.

We assume throughout that the reservation price coincides with income, so that market demand is

$$Q(p, \theta) = 1 - F(p, \theta) \tag{3}$$

where population has been normalized to unity. This is clearly the sharpest way to model the relationship between income, reservation prices and demand. The weaker assumption of strict proportionality of reservation prices to income comes out, e.g., in models for durabes, such as that suggested by Deaton and Muellbauer (1980, pp.366-69). In this case the distribution

[3] To quote some examples, Beta, Gamma, Chi square, F, Lognormal, all follow this general pattern (one maximum, one zero crossing and one minimum) if subject to appropriately defined mean preserving spreads. Clearly, asymmetric distributions, while preserving the pattern, trace out a more irregular function than that plotted in the figure.

of reservation prices is isomorphic to the income distribution, when the latter is lognormal (Cowell 1995, pp.71-78). One should however notice that, independently of the specific form of the income distribution, our general argument rests only on the idea that a mean preserving spread on incomes generates single and double crossings of cumulative and density distributions of the reservation prices – which is always the case if the reservation price is monotonically increasing in income.[4]

Our demand function is clearly continuous and continuously differentiable. Moreover,

$$\begin{array}{l} Q_p = -f(p, \theta) \\ Q_{pp} < 0 \quad for \quad p < h(\theta); \quad Q_{pp} > 0 \quad for \quad p > h(\theta); \\ Q_{pp} = 0 \quad for \quad p = h(\theta) \end{array}$$

This definition of market demand makes it clear that both its position and its features depend on the parameter θ.

Our focus is on studying how changes in income dispersion affect the optimal behaviour of noncompetitive firms. As is well known, changes of this kind – which have apparently taken place in many countries – may be attributed to long-run structural factors (such as the skill distribution of workers, the insitutional framework of wage negotiations, the relative weight of capitale vs labour income, etc.), as well as to changes in the shorter-run redistributive effects of fiscal policy. As to the latter, one might think of y as disposable income, and accordingly interpret changes in θ as (equal yield) changes, e.g., in the degree of progression of the income tax schedule as measured by the residual progression index.[5]

The crucial issue we are interested in is then how changes in θ translate into comovements of demand and its elasticity. This may have some relevance in its own, as pointing out a mechanism through which income distribution affects the degree of competition; however, it should also be recalled that in a non-competitive general equilibrium setting the pro- or counter-cyclical

[4] In this sense the assumption $p = y$ drastically simplifies the exposition, with no substantial loss of generality. On the other hand, the binary-choice model of demand is widely applied in the literature (e.g., Anderson *et al.*, 1992); in the analysis of the relationship between income distribution and demand it is very convenient, as continuous individual demand curves are standardly derived from omothetic preferences, which prevent any discussion of distributional issues.

[5] As is well known, residual progression is (inversely) measured by the elasticity of post-tax income to pre-tax income; an increase in this index shifts unambiguously down the concentration curve (see, e.g., Lambert, 1990, chs 7 and 9). This Lorenz dominance is equivalent, under a equal-yield constraint, to a mean preserving spread of the distribution of disposable income.

behaviour of demand elasticity is the key element to assess the role of demand in determining the equilibrium output.[6]

Given market demand (3), the (positive) price elasticity of demand is given by

$$\eta(p,\theta) = \frac{pf(p,\theta)}{1-F(p,\theta)} \tag{4}$$

By differentiating with respect to θ one easily obtains

$$\eta_\theta(p,\theta) = \left(\frac{f_\theta(p,\theta)}{f(p,\theta)} + \frac{F_\theta(p,\theta)}{1-F(p,\theta)}\right)\eta(p,\theta) \tag{5}$$

Simple inspection of (5) reveals that changes in θ affect η differently, depending on where p lies in the four intervals A, B, C and D identified above.

A priori, the sign of η_θ is clearly unambiguous whenever f_θ and F_θ have the same sign, ambiguous otherwise: hence we can say that $\eta_\theta > 0$ for $p \in A$ and $\eta_\theta < 0$ for $p \in C$; in intervals B and D the sign of η_θ is potentially ambiguous. However, since η_θ changes sign going from A to C, it follows trivially by continuity that at least one point $\widehat{p}$ exists in the interior of B such that $\eta_\theta = 0$, and hence an interval $\widehat{B} \subset B$ exists where $\eta_\theta < 0$ – the left boundary of $\widehat{B}$ being $\widehat{p}$. This allows us to establish the following

Proposition 1 For all distributions obeying (1) and (2), and such that a change in θ generates single crossing of distributions and double crossing of densities, there exists a non-empty interval $\widehat{B}$ where the normalized demand function (3) and its elasticity (4) move in the same direction following a change in θ.
Proof Follows trivially from the fact that, by the definition of B, $Q_\theta(p,\theta) = -F_\theta(p,\theta) < 0$ for $p \in B$, while by the definition of $\widehat{B} \subset B$, $\eta_\theta(p,\theta) < 0$ for $p \in \widehat{B}$. □

An immediate implication of this proposition is that, if the income distribution is subject to a change in dispersion as measured by θ, firms whose equilibrium price lies in the specified range face a positive comovement in demand and price elasticity. In particular, as incomes become less dispersed, for all initial prices $p \in \widehat{B}$ firms experience an increase in both the level and the elasticty of demand. Of course, Proposition 1 is a simple existence proof for $\widehat{p}$, which says nothing as to uniqueness in the set B and, *a fortiori*, over

[6] For a general discussion of different perspectives on the comovements of market demand and its price elasticity, see Benassi *et al.* (1994, ch 5) and the references therein.

the whole range of p – the sign of η_θ in area D is clearly still ambigous. Sorting this out would enable us to determine the behaviour of η over the whole range of p. The properties of the income distribution which deliver uniqueness of $\widehat{p}$ are discussed in the next section.

3 Income share elasticity and the price elasticity of demand

Ideally, one would expect to pin down a unique value $\widehat{p}$, such that $\eta_\theta > 0$ for all $p < \widehat{p}$ and $\eta_\theta < 0$ for all $p > \widehat{p}$. Given that $\eta_\theta > 0$ for $p \in A$ and $\eta_\theta < 0$ for $p \in \widehat{B}$, η_θ crosses zero from above at the left boundary of $\widehat{B}$, i.e.at $\widehat{p}$. In order to define the conditions for $\widehat{p}$ to be unique, we first notice that the derivative of η_θ with respect to p is

$$\begin{aligned}\eta_{\theta p} &= \frac{\eta(p,\theta)}{f(p,\theta)}\left(f_{p\theta}(p,\theta) - \frac{f_p(p,\theta)}{f(p,\theta)}f_\theta(p,\theta)\right) \\ &\quad + \left(\eta_p + \frac{\eta^2}{p}\right)\left[\frac{f_\theta(p,\theta)}{f(p,\theta)} + \frac{F_\theta(p,\theta)}{1-F(p,\theta)}\right]\end{aligned} \tag{6}$$

which, for $\eta_\theta = 0$ collapses to

$$\eta_{\theta p|\eta_\theta=0} = \frac{\eta(p,\theta)}{p}\Pi_\theta\,(p,\theta) \tag{7}$$

where $\Pi_\theta\,(y,\theta)$ is the derivative with respect to θ of the income share elasticity (Esteban, 1986). The latter is defined as

$$\Pi\,(y,\theta) = 1 + \frac{yf_y\,(y,\theta)}{f\,(y,\theta)}$$

and measures the percentage change of the income share accruing to individuals of income y, given a marginal change in y.[7] Esteban shows that there is a one-to-one correspondence between $f\,(y,\cdot)$ and $\Pi\,(y,\cdot)$, so that any given distribution can be characterized in terms of Π.

Therefore, given (7),

$$\text{at } \eta_\theta = 0, \quad sign\left[\eta_{\theta p}(p,\theta)\right] = sign\left[\Pi_\theta(p,\theta)\right] \tag{8}$$

This is particularly convenient, as the Π function typically exhibits some useful regularity properties.

[7] Formally, $\Pi = \lim_{h\to 0}\frac{1}{\mu}\int_y^{y+h} xf\,(x,\theta)\,dx$, where μ is the mean income (Esteban 1986, p.441).