Chapter 5

Stability of MHD laminar flows in a porous medium with Brinkman law

5.1 Mathematical formulation of the problem

This chapter is devoted to the study of the stability of the laminar flows in a homogeneous, incompressible, electrically conducting fluid saturating an infinite horizontal porous layer embedded in a constant magnetic field. This problem has been studied by Rudraiah and Mariyappa in [78] [1] in order to investigate the effect of the geomagnetic field on non-convective flows in the geothermal region. It is known that in the geothermal region the sub-surface ground water posseses a general upward convective drift due to buoyancy induced by the high underground temperature. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in the surface streams, an unstable state may be induced and complicated convective motions appear in the layers near the surface. In those circumstances it is of practical interest to consider the effect of the geomagnetic field on such flows and see whether the magnetic field inhibits this instability. In particular in [78] Rudraiah and Mariyappa studied the stability of steady hydromagnetic flows in a porous medium by assuming the fluid with a finite electrical conductivity, valid the Oberbeck-Boussinesq approximation and neglecting the effects of its viscosity with respect to the friction that manifests itself at the pores. Here, instead, we include the frictional forces in the fluid by considering the unsteady Brinkman model for flows of a viscous fluid in a porous medium.

[1] The stability of MHD laminar flows, in different situations, is also studied in [42, 43, 74].

Let $Oxyz$ be a cartesian frame of reference with unit vector fields $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward, and let $\Omega_d = \mathbb{R}^2 \times (-d, d)$ ($d = \text{const} > 0$) be a horizontal porous layer bounded by the planes $z = \pm d$, assumed both rigid, electrically non conducting and at rest or in motion with velocity parallel to the plane $z = 0$. As the fluid filling Ω_d is concerned we assume that it is homogeneous, electrically conducting, embedded in a constant magnetic field $\mathbf{H}_0 = H_0\mathbf{k}$ and submitted to a conservative force $\mathbf{F}$ with potential $\mathcal{U}$. By following the same arguments in sections 2.5 and 2.8, the equations of non-relativistic magnetohydrodynamics in a porous medium in the isothermal case are the usual equations governing the fluid flow in a porous matrix suitably modified to take into account of the Lorentz force, to which equations $(2.63)_4$ and $(2.63)_5$ are added:

$$\begin{cases} \frac{\rho_0}{\varphi}\mathbf{v}_t = -\nabla P - \frac{\mu_1}{\varphi K}\mathbf{v} + \frac{\mu_2}{\varphi}\Delta\mathbf{v} + \mu_m(H_0\mathbf{k} + \mathbf{H}) \cdot \nabla\mathbf{H} \\ \text{div}\,\mathbf{v} = \text{div}\,\mathbf{H} = 0 \\ \mathbf{H}_t + \nabla \times [(H_0\mathbf{k} + \mathbf{H}) \times \mathbf{v}] = \eta\Delta\mathbf{H} \end{cases} \tag{5.1}$$

where

$\mathbf{v} = (U, V, W)$ the seepage velocity,	ρ_0 the density of the fluid,
$\mathbf{H}$ the induced magnetic field,	μ_m the magnetic permeability,
μ_i $(i = 1, 2)$ viscosity coefficients,	η the magnetic viscosity of the fluid,
φ the porosity of the medium,	K the permeability of the medium

and

$$P = p + \frac{\mu_m}{2}|H_0\mathbf{k} + \mathbf{H}|^2 - \rho_0\mathcal{U}$$

is the generalized pressure.

To equations (5.1) we append the boundary conditions

$$\begin{cases} U(x, y, -d, t) = U_1(x, y, t), \quad U(x, y, d, t) = U_2(x, y, t) \\ V(x, y, -d, t) = V_1(x, y, t), \quad V(x, y, d, t) = V_2(x, y, t) \\ W(x, y, -d, t) = W(x, y, d, t) = 0 \\ \mathbf{H}(x, y, -d, t) = \mathbf{H}(x, y, d, t) = \mathbf{0} \end{cases} \tag{5.2}$$

with U_i, V_i $(i = 1, 2)$ assigned regular fields on $\mathbb{R}^2 \times [0. +\infty)$. Boundary conditions $(5.2)_1$-$(5.2)_3$ tell us that the fluid adheres to the impermeable plates $z = \pm d$ whereas $(5.2)_4$ yields that the magnetic field is continuous at the boundaries as the bounding surfaces are electrically non-conducting [1].

In order to non-dimensionalize equations (5.1) we introduce the following non-dimensional quantities

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \; t^* = \frac{\mu_2}{\rho_0 d^2}t, \; \mathbf{v}^* = \frac{\rho_0 d}{\mu_2}\mathbf{v}, \; \mathbf{H}^* = \frac{\mathbf{H}}{H_0}, \; P^* = \frac{\varphi\rho_0 K}{\mu_1\mu_2}P,$$

substitute them in (5.1) and get the dimensionless equations governing the motion (omitting the asterisks)

$$\begin{cases} \tilde{D}a\mathbf{v}_t = -\nabla P - \mathbf{v} + \tilde{D}a\Delta\mathbf{v} + \tilde{D}aP_m(\mathbf{H}\cdot\nabla\mathbf{H} + \mathbf{H}_z) \\ \text{div}\,\mathbf{v} = \text{div}\,\mathbf{H} = 0 \\ \mathbf{H}_t + \nabla\times[(\mathbf{k}+\mathbf{H})\times\mathbf{v}] = \dfrac{1}{R_m}\Delta\mathbf{H} \end{cases} \tag{5.3}$$

where

$$\tilde{D}a = \frac{K\mu_2}{d^2\mu_1},\ \ P_m = \frac{\varphi\mu_m H_0^2 d^2\rho_0}{\mu_2^2},\ \ R_m = \frac{\mu_2}{\varphi\eta\rho_0},\ \ M = \sqrt{P_m R_m}$$

are, respectively, the Darcy, the magnetic pressure, the magnetic Reynolds and the Hartmann numbers. Of course, the initial and the boundary conditions must be modified according to the chosen scalings.

5.2 Laminar MHD flows

Looking for the one-dimensional laminar flows of the type

$$\begin{cases} \mathbf{v} = U(z)\mathbf{i} \\ \mathbf{H} = H(z)\mathbf{i}, \end{cases}$$

from (5.1), it turns out that (U, H) have to fulfil the following system:

$$\begin{cases} \nabla P = \left(-U + \tilde{D}a\dfrac{\mathrm{d}^2U}{\mathrm{d}z^2} + \tilde{D}aP_m\dfrac{\mathrm{d}H}{\mathrm{d}z}\right)\mathbf{i} \\ \dfrac{\mathrm{d}^2H}{\mathrm{d}z^2} + R_m\dfrac{\mathrm{d}U}{\mathrm{d}z} = 0 \end{cases} \tag{5.4}$$

with boundary conditions

$$\begin{cases} U(-1) = U_1, \quad U(1) = U_2 \\ H(\pm 1) = 0 \end{cases} \tag{5.5}$$

where U_1 and U_2 are assigned constants. It is easy to check that the solutions of the boundary value problem (5.4)-(5.5) are given by

$$\begin{cases} U(z) = A_1\mathrm{e}^{\tau z} + A_2\mathrm{e}^{-\tau z} - \dfrac{A_0 + \tilde{D}aP_mB_1}{\tau^2\tilde{D}a} \\ H(z) = -R_m\left[\dfrac{A_1\mathrm{e}^{\tau z} - A_2\mathrm{e}^{-\tau z}}{\tau} - \dfrac{A_0 z}{\tau^2\tilde{D}a}\right] - \dfrac{B_1 z}{\tau^2\tilde{D}a} + B_2 \\ P = A_0 x + p_0 \end{cases} \tag{5.6}$$

where

$$\begin{cases} \tau = \sqrt{\dfrac{1+\tilde{D}aM^2}{\tilde{D}a}} \\ A_1 = \dfrac{U_2-U_1}{4\sinh\tau} + \dfrac{(U_1+U_2+2A_0)\tau}{4(\tau\cosh\tau+\tilde{D}aM^2\sinh\tau)} \\ A_2 = -\dfrac{U_2-U_1}{4\sinh\tau} + \dfrac{(U_1+U_2+2A_0)\tau}{4(\tau\cosh\tau+\tilde{D}aM^2\sinh\tau)} \\ B_1 = R_m\dfrac{2A_0\tau\cosh\tau - [\tau^2\tilde{D}a(U_1+U_2)+2A_0]\sinh\tau}{2(\tau\cosh\tau+\tilde{D}aM^2\sinh\tau)} \\ B_2 = \dfrac{R_m(U_2-U_1)}{2\tau}\coth\tau, \end{cases}$$

A_0 and p_0 are real constants.

5.2.1 Hartmann flow

As special case of (5.6), for $U(\pm1)=0$ and $A_0 \neq 0$, one obtains the Hartmann flow

$$\begin{cases} U(z) = \dfrac{A_0\tau\cosh\tau}{\tau\cosh\tau+\tilde{D}aM^2\sinh\tau}\left[\dfrac{\cosh(\tau z)}{\cosh\tau}-1\right] \\ H(z) = \dfrac{R_mA_0\sinh\tau}{\tau\cosh\tau+\tilde{D}aM^2\sinh\tau}\left[z-\dfrac{\sinh(\tau z)}{\sinh\tau}\right] \\ P = A_0x+p_0. \end{cases} \tag{5.7}$$

In Figure 5.1 we have plotted the normalized velocity profiles in Hartmann flow for different values of the parameter τ. The velocity U has been normalized by dividing $(5.7)_1$ by the velocity at the centre of the channel

$$V = \frac{A_0\tau(1-\cosh\tau)}{\tau\cosh\tau+\tilde{D}aM^2\sinh\tau}.$$

If τ is small, viscosity dominates the induction drag and the velocity profile is nearly parabolic. If τ is large, on the other hand, viscosity is unimportant save in thin boundary layers (thickness $\sim 1/\tau$) near the walls; away from the walls U is nearly constant. Concerning the induced magnetic field, from $(5.7)_2$ we deduce that it tends to zero as $\tau \to +\infty$ and then, if the magnetic Reynolds number is small, the embedding magnetic field lines are not greatly destorted by the flow.

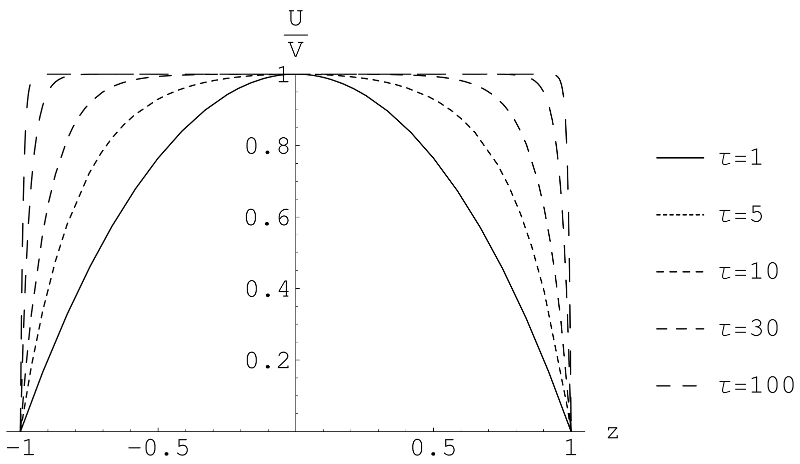

Figure 5.1: Normalized velocity profiles in Hartmann flow for different values of τ.

5.2.2 Couette flow

For $A_0 = 0$, $U(-1) = -V$ and $U(1) = V$, with $V = const$, the magnetic Couette flow is obtained

$$\begin{cases} U(z) = \dfrac{V \sinh(\tau z)}{\sinh \tau} \\ H(z) = \dfrac{R_m V \coth \tau}{\tau} \left[1 - \dfrac{\cosh(\tau z)}{\cosh \tau} \right] \\ P = p_0. \end{cases} \tag{5.8}$$

Normalized velocity profiles in Couette flow for different values of the parameter τ are shown in Figure 5.2. Velocity has been normalized by dividing $(5.8)_1$ by the velocity V of the upper plate.

If τ is small the velocity profile is nearly linear, while, if it is large, the seepage velocity is constant except for thin boundary layers (thickness $\sim 1/\tau$) near the walls where viscosity dominates the induction drag. From $(5.8)_2$, in the limit as $\tau \to +\infty$, the induced magnetic field tends to zero as in Hartmann flow.

Finally it is interesting to observe that for $M \to 0$, $(5.7)_1$ and $(5.8)_1$ give the Poiseuille and Couette flows found by Kaviany in [32].

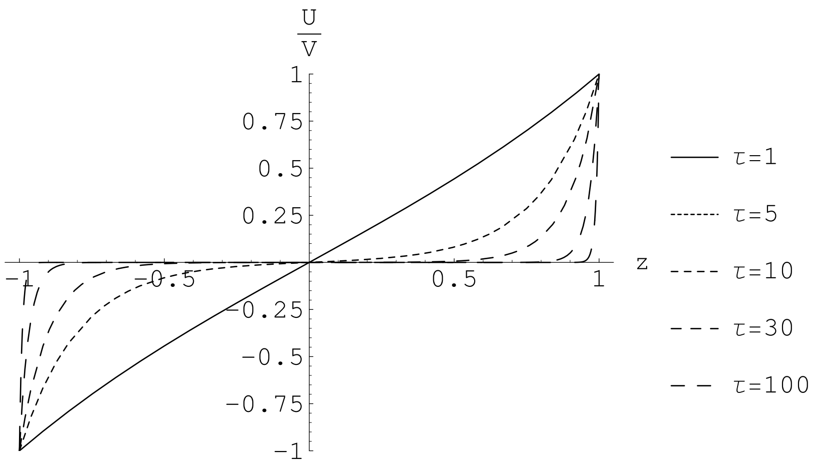

Figure 5.2: Normalized velocity profiles in Couette flow for different values of τ.

5.3 Sufficient condition for linear stability

The evolution equations of a perturbation $(\mathbf{u}, \mathbf{h}, p_1)$ to the basic laminar flow $m_0 = (\mathbf{v}, \mathbf{H}, P)$ given by (5.6) are

$$\begin{cases} \tilde{D}a\mathbf{u}_t = -\nabla p_1 - \mathbf{u} + \tilde{D}a\Delta\mathbf{u} \\ \qquad +\tilde{D}aP_m\left[\mathbf{h}\cdot\nabla\mathbf{h} + H'(z)h_3\mathbf{i} + H(z)\mathbf{h}_x + \mathbf{h}_z\right] \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \mathbf{h}_t = -\mathbf{u}\cdot\nabla\mathbf{h} - H'(z)u_3\mathbf{i} + \mathbf{h}\cdot\nabla\mathbf{u} + H(z)\mathbf{u}_x \\ \qquad +\mathbf{u}_z - U(z)\mathbf{h}_x + U'(z)h_3\mathbf{i} + \frac{1}{R_m}\Delta\mathbf{h} \end{cases} \tag{5.9}$$

in $\mathbb{R}^2 \times (-1, 1) \times (0, +\infty)$, under the initial and boundary conditions:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{h}(\mathbf{x}, 0) = \mathbf{h}_0(\mathbf{x})$$

$$\mathbf{u}(x, y, \pm 1, t) = \mathbf{h}(x, y, \pm 1, t) = \mathbf{0}, \tag{5.10}$$

with $\mathbf{u}_0$, $\mathbf{h}_0$ assigned divergence-free regular fields.

Linearizing (5.9) we obtain

$$\begin{cases} \tilde{D}a\mathbf{u}_t = -\nabla p_1 - \mathbf{u} + \tilde{D}a\Delta\mathbf{u} + \tilde{D}aP_m\left[H'(z)h_3\mathbf{i} + H(z)\mathbf{h}_x + \mathbf{h}_z\right] \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \mathbf{h}_t = -H'(z)u_3\mathbf{i} + H(z)\mathbf{u}_x + \mathbf{u}_z - U(z)\mathbf{h}_x + U'(z)h_3\mathbf{i} + \frac{1}{R_m}\Delta\mathbf{h} \end{cases} \tag{5.11}$$

and, as in section 4.3, look for periodic solutions in the x, y directions of periods $2\pi/a_x$ and $2\pi/a_y$ ($a_x > 0$, $a_y > 0$), respectively,

$$\begin{cases} \mathbf{u}'(x,y,z,t) = \hat{\mathbf{u}}(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ \mathbf{h}'(x,y,z,t) = \hat{\mathbf{h}}(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ p_1'(x,y,z,t) = \hat{p}_1(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \end{cases} \tag{5.12}$$

with complex wave speed $c = c_r + \mathrm{i}c_i$ and two-dimensional wave number $a = (a_x^2 + a_y^2)^{1/2}$.
Denoting by D the differential operator d/dz, we substitute the expressions (5.12) into (5.11) and obtain the following system of ordinary differential equations:

$$\begin{cases} \tilde{D}ac\hat{u}_1 + \hat{u}_1 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_1 - \tilde{D}aP_m\big[\mathrm{D}\hat{h}_1 \\ \qquad + \mathrm{i}a_x H(z)\hat{h}_1 + H'(z)\hat{h}_3\big] = -\mathrm{i}a_x\hat{p}_1 \\ \tilde{D}ac\hat{u}_2 + \hat{u}_2 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_2 - \tilde{D}aP_m(\mathrm{D}\hat{h}_2 \\ \qquad + \mathrm{i}a_x H(z)\hat{h}_2) = -\mathrm{i}a_y\hat{p}_1 \\ \tilde{D}ac\hat{u}_3 + \hat{u}_3 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_3 - \tilde{D}aP_m\big[\mathrm{D}\hat{h}_3 \\ \qquad + \mathrm{i}a_x H(z)\hat{h}_3\big] = \mathrm{D}\hat{p}_1 \\ \mathrm{D}\hat{u}_3 = -\mathrm{i}a_x\hat{u}_1 - \mathrm{i}a_y\hat{u}_2 \\ \mathrm{D}\hat{h}_3 = -\mathrm{i}a_x\hat{h}_1 - \mathrm{i}a_y\hat{h}_2 \\ c\hat{h}_1 + \mathrm{i}a_x U(z)\hat{h}_1 + H'(z)\hat{u}_3 - U'(z)\hat{h}_3 - \mathrm{D}\hat{u}_1 \\ \qquad - \mathrm{i}a_x H(z)\hat{u}_1 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_1 = 0 \\ c\hat{h}_2 + \mathrm{i}a_x U(z)\hat{h}_2 - \mathrm{D}\hat{u}_2 - \mathrm{i}a_x H(z)\hat{u}_2 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_2 = 0 \\ c\hat{h}_3 + \mathrm{i}a_x U(z)\hat{h}_3 - \mathrm{D}\hat{u}_3 - \mathrm{i}a_x H(z)\hat{u}_3 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_3 = 0 \end{cases} \tag{5.13}$$

with boundary conditions

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \mathbf{0}, \quad \hat{\mathbf{h}} = (\hat{h}_1, \hat{h}_2, \hat{h}_3) = \mathbf{0} \quad \text{at } z = \pm 1. \tag{5.14}$$

In deriving equations (5.13) we have considered general three-dimensional disturbances and we now show that the three-dimensional problem defined by (5.13) and (5.14) can be reduced to an equivalent two-dimensional one. To this purpose we introduce the Squire type transformations [17]

$$\begin{cases} \tilde{u} = a_x\hat{u}_1 + a_y\hat{u}_2, \quad \tilde{w} = a\hat{u}_3, \\ \tilde{h} = a_x\hat{h}_1 + a_y\hat{h}_2, \quad \tilde{k} = a\hat{h}_3, \\ a_x U(z) = a\tilde{U}(z), \quad a_x H(z) = a\tilde{H}(z), \\ \tilde{p} = a\hat{p}_1, \end{cases} \tag{5.15}$$

then equations (5.13) can be combined to give

$$\begin{cases} \big[\tilde{D}ac + 1 - \tilde{D}a\left(\mathrm{D}^2 - a^2\right)\big]\tilde{u} - \tilde{D}aP_m\big[\big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{h} \\ \qquad\qquad + \tilde{H}'(z)\tilde{k}\big] = -\mathrm{i}a\tilde{p} \\ \big[\tilde{D}ac + 1 - \tilde{D}a\left(\mathrm{D}^2 - a^2\right)\big]\tilde{w} - \tilde{D}aP_m\big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{k} = -\mathrm{D}\tilde{p} \\ \mathrm{D}\tilde{w} = -\mathrm{i}a\tilde{u} \\ \mathrm{D}\tilde{k} = -\mathrm{i}a\tilde{h} \\ \big[c + \mathrm{i}a\tilde{U}(z) - R_m^{-1}\big(\mathrm{D}^2 - a^2\big)\big]\tilde{h} + \tilde{H}'(z)\tilde{w} - \tilde{U}'(z)\tilde{k} \\ \qquad\qquad -\big[\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big]\tilde{u} = 0 \\ \big[c + \mathrm{i}a\tilde{U}(z) - R_m^{-1}\big(\mathrm{D}^2 - a^2\big)\big]\tilde{k} - \big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{w} = 0 \end{cases} \tag{5.16}$$

and the boundary conditions are

$$\tilde{u} = \tilde{w} = \tilde{h} = \tilde{k} = 0 \quad \text{at} \quad z = \pm 1.$$

These equations have exactly the same mathematical form as the original equations (5.13) with $a_y = u_2 = h_2 = 0$ and they define the equivalent two-dimensional problem. It is sufficient, therefore, to consider only two-dimensional disturbances; for, once the solution of equations (5.13) and (5.14) with $a_y = u_2 = h_2 = 0$ has been obtained, we can immediately obtain the corresponding solution of the equivalent two-dimensional problem by a trivial change in notation and from this, by means of transformations (5.15), we can then obtain the solution of the original three-dimensional problem. Therefore, in seeking sufficient criteria for linear stability of the basic laminar motion m_0, we may consider only two-dimensional perturbations $\mathbf{u}' = (u_1', 0, u_3')$, $\mathbf{h}' = (h_1', 0, h_3')$, and it is then convenient to introduce the stream functions ψ_1 and ψ_2 such that

$$u_1' = \frac{\partial \psi_1}{\partial z}, \quad u_3' = -\frac{\partial \psi_1}{\partial x},$$

$$h_1' = \frac{\partial \psi_2}{\partial z}, \quad h_3' = -\frac{\partial \psi_2}{\partial x}.$$

If we next let

$$\begin{cases} \psi_1(x,z,t) = \phi_1(z)\exp(\mathrm{i}ax + ct) \\ \psi_2(x,z,t) = \phi_2(z)\exp(\mathrm{i}ax + ct) \end{cases}$$

then

$$\begin{cases} \tilde{u} = \mathrm{D}\phi_1, \quad \tilde{w} = -\mathrm{i}a\phi_1, \\ \tilde{h} = \mathrm{D}\phi_2, \quad \tilde{k} = -\mathrm{i}a\phi_2, \end{cases}$$

and, substituting into (5.16), we have

$$\begin{cases} \left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right]\mathrm{D}\phi_1-\tilde{D}aP_m\left[\mathrm{D}+\mathrm{i}aH(z)\right]\mathrm{D}\phi_2 \\ \qquad +\tilde{D}aP_m\mathrm{i}aH'(z)\phi_2=-\mathrm{i}a\tilde{p} \\ -\mathrm{i}a\left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right]\phi_1 \\ \qquad +\mathrm{i}a\tilde{D}aP_m[\mathrm{D}+\mathrm{i}aH(z)]\phi_2=-\mathrm{D}\tilde{p} \\ \left[c+\mathrm{i}aU(z)-R_m^{-1}(\mathrm{D}^2-a^2)\right]\mathrm{D}\phi_2-\mathrm{i}aH'(z)\phi_1 \\ \qquad +\mathrm{i}aU'(z)\phi_2-\left[\mathrm{D}+\mathrm{i}aH(z)\right]\mathrm{D}\phi_1=0 \\ \left[c+\mathrm{i}aU(z)-R_m^{-1}\left(\mathrm{D}^2-a^2\right)\right]\phi_2-[\mathrm{D}+\mathrm{i}aH(z)]\phi_1=0, \end{cases} \tag{5.17}$$

under the boundary conditions

$$\phi_j(\pm 1)=\mathrm{D}\phi_j(\pm 1)=0, \quad j=1,2. \tag{5.18}$$

Eliminating the pressure $\tilde{p}$ between equations $(5.17)_1$, $(5.17)_2$ and applying the operator D^2-a^2 to equation $(5.17)_4$, we readily get

$$\begin{cases} \left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right](\mathrm{D}^2-a^2)\phi_1 \\ \quad =\tilde{D}aP_m\left[(\mathrm{D}+\mathrm{i}aH(z))(\mathrm{D}^2-a^2)-\mathrm{i}aH''(z)\right]\phi_2 \\ \left\{\left[c-R_m^{-1}(\mathrm{D}^2-a^2)\right](\mathrm{D}^2-a^2)+\mathrm{i}a[U''(z)-a^2U(z)]\right. \\ \quad \left.+2\mathrm{i}aU'(z)\mathrm{D}+\mathrm{i}aU(z)\mathrm{D}^2\right\}\phi_2=\left\{(\mathrm{D}^2-a^2)\mathrm{D}\right. \\ \quad \left.+\mathrm{i}a[H''(z)-a^2H(z)]+2\mathrm{i}aH'(z)\mathrm{D}+\mathrm{i}aH(z)\mathrm{D}^2\right\}\phi_1. \end{cases} \tag{5.19}$$

Denoting by ϕ_j^* the complex conjugate of ϕ_j $(j=1,2)$, multiplying $(5.19)_1$ by ϕ_1^*, $(5.19)_2$ by $\tilde{D}aP_m\phi_2^*$, summing and integrating over the interval $(-1,1)$ we obtain the eigenvalue relation

$$\left[(\tilde{D}ac+1)(I_1^2+a^2I_0^2)+\tilde{D}a(I_2^2+2a^2I_1^2+a^4I_0^2)\right]$$
$$+\tilde{D}aP_m\left[c(J_1^2+a^2J_0^2)+R_m^{-1}(J_2^2+2a^2J_1^2+a^4J_0^2)\right]=\tilde{D}aP_m(Q+R),$$

with

$$I_n^2=\int_{-1}^{1}|\mathrm{D}^n\phi_1|^2\mathrm{d}z, \quad J_n^2=\int_{-1}^{1}|\mathrm{D}^n\phi_2|^2\mathrm{d}z \quad (n=0,1,2),$$

$$Q=-\int_{-1}^{1}\mathrm{D}^3\phi_2\phi_1^*\mathrm{d}z+a^2\int_{-1}^{1}\mathrm{D}\phi_2\phi_1^*\mathrm{d}z$$
$$-2\mathrm{i}a\int_{-1}^{1}H'(z)\phi_2\mathrm{D}\phi_1^*\mathrm{d}z-\mathrm{i}a\int_{-1}^{1}H(z)\phi_2\mathrm{D}^2\phi_1^*\mathrm{d}z+\mathrm{i}a^3\int_{-1}^{1}H(z)\phi_2\phi_1^*\mathrm{d}z$$

and

$$
\begin{aligned}
R = {} & \mathrm{i}a \int_{-1}^{1} \left[U''(z) - a^2 U(z)\right] |\phi_2|^2 \mathrm{d}z + 2\mathrm{i}a \int_{-1}^{1} U'(z) \mathrm{D}\phi_2 \phi_2^* \mathrm{d}z \\
& - \int_{-1}^{1} \mathrm{D}^3 \phi_1 \phi_2^* \mathrm{d}z + \mathrm{i}a \int_{-1}^{1} U(z) \mathrm{D}^2 \phi_2 \phi_2^* \mathrm{d}z \\
& - \mathrm{i}a \int_{-1}^{1} \left[H''(z) - a^2 H(z)\right] \phi_1 \phi_2^* \mathrm{d}z - 2\mathrm{i}a \int_{-1}^{1} H'(z) \mathrm{D}\phi_1 \phi_2^* \mathrm{d}z \\
& + a^2 \int_{-1}^{1} \mathrm{D}\phi_1 \phi_2^* \mathrm{d}z - \mathrm{i}a \int_{-1}^{1} H(z) \mathrm{D}^2 \phi_1 \phi_2^* \mathrm{d}z.
\end{aligned}
$$

Thus

$$
\begin{aligned}
c_r = & \Big\{ \tilde{D}aP_m \mathrm{Re}(Q+R) - \big[(I_1^2 + a^2 I_0^2) + \tilde{D}a(I_2^2 + 2a^2 I_1^2 + a^4 I_0^2) \\
& + \frac{\tilde{D}aP_m}{R_m}(J_2^2 + 2a^2 J_1^2 + a^4 J_0^2)\big] \Big\} \Big\{ \tilde{D}a(I_1^2 + a^2 I_0^2) + \tilde{D}aP_m(J_1^2 + a^2 J_0^2) \Big\}^{-1},
\end{aligned} \tag{5.20}
$$

$$
c_i = \frac{P_m \mathrm{Im}(Q+R)}{I_1^2 + a^2 I_0^2 + P_m(J_1^2 + a^2 J_0^2)},
$$

with

$$
\mathrm{Re}(Q+R) = \frac{\mathrm{i}a}{2} \int_{-1}^{1} U'(z)(\mathrm{D}\phi_2 \phi_2^* - \mathrm{D}\phi_2^* \phi_2) \mathrm{d}z - \frac{\mathrm{i}a}{2} \int_{-1}^{1} H''(z)(\phi_1 \phi_2^* - \phi_1^* \phi_2) \mathrm{d}z
$$

and

$$
\begin{aligned}
\mathrm{Im}(Q+R) = {} & \int_{-1}^{1} (\phi_1 \mathrm{D}^3 \phi_2^* - \phi_1^* \mathrm{D}^3 \phi_2) \mathrm{d}z \\
& - a^2 \int_{-1}^{1} (\phi_1 \mathrm{D}\phi_2^* - \phi_1^* \mathrm{D}\phi_2) \mathrm{d}z - \mathrm{i}a \int_{-1}^{1} H(z)(\mathrm{D}\phi_1 \phi_2^* + \mathrm{D}\phi_1^* \phi_2) \mathrm{d}z \\
& + \mathrm{i}a \int_{-1}^{1} H(z)(\mathrm{D}\phi_1 \mathrm{D}\phi_2^* + \mathrm{D}\phi_1^* \mathrm{D}\phi_2) \mathrm{d}z + \mathrm{i}a^3 \int_{-1}^{1} H(z)(\phi_1 \phi_2^* + \phi_1^* \phi_2) \mathrm{d}z \\
& + \mathrm{i}a \int_{-1}^{1} [U''(z) - a^2 U(z)] |\phi_2|^2 \mathrm{d}z - \mathrm{i}a \int_{-1}^{1} U(z) |\mathrm{D}\phi_2|^2 \mathrm{d}z \\
& + \frac{\mathrm{i}a}{2} \int_{-1}^{1} U'(z)(\mathrm{D}\phi_2 \phi_2^* + \mathrm{D}\phi_2^* \phi_2) \mathrm{d}z + \frac{\mathrm{i}a}{2} \int_{-1}^{1} H'(z)(\phi_1 \mathrm{D}\phi_2^* + \phi_1^* \mathrm{D}\phi_2) \mathrm{d}z \\
& + \frac{\mathrm{i}a}{2} \int_{-1}^{1} H'(z)(\mathrm{D}\phi_1 \phi_2^* + \mathrm{D}\phi_1^* \phi_2) \mathrm{d}z.
\end{aligned}
$$

Setting

$$
\mathcal{I} = \tilde{D}a \mathrm{Re}(Q+R)
$$

and

$$\mathcal{D} = R_m(I_1^2 + a^2 I_0^2) + R_m\tilde{D}a(I_2^2 + 2a^2 I_1^2 + a^4 I_0^2) + \tilde{D}aP_m(J_2^2 + 2a^2 J_1^2 + a^4 J_0^2),$$

(5.20) becomes

$$c_r = \frac{1}{\tilde{D}a}\left(M^2\frac{\mathcal{I}}{\mathcal{D}} - 1\right)\mathcal{D}\left[I_1^2 + a^2 I_0^2 + P_m(J_1^2 + a^2 J_0^2)\right]^{-1} \tag{5.21}$$

with ϕ_1 and ϕ_2 belonging to the space of the kinematically admissible fields

$$\mathcal{H} = \left\{(\phi_1, \phi_2) \in \left(\mathrm{H}^2(-1,1)\right)^2 : \phi_i(\pm 1) = \mathrm{D}\phi_i(\pm 1) = 0 \quad \forall i = 1,2\right\}.$$

If we now define

$$\frac{1}{M_L^2(a)} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}, \tag{5.22}$$

from (5.21) we readily deduce that the modes of wave number a are linearly stable for $M \leq M_L(a)$.

The Euler-Lagrange equations corresponding to the variational problem (5.22) are:

$$\begin{cases} R_m\left[-1 + \tilde{D}a(\mathrm{D}^2 - a^2)\right](\mathrm{D}^2 - a^2)\phi_1 + \mathrm{i}a\sigma H''\phi_2 = 0 \\ \tilde{D}aP_m(\mathrm{D}^2 - a^2)^2\phi_2 + \mathrm{i}a\sigma\left(-H''\phi_1 + 2U'\mathrm{D}\phi_2 + U''\phi_2\right) = 0, \end{cases} \tag{5.23}$$

with boundary conditions (5.18). $M_L^2(a)$ is the least positive eigenvalue of the characteristic value problem (5.23) with boundary conditions (5.18).

Finally, if

$$M \leq M_c = \min_{a>0} M_L(a), \tag{5.24}$$

then all modes are stable and hence the basic laminar motion m_0 is linearly stable.

5.4 Sufficient condition for global non linear exponential stability

In order to study the nonlinear stability of m_0 by employing the energy method we introduce the Liapunov function

$$E(t) = \frac{1}{2}(\tilde{D}a\|\mathbf{u}\|^2 + \tilde{D}aP_m\|\mathbf{h}\|^2)$$

where $\|\cdot\|$ denotes as usual the $\mathrm{L}^2(\Omega)$ norm and

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [-1, 1]$$

is the period cell. Taking into account the periodicity and the boundary conditions, we find along the solutions of (5.9)

$$\dot{E}(t) = \frac{1}{R_m}\left[M^2\frac{\mathscr{I}(\mathbf{u},\mathbf{h})}{\mathscr{D}(\mathbf{u},\mathbf{h})} - 1\right]\mathscr{D}(\mathbf{u},\mathbf{h}), \tag{5.25}$$

where

$$\mathscr{I}(\mathbf{u},\mathbf{h}) = \tilde{D}a\left[\int_\Omega U'(z)h_1h_3\mathrm{d}\Omega + \int_\Omega H'(z)(h_3u_1 - h_1u_3)\mathrm{d}\Omega\right],$$

$$\mathscr{D}(\mathbf{u},\mathbf{h}) = R_m\left(\|\mathbf{u}\|^2 + \tilde{D}a\|\nabla\mathbf{u}\|^2\right) + \tilde{D}aP_m\|\nabla\mathbf{h}\|^2,$$

and the perturbations $\mathbf{u}$, $\mathbf{h}$ belong to the kinematically admissible space

$$\mathscr{W} = \{(\mathbf{u},\mathbf{h}) \in (\mathrm{H}^1(\Omega))^6 : \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0,\ \mathbf{u}(x,y,\pm1) = \mathbf{h}(x,y,\pm1) = 0\}.$$

Let now

$$\frac{1}{M_E^2} = \max_{\mathscr{W}}\frac{\mathscr{I}}{\mathscr{D}}. \tag{5.26}$$

The existence of the maximum of the functional $\mathscr{I}/\mathscr{D}$ can proved by following the proof of Theorem 4.2.

By assuming $M < M_E$, from (5.25) and since, by Poincaré inequality,

$$\mathscr{D}(\mathbf{u},\mathbf{h}) \geq \frac{1}{4}\left[R_m\left(4 + \tilde{D}a\pi^2\right)\|\mathbf{u}\|^2 + \tilde{D}aP_m\pi^2\|\mathbf{h}\|^2\right],$$

we obtain the energy inequality

$$\dot{E}(t) \leq \frac{1}{R_m}\left(\frac{M^2}{M_E^2} - 1\right)\nu_0E(t), \tag{5.27}$$

where

$$\nu_0 = \frac{1}{2}\min\left\{\frac{R_m(4 + \tilde{D}a\pi^2)}{\tilde{D}a}, \pi^2\right\}. \tag{5.28}$$

Integrating (5.27) we have global nonlinear exponential stability of the basic motion m_0 according to the following inequality

$$E(t) \leq E(0)\exp\left[\frac{1}{R_m}\left(\frac{M^2}{M_E^2} - 1\right)\nu_0t\right].$$

The Euler-Lagrange equations corresponding to the variational problem (5.26) are

$$\begin{cases} \lambda\left[\tilde{D}a\,(H'h_3\mathbf{i} - H'h_1\mathbf{k})\right] + R_m(\mathbf{u} - \tilde{D}a\Delta\mathbf{u}) = -\nabla p' \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \lambda\,(H'u_1\mathbf{k} - H'u_3\mathbf{i} + U'h_3\mathbf{i} + U'h_1\mathbf{k}) - P_m\Delta\mathbf{h} = -\nabla\chi \end{cases} \tag{5.29}$$

where p', χ are Lagrange multipliers associated with the divergence constraints. This system must be solved subject to the boundary conditions (5.10) and M_E^2 is then obtained as the least positive eigenvalue λ of the characteristic-value problem (5.29) and (5.10).

Remark 5.1 (An estimate of M_E)**.** By Cauchy and Poincarè inequalities we have

$$\begin{aligned}\frac{\mathscr{I}(\mathbf{u},\mathbf{h})}{\mathscr{D}(\mathbf{u},\mathbf{h})} &\leq 4\tilde{D}a\frac{q\|\mathbf{u}\|^2+(r+q)\|\mathbf{h}\|^2}{R_m(4+\tilde{D}a\pi^2)\|\mathbf{u}\|^2+\tilde{D}aP_m\pi^2\|\mathbf{h}\|^2} \\ &\leq 4\max\left\{\frac{q\tilde{D}a}{R_m(4+\tilde{D}a\pi^2)},\frac{r+q}{P_m\pi^2}\right\},\end{aligned} \tag{5.30}$$

where

$$2r=\max_{z\in[-1,1]}|U'(z)|,\quad 2q=\max_{z\in[-1,1]}|H'(z)|.$$

Consequently, from (5.26) and (5.30) we deduce that

$$M_E\geq M_E^*=\frac{1}{2}\sqrt{\min\left\{\frac{R_m(4+\tilde{D}a\pi^2)}{q\tilde{D}a},\frac{P_m\pi^2}{r+q}\right\}},$$

and therefore the condition $M<M_E^*$ implies that m_0 is globally nonlinearly exponentially stable.

Let us consider now the set of two-dimensional disturbances in the xz-plane

$$\mathscr{V}=\left\{(\mathbf{u},\mathbf{h})\in\mathscr{W}:\mathbf{u}=\frac{\partial\psi_1}{\partial z}\mathbf{i}-\frac{\partial\psi_1}{\partial x}\mathbf{k},\ \ \mathbf{h}=\frac{\partial\psi_2}{\partial z}\mathbf{i}-\frac{\partial\psi_2}{\partial x}\mathbf{k}\right\}.$$

Since $\mathscr{V}$ is a closed subset of $\mathscr{W}$

$$\exists\max_{\mathscr{V}}\frac{\mathscr{I}}{\mathscr{D}}=\frac{1}{M_E'^2}\leq\frac{1}{M_E^2}.$$

Therefore $M_E\leq M_E'$ and hence two-dimensional disturbances in the xz-plane are more stable than three-dimensional ones. For such disturbances the Euler-Lagrange equations (5.29) reduce to

$$\begin{cases} R_m\left(-\Delta\psi_1+\tilde{D}a\Delta^2\psi_1\right)+\lambda\tilde{D}aH''\dfrac{\partial\psi_2}{\partial x}=0 \\ P_m\Delta^2\psi_2+\lambda\left(-H''\dfrac{\partial\psi_1}{\partial x}+2U'\dfrac{\partial^2\psi_2}{\partial x\partial z}+U''\dfrac{\partial\psi_2}{\partial x}\right)=0\end{cases}$$

whose normal mode form is identical with equations (5.23) and thus, for disturbances of this type, the linear theory and the energy method lead to identical results.

5.4.1 Non linear stability of the MHD laminar flows with respect to disturbances normal to the embedding magnetic field

Let $\mathcal{S}$ be the set of the two-dimensional perturbations considered by Rionero and Maiellaro in [74]

$$\begin{cases} \mathbf{u}(z,t) = u_1(z,t)\mathbf{i} + u_2(z,t)\mathbf{j} \\ \mathbf{h}(z,t) = h_1(z,t)\mathbf{i} + h_2(z,t)\mathbf{j}. \end{cases} \tag{5.31}$$

Introducing the energy

$$\mathcal{E}(t) = \frac{\tilde{D}a}{2}\int_{-1}^{1} |\mathbf{u}(z,t)|^2 \mathrm{d}z + \frac{\tilde{D}aP_m}{2}\int_{-1}^{1} |\mathbf{h}(z,t)|^2 \mathrm{d}z,$$

the following energy inequality is easily obtained

$$\dot{\mathcal{E}}(t) \le -\frac{\nu_0}{R_m}\mathcal{E}(t), \tag{5.32}$$

with ν_0 defined through (5.28).

Integrating (5.32) one can prove that m_0 is globally nonlinearly exponentially stable with respect to laminar disturbances (5.31) for all Hartmann numbers according to

$$\mathcal{E}(t) \le \mathcal{E}(0)\exp\left(-\frac{\nu_0}{R_m}t\right).$$

5.5 The convergence of the Galerkin method

In using the Galerkin method to solve the eigenvalue problem (5.23) with boundary conditions (5.18), we expand the eigenfunctions (ϕ_1, ϕ_2) in terms of the set $\{C_n, S_n\}_{n\in\mathbb{N}}$ introduced by Harris and Reid in [26] which is orthonormal with respect to the $\mathrm{L}^2[-1,1]$ inner product

$$(f,g) = \int_{-1}^{1} f(z)g(z)\mathrm{d}z$$

and complete in the space $\left\{f \in \mathrm{H}^2[-1,1]/f(\pm 1) = \mathrm{D}f(\pm 1) = 0\right\}$ with the second derivative norm.

$$C_n(z) = \frac{1}{\sqrt{2}}\left[\frac{\cosh(\lambda_n z)}{\cosh\lambda_n} - \frac{\cos(\lambda_n z)}{\cos\lambda_n}\right]$$

and

$$S_n(z) = \frac{1}{\sqrt{2}}\left[\frac{\sinh(\mu_n z)}{\sinh\mu_n} - \frac{\sin(\mu_n z)}{\sin\mu_n}\right],$$

λ_n and μ_n being roots of the characteristic equations

$$\tanh\lambda + \tan\lambda = 0 \quad \text{and} \quad \coth\mu - \cot\mu = 0, \tag{5.33}$$

respectively, are solutions of the Sturm-Liouville problem

$$\mathrm{D}^4 y - \nu^4 y = 0$$

with boundary conditions $y(\pm 1) = \mathrm{D}y(\pm 1) = 0$. $2\lambda_n$ and $2\mu_n$ are listed in [11], page 636[2].

We now expand ϕ_1 and ϕ_2 in terms of $\{C_n, S_n\}_{n\in\mathbb{N}}$

$$\begin{cases} \phi_1 = \sum_{n=1}^{+\infty} \left[A_n^{(C)} C_n(z) + A_n^{(S)} S_n(z)\right], \\ \phi_2 = \sum_{n=1}^{+\infty} \left[B_n^{(C)} C_n(z) + B_n^{(S)} S_n(z)\right], \end{cases} \tag{5.34}$$

insert these series into system (5.23) and by requiring that the error in the differential equations (5.23) be orthogonal to C_r and S_r for each positive integer r, we obtain an infinite system of linear homogeneous equations for the constants $A_n^{(C)}$, $A_n^{(S)}$, $B_n^{(C)}$ and $B_n^{(S)}$. In order that these constants do not vanish identically, the determinant of the system must vanish, and this condition yields the secular determinant

$$\begin{vmatrix} Z_{nr} & 0 & \mathrm{i}a\sigma E_{nr}^{(C)} & \mathrm{i}a\sigma E_{nr}^{(S)} \\ 0 & F_{nr} & \mathrm{i}a\sigma G_{nr}^{(C)} & \mathrm{i}a\sigma G_{nr}^{(S)} \\ -\mathrm{i}a\sigma E_{nr}^{(C)} & -\mathrm{i}a\sigma E_{nr}^{(S)} & L_{nr} + \mathrm{i}a\sigma X_{nr}^{(C)} & \mathrm{i}a\sigma X_{nr}^{(S)} \\ -\mathrm{i}a\sigma G_{nr}^{(C)} & -\mathrm{i}a\sigma G_{nr}^{(S)} & \mathrm{i}a\sigma Y_{nr}^{(C)} & Q_{nr} + \mathrm{i}a\sigma Y_{nr}^{(S)} \end{vmatrix} = 0 \tag{5.35}$$

where

$$Z_{nr} = R_m \left\{ [a^2 + \tilde{D}a(\lambda_n^4 + a^4)]\delta_{nr} - (1 + 2a^2\tilde{D}a) \int_{-1}^{1} C_n''(z) C_r(z) \mathrm{d}z \right\},$$

$$F_{nr} = R_m \left\{ [a^2 + \tilde{D}a(\mu_n^4 + a^4)]\delta_{nr} - (1 + 2a^2\tilde{D}a) \int_{-1}^{1} S_n''(z) S_r(z) \mathrm{d}z \right\},$$

$$E_{nr}^{(C)} = \int_{-1}^{1} H''(z) C_n(z) C_r(z) \mathrm{d}z, \quad E_{nr}^{(S)} = \int_{-1}^{1} H''(z) S_n(z) C_r(z) \mathrm{d}z,$$

$$G_{nr}^{(C)} = E_{rn}^{(S)}, \quad G_{nr}^{(S)} = \int_{-1}^{1} H''(z) S_n(z) S_r(z) \mathrm{d}z,$$

$$L_{nr} = \tilde{D}aP_m \left[(\lambda_n^4 + a^4)\delta_{nr} - 2a^2 \int_{-1}^{1} C_n''(z) C_r(z) \mathrm{d}z \right],$$

[2] In [11] the roots of the characteristic equations (5.33) are derived for the interval $[-1/2, 1/2]$ and here we have re-calculated them for the interval [-1,1].

$$Q_{nr} = \tilde{D}aP_m \left[(\mu_n^4 + a^4)\delta_{nr} - 2a^2 \int_{-1}^{1} S_n''(z) S_r(z) \mathrm{d}z \right],$$

$$X_{nr}^{(C)} = \int_{-1}^{1} U'(z) \left[C_n'(z) C_r(z) - C_n(z) C_r'(z) \right] \mathrm{d}z,$$

$$X_{nr}^{(S)} = \int_{-1}^{1} U'(z) \left[S_n'(z) C_r(z) - S_n(z) C_r'(z) \right] \mathrm{d}z,$$

$$Y_{nr}^{(C)} = -X_{rn}^{(S)} \text{ and } Y_{nr}^{(S)} = \int_{-1}^{1} U'(z) \left[S_n'(z) S_r(z) - S_n(z) S_r'(z) \right] \mathrm{d}z.$$

By considering only a finite number of terms (say N) in the expansions (5.34), (5.35) reduces to an algebraic equation in the unknown σ whose least positive solution is the so-called N-*th* approximation of $M_L^2(a)$. With $N = 1, 2, ...,$ we get a non-increasing sequence of approximations of the critical Hartmann number which, as we shall prove, converges to the exact value of $M_L(a)$. The convergence of the Galerkin method is based on the following Mikhlin Theorem [14, 49].

Theorem 5.1 (Mikhlin)**.** *Let λ be a parameter in the equation*

$$\mathcal{A}u - \lambda \mathcal{K} u = 0, \tag{5.36}$$

where $\mathcal{A}$ and $\mathcal{K}$ are linear operators, and the domain of $\mathcal{A}$, $D_{\mathcal{A}}$, is a linear space which is dense in a Hilbert space $\mathcal{H}$ with inner product $< \cdot, \cdot >$. Let $D_{\mathcal{A}}$ be contained in the domain of $\mathcal{K}$, $D_{\mathcal{K}}$, and assume that

1) $\mathcal{A}$ is a positive-definite self-adjoint operator,

2) the operator $\mathcal{A}^{-1}\mathcal{K}$ can be extended to be completely continuous on the Hilbert space $\mathcal{H}_0$ which is the completion of $D_{\mathcal{A}}$ under the norm $< \mathcal{A}\cdot, \cdot >^{1/2}$.

Then the Galerkin method for calculating the eigenvalues of (5.36) *is a convergent process in $\mathcal{H}_0$.*

The eigenvalue problem (5.23) with boundary conditions (5.18) can be cast in a form such that the conditions of Mikhlin Theorem are satisfied. In fact, by setting $\lambda = -\mathrm{i}a\sigma$,

$$\mathcal{A} = \begin{pmatrix} R_m \left[-1 + \tilde{D}a(\mathrm{D}^2 - a^2) \right] (\mathrm{D}^2 - a^2) & 0 \\ 0 & \tilde{D}aP_m(\mathrm{D}^2 - a^2)^2 \end{pmatrix} \tag{5.37}$$

and

$$\mathcal{K} = \begin{pmatrix} 0 & H''(z) \\ -H''(z) & 2U'(z)\mathrm{D} + U''(z) \end{pmatrix}, \tag{5.38}$$

equations (5.23) can be rewritten as

$$\mathcal{A}\mathbf{u} = \lambda \mathcal{K}\mathbf{u}$$

where $\mathbf{u}$ is the column vector with components u_1 and u_2.

The domain of the linear operator $\mathcal{A}$ is the vector space

$$D_{\mathcal{A}} = \Big\{\mathbf{u} = (u_1, u_2) \in \big(\mathrm{C}^4([-1,1],\mathbb{C})\big)^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \ \forall j = 1,2\Big\}$$

which is dense in the Hilbert space $\big(\mathrm{L}^2([-1,1],\mathbb{C})\big)^2$ with inner product

$$(\mathbf{u},\mathbf{v}) = \int_{-1}^{1} (u_1 v_1^* + u_2 v_2^*)\,\mathrm{d}z$$

and norm

$$\|\mathbf{u}\| = \left[\int_{-1}^{1} (|u_1|^2 + |u_2|^2)\mathrm{d}z\right]^{1/2}.$$

It can be readily established by direct integration by parts that $\mathcal{A}$ is positive-definite on the space $D_{\mathcal{A}}$. In fact

$$\begin{aligned}(\mathcal{A}\mathbf{u},\mathbf{u}) =& R_m \int_{-1}^{1} [-1 + \tilde{D}a(\mathrm{D}^2 - a^2)](\mathrm{D}^2 - a^2)u_1 u_1^* \mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} (\mathrm{D}^2 - a^2)^2 u_2 u_2^* \mathrm{d}z \\ &= R_m \int_{-1}^{1} [|\mathrm{D}u_1|^2 + a^2|u_1|^2 + \tilde{D}a|(\mathrm{D}^2 - a^2)u_1|^2]\mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} |(\mathrm{D}^2 - a^2)u_2|^2 \mathrm{d}z \geq 0 \quad \forall \mathbf{u} \in D_{\mathcal{A}}\end{aligned}$$

and the equality holds if and only if $\mathbf{u} = \mathbf{0}$. Similarly we can prove that

$$(\mathcal{A}\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathcal{A}\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in D_{\mathcal{A}},$$

viz $\mathcal{A}$ is self-adjoint on $D_{\mathcal{A}}$.

It is easy to check that $(\mathcal{A}\mathbf{u},\mathbf{v})$ defines an inner product on $D_{\mathcal{A}}$ with the corresponding norm

$$\begin{aligned}\|\mathbf{u}\|_{\mathcal{A}}^2 := (\mathcal{A}\mathbf{u},\mathbf{u}) =& R_m \int_{-1}^{1} [|\mathrm{D}u_1|^2 + a^2|u_1|^2 + \tilde{D}a|(\mathrm{D}^2 - a^2)u_1|^2]\mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} |(\mathrm{D}^2 - a^2)u_2|^2 \mathrm{d}z,\end{aligned}$$

and let $\mathcal{H}_0$ be the completion of $D_{\mathcal{A}}$ under the norm $\|\cdot\|_{\mathcal{A}}$.
Let us consider now the Hilbert space of the kinematically admissible fields introduced in section 5.3

$$\mathcal{H} = \left\{\mathbf{u} = (u_1, u_2) \in (\mathrm{H}^2([-1,1],\mathbb{C})^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \;\forall j = 1,2\right\}$$

and observe that it is the completion of $D_{\mathcal{A}}$ under the norm

$$\|\mathbf{u}\|_{\mathcal{H}} = \left(\sum_{j=1}^{2} \|\mathrm{D}^2 u_j\|_2^2\right)^{\frac{1}{2}},$$

$\|\cdot\|_2$ denoting the standard $\mathrm{L}^2([-1,1],\mathbb{C})$-norm. The well-known isoperimetric inequalities

$$\|\mathrm{D}^2 w\|_2^2 \geq \lambda_1^4 \|w\|_2^2 \quad \forall w \in \mathrm{H}^2([-1,1],\mathbb{C}),\ w(\pm 1) = \mathrm{D}w(\pm 1) = 0, \tag{5.39}$$

and

$$\|\mathrm{D}w\|_2^2 \geq \frac{\pi^2}{4}\|w\|_2^2 \quad \forall w \in \mathrm{H}^1([-1,1],\mathbb{C}),\ w(\pm 1) = 0, \tag{5.40}$$

with λ_1 the least positive root of the characteristic equation $(5.33)_1$, give

$$\sqrt{\tilde{D}a \min\{R_m, P_m\}}\|\mathbf{u}\|_{\mathcal{H}} \leq \|\mathbf{u}\|_{\mathcal{A}}$$
$$\leq \sqrt{R_m\left(\frac{4}{\pi^2} + \frac{a^2}{\lambda_1^4}\right) + \tilde{D}a \max\{R_m, P_m\}\left(1 + \frac{8a^2}{\pi^2} + \frac{a^4}{\lambda_1^4}\right)}\|\mathbf{u}\|_{\mathcal{H}},$$

for all $\mathbf{u} \in D_{\mathcal{A}}$, viz $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms on the space $D_{\mathcal{A}}$ and hence $\mathcal{H}_0 = \mathcal{H}$.

Concerning the linear operator $\mathcal{K}$ defined in (5.38), since U and H are smooth functions in $[-1,1]$, the domain of $\mathcal{K}$ is the Banach space

$$D_{\mathcal{K}} = \mathrm{L}^2([-1,1],\mathbb{C}) \times \mathrm{H}^1([-1,1],\mathbb{C})$$

endowed with the norm $\|\mathbf{u}\|_{D_{\mathcal{K}}} = \|u_1\|_2 + \|u_2\|_2 + \|\mathrm{D}u_2\|_2$, and it can be readily shown that $\mathcal{K}$ is bounded. Obviously $D_{\mathcal{A}}$ is contained in $D_{\mathcal{K}}$.

Let now $\mathcal{T} = \mathcal{A}^{-1}\mathcal{K}$:

$$\mathcal{T} : \mathbf{u} \in D_{\mathcal{K}} \mapsto \mathcal{T}\mathbf{u} = \int_{-1}^{1} \mathbf{G}(z,\xi)\mathcal{K}(\xi)\mathbf{u}(\xi)\mathrm{d}\xi, \tag{5.41}$$

where $\mathbf{G}$ is the matrix Green function corresponding to the matrix differential operator $\mathcal{A}$ with boundary conditions (5.18) (see Appendix A for details). Since the elements G_{11} and G_{22} in the matrix Green function $\mathbf{G}$ belong to $\mathrm{C}^2([-1,1] \times [-1,1])$ and vanish with their first derivatives with respect to z at the boundaries $z = \pm 1$, the range of $\mathcal{T}$ is contained in the Banach space

$$\mathscr{H} = \left\{ \mathbf{u} = (u_1, u_2) \in \left(\mathrm{C}^2([-1,1],\mathbb{C})\right)^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \, \forall j = 1,2 \right\}$$

endowed with the standard norm

$$\|\mathbf{u}\|_{\mathscr{H}} = \sum_{h,j=1}^{2} \max_{z \in [-1,1]} |\mathrm{D}^h u_j(z)|.$$

Now we prove that $\mathcal{T}$ is completely continuous, i.e. $\mathcal{T}$ takes bounded sequences in $D_{\mathcal{K}}$ into sequences in $\mathscr{H}$ with a convergent subsequence. Let $\{\mathbf{u}_n = (u_{1,n}, u_{2,n})\}_{n\in\mathbb{N}}$ be a bounded sequence in $D_{\mathcal{K}}$ and $\epsilon > 0$, then there exist c, l, $\delta_\epsilon \in \mathbb{R}^+$ such that

$$\|\mathbf{u}_n\|_{D_{\mathcal{K}}} = \|u_{1,n}\|_2 + \|u_{2,n}\|_2 + \|\mathrm{D}u_{2,n}\|_2 \le c \;\; \forall n \in \mathbb{N},$$

$$\left| \frac{\partial^h G_{jj}}{\partial z^h}(z,\xi) \right| \le l \quad \forall j = 1,2, \;\; \forall h = 0,1,2$$

and

$$\left| \frac{\partial^h G_{jj}}{\partial z^h}(z',\xi) - \frac{\partial^h G_{jj}}{\partial z^h}(z,\xi) \right| < \epsilon \;\; \forall j = 1,2, \;\; \forall |z'-z| < \delta_\epsilon, \;\; \forall h = 0,1,2.$$

Then, setting $\mathcal{T}\mathbf{u}_n = ((\mathcal{T}\mathbf{u}_n)_1, (\mathcal{T}\mathbf{u}_n)_2)$, we have

$$|\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z)| \le lc \sup_{\|\mathbf{w}\|_{D_{\mathcal{K}}}=1} \|\mathcal{K}\mathbf{w}\| \quad \forall j = 1,2, \;\; \forall z \in [-1,1],$$
$$\forall h = 0,1,2, \;\; \forall n \in \mathbb{N}$$

and

$$|\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z') - \mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z)| \le \epsilon c \sup_{\|\mathbf{w}\|_{D_{\mathcal{K}}}=1} \|\mathcal{K}\mathbf{w}\| \quad \forall j = 1,2,$$
$$\forall |z'-z| < \delta_\epsilon, \;\; \forall h = 0,1,2, \; \forall n \in \mathbb{N}.$$

Therefore $\{\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j\}_{n\in\mathbb{N}}$ is a uniformly bounded, equicontinuous sequence for all $j = 1,2$ and $h = 0,1,2$, and so, by Ascoli-Arzelà Theorem, there exists a subsequence $\{\mathcal{T}\mathbf{u}_{k_n}\}_{n\in\mathbb{N}} \subset \{\mathcal{T}\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that $\{\mathrm{D}^h(\mathcal{T}\mathbf{u}_{k_n})_j\}_{n\in\mathbb{N}}$ converges uniformly in $[-1,1]$ for all $j = 1,2$ and $h = 0,1,2$. Then $\{\mathcal{T}\mathbf{u}_{k_n}\}_{n\in\mathbb{N}}$ converges in $\mathscr{H}$.

Finally, since the Hilbert space $\mathcal{H}$ is contained in $D_{\mathcal{K}}$ with[3]

$$\|\mathbf{u}\|_{\mathcal{H}} \ge \frac{\pi}{4}\|\mathbf{u}\|_{D_{\mathcal{K}}} \quad \forall \mathbf{u} \in \mathcal{H}, \tag{5.42}$$

[3]The inequality (5.42) is obtained combining the inequalities (5.39) and (5.40).

and since the uniform convergence in $[-1,1]$ implies the convergence in $L^2[-1,1]$-norm we deduce that $\mathcal{T}:\mathcal{H}\to\mathcal{H}$ is completely continuous. Therefore the conditions of Mikhlin Theorem are established and thus the Galerkin method for computing the eigenvalues of the characteristic-value problem (5.23)-(5.18) is a convergent process in the space of the kinematically admissible fields $\mathcal{H}$. In particular the sequence of the approximations of the critical Hartmann number M_c defined in (5.24) converges to the exact value.