Chapter 1

Basic concepts and mathematical methods

1.1 Evolution equations. Dynamical systems

Let $\mathcal{F}$ be a phenomenon occurring in a domain Ω of the physical three-dimensional space $\mathbb{R}^3$ and let $v_i(\mathbf{x}, t)$ - with $i = 1, 2, ..., n$ ($n \in \mathbb{N}$), $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$ an instant of time - represent the relevant quantities describing the state of $\mathcal{F}$. The vector $\mathbf{v} \equiv (v_1, v_2, ..., v_n)$ is the *state vector*. The phenomenon $\mathcal{F}$ is modelled by a P.D.E. if one can establish the existence of a function

$$\mathbf{F}\left(\mathbf{x}, t, \mathbf{v}, \frac{\partial v_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, ...\right), \quad i, j = 1, 2, ..., n; \; r, s = 1, 2, 3$$

which governs the behaviour of the time derivative of $\mathbf{v}$, viz, for any $T > 0$,

$$\mathbf{v}_t = \mathbf{F} \qquad \text{in } \; \Omega \times (0, T) \tag{1.1}$$

subject to the initial data

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega \tag{1.2}$$

and suitable boundary conditions

$$A(\mathbf{v}, \nabla \mathbf{v}) = \hat{\mathbf{v}} \quad \text{on } \partial\Omega \times (0, T) \tag{1.3}$$

where $\mathbf{v}_0(\mathbf{x}, t)$ and $\hat{\mathbf{v}}(\mathbf{x}, t)$ are prescribed functions and A is an assigned operator.

The initial-boundary value problem (I.B.V.P.) (1.1)-(1.3) is a mathematical model for the evolution of the state vector $\mathbf{v}$ of the phenomenon $\mathcal{F}$ and therefore it represents the *evolution equation* of $\mathcal{F}$. The space X of vector valued functions defined in Ω and satisfying the prescribed boundary conditions, endowed with an appropriate metric, is called the *state space.* The choice of the metric is the core of the problem and has to be linked to the physics of the phenomenon (see [20] for a detailed discussion).

Let X be the state space of the evolution equation (1.1) endowed with a metric d suitably chosen. As a first indication that the model of the real world phenomenon $\mathcal{F}$ is correct, one requires the *well posedness* in the sense of Hadamard [97]. Hadamard's conditions for a well posed problem are:

i) the existence of a solution;

ii) the uniqueness of the solution;

iii) the continuous dependence of the solution on the data.

The first two conditions require that the I.B.V.P. (1.1)-(1.3) admits one and only one global (in time) solution, that is the solution exists for every finite interval of time. The third condition states that a slight variation of the data for the problem should cause the solution to vary only slightly. Thus since data are generally obtained experimentally and may be subject to numerical approximations, one requires that the solution be stable under small variations in initial and/or boundary data. We shall now make this last requirement clearer by formalizing it in a mathematically rigorous way.

Let $\mathbf{v}(\mathbf{v}_0, t)$ be a global solution to the problem (1.1)-(1.3). $\mathbf{v}$ is a dynamical system according to the following definition [34, 94]

Definition 1.1. *A dynamical system on a metric space X is a map*

$$\mathbf{v} : (\mathbf{v}_0, t) \in X \times \mathbb{R} \mapsto \mathbf{v}(\mathbf{v}_0, t) \in X$$

such that $\mathbf{v}(\mathbf{v}_0, 0) = \mathbf{v}_0$.

Given $\mathbf{v}_0 \in X$, for a dynamical system $\mathbf{v}$, the function

$$\mathbf{v}(\mathbf{v}_0, \cdot) : t \in \mathbb{R} \mapsto \mathbf{v}(\mathbf{v}_0, t) \in X$$

is called a *motion* associated with the initial data $\mathbf{v}_0$. If

$$\mathbf{v}(t) = \mathbf{v}_0 \quad \forall t \in \mathbb{R}^+,$$

the motion is *stationary* or *steady* and $\mathbf{v}_0$ is an *equilibrium* or *critical point.* The set $\{(t, \mathbf{v}(t)) : t \in \mathbb{R}^+\}$ is the *positive graph* of the motion $\mathbf{v}$, and its projection into X, i.e. the subset $\gamma(\mathbf{v}_0) = \{\mathbf{v}(t) : t \in \mathbb{R}^+\}$, is the *positive orbit* or the *trajectory* starting at $\mathbf{v}_0$. A subset $I \subset X$ is *positively invariant* if $\mathbf{v}_0 \in I \Rightarrow \gamma(\mathbf{v}_0) \subset I$. If there exists $T > 0$ such that $\mathbf{v}(t + T) = \mathbf{v}(t)$ $\forall t \in \mathbb{R}$, the motion $\mathbf{v}$ is *periodic in time with period T.*

Returning to the mathematical formalization of the reqirement *iii*) and denoting by $S(\mathbf{x}, r)$ $(r > 0)$ the open ball in the metric space (X, d) of centre $\mathbf{x}$ and radius r,

$$S(\mathbf{x}, r) = \{y \in X : d(\mathbf{x}, \mathbf{y}) < r\},$$

we state the following

Definition 1.2. *A motion* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *depends contionuously on the initial data if and only if*

$$\begin{aligned} \forall T > 0,\ \forall \epsilon > 0\ \exists \delta(\epsilon, T) > 0\ :\ \mathbf{v}_1 \in S(\mathbf{v}_0, \delta) \Rightarrow \\ \mathbf{v}(\mathbf{v}_1, t) \in S(\mathbf{v}(\mathbf{v}_0, t), \epsilon)\ \forall t \in [0, T]. \end{aligned} \tag{1.4}$$

1.2 Ill-posed problems

A problem which is not well posed is said to be *ill posed*. Ill posedness is then due to the lack of one of the requirements *i*), *ii*), *iii*) in the previous section.

Example 1.1 (Lack of existence)**.** Let

$$\sum_{|\alpha| \le k} a_\alpha(x_1, \dots, x_n) \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = F(x_1, \dots, x_n), \tag{1.5}$$

be a partial differential equation in which $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index of length $|\alpha| = \alpha_1 + \dots + \alpha_n$, $k \in \mathbb{N}$, the coefficients a_α and the datum F are assigned analytic functions of the real variables $x_1, \dots, x_n$. Assume that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ at which at least one of the functions a_α, with $|\alpha| = k$, does not vanish. Then, the Cauchy-Kovalewski Theorem ensures the existence of a solution to (1.5) in a neighborhood of $\mathbf{x}_0$. If we weaken the hypothesis of analyticity of the datum by assuming $F \in \mathrm{C}^\infty(\mathbb{R}^n)$, the existence of a classical solution is not guaranteed. To this end we report the celebrated Lewy's example [37]

$$u_x + \mathrm{i} u_y - 2\mathrm{i}(x + \mathrm{i}y) u_t = F(x, y, t) \quad (x, y, t) \in \mathbb{R}^3, \tag{1.6}$$

where i is the imaginary unit. Although the coefficients in equation (1.6) are all analytic functions, Lewy was able to show that there exists $F \in \mathrm{C}^\infty(\mathbb{R}^3)$ such that (1.6) has no C^1 solution anywhere in $\mathbb{R}^3$. Therefore, by adding to (1.6) an initial condition

$$u(x, y, 0) = u_0(x, y) \in \mathrm{C}^1(\mathbb{R}^2),$$

we have an example of ill posed I.V.P. in the state space $\mathrm{C}^1(\mathbb{R}^2)$ as it does not admit a classical solution.

Lack of uniqueness implies ill posedness also because it guarantees that continuous dependence cannot be obtained. In fact, if $\mathbf{v}$ is a dynamical system on a metric space X according to Definition 1.1, the following Theorem holds.

Theorem 1.1. *A motion which is not unique cannot depend contionuously on the initial data.*

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be two motions associated with the same initial data, i.e. $\mathbf{v}(0) = \mathbf{w}(0)$, such that there exists $t^* > 0$ for which $d(\mathbf{v}(t^*), \mathbf{w}(t^*)) = \epsilon^* > 0$. Then, for $T > t^*$ and $0 < \epsilon < \epsilon^*$, (1.4) does not hold. □

Example 1.2 (Lack of uniqueness)**.** We shall now present a counterexample to uniqueness in fluid mechanics. The Navier-Stokes equations

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{b} \\ \\ \operatorname{div} \mathbf{v} = 0 \end{cases} \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \tag{1.7}$$

are a mathematical model describing the motion of an incompressible homogeneous viscous fluid occurring in a fixed region $\Omega \subseteq \mathbb{R}^3$. In (1.7) $\mathbf{x} = (x, y, z) \in \Omega$ is the space variable, $t \in \mathbb{R}^+$ the time, $\mathbf{v}$ the velocity field, p the pressure field divided by the constant density of the fluid, $\nu(> 0)$ the coefficient of kinematic viscosity and $\mathbf{b}$ the body force acting on the fluid. When the fluid adheres completely to the boundary $\partial\Omega$, the initial-boundary condition to append to (1.7) are:

$$\begin{cases} \mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{v}(\mathbf{x}, t) = \mathbf{v}^*(\mathbf{x}, t) & (\mathbf{x}, t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.8}$$

where $\mathbf{v}_0$ and $\mathbf{v}^*$ are prescribed vector functions, $\mathbf{v}_0$ being divergence-free. Let us consider the case

$$\Omega \equiv \mathbb{R}^3, \quad \mathbf{b} = \mathbf{0} \quad \text{and} \quad \mathbf{v}_0 = \mathbf{0}.$$

Then one readily obtains that the I.V.P. (1.7)-(1.8)$_1$ (there is no boundary when $\Omega \equiv \mathbb{R}^3$) admits at least the following three solutions (see [22, 55] for other solutions):

$$\begin{aligned} &\mathbf{v} = \mathbf{0}, && p = p_0(t); \\ &\mathbf{v} = t(y\mathbf{i} + x\mathbf{j}), && p = p_1(t) - xy - \frac{x^2 + y^2}{2} t^2; \\ &\mathbf{v} = \sin t\,\mathbf{i} + \sinh t(z\mathbf{j} + y\mathbf{k}), && p = p_2(t) - x\cos t - \frac{y^2 + z^2}{2} \sinh^2 t - yz\cosh t; \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the x, y and z axes, respectively, and $p_i(t)$ $(i = 0, 1, 2)$ are arbitrary functions.

Example 1.3 (Lack of contionuous dependence)**.** Let us consider the *backward heat equation*

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t < 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

where u_0 is a prescribed $\mathrm{C}^2(\mathbb{R})$ function. Then we take $\mathrm{C}^2(\mathbb{R})$ endowed with the L^∞-norm,

$$\|f\|_\infty = \max_{x\in\mathbb{R}} |f(x)|,$$

as the state space. For $u_0 \equiv 0$, (1.9) admits the zero solution which, as we shall soon show, does not depend continuously on the initial datum. In fact, let

$$u_0 = u_{0n} = \frac{1}{n}\sin(nx) \quad (n \in \mathbb{N});$$

then

$$u_n = \frac{\mathrm{e}^{-n^2 t}}{n}\sin(nx)$$

is the solution to (1.9) and, for all $n \in \mathbb{N}$,

$$\|u_{0n}\|_\infty = \frac{1}{n}, \quad \|u_n\|_\infty = \frac{\mathrm{e}^{n^2|t|}}{n} > 1\ \forall |t| > \frac{1}{2\mathrm{e}}.$$

Therefore, for $\epsilon = 1$ and $T > (2\mathrm{e})^{-1}$, (1.4) does not hold.

1.3 Liapunov stability

The Liapunov stability of a motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ of a dynamical system $\mathbf{v}$ extends the requirements of continuous dependence to the infinite interval of time $(0,+\infty)$.

Definition 1.3. *A motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ is stable in the sense of Liapunov with respect to perturbations in the initial data if and only if*

$$\begin{aligned} &\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0 \ :\ \mathbf{v}_1 \in S(\mathbf{v}_0,\delta) \Rightarrow \\ &\qquad\qquad \mathbf{v}(\mathbf{v}_1,t) \in S(\mathbf{v}(\mathbf{v}_0,t),\epsilon) \quad \forall t \in \mathbb{R}^+. \end{aligned} \tag{1.10}$$

A motion is *unstable* if it is not stable. Obviously (1.10) implies (1.4) and hence, by means of Theorem 1.1, a motion which is stable is also unique.

Definition 1.4. *A motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ is said to be an attractor (or attractive) on a set $Y \subseteq X$ if:*

$$\mathbf{v}_1 \in Y \Rightarrow \lim_{t\to+\infty} d[\mathbf{v}(\mathbf{v}_0,t),\mathbf{v}(\mathbf{v}_1,t)] = 0. \tag{1.11}$$

The biggest set Y on which (1.11) *holds is called the domain of attraction of* $\mathbf{v}(\mathbf{v}_0,\cdot)$.

Definition 1.5. *A motion* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is asymptotically stable if it is stable and there exists* $\delta > 0$ *such that* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is attractive on* $S(\mathbf{v}_0, \delta)$*. In particular* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is exponentially stable if*

$$\exists \delta,\ \lambda(\delta),\ M(\delta) \in \mathbb{R}^+ \ : \ \forall \mathbf{v}_1 \in X,\ d(\mathbf{v}_1, \mathbf{v}_0) < \delta \Rightarrow \\ d[\mathbf{v}(\mathbf{v}_1, t), \mathbf{v}(\mathbf{v}_0, t)] \leq M \mathrm{e}^{-\lambda t} d(\mathbf{v}_1, \mathbf{v}_0) \quad \forall t \in \mathbb{R}^+.$$

If $\delta = +\infty$*, then* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is globally asymptotically (or exponentially) stable.*

Liapunov stability of a set is of fundamental interest expecially in connection with the asymptotic behaviour of motions. In order to formalize this notion we recall the definition of distance between two subsets A, B of a metric space X:

$$d(A, B) = \inf_{x \in A,\, y \in B} d(x, y) \tag{1.12}$$

and denote by $S(A, r)$, $r > 0$, the open set $\{x \in X : d(x, A) < r\}$, where $d(x, A) = d(\{x\}, A)$ according to (1.12).

Definition 1.6. *A set* $A \subset X$ *is Liapunov stable with respect to the dynamical system* $\mathbf{v}$ *if*

$$\forall \epsilon > 0\, \exists\, \delta(\epsilon) > 0 \ : \ \mathbf{v}_0 \in S(A, \delta) \Rightarrow \gamma(\mathbf{v}_0) \subset S(A, \epsilon).$$

A set is *unstable* if it is not stable.

Definition 1.7. *A set* A *is said to be an attractor or attractive on an open set* $B \supset A$ *with respect to the dynamical system* $\mathbf{v}$ *if it is positive invariant and*

$$\mathbf{v}_0 \in B \Rightarrow \lim_{t \to +\infty} d[\mathbf{v}(\mathbf{v}_0, t), A] = 0. \tag{1.13}$$

The largest open set on which (1.13) *holds is the domain of attraction of* A*. If* $B = X$*, then* A *is a global attractor.*

Definition 1.8. *A set* A *is asymptotically stable if it is stable and if there exists* $\delta > 0$ *such that* A *is attractive on* $S(A, \delta)$*. In particular* A *is exponentially stable if*

$$\exists \delta, \lambda(\delta), M(\delta) \in \mathbb{R}^+ : \mathbf{v}_0 \in S(A, \delta) \Rightarrow d[\mathbf{v}(\mathbf{v}_0, t), A] \leq M \mathrm{e}^{-\lambda t} d(\mathbf{v}_0, A) \forall t > 0.$$

If the domain of attraction is the whole space X*, i.e.* $\delta = +\infty$*, then the asymptotic (or exponential) stability is said to be global.*

Remark 1.1. Let X be a normed linear space, $d : (x, y) \in X \times X \mapsto \|x - y\|$ the metric induced by the norm $\| \cdot \|$, $\mathbf{v}$ a dynamical system on X and

$$\mathbf{u}(\mathbf{u}_0, t) = \mathbf{v}(\mathbf{v}_1, t) - \mathbf{v}(\mathbf{v}_0, t) \quad (\mathbf{v}_1 = \mathbf{u}_0 + \mathbf{v}_0)$$

the perturbation at time t to the basic motion $\mathbf{v}(\mathbf{v}_0, \cdot)$. Then (1.10) is equivalent to

$$\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0 \ : \ \|\mathbf{u}_0\| < \delta \Rightarrow \ \|\mathbf{u}(\mathbf{u}_0, t)\| < \epsilon \ \forall t \in \mathbb{R}^+,$$

viz the stability of a given basic motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ may be expressed through the stability of the zero solution of the perturbed dynamical system

$$\mathbf{u} : (\mathbf{u}_0, t) \in X \times \mathbb{R}^+ \mapsto \mathbf{v}(\mathbf{v}_0 + \mathbf{u}_0, t) - \mathbf{v}(\mathbf{v}_0, t).$$

Remark 1.2. On a set X, a functional

$$\rho : X \times X \to \mathbb{R}$$

is a positive-definite function if it satisfies

a) $\rho(x, y) \geq 0 \quad \forall x, y \in X$,

b) $\rho(x, y) = 0 \Leftrightarrow x = y$.

A metric is obviously a positive-definite function but the converse is true when additionally there holds

1) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$,

2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in X$.

Furthermore we note that a positive-definite function does not define a topology. Nevertheless we define *open ball* in X with centre x and radius $r(> 0)$ the set

$$S_\rho(x, r) = \{y \in X : \rho(x, y) < r\}.$$

Sometimes a positive-definite function is chosen as a measure of the perturbations [35].

1.4 Topology dependent stability

In the applications the state space X is often a normed linear space $(X, \|\cdot\|)$ and a metric d is induced by the norm $\|\cdot\|$ as in Remark 1.1. It is well known that two equivalent norms induce two equivalent metrics and then the same topology [88]. Therefore stability (resp. instability) with respect to a fixed norm implies stability (resp. instability) with respect to an equivalent one. But only on a linear finite dimensional space all norms are equivalent [40] and consequently stability does not depend on the chosen norm. On an infinite dimensional space instead, it can turn out that a solution is stable with a choice of the metric and unstable with another one.

Example 1.4 (Fichera [19]). Let us consider the Cauchy problem

$$\begin{cases} u_t = \left(\frac{2}{t} - 6t^5x^2\right) u(x,t) & (x,t) \in [-1,1] \times [1,+\infty[\\ u(x,1) = f(x) & x \in [-1,1] \end{cases} \tag{1.14}$$

which, for $f \equiv 0$, admits the trivial solution $u \equiv 0$. By taking $\mathrm{C}^0[-1,1]$ as the state space and considering on it both the $\mathrm{L}^1[-1,1]$-norm,

$$\|w\|_1 = \int_{-1}^{1} |w(x)| \mathrm{d}x,$$

and the $\mathrm{L}^\infty[-1,1]$-norm, we shall show that the zero solution is stable with respect to the $\mathrm{L}^1[-1,1]$-norm and unstable with respect to the $\mathrm{L}^\infty[-1,1]$-norm. It is easy to check that

$$u(x,t) = f(x)t^2 \mathrm{e}^{-(t^6-1)x^2} \tag{1.15}$$

is the solution to (1.14) and thus

$$\begin{aligned} \|u(\cdot,t)\|_1 &= \int_{-1}^{1} |u(x,t)| \mathrm{d}x = \int_{-1}^{1} |f(x)| t^2 \mathrm{e}^{-(t^6-1)x^2} \mathrm{d}x \\ &\le \|f\|_\infty t^2 \int_{-1}^{1} \mathrm{e}^{-(t^6-1)x^2} \mathrm{d}x = \|f\|_\infty \frac{2t^2}{\sqrt{t^6-1}} \int_0^{\sqrt{t^6-1}} \mathrm{e}^{-\xi^2} \mathrm{d}\xi \\ &< \|f\|_\infty \frac{t^2}{\sqrt{t^6-1}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\xi^2} \mathrm{d}\xi = \|f\|_\infty t^2 \sqrt{\frac{\pi}{t^6-1}} \to 0 \quad \text{as } t \to +\infty \end{aligned}$$

which implies the stability of the zero solution with respect to the $\mathrm{L}^1[-1,1]$-norm. On the other hand, if we choose as perturbation to the initial datum

$$f(x) = f_n(x) = \frac{\mathrm{e}^{-x^2}}{n} \quad (n \in \mathbb{N}),$$

the solution (1.15) to (1.14) becomes

$$u_n(x,t) = \frac{t^2}{n} \mathrm{e}^{-t^6x^2}.$$

Because of

$$\|f_n\|_\infty = \frac{1}{n}$$

the data tend to zero as $n \to +\infty$, while

$$\|u_n\|_\infty = \max_{x \in [-1,1]} \left| \frac{t^2}{n} \mathrm{e}^{-t^6x^2} \right| = \frac{t^2}{n} \to +\infty \quad \text{as } t \to +\infty \quad \forall n \in \mathbb{N},$$

by which the instability of the zero solution with respect to the $\mathrm{L}^\infty[-1,1]$-norm follows.

Example 1.5 (Hadamard)**.** We consider the initial value problem for the Laplace equation

$$\begin{cases} u_{tt} + u_{xx} = 0 & x \in [0,1],\ t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = u^*(x), \end{cases} \tag{1.16}$$

with u^* a prescribed $\mathrm{C}^2[0,1]$ function vanishing at $x=0$ and $x=1$, and we take the linear space

$$X = \{f \in \mathrm{C}^2[0,1] : f(0) = f(1) = 0\},$$

endowed with the $\mathrm{L}^\infty[0,1]$-norm, as the state space.

For $u^* = 0$ the I.V.P. (1.16) admits the trivial solution. This solution is unstable with respect to the $\mathrm{L}^\infty[0,1]$-norm. In fact, by choosing

$$u_n^* = \frac{1}{n}\sin(n\pi x) \quad (n \in \mathbb{N})$$

as perturbation to the initial datum, the explicit solution to (1.16) is

$$u_n(x,t) = \frac{1}{n^2\pi}\sinh(n\pi t)\sin(n\pi x).$$

Thus, since

$$\|u_n^*\|_\infty = \frac{1}{n}$$

and

$$\|u_n\|_\infty = \frac{\sinh(n\pi t)}{n^2\pi},$$

it follows that, as $n \to +\infty$, the data tend uniformly to zero while $\|u_n\|_\infty$ tends to $+\infty$.

Recalling that the set of functions $\{\sin(k\pi x)\}_{k\in\mathbb{N}}$ is complete in the state space X under the $\mathrm{L}^\infty[0,1]$-norm, let $u^* \in X$,

$$u^*(x) = \sum_{k=1}^{+\infty} a_k \sin(k\pi x) \quad \text{with} \quad a_k = 2\int_0^1 u^*(x)\sin(k\pi x)\mathrm{d}x.$$

Then

$$u(x,t) = \sum_{k=1}^{+\infty} \frac{a_k}{k\pi}\sinh(k\pi t)\sin(k\pi x) \tag{1.17}$$

is the explicit solution to (1.16). Furthermore, we observe that, along the solutions (1.17), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\mathrm{d}x = -\int_0^1 u_t u_{xx}\mathrm{d}x = -[u_t u_x]_0^1 + \int_0^1 u_x u_{tx}\mathrm{d}x,$$

and hence, since u_t vanishes at $x = 0$ and $x = 1$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x = 0,$$

by which

$$\int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x = \text{constant} = \int_0^1 \left(u_t^2 - u_x^2\right)_{t=0} \mathrm{d}x = \int_0^1 u^{*2}(x)\mathrm{d}x.$$

Therefore on the class of solutions (1.17), $1/2 \int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x$ can be assumed as measure of the perturbations (see Remark 1.2) and the stability of the trivial solution is then recovered.

1.5 Normal modes analysis

We consider a basic steady solution $\mathbf{v}$ to the I.B.V.P. (1.1)-(1.3) and let $\mathbf{u}$ denote a perturbation to $\mathbf{v}$. The altered motion $\mathbf{v} + \mathbf{u}$ must satisfy the evolution equation (1.1), the same boundary conditions as $\mathbf{v}$, and the initial condition

$$\mathbf{v}(\mathbf{x}, 0) + \mathbf{u}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) + \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega.$$

Thus, for any $T > 0$, the disturbance $\mathbf{u}$ fulfils the *evolution equation of the perturbation*

$$\begin{cases} \mathbf{u}_t = \mathbf{G} & \text{in } \Omega \times (0, T) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega \\ A[\mathbf{u} + \mathbf{v}, \nabla(\mathbf{u} + \mathbf{v})] = \mathbf{0} & \text{on } \partial\Omega \times [0, T] \end{cases} \tag{1.18}$$

where

$$\begin{aligned} \mathbf{G} &= \mathbf{G}\left(\mathbf{x}, t, \mathbf{v}, \mathbf{u}, \frac{\partial v_i}{\partial x_r}, \frac{\partial u_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, \frac{\partial^2 u_j}{\partial x_r \partial x_s}, ...\right) \\ &= \mathbf{F}\left[\mathbf{x}, t, \mathbf{v} + \mathbf{u}, \frac{\partial(v_i + u_i)}{\partial x_r}, \frac{\partial^2(v_j + u_j)}{\partial x_r \partial x_s}, ...\right] - \mathbf{F}\left[\mathbf{x}, t, \mathbf{v}, \frac{\partial v_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, ...\right]. \end{aligned}$$

We now assume that the evolution equations of the perturbations may be linearized for sufficiently small disturbances. The linearization of (1.18) is straightforward in principle and in practise: all products and powers (higher than the first) of the increments are neglected while only the terms which are linear in them are retained. Thereby a linear homogeneous system of partial differential equations is obtained. These have coefficients that may vary in space but not in time because the basic motion is steady. Experience with the method of separation of variables and the Laplace transforms suggests that, in general, the solutions of the linearized system can be expressed as the real parts of integrals of components, each component varying with time like e^{ct} for some complex number $c = c_r + ic_i$. The linear system will determine the values of c and the spacial variation of corresponding components as eigenvalues and eigenfunctions.

If the basic motion has some simple simmetry, the linear system may be transformed with respect to some of the space variables as well as the time. For example, consider a mechanical system confined between two parallel planes and in which the physical variables in the stationary motion are functions of the coordinate normal to the planes (say z). In this case the Laplace transform of the system with respect to t, the Fourier transforms with respect to x and y may be taken to express the perturbation $\mathbf{u}$ in the form

$$\mathbf{u}(x, y, z, t) = \mathrm{Re} \int_{-\infty}^{+\infty} \mathrm{d}a_x \int_{-\infty}^{+\infty} \mathrm{d}a_y \int_L \hat{\mathbf{u}}(z) \exp[ct + \mathrm{i}(a_x x + a_y y)] \mathrm{d}c$$

where L is the path for the inversion of the Laplace transform. $\hat{\mathbf{u}}$ is to be found from the initial data and the transformed system of ordinary differential equations in z and of the boundary conditions. This system gives an *eigenvalue relation* of the form

$$\mathcal{G}(c, a_x, a_y, X_1, X_2, ..., X_m) = 0$$

which involves the *complex wave speed* c, the *wave numbers* a_x and a_y along the directions x and y, respectively, and other $m(\in \mathbb{N})$ non-dimensional numbers X_i $(i = 1, 2, ..., m)$ related to the basic stationary motion, and further it yields the eigenfunctions $\hat{\mathbf{u}}$ except for an arbitrary function of c, a_x and a_y that may be specified by the initial conditions. This is the method of *normal modes*, whereby small disturbances are expanded in terms of a complete set of modes, which may be treated separately because each satisfies the linear system.

If $c_r > 0$ for a mode, then the corresponding disturbance will be amplified, growing exponentially until it is so large that nonlinearities in the evolution equation of the perturbation become significant. If $c_r = 0$ the mode is said to be *neutrally stable* since the corresponding disturbance will remain small for all time $t > 0$. Finally if $c_r < 0$ the mode is said *strictly stable* or *stable* and the magnitude of corresponding disturbance will tend exponentially to zero as $t \to +\infty$. A small disturbance of the basic motion will in general excite all modes, so that if $c_r > 0$ for at least one mode then the motion is *linearly unstable*. Conversely, if $c_r \leq 0$ for all a complete set of modes then the flow is *linearly stable*. A mode is *marginally stable* if $c_r = 0$ for critical values of the parameters on which the eigenvalue c depends but

$c_r > 0$ for some neighbouring values of the parameters. By definition a marginal stable mode is neutral stable but the converse is not true since, for a neutral stable mode, c_r is not necessarily positive for any neighbouring values of the parameters. The values of the parameters for marginal stability are often sought to give a criterion of stability. The critical relationship among the parameters is the equation of *marginal curve* (or *surface*).

Marginal stable modes can be one of two kinds. The two kinds correspond to the two ways in which the amplitudes of a small perturbation can grow or be damped: they can grow (or be damped) aperiodically; or they can grow (or be damped) by oscillations of increasing (or decreasing) amplitude. In the former case $c = 0$ at marginal stability, i.e. $c_r = c_i = 0$, and the transition from stability to instability takes place via a marginal state exhibiting a stationary pattern of motions. In the latter case $c_i \neq 0$ at marginal stability and the transition takes place via a marginal state exhibiting oscillatory motions with a certain definite characteristic frequency.

If $c_{\mathrm{i}} \neq 0 \Rightarrow c_r < 0$ it then is said that the *Principle of exchange of stabilities* holds. Therefore at the onset of instability a stationary pattern of motion prevails and instability sets in as *steady secondary flow.* Of course if $c_{\mathrm{i}} = 0$ always, i.e. $c \in \mathbb{R}$, then exchange of stabilities always holds such as in the case of the convection cells that arise in a fluid heated from below [11, 17, 87]. On the other hand if $c_{\mathrm{i}} \neq 0$ and exchange of stabilities does not hold, then at the onset of instability oscillatory motions prevail, then one says, according to a definition due to Eddington [18], that one has a case of *overstability.*

1.6 Fundamental topics of nonlinear stability

In this section we shall illustrate some elementary concepts of the theories by Landau and by Hopf by means of the following three examples.

Example 1.6 (Supercritical stability)**.** Let us consider the boundary value problem

$$\begin{cases} u_t - u + u^3 = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } u = 0 \text{ and } \pi, \end{cases} \tag{1.19}$$

where $R \in \mathbb{R}^+$. We consider the basic steady solution $\bar{u} \equiv 0$. Linearizing the perturbations to the trivial solution, we find

$$\begin{cases} u_t - u = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } x = 0 \text{ and } \pi, \end{cases}$$

whose solution can easily be represented as sum of the normal modes,

$$u = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{s_n t} \sin(nx),$$

where

$$s_n = 1 - n^2/R \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

and hence the principle of exchange of stability holds. The n-th mode is stable if and only if $R \leq n^2$. Therefore the zero solution is linearly stable if and only if all the modes are stable, i.e.

$$R \leq R_c = \min_{n\in\mathbb{N}} n^2 = 1.$$

Next we examine the nonlinear stability when the parameter R is nearly critical

$$R = R_c + \epsilon = 1 + \epsilon, \quad 0 < |\epsilon| \ll 1.$$

In particular, if R is just supercritical ($\epsilon > 0$), then all the normal modes except the first decay exponentially in time, so it is plausible to ignore the higher modes in the linearized initial-value problem. Accordingly, we approximate the linearized solution by

$$u \cong A_1 \mathrm{e}^{s_1 t} \sin x. \tag{1.20}$$

This solution grows very slowly so that, however small the disturbance is initially, it will cease to be small only after a long time of the order of $-(\ln A_1)/s_1 \sim -(\ln A_1)/\epsilon$ as $R \downarrow 1$. By this time nonlinearity will have modified the exponential growth and the solution given by (1.20) will have become invalid. To approximate the solution uniformly over so long a time, we anticipate that the nonlinear solution satisfies

$$u \sim u_1 \equiv A(t) \sin x \quad \text{as } R \to 1, \ A \to 0 \tag{1.21}$$

for all time, where the amplitude equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = a_1 A + a_2 A^2 + a_3 A^3 + \cdots . \tag{1.22}$$

Moreover, we assume that the exact solution can be expanded as

$$u = u_1 + u_2 + u_3 + \cdots , \tag{1.23}$$

where the fundamental mode at marginal stability is given by equation (1.21) and

$$u_r = O(A^r) \quad \text{as } A \to 0 \text{ for } r = 2, 3, \ldots .$$

To find the expansions (1.22) and (1.23) by iteration, we first transfer the small terms of equation $(1.19)_1$ to the right hand side, writing

$$u_{xx} + u = u_t + u^3 + \frac{\epsilon}{1+\epsilon} u_{xx}. \tag{1.24}$$

Note that each of the three terms on the right-hand side is small, the first because the disturbance varies slowly, the second because the nonlinearity is weak, and the third because the parameter R is nearly critical. Note also that the linear operator $\mathrm{L} = \partial^2/\partial x^2 + 1$ associated with the left-hand side in (1.24) is such that $\mathrm{L}u_1 = 0$, where u_1 is the most unstable mode given by equation (1.21) at marginal stability.

Checking the first approximation, we equate all terms of order A in equation (1.24) to find

$$0 = \mathrm{L}u_1 = \left(a_1 - \frac{\epsilon}{1+\epsilon}\right) A \sin x.$$

Therefore we identify $a_1 = s_1 \sim \epsilon$ as $R \to 1$, in agreement with the linear theory.

For the next approximation, we equate terms of order A^2 in equation (1.24) to find in the limit as $R \to 1$ that

$$\mathrm{L}u_2 = a_2 A^2 \sin x. \tag{1.25}$$

Similarly, the boundary conditions $(1.19)_2$ give

$$u_2 = 0 \quad \text{at } x = 0, \pi. \tag{1.26}$$

If the solution u_2 of the linear inhomogeneous problem (1.25)-(1.26) exists, we may multiply (1.25) by u_1, integrate from 0 to π and deduce

$$a_2 A^3 \int_0^\pi \sin^2 x \mathrm{d}x = \int_0^\pi (\mathrm{L}u_2) u_1 \mathrm{d}x = \int_0^\pi u_2 (\mathrm{L}u_1) \mathrm{d}x = 0$$

on integration by parts, and use of the boundary conditions (1.26) and of $\mathrm{L}u_1 = 0$. Therefore

$$a_2 = 0.$$

This is called the *solvability condition* of equations (1.25) and (1.26), it being necessary for the existence of the solution u_2.

We now go back to solve equations (1.25) and (1.26), seeing trivially that

$$u_2 = 0, \tag{1.27}$$

i.e. that the second harmonic happens not to be excited. Of course any multiple of u_1 could be added to this solution u_2, but such an addition could be transferred to the fundamental solution u_1 by re-definition of the amplitude A. So we may take the solution (1.27) without loss of generality. This choice of normalization can be systematized by imposition of the orthogonality condition,

$$\int_0^\pi u_1(u-u_1)\mathrm{d}x = 0. \tag{1.28}$$

For the next approximation, we equate terms of order A^3 in equations (1.24) and $(1.19)_2$, finding in the limit as $R \to 1$ that

$$\mathrm{L}u_3 = (a_3 \sin x + \sin^3 x)A^3 = \left[\left(a_3 + \frac{3}{4}\right)\sin x - \frac{1}{4}\sin(3x)\right]A^3 \tag{1.29}$$

and

$$u_3 = 0 \quad \text{at } z = 0, \pi. \tag{1.30}$$

Multiplying (1.29) by u_1, integrating from 0 to π, etc., we get the solvability condition

$$a_3 = -\frac{3}{4}.$$

Then one may go back to equations (1.29) and (1.30) and show that their solution is

$$u_3 = \frac{A^3}{32}\sin(3x).$$

Although one could go on and find a_4, u_4, a_5, etc. in turn[1] we stop the iteration here, having found the *Landau equation* to the cubic approximation for $0 < |\epsilon| \ll 1$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \epsilon A - \frac{3}{4}A^3 \tag{1.31}$$

whose explicit general solution is

$$A^2 = \frac{4\epsilon A_0^2}{(4\epsilon - 3A_0^2)\mathrm{e}^{-2\epsilon t} + 3A_0^2}, \tag{1.32}$$

where A_0 is the amplitude at $t = 0$.

First we consider the case $\epsilon < 0$, i.e. R is just subcritical. Then equation (1.32) confirms that the disturbance decays in accord with the linear theory, i.e. $|A| \sim A_0\mathrm{e}^{\epsilon t}$ as $t \to +\infty$ and $A_0 \to 0$. In this case the term $-(3A^3)/4$ in equation (1.31) due to the nonlinearity remains small if it is initially small.

[1] It is easy to show that

$$a_n = 0 \quad \text{and} \quad u_n = 0 \quad \text{for } n = 4, 6, 8, \ldots.$$

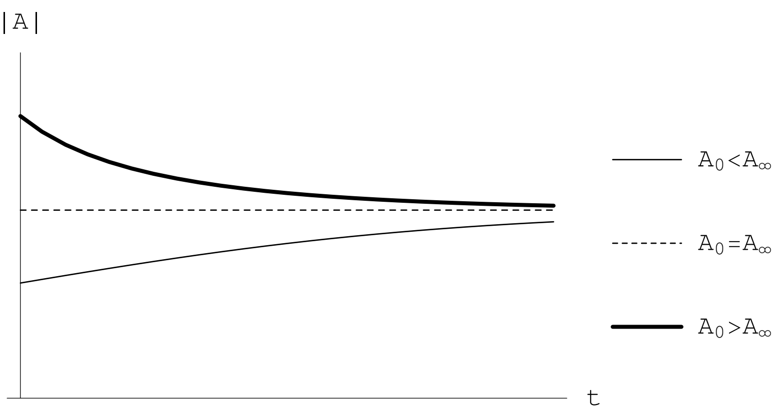

Figure 1.1: Supercritical stability for $0 < \epsilon \ll 1$: the development of $|A|$ as a function of time for two initial values A_0.

If $\epsilon > 0$, i.e. R is just supercritical, it is easy to check that (1.31) admits the stationary solution $|A| = A_\infty = 2\sqrt{\epsilon/3}$ by which we may rewrite (1.32) as

$$A^2 = \frac{A_\infty^2}{\left(\dfrac{A_\infty^2}{A_0^2} - 1\right) \mathrm{e}^{-2\epsilon t} + 1}$$

and deduce that

$$|A| \to A_\infty \quad \text{as } t \to +\infty,$$

whatever the value of A_0. This is called *supercritical stability*, the basic solution $\bar{u} \equiv 0$ being linearly unstable for $R > 1$ but settling down as a new steady solution which is, moreover, indipendent of the initial conditions. The development of $|A|$ with time is sketched in Figure 1.1 and the dipendence of the steady solutions $|A| = 0$, $|A| = A_\infty$ upon R in Figure 1.2. The branching of the curve of the equilibrium solutions at $R = R_c = 1$ is called a *bifurcation*.

Example 1.7 (Subcritical instability)**.** We now consider the boundary value problem

$$\begin{cases} u_t - u - u^3 = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } u = 0 \text{ and } \pi, \end{cases}$$

which differs from (1.19) only for the sign of the cubic term. By following the previous arguments, the Landau equation to the cubic approximation for $0 < |\epsilon| \ll 1$ is now given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \epsilon A + \frac{3}{4} A^3, \tag{1.33}$$

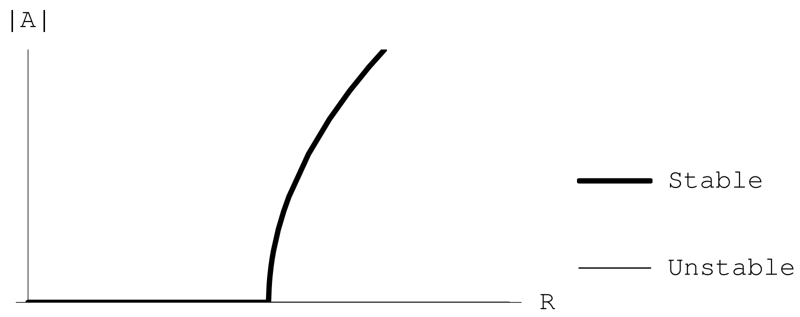

Figure 1.2: The bifurcation curve: the amplitude of the equilibrium solution as a function of R.

whose general solution is

$$A^2 = \frac{4\epsilon A_0^2}{(4\epsilon + 3A_0^2)\mathrm{e}^{-2\epsilon t} - 3A_0^2}, \tag{1.34}$$

where, as before, A_0 is the initial amplitude.

For $\epsilon > 0$ the solution (1.34) breaks down after a finite time, $|A|$ becoming infinite at $t = (2\epsilon)^{-1}\ln[1 + 4\epsilon/(3A_0^2)]$ strengthening the predictions of the linear theory.

For $\epsilon < 0$ (1.33) admits the steady solution $|A| = A_\infty = 2\sqrt{-\epsilon/3}$ and (1.34) may be rewritten as

$$A^2 = \frac{A_\infty^2}{\left(\dfrac{A_\infty^2}{A_0^2} - 1\right)\mathrm{e}^{-2\epsilon t} + 1}$$

by which we readily deduce that

- if $0 < A_0 < A_\infty$, then $|A| \to 0 \quad \text{as } t \to +\infty$;
- if $A_0 = A_\infty$, then $|A| = A_\infty \ \forall t \geq 0$;
- if $A_0 > A_\infty$, then $|A| \to +\infty \quad \text{as } t \to \dfrac{1}{2\epsilon}\ln\left(1 - \dfrac{A_\infty^2}{A_0^2}\right)$.

This case is called *subcritical instability*, because instability occurs with finite amplitude $|A_0| > A_\infty$ when all the infinitesimal disturbances are stable; it is also called *metastability* by physicists. The development of $|A|$ as a function of time is shown in Figure 1.3 and the equilibrium solutions $|A|$ as functions of R in Figure 1.4.

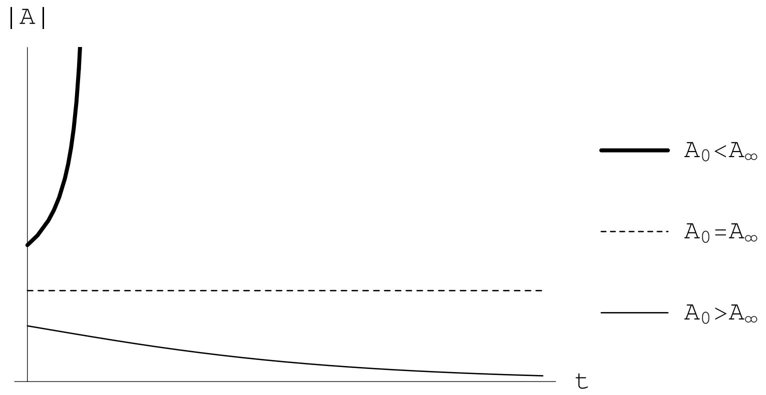

Figure 1.3: Subcritical instability for $0 < -\epsilon \ll 1$: the development of $|A|$ as a function of time for two initial values A_0.

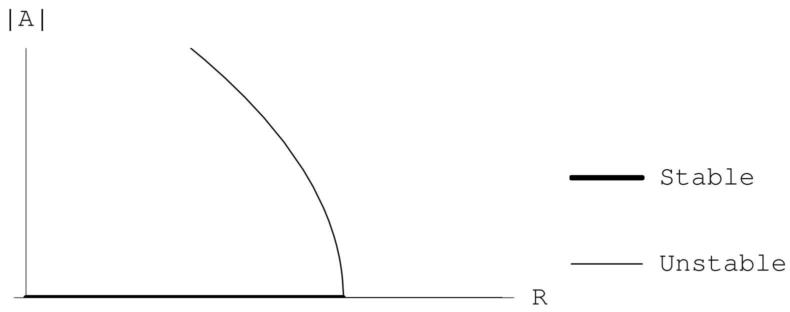

Figure 1.4: The amplitude of the equilibrium solution as a function of R.

Example 1.8 (Hopf bifurcation)**.** We shall now describe the Hopf bifurcation whereby a periodic, rather than a steady, solution may bifurcate at the margin of stability of a steady basic solution. We take as an example the simple system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (R - x^2 - y^2)x - y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + (R - x^2 - y^2)y, \end{cases} \tag{1.35}$$

where R is a real parameter.

The linear stability of the null solution $x = y = 0$ is governed by the linearized system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Rx - y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + Ry. \end{cases} \tag{1.36}$$

Next we find solutions to (1.36) of the type

$$\begin{cases} x = \gamma_1 \mathrm{e}^{ct} \\ y = \gamma_2 \mathrm{e}^{ct} \end{cases} \tag{1.37}$$

with $\gamma_1,\ \gamma_2 \in \mathbb{R}$ and $c = c_r + \mathrm{i}c_i \in \mathbb{C}$. Putting (1.37) into (1.36) yields the following linear algebraic system in the unknowns $\gamma_1,\ \gamma_2$

$$\begin{cases} (c - R)\gamma_1 + \gamma_2 = 0 \\ -\gamma_1 + (c - R)\gamma_2 = 0. \end{cases}$$

For a non-zero solution to this system, we require that

$$(c - R)^2 + 1 = 0$$

by which we get the following eigenvalue relation

$$c = R \pm \mathrm{i}.$$

Then the null solution is linearly stable if $R \leq 0$, unstable if $R > 0$. As R increases through the critical value $R_c = 0$ the real part of the two complex conjugate eigenvalues increases through zero. To examine the nonlinear stability of the trivial solution it seems easiest to use the polar coordinates r and θ, by which $x = r\cos\theta$ and $y = r\sin\theta$, because then the system may be solved explicitly. In fact, equations (1.35) become

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = r(R - r^2) \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 1. \end{cases} \tag{1.38}$$

The first of these equation is a Landau equation (see equation (1.31)), and one may regard x and y as the real and the imaginary part of a complex amplitude $A = r\mathrm{e}^{\mathrm{i}\theta}$. The general solution of system (1.38) is

$$\begin{cases} r^2 = \dfrac{Rr_0^2}{r_0^2 + (R - r_0^2)\mathrm{e}^{-2Rt}} \\ \theta = t + \theta_0 \end{cases} \tag{1.39}$$

where r_0 and θ_0 are the initial values. For $R \leq 0$ all solutions tend to the trivial one as $t \to +\infty$, whatever the initial conditions are, viz the null solution is a global attractor. For $R > 0$, from $(1.39)_1$ we deduce that, if $r_0 = 0$, then $r = 0$ for all $t > 0$, whereas

$$r \to \sqrt{R} \quad \text{as} \quad t \to +\infty \tag{1.40}$$

for $r_0 \neq 0$. Therefore, for $R > 0$, there exists the stable solution

$$\begin{cases} x = \sqrt{R}\cos(t + \theta_0) \\ y = \sqrt{R}\sin(t + \theta_0) \left(= \sqrt{R}\cos(t + \theta_0 - \pi/2)\right) \end{cases} \tag{1.41}$$

as well as the unstable null solution. Note that the stable solution (1.41) depends on initial data only through the phase of the complex amplitude A. This is due to the fact that, in this case, the principle of exchange of stabilities does not hold. Finally, from (1.39) and (1.40), for $R > 0$ (x, y) tends to the circle $r = \sqrt{R}$ as $t \to +\infty$ if it initially is any point other than the origin, that is the domain of attraction of the circle $r = \sqrt{R}$ is the whole plane except the origin.

1.7 Liapunov direct method

In 1893 Liapunov introcuced the so-called *direct method* [38] in order to establish conditions ensuring stability of solutions of O.D.E.s and, only in the second half of the fifties, it was generalized to P.D.E.s by Movchan [51, 52]. This approach requires no explicit knowledge of the solutions, but instead it uses an auxiliary function.

Definition 1.9. *Let* $\mathbf{v}$ *a dynamical system on a metric space* X*. A functional* $V : X \to \mathbb{R}$ *is a Liapunov function on a subset* $I \subset X$ *if*

a) V *is continuous on* I*,*

b) $\forall \mathbf{v}_0 \in I \;:\; V[\mathbf{v}(\mathbf{v}_0, \cdot)]$ *is a non-increasing function of time.*

As observed in Remark 1.1 the stability of a motion on a normed linear space may be expressed through the stability of the zero solution of the perturbed dynamical system. For this reason, one can employ the direct method to investigate the stability of an equilibrium point. Assuming X a normed linear space, denoting by $\mathcal{F}_r$, $r > 0$, the set

$$\mathcal{F}_r = \left\{ f \in \mathrm{C}^0([0,r)) : f(0) = 0,\ f \text{strictly increasing} \right\},$$

and by E_α, $\alpha \in \mathbb{R}$, the set

$$E_\alpha = \{\mathbf{x} \in X : V(\mathbf{x}) < \alpha\},$$

then the Liapunov direct method can be summarized by the following two Theorems.

Theorem 1.2. *Let $\mathbf{u}$ be a dynamical system on a normed linear space X and let $\mathbf{0}$ be an equilibrium point. If V is a Liapunov function on the open ball $S(\mathbf{0}, r)$, for some $r > 0$, such that:*

i) $V(\mathbf{0}) = 0$,

ii) $\exists f \in \mathcal{F}_r \ : \ V(x) \geq f(\|x\|) \ \forall \mathbf{x} \in S(\mathbf{0}, r)$,

then $\mathbf{0}$ is stable. If, in addition,

iii) $\forall \mathbf{x} \in S(\mathbf{0}, r)$ $V[\mathbf{u}(\mathbf{x}, \cdot)]$ *is differentiable with respect to time,*

iv) $\exists g \in \mathcal{F}_r : \dot{V}[\mathbf{u}(\mathbf{x}, t)] \leq -g(\|\mathbf{u}(\mathbf{x}, t)\|) \ \forall \mathbf{x} \in S(\mathbf{0}, r),\ \forall t \in \mathbb{R}^+$,

then $\mathbf{0}$ is asymptotically stable.

Proof. Let us consider $0 < \epsilon < r$ and introduce $\alpha \in \mathbb{R}$,

$$0 < \alpha < f(\epsilon) \leq \inf_{\|\mathbf{x}\| = \epsilon} V(\mathbf{x}).$$

By *ii*) and the continuity of V on $S(\mathbf{0}, r)$ we readily deduce that E_α is an open positive invariant subset of the open ball $S(\mathbf{0}, \epsilon)$. The stability is then immediately obtained observing that, by the assumption *i*) and the continuity of V in $\mathbf{0}$, there exists $\delta(\epsilon) > 0$ such that $S(\mathbf{0}, \delta) \subset E_\alpha$ and so $\mathbf{u}_0 \in S(\mathbf{0}, \delta) \Rightarrow \gamma(\mathbf{u}_0) \subset E_\alpha \subset S(\mathbf{0}, \epsilon)$.

Concerning the asymptotic stability, choosing $\mathbf{u}_0 \in S(\mathbf{0}, \delta)$, by *ii*)-*iv*) it follows that

$$\begin{aligned} 0 \leq f(\|\mathbf{u}(\mathbf{u}_0, t)\|) \leq V[\mathbf{u}(\mathbf{u}_0, t)] &= V(\mathbf{u}_0) + \int_0^t \dot{V}[\mathbf{u}(\mathbf{u}_0, \tau)] \mathrm{d}\tau \\ &\leq V(\mathbf{u}_0) - \int_0^t g(\|\mathbf{u}(\mathbf{u}_0, \tau)\|) \mathrm{d}\tau \quad \forall t \in \mathbb{R}^+. \end{aligned} \tag{1.42}$$

Since $V[\mathbf{u}(\mathbf{u}_0,\cdot)] : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded non-increasing function, there exists $\beta \in \mathbb{R}$ such that

$$0 \le \inf_{t\ge 0} V[\mathbf{u}(\mathbf{u}_0,t)] = \beta \le V(\mathbf{u}_0) < \alpha.$$

But $\beta > 0$ implies $\gamma(\mathbf{u}_0) \cap E_\beta = \emptyset$ and hence, since E_β is an open set, there exists $r^* > 0$ such that $S(\mathbf{0}, r^*) \subset E_\beta$ and

$$\|\mathbf{u}(\mathbf{u}_0,t)\| \ge r^*, \quad g(r^*) \le g(\|\mathbf{u}(\mathbf{u}_0,t)\|) \ \forall t \ge 0.$$

Consequently (1.42) gives

$$0 < V[\mathbf{u}(\mathbf{u}_0,t)] \le V(\mathbf{u}_0) - \int_0^t g(r^*)\mathrm{d}\tau = V(\mathbf{u}_0) - tg(r^*) < 0$$

for $t > V(\mathbf{u}_0)/g(r^*)$, which is impossible. Therefore $\beta = 0$ and the asymptotic stability is then achieved. □

Theorem 1.3. *Let* $\mathbf{u}$ *be a dynamical system on the normed linear space* X *and* $\mathbf{0}$ *be an equilibrium point. If* V *is a Liapunov function on the open ball* $S(\mathbf{0}, r)$*, for some* $r > 0$*, and*

i) $V(\mathbf{0}) = 0$,

ii) $\forall \epsilon \in]0, r]$ *the open set* $A_\epsilon = S(\mathbf{0}, \epsilon) \cap E_0$ *is non-empty,*

iii) $\forall \mathbf{x} \in A_r$ $V[\mathbf{u}(\mathbf{x},\cdot)]$ *is differentiable with respect to time,*

iv) $\exists g \in \mathcal{F}_{\bar r}$, $\bar r > -\inf_{A_r} V$*, such that*

$$\dot V[\mathbf{u}(\mathbf{x},t)] \le -g[-V[\mathbf{u}(\mathbf{x},t)]] \quad \forall \mathbf{x} \in A_r,\ \forall t \in \mathbb{R}^+,$$

then $\mathbf{0}$ *is unstable.*

Proof. Because of i) and the continuity of V in $\mathbf{0}$, there exists $\epsilon \in]0, r]$ such that $\mathbf{x} \in S(\mathbf{0}, \epsilon) \Rightarrow V(\mathbf{x}) > -1$. The equilibrium point $\mathbf{0}$ cannot be stable because, otherwise, there would exist $\delta \in]0, \epsilon]$ such that $\mathbf{u}_0 \in S(\mathbf{0}, \delta) \Rightarrow \gamma(\mathbf{u}_0) \subset S(\mathbf{0}, \epsilon)$ and hence, taking into account ii), $\mathbf{u}_0 \in A_\delta \Rightarrow \gamma(\mathbf{u}_0) \subset A_\epsilon$. Next, chosen $\mathbf{u}_0 \in A_\delta$, iii) and iv) give

$$\begin{aligned}
-1 < V[\mathbf{u}(\mathbf{u}_0,t)] &= V(\mathbf{u}_0) + \int_0^t \dot V[\mathbf{u}(\mathbf{u}_0,\tau)]\mathrm{d}\tau \\
&\le V(\mathbf{u}_0) - \int_0^t g[-V(\mathbf{u}(\mathbf{u}_0,\tau))]\mathrm{d}\tau \\
&\le V(\mathbf{u}_0) - \int_0^t g[-V(\mathbf{u}_0)]\mathrm{d}\tau = V(\mathbf{u}_0) - tg[-V(\mathbf{u}_0)] < -1
\end{aligned}$$

for $t > \dfrac{1 + V(\mathbf{u}_0)}{g[-V(\mathbf{u}_0)]}$, which is impossible. Therefore $\mathbf{0}$ is unstable. □

Remark 1.3. Let $\mathbf{u}$ be a dynamical system on a normed linear space X and $\mathbf{0}$ be an equilibrium point. If V is a Liapunov function on $S(\mathbf{0}, r)$ and satisfies

$$V(\mathbf{0}) = 0, \quad V(\mathbf{x}) > 0 \ \forall \mathbf{x} \neq \mathbf{0},$$

then $\mathbf{0}$ is stable with respect to the measure V of the perturbation (see Remark 1.2). Moreover, if there exists $c > 0$ such that

$$\dot{V} \leq -cV$$

along the motions with initial data in $S(\mathbf{0}, r)$, then $\mathbf{0}$ is asymptotically exponentially stable with respect to the measure V of pertubation according to the following inequality

$$V[\mathbf{u}(\mathbf{u}_0, t)] \leq V(\mathbf{u}_0)\mathrm{e}^{-ct} \quad \forall \mathbf{u}_0 \in S(\mathbf{0}, r),\ \forall t \in \mathbb{R}^+.$$

1.8 The norm as Liapunov function: the energy method

Although *energy method* originated in the works of Reynolds [71] and Orr [59], its modern version can be considered to be a particular case of the Liapunov direct method (see [23, 81, 87]).

Let us consider a basic solution $\mathbf{v}$ to the I.B.V.P. (1.1)-(1.3) and deduce the evolution equations of the perturbation (1.18) as in section 1.5. Then we take a linear subspace $\mathcal{H}$ of $\mathrm{L}^2(\Omega)$, endowed with the standard L^2-norm

$$\|f\|^2 = \int_\Omega f^2 \mathrm{d}\Omega,$$

as state space and define the *energy* E of the perturbation $\mathbf{u}$ through

$$E(t) = \frac{1}{2}\|\mathbf{u}\|^2.$$

For the fluid motions treated in this thesis multiplying $(1.18)_1$ by $\mathbf{u}$ and integrating over Ω yield a relation of the type

$$\dot{E}(t) = R\mathcal{I} - \mathcal{D}$$

where R is a non-dimensional number related to the physics of the phenomenon $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{D}$ are quadratic integral functionals involving $\mathbf{u}$ and $\nabla\mathbf{u}$, $\mathcal{D}$ being definite positive. The stability of the basic solution $\mathbf{v}$ is then linked to the variational problem

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}.$$

Indeed, if a Poincaré type inequality holds, that is

$$\exists \gamma \in \mathbb{R}^+ \ : \ \|\mathbf{u}\| \leq \gamma \mathcal{D} \quad \forall \mathbf{u} \in \mathcal{H},$$

one obtains the *energy inequality*

$$\dot{E}(t) \leq -\frac{2}{\gamma}\frac{R_E - R}{R_E} E(t)$$

which integrates to

$$E(t) \leq E(0) \exp\left(-\frac{2}{\gamma}\frac{R_E - R}{R_E} t\right)$$

and consequently the global exponential stability of the basic motion $\mathbf{v}$ is achieved (see Remark 1.3).

Example 1.9. We end this introductory chapter by illustrating the energy method on a simple example.

We wish to examine the stability of the zero solution to the I.B.V.P. for the diffusion equation with a linear source term and a convective term

$$\begin{cases} u_t + u u_x = u_{xx} + au & x \in (0,d), \ \ t \in \mathbb{R}^+ \\ u(0,t) = u(d,t) = 0 & \forall t \geq 0 \\ u(x,0) = u_0(x) & \forall x \in (0,d) \end{cases} \tag{1.43}$$

where a is a non-negative constant.

If we attempt a linear analysis, that is linearize $(1.43)_1$ about the trivial solution $u \equiv 0$, we obtain

$$\begin{cases} u_t = u_{xx} + au & x \in (0,d), \ \ t \in \mathbb{R}^+ \\ u(0,t) = u(d,t) = 0 & \forall t \geq 0 \\ u(x,0) = u_0(x) & \forall x \in (0,d) \end{cases}$$

whose solution may be represented as an infinite sum of normal modes

$$u = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{s_n t} \sin \frac{n\pi x}{d}$$

where

$$s_n = a - \frac{n^2\pi^2}{d^2} \quad \text{and} \quad A_n = \frac{2}{d}\int_0^d u_0(x) \sin\frac{n\pi x}{d} \mathrm{d}x.$$

The n-th mode is stable if and only if $a \leq n^2\pi^2/d^2$ and so the zero solution is linearly stable if and only if all modes are stable, namely if and only if

$$a \leq \min_{n\in\mathbb{N}} \frac{n^2\pi^2}{d^2} = \frac{\pi^2}{d^2}. \tag{1.44}$$

In order to find a sufficient condition for global nonlinear stability of the zero solution we employ the energy method. Then we multiply the differential equation $(1.43)_1$ by u, integrate over the interval $(0, d)$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = \int_0^d u\frac{\partial^2 u}{\partial x^2}\mathrm{d}x + a\|u\|^2$$

where $\|\cdot\|$ denotes the standard $\mathrm{L}^2(0,d)$-norm. Note that the convective term integrates to zero, since, from $(1.43)_2$,

$$\int_0^d u^2\frac{\partial u}{\partial x}\mathrm{d}x = \frac{1}{3}u^3\big|_0^d = 0.$$

Again, as before, integrating by parts yields

$$\int_0^d u\frac{\partial^2 u}{\partial x^2}\mathrm{d}x = -\|u_x\|^2.$$

So we get the energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -\|u_x\|^2 + a\|u\|^2 \leq -\|u_x\|^2\left(1 - a\max_{\mathcal{H}}\frac{\|u\|^2}{\|u_x\|^2}\right), \tag{1.45}$$

where $\mathcal{H} = \{u \in \mathrm{H}^1(0,d) : u(0) = u(d) = 0\}$ is *the space of the admissible functions* over which we seek a maximum. We now define R_E by

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{\|u\|^2}{\|u_x\|^2}, \tag{1.46}$$

so that the energy inequality (1.45) may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -2\|u_x\|^2\left(1 - \frac{a}{R_E}\right).$$

If $a < R_E$, then $1 - a/R_E > 0$ and, by using the Poincaré inequality, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -2\frac{\pi^2}{d^2}\left(1 - \frac{a}{R_E}\right)\|u\|^2$$

and consequently

$$\|u\|^2 \leq \|u_0\|^2\exp\left[-2\frac{\pi^2}{d^2}\left(1 - \frac{a}{R_E}\right)t\right].$$

We have thus shown that if $a < R_E$, $\|u\| \to 0$ as $t \to +\infty$ with the decay at least exponential in time.

The problem remains to find R_E. The Euler-Lagrange equation associated to the variational problem (1.46) is found as follows. Let $u \in \mathcal{H}$ be the function on which $\|u\|^2/\|u_x\|^2$ attains its maximum. Then, letting ϵ be a non-negative parameter, for any $\phi \in \mathcal{H}$, the function

$$F(\epsilon) = \frac{\|u + \epsilon\phi\|^2}{\|u_x + \epsilon\phi_x\|^2}$$

attains its maximum for $\epsilon = 0$ and hence one has

$$\begin{aligned} 0 = \left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= \frac{2}{\|u_x\|^2}\left(\int_0^d u\phi \mathrm{d}x - \frac{\|u\|^2}{\|u_x\|^2}\int_0^d u_x\phi_x \mathrm{d}x\right) \\ &= \frac{2}{\|u_x\|^2}\int_0^d \left(u\phi - R_E^{-1}u_x\phi_x\right)\mathrm{d}x = \int_0^d \phi\left(u + R_E^{-1}u_{xx}\right)\mathrm{d}x. \end{aligned}$$

Since ϕ is an arbitrary function belonging to $\mathcal{H}$, we have

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + R_E u = 0, \quad u(0) = u(d) = 0 \tag{1.47}$$

which gives an eigenvalue problem for R_E.

It is easy to show that the eigenvalues of (1.47) are given by

$$R_E = \frac{n^2\pi^2}{d^2}, \quad n \in \mathbb{N}.$$

For stability, we need $a < \min_{n\in\mathbb{N}} R_E = \pi^2/d^2$. Therefore the criterion we have just derived by employing the energy method is the same as that found by normal modes analysis and so (1.44) is a necessary and sufficient condition for stability of the zero solution to the I.B.V.P. (1.43). Furthermore, since the condition $a < \pi^2/d^2$ ensures global nonlinear exponential stability of the zero solution, no subcritical instability is allowable (see Example 1.7).