Chapter 4

Best decomposition

The representation of the solutions of parabolic problems by means of iterates of approximating operators may be more effective if we choose appropriately the sequence of operators. Even the quantitative estimates between the semigroup and the iterates may be affected by this choice. In this chapter we introduce a method which can be useful in order to consider a combinations of different sequences of operators in the approximation of the same problem.

Using a general procedure we consider some combination of different approximation processes by means of projections on orthogonal subspaces. We concentrate our attention on some particular positive approximation processes in spaces of L^2-real functions in order to satisfy a prescribed Voronovskaja-type formula. Some similar questions have also been considered in [31] and in [32].

The results in this chapter are contained in [40]

4.1 Direct sums of approximation processes

We are mainly interested in the application of a general and simple method which consists in constructing a new approximation process starting with a decomposition of a Hilbert space into the direct sum of orthogonal subspaces and associating to each subspace an assigned approximation process.

In this way we obtain some noteworthy results regarding the possibility of obtaining new Voronovskaja-type formulas from assigned ones and extending the class of differential problems under consideration.

The general method can actually be applied in different settings. Indeed, we may have the necessity of using different approximation processes on orthogonal subspaces as done in Section 4.2 in connection with Bernstein-Kantorovich and Bernstein-Durrmeyer operators; this may happen for example in studying diffusion models in population genetics where different factors may depend on the subspace containing the initial condition. Indeed, it is well-known that the differential operator arising from the Voronovskaja's formula for both Bernstein-Kantorovich and Bernstein-Durrmeyer operators describes the evolution process associated with some diffusion models in population genetics through the representation given in (4.2.13) which depends only on the initial condition u_0 in (4.2.14). Hence the method used in Section 4.2 allows us to arrange better the choice of the subspace V and the approximating operators to the initial condition. A different motivation can be the preservation of some functions by a modified classical approximation process; this was already realized in [31] for some sequences of algebraic polynomials and now we have also considered an example concerned with convolution operators in Section 4.3. Different applications to projections onto splines can also be considered; here we have not dealt with this case due to the large literature already existing in this field (see [46, Section 13.4]) and also because we are only interested in the possibility of approximating the solution of wider classes of differential problems and consequently only to more general Voronovskaja-type formulas.

The method is based on some simple properties of Hilbert spaces. Consider an Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a decomposition

$$\mathcal{H} = \bigoplus_{i\in I} V_i$$

of $\mathcal{H}$ into the direct sum of orthogonal closed subspaces V_i, $i \in I$ and for every $i \in I$ denote by P_i the canonical orthogonal projection onto the subspace V_i.

Now, let $(L_i)_{i\in I}$ be a family of linear operators from $\mathcal{H}$ into itself and consider the linear operator $L : \mathcal{H} \to \mathcal{H}$ defined by setting, for every $u \in \mathcal{H}$,

$$L(u) = \sum_{i\in I} P_i(L_i(u)) \; . \tag{4.1.1}$$

In this way we associate the operator L to the families $(V_i)_{i\in I}$ and $(L_i)_{i\in I}$.

Observe that if $u, v \in \mathcal{H}$ and $L_i(u) = v$ for every $i \in I$ then we have $L(u) = v$ too. In particular if all the operators L_i, $i \in I$ coincide with an operator T we also have $L = T$.

Moreover, it is also interesting to observe that we can also study perturbations of an operator L having the form (4.1.1) by modifying some of its components L_i; this will be performed in Section 4.3 in connection with Jackson convolution operators.

At this point, we apply the preceding procedure to a sequence of families $(L_{i,n})_{i\in I}$ of linear operators and using (4.1.1) we define the new sequence $(L_n)_{n\in\mathbb{N}}$ of linear operators given by

$$L_n(u) = \sum_{i\in I} P_i(L_{i,n}(u)) \; . \tag{4.1.2}$$

It is immediate to check that if every sequence $(L_{i,n})_{n\in\mathbb{N}}$, $i \in I$, is an approximation process on $\mathcal{H}$, then $(L_n)_{n\in\mathbb{N}}$ satisfies the same property. Moreover, if every sequence $(L_{i,n})_{n\geq 1}$ satisfies an abstract Voronovskaja-type formula

$$\lim_{n\to+\infty} n(L_{i,n}u - u) = A_i(u) \ , \qquad u \in D \ , \tag{4.1.3}$$

where $A_i : D \to \mathcal{H}$ is a linear operator and D is a subspace of $\mathcal{H}$, then the sequence $(L_n)_{n\geq 1}$ satisfies the Voronovskaja's formula

$$\lim_{n\to+\infty} n(L_n u - u) = \sum_{i\in I} P_i(A_i(u)) \ , \qquad u \in D \ . \tag{4.1.4}$$

Using this general scheme, we pass to consider some cases of particular interest in different settings where we can add more details on the convergence of the constructed operators and their Voronovskaja-type formulas.

It will be useful to observe that if a finite-dimensional subspace V of $\mathcal{H}$ is generated by the independent system $\{\alpha_1, \ldots, \alpha_m\}$, then the projection P_V of $\mathcal{H}$ onto V can be easily obtained by considering the square matrix $A := (\langle \alpha_i, \alpha_j \rangle)_{i,j=1,\ldots,m}$ and taking into account that for every $f, g \in \mathcal{H}$ we have $P_V(f) = g$ if and only if $AG = F$ where F is the column vector with components $(\langle f, \alpha_i \rangle)_{i=1,\ldots,m}$ and $G = (g_i)_{i=1,\ldots,m}$ is the vector of the components of $P_V(f)$ in the subspace V, i.e. $P_V(f) = \sum_{i=1}^m g_i \alpha_i$; imposing $\langle P_V(f) - f, \alpha_i \rangle = 0$ for every $i = 1, \ldots, m$ we find

$$G = A^{-1} \cdot F \tag{4.1.5}$$

and in particular, if $\{\alpha_1, \ldots, \alpha_m\}$ is an orthogonal system

$$g_i = \frac{\langle f, \alpha_i \rangle}{\|\alpha_i\|^2} \ . \tag{4.1.6}$$

4.2 Bernstein-Kantorovich-Durrmeyer operators

In this section we split the space $L^2(0,1)$ into two components and consider a combination of the classical Bernstein-Kantorovich and Bernstein-Durrmeyer operators. Obviously the same construction may be carried on by considering different orthogonal subspaces of $L^2(0,1)$ or different sequences of operators.

First, we recall that for every $n \geq 1$ the n-th Bernstein-Kantorovich $K_n : L^2(0,1) \to L^2(0,1)$ and respectively the n-th Bernstein-Durrmeyer operator $M_n : L^2(0,1) \to L^2(0,1)$ are defined by setting, for every $f \in L^2(0,1)$ and $x \in [0,1]$,

$$K_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \,, \tag{4.2.1}$$

and respectively

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt \,, \tag{4.2.2}$$

where, as usual, $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$.

We also recall that (see, e.g., [58, p. 31] and [9, Section 5.3.7, 5.3.8])

$$\begin{aligned}
K_n(\mathbf{1}) &= \mathbf{1}\,, \quad K_n(\mathrm{id})(x) = \frac{2nx+1}{2(n+1)}\,, &(4.2.3)\\
K_n(\mathrm{id}^2)(x) &= \frac{3n(n-1)x^2+6nx+1}{3(n+1)^2}\,,\\
M_n(\mathbf{1}) &= \mathbf{1}\,, \quad M_n(\mathrm{id})(x) = \frac{nx+1}{n+2}\,, &(4.2.4)\\
M_n(\mathrm{id}^2)(x) &= \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)}\,,
\end{aligned}$$

for every $x \in [0,1]$ and these formulas ensure the convergence of the sequences $(K_n)_{n\geq1}$ and $(M_n)_{n\geq1}$ to the identity operator by the classical Korovkin's theorem (see e.g. [9, Theorem 4.2.7]).

Moreover, estimates of the convergence can be found with respect to the classical modulus of continuity $\omega(f,\delta)$ in spaces of continuous functions (see [9, (5.3.38)–(5.3.42) and (5.3.51)–(5.3.53)])

$$|K_n f(x) - f(x)| \leq 2\omega\left(f, \frac{\sqrt{(n-1)\,x(1-x)}}{n+1}\right)\,,$$

$$|M_n f(x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{2(n-3)\,x(1-x)+2}{(n+2)(n+3)}}\right)$$

which give

$$\|K_n f(x) - f(x)\| \le 2\omega\left(f, \frac{1}{\sqrt{n}}\right) , \quad \|M_n f(x) - f(x)\| \le 2\omega\left(f, \frac{1}{\sqrt{n}}\right)$$

for every $f \in C([0,1])$ and with respect to the averaged modulus of smoothness $\tau(f,\delta)_2 := \left(\int_0^1 \omega(f,\delta,x)^2\,dx\right)^{1/2}$

$$\|K_n f - f\|_2 \le 748\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 , \tag{4.2.5}$$

$$\|M_n f - f\|_2 \le 748\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 \tag{4.2.6}$$

for every $f \in L^2(0,1)$.

Finally, we also recall the following Voronovskaja-type formulas

$$\lim_{n\to+\infty} n(K_n(f) - f) = \frac{1}{2} A(f) , \tag{4.2.7}$$

$$\lim_{n\to+\infty} n(M_n(f) - f) = A(f) , \tag{4.2.8}$$

which are satisfied for every $f \in C^2([0,1])$, where $A : C^2([0,1]) \to C([0,1])$ denotes the differential operator defined by

$$Au(x) := \frac{d}{dx}\left(x(1-x)\,u'(x)\right) , \qquad u \in C^2([0,1]) , \quad x \in [0,1] .$$

Now, let V be the subspace of $L^2(0,1)$ consisting of all linear functions on $[0,1]$ and its orthogonal subspace given by

$$W := \left\{ v \in L^2(0,1) \mid \int_0^1 (a+bt)\,v(t)\,dt = 0 \text{ for every } a,b \in \mathbb{R} \right\} ;$$

it is easy to recognize that

$$\begin{aligned} W &= \left\{ v \in L^2(0,1) \mid \int_0^1 v(t)\,dt = 0 , \int_0^1 t\,v(t)\,dt = 0 \right\} \\ &= \left\{ v \in L^2(0,1) \mid \int_0^1 t\,v(t)\,dt = 0 , \int_0^1 (1-t)\,v(t)\,dt = 0 \right\} ; \end{aligned}$$

moreover, P_V and P_W denote the orthogonal projections onto the subspaces V and respectively W.

According to the general procedure, we can define the new sequence $(L_n)_{n\ge1}$ of linear operators on $L^2(0,1)$ by setting

$$L_n f(x) := P_V(K_n(f))(x) + P_W(M_n(f))(x) , \quad f \in L^2(0,1) , \ x \in [0,1] . \tag{4.2.9}$$

In order to write a more explicit expression of the operators L_n, we consider the orthogonal basis of V consisting of the two functions $\mathbf{1}$ and $\mathbf{1} - 2\mathrm{id}$.

Using (4.1.5), for every $f \in L^2(0,1)$ and $x \in [0,1]$, we get

$$\begin{aligned}
P_V(K_n(f))(x) &= \int_0^1 K_n f(t)\,dt + \frac{\int_0^1 (1-2t) K_n f(t)\,dt}{\int_0^1 (1-2t)^2\,dt}\ (1-2x) \\
&= (n+1) \sum_{k=0}^{n} \binom{n}{k} \frac{k!(n-k)!}{(n+1)!} \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds \\
&\quad +3(n+1) \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k!(n-k)!}{(n+1)!} - 2\frac{(k+1)!(n-k)!}{(n+2)!} \right) \\
&\quad \times \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\ (1-2x) \\
&= \int_0^1 f(s)\,ds + 3 \sum_{k=0}^{n} \left(1 - 2\,\frac{k+1}{n+2}\right) \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\ (1-2x) \\
&= (4-6x) \int_0^1 f(s)\,ds - \frac{6}{n+2} \sum_{k=0}^{n} (k+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\ (1-2x)
\end{aligned}$$

and consequently

$$\begin{aligned}
P_W(M_n(f))(x) &= M_n f(x) - P_V(M_n(f))(x) \\
&= M_n f(x) - \int_0^1 M_n f(t)\,dt - \frac{\int_0^1 (1-2t) M_n f(t)\,dt}{\int_0^1 (1-2t)^2\,dt}\ (1-2x) \\
&= M_n f(x) - \sum_{k=0}^{n} \int_0^1 p_{n,k}(s) f(s)\,ds \\
&\quad -3 \sum_{k=0}^{n} \left(1 - 2\,\frac{k+1}{n+2}\right) \int_0^1 p_{n,k}(s)\, f(s)\,ds\ (1-2x) \\
&= M_n f(x) - \int_0^1 f(s)\,ds - 3 \sum_{k=0}^{n} \int_0^1 p_{n,k}(s)\, f(s)\,ds\ (1-2x) \\
&\quad + \frac{6}{n+2} \sum_{k=0}^{n} (k+1) \int_0^1 p_{n,k}(s) f(s)\,ds\ (1-2x) \\
&= M_n f(x) + (-4+6x) \int_0^1 f(s)\,ds + \frac{6}{n+2} \int_0^1 (ns+1) f(s)\,ds\ (1-2x)\ .
\end{aligned}$$

Hence, from (4.2.9) we obtain

$$\begin{aligned} L_n f(x) &= M_n f(x) \qquad (4.2.10)\\ &+\frac{6(1-2x)}{n+2}\left(\int_0^1 (ns+1)f(s)\,ds - \sum_{k=0}^{n}(k+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\right)\\ &= M_n f(x) + \frac{6n(1-2x)}{n+2}\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)}\left(s-\frac{k}{n}\right) f(s)\,ds \end{aligned}$$

for every $f \in L^2(0,1)$ and $x \in [0,1]$.

The convergence of $(L_n)_{n\geq 1}$ to the identity operator on $L^2(0,1)$ is ensured by (4.2.9) and the analogous properties of the sequences $(K_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$.

As regards to a quantitative estimate of the convergence, again from (4.2.9) and (4.2.5)–(4.2.6) we get, for every $f \in L^2(0,1)$,

$$\|L_n f - f\|_2 \leq 1496\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 .$$

We explicitly observe that

$$\begin{aligned} L_n\mathbf{1} &= \mathbf{1}\,,\\ L_n\mathrm{id}(x) &= \frac{nx+1}{n+2} + \frac{n(2x-1)}{2(n+1)(n+2)} = \frac{n}{n+1}\,x + \frac{1}{2(n+1)}\,,\\ L_n\mathrm{id}^2(x) &= \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)} + \frac{n(2x-1)}{2(n+1)(n+2)}\\ &= \frac{n(n-1)}{(n+2)(n+3)}\,x^2 + \frac{n(5n+7)}{(n+1)(n+2)(n+3)}\,x\\ &\quad -\frac{n^2-n-4}{2(n+1)(n+2)(n+3)}\,. \end{aligned}$$

Moreover, the following result establishes a Voronovskaja's formula for the sequence $(L_n)_{n\geq 1}$.

Theorem 4.2.1 *For every $f \in C^2([0,1])$, we have*

$$\lim_{n\to+\infty} n(L_n f(x) - f(x)) = Af(x) + 3(1-2x)\int_0^1 (1-2t)\,f(t)\,dt\,. \qquad (4.2.11)$$

uniformly with respect to $x \in [0,1]$.

Proof. Indeed, from (4.1.4) and (4.2.7)–(4.2.8) and using twice the integration by parts, for every $f \in C^2([0,1])$ and $x \in [0,1]$ we have

$$\begin{aligned}
\lim_{n\to+\infty} n(L_n f(x) - f(x)) &= P_V\left(\frac{1}{2}Af\right)(x) + P_W\,(Af)\,(x) \\
&= \frac{1}{2}\int_0^1 \big(t(1-t)\,f'(t)\big)'\,dt + \frac{3}{2}\,(1-2x)\int_0^1 (1-2t)\,\big(t(1-t)\,f'(t)\big)'\,dt \\
&\quad + Af(x) - \int_0^1 \big(t(1-t)\,f'(t)\big)'\,dt \\
&\quad -3(1-2x)\int_0^1 (1-2t)\,\big(t(1-t)\,f'(t)\big)'\,dt \\
&= Af(x) - 3(1-2x)\int_0^1 t(1-t)\,f'(t)\,dt \\
&= Af(x) + 3(1-2x)\int_0^1 (1-2t)\,f(t)\,dt
\end{aligned}$$

and this completes the proof. □

Finally, we observe that the differential operator $B : C^2([0,1]) \to C([0,1])$ defined by

$$Bu(x) := Au(x) + 3(1-2x)\int_0^1 (1-2t)\,u(t)\,dt\ , \qquad u \in C^2([0,1])\ , \quad x \in [0,1]\ ,$$

may be considered as a bounded perturbation of the operator A since

$$\int_0^1 \left(3(1-2x)\int_0^1 (1-2t)\,u(t)\,dt\right)^2 dx \le 9\left(\int_0^1 u(t)\,dt\right)^2 \le 9\|u\|_2^2\ ;$$

hence $A - B$ is bounded and $\|A - B\| \le 3$.

It is well-known that the closure $(A, D(A))$ of $(A, C^2([0,1]))$ is defined on the domain

$$\begin{aligned}
D(A) := \{f \in L^2(0,1) \quad | \quad & f \text{ is locally absolutely continuous in }]0,1[\\
& \text{and } x(1-x)\,f'(x) \in W_0^{1,2}(0,1)\}\ ,
\end{aligned}$$

and generates a C_0-semigroup $(T(t))_{t\ge0}$ of contraction on $L^2(0,1)$ which is analytic (with angle $\pi/2$) and immediately compact (see e.g. [1, Theorem 2.3]). From the classical perturbation theory of C_0-semigroup (see e.g. [48, Section III.1] or also [64, Section 3.1]) we conclude that also $(B, D(A))$ generates an analytic C_0-semigroup $(S(t))_{t\ge0}$ on $L^2(0,1)$ with angle $\pi/2$ on the same domain $D(A)$. From this it also follows that $C^2([0,1])$ is a core for $(B, D(A))$ and further

$$\|S(t)\| \le e^{\|A-B\|t}\|T(t)\| \le e^{3t}\ . \qquad (4.2.12)$$

Moreover, in connection with the operators L_n we have the following representation of the semigroup $(S(t))_{t\ge0}$.

Theorem 4.2.2 *For every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$\lim_{n\to+\infty} L_n^{k(n)} = S(t) \qquad \text{strongly on } L^2(0,1)\ . \tag{4.2.13}$$

Proof. Since $(B, D(A))$ generates a C_0-semigroup in $L^2(0,1)$ with growth bound ≤ 3, the range of $\lambda - B$ coincides with $L^2(0,1)$ for every $\lambda > 3$. Moreover, for every $n \geq 1$ we have

$$\begin{aligned}\|L_n(f) - M_n(f)\|_2^2 &\leq 36 \left(\sum_{k=0}^{n} \int_{k/(n+1)}^{(k+1)/(n+1)} \left(s - \frac{k}{n}\right) f(s)\, ds \right)^2 \\ &\leq \frac{36}{(n+1)^2} \left(\int_0^1 f(s)\, ds \right)^2 \leq \frac{36}{(n+1)^2} \|f\|_2^2\end{aligned}$$

and consequently $\|L_n\| \leq \|M_n\| + 6/(n+1) \leq 1 + 6/(n+1)$ which yields, for every $k \geq 1$,

$$\|L_n^k\| \leq \left(1 + \frac{6}{n+1}\right)^k = \left(\left(1 + \frac{6}{n+1}\right)^n\right)^{k/n} \leq e^{6k/n}\ .$$

Hence, the stability condition in Trotter's Theorem II.1.1 is satisfied and its application yields completely the proof. □

The preceding result ensures the possibility of approximating the solutions of the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial u}{\partial x}\left(x(1-x)\, \dfrac{\partial u}{\partial x}(t,x)\right) + 3(1-2x) \displaystyle\int_0^1 (1-2s)\, u(t,s)\, ds\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq 0\ ,\ x \in [0,1]\ , \\ u(0,x) = u_0(x)\ , \qquad\qquad\qquad\qquad\qquad u_0 \in L^2(0,1)\ , \end{cases} \tag{4.2.14}$$

using iterates of the operators L_n applied to the initial condition; namely, for every $t \geq 0$ and $x \in [0,1]$ we have

$$u(x,t) = S(t)u_0(x) = \lim_{n\to+\infty} L_n^{[nt]} u_0(x)\ ,$$

in the norm L^2 with respect to $x \in [0,1]$ and uniformly in compact intervals with respect to $t \geq 0$.

Quantitative estimates of the above convergence formulas can be obtained on suitable subspaces using the results in Chapter 1, provided that we have a quantitative versions of (4.2.7) and (4.2.8); for the sake of brevity we state it only for Bernstein-Durrmeyer operators, since the same methods can be applied to obtain a similar estimate for Bernstein-Kantorovich operators.

Lemma 4.2.3 *We have*

1. $M_n\mathbf{1}(x) - 1 = 0$,

2. $M_n(\mathrm{id} - x)(x) = \left(\frac{nx+1}{n+2} - x\right)$,

3. $M_n((\mathrm{id} - x)^2)(x) = \left(\frac{n(n-1)x^2 + 4nx + 2}{(n+2)(n+3)} - 2x\frac{nx+1}{n+2} + x^2\right)$,

4. $|M_n((\mathrm{id} - x)^4)(x)| \leq \frac{C}{n^2}$.

Proof. The statements *1*, *2* and *3* follow easily from (4.2.4).
Now a straightforward calculus gives, for every $m \geq 1$,

$$\begin{aligned} M_n(\mathrm{id}^m)(x) &= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 \binom{n}{k} t^k(1-t)^{n-k}t^m\,dt \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\binom{n}{k}\beta(k+m+1, n-k+1) \\ &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)\cdots(k+m)}{(n+2)\cdots(n+m+1)}\ , \end{aligned}$$

where $\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Euler's beta function. So we have

$$\begin{aligned} M_n(\mathrm{id}^3)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)(k+2)(k+3)}{(n+2)(n+3)(n+4)} \\ &= \frac{n^3B_n(\mathrm{id}^3)(x) + 6n^2B_n(\mathrm{id}^2)(x) + 11nB_n(\mathrm{id})(x) + 6}{(n+2)(n+3)(n+4)} \\ &= \frac{nx(1+3x(n-1)+x^2(n-1)(n-2)) + 6nx(1+x(n-1)) + 11nx + 6}{(n+2)(n+3)(n+4)} \end{aligned}$$

and

$$\begin{aligned} M_n(\mathrm{id}^4)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)(k+2)(k+3)(k+4)}{(n+2)(n+3)(n+4)(n+5)} \\ &= \frac{n^4B_n(\mathrm{id}^4)(x) + 10n^3B_n(\mathrm{id}^3)(x) + 35n^2B_n(\mathrm{id}^2)(x) + 50nB_n(\mathrm{id})(x) + 24}{(n+2)(n+3)(n+4)(n+5)} \\ &= \frac{1}{(n+2)(n+3)(n+4)(n+5)}\times \\ &\times \big[nx(1+7x(n-1)+6x^2(n-1)(n-2)+x^3(n-1)(n-2)(n-3)) + \\ &10nx(1+3x(n-1)+x^2(n-1)(n-2)) + 35nx(1+x(n-1)) + 50nx + 24\big] \end{aligned}$$

and consequently, using (4.2.4) we obtain

$$
\begin{aligned}
&M_n((\mathrm{id}-x)^4)(x)\\
&= M_n(\mathrm{id}^4)(x) - 4xM_n(\mathrm{id}^3)(x) + 6x^2M_n(\mathrm{id}^2)(x) - 4x^3M_n(\mathrm{id})(x) + x^4\\
&= x^4\Big(1 - \frac{4\,n}{n+2} + \frac{6\,n(n-1)}{(n+2)(n+3)} - \frac{4\,n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{n(n-1)(n-2)(n-3)}{(n+2)(n+3)(n+4)(n+5)}\Big) + x^3\Big(-\frac{4}{n+2} + \frac{24\,n}{(n+2)(n+3)}\\
&\quad - \frac{36\,n(n-1)}{(n+2)(n+3)(n+4)} + \frac{16\,n(n-1)(n-2)}{(n+2)(n+3)(n+4)(n+5)}\Big)\\
&\quad + x^2\Big(\frac{12}{(n+2)(n+3)} - \frac{72\,n}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{72n(n-1)}{(n+2)(n+3)(n+4)(n+5)}\Big) + x\Big(\frac{24}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{96}{(n+2)(n+3)(n+4)(n+5)}\Big) + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\\
&= \frac{12\,n^2}{(n+2)(n+3)(n+4)(n+5)}\,x^2(1-x)^2 + o\left(\frac{1}{n^2}\right)\\
&\le \frac{3}{4n^2} + \frac{C_1}{n^3} \le \frac{C}{n^2}\,.
\end{aligned}
$$

□

Proposition 4.2.4 *Let $0 < \alpha \le 1$; then there exist $C_1, C_2 > 0$ such that, for every $f \in C^{2,\alpha}([0,1])$*

$$\|n(M_n(f) - f) - Af\| \le C\frac{M_f}{n^{\alpha/2}}\,,$$

where M_f is the seminorm defined by

$$M_f := \|f'\| + \|f''\| + L_{f''} \tag{4.2.15}$$

Proof. Let $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ and let $\mathcal{A}_{M_n}$ be the operator defined by (2.2.2) taking $L = M_n$. From Lemma 4.2.3 we get

$$
\begin{aligned}
\mathcal{A}_{M_n}f(x) &= f'(x)M_n(\mathrm{id}-x)(x) + \frac{1}{2}f''(x)M_n\left((\mathrm{id}-x)^2\right)(x)\\
&= f'(x)\left(\frac{nx+1}{n+2} - x\right)\\
&\quad + f''(x)\frac{1}{2}\left(\frac{n(n-1)x^2 + 4nx + 2}{(n+2)(n+3)} - 2x\frac{nx+1}{n+2} + x^2\right)\,.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&|n(M_nf(x) - f(x)) - Af(x)| \qquad (4.2.16)\\
&\quad \le |n(L_nf(x) - f(x) - \mathcal{A}_{M_n}f(x)| + |n\mathcal{A}_{M_n}f(x) - Af(x)|\,.
\end{aligned}
$$

In regard to the first term in (4.2.16) we use Theorem 1.1.2 and taking into account Lemma 4.2.3, we obtain the existence of $C_1, C_2 > 0$ such that

$$M_n\left((\mathrm{id}-x)^2\right)(x) \leq \frac{C_1}{n}\,, \quad M_n\left((\mathrm{id}-x)^4\right)(x) \leq \frac{C_2}{n^2}\,.$$

Thus

$$|n\left(L_n f(x) - f(x) - \mathcal{A}_{M_n} f(x)\right)| \leq n\frac{L_{f''}}{2}\left(\frac{C_1}{n}\right)^{\alpha/2}\left(\frac{C_1^2}{n^2}+\frac{C_2}{n^2}\right)^{1/2}\,.$$

As regards the second term in (4.2.16) we have

$$\begin{aligned}
&n\mathcal{A}_{M_n} f(x) - Af(x)\\
&\quad = f'(x)n\left(\frac{nx+1}{n+2}-x\right) + f''(x)\frac{n}{2}\left(\frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)} - 2x\frac{nx+1}{n+2}+x^2\right)\\
&\qquad - (1-2x)f'(x) - x(1-x)f''(x)\\
&\quad = f'(x)2\frac{2x-1}{n+1} + f''(x)\frac{n(8x^2-8x+1)-6x(1-x)}{(n+2)(n+3)}\,,
\end{aligned}$$

and this yields

$$\|n\mathcal{A}_{M_n} f - Af\| \leq \frac{16}{n}\|f''\| + \frac{6}{n}\|f'\|\,.$$

Finally collecting the above inequalities we obtain

$$\begin{aligned}
&|n(M_n f(x) - f(x)) - Af(x)|\\
&\quad \leq n\frac{L_{f''}}{2}\left(\frac{C_1}{n}\right)^{\alpha/2}\left(\frac{C_1^2}{n^2}+\frac{C_2}{n^2}\right)^{1/2} + \frac{16}{n}\|f''\| + \frac{6}{n}\|f'\|\\
&\quad \leq C\frac{M_f}{n^{\alpha/2}}\,,
\end{aligned}$$

where M_f is the seminorm defined by (4.2.15). □

A similar estimate holds for Bernstein-Kantorovich operators (with different constants) and the same estimates continue to hold for both operators with respect to the L^2-norm.

Hence, for every $f \in C^{2,\alpha}([0,1])$ and $n \geq 1$, we have

$$\|n(L_n(f)-f) - Bf\|_w \leq \psi_n(f)\,, \qquad \|n(L_n(f)-f)\|_w \leq \varphi_n(f)\,, \quad (4.2.17)$$

where

$$\psi_n(f) := C\frac{M_f}{n^{\alpha/2}}\,, \qquad \varphi_n(f) := \|B(f)\|_w + C\frac{M_f}{n^{\alpha/2}}\,.$$

From (4.2.12) the growth bound of the semigroup $(S(t))_{t\geq 0}$ is less or equal than 3 (and constant $M = 1$) and therefore, applying Theorem 1.1.2, we obtain the following result.

Proposition 4.2.5 *For every $t \geq 0$, $(k(n))_{n\geq 1}$ sequence of positive integers and $f \in C^{2,\alpha}([0,1])$, we have*

$$\left\| L_n^{k(n)} u - S(t)u \right\|_w \leq t \exp(3\,e^{3/n}\,t)\,\psi_n(u) \tag{4.2.18}$$
$$+ \left(\exp(3\,e^{3/n}\,t_n) \left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, e^{3\,k(n)/n}\, \frac{\sqrt{k(n)}}{n} \right.$$
$$\left. + \frac{3}{n}\,\frac{k(n)}{n}\, \exp\left(3\,e^{3/n}\,\frac{k(n)}{n}\right) \right) \varphi_n(u)\ ,$$

where $t_n := \sup\{t, k(n)/n\}$.

In particular, if we take $k(n) = [nt]$, we obviously have $t_n = t$ and $\left| \frac{[nt]}{n} - t \right| = \frac{nt}{n} - \frac{[nt]}{n} \leq \frac{1}{n}$. Hence (4.2.18) yields

$$\left\| L_n^{k(n)} u - S(t)u \right\|_2 \leq t \exp(3\,e^{3/n}\,t)\,\psi_n(u) \tag{4.2.19}$$
$$+ \frac{1}{\sqrt{n}} \left(\frac{\exp(3\,e^{3/n}\,t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\, e^{3\,t} + \frac{3\,t}{\sqrt{n}} \exp\left(3\,e^{3/n}\,t\right) \right) \varphi_n(u)\ .$$

Of course the definition of $(L_n)_{n\geq 1}$ depends also on the decomposition of the space $L^2(0,1)$. Using different decompositions, we can describe the solution of different evolution problems in terms of iterates of suitable operators.

A different interesting example can be performed using the one-dimensional subspace X generated by the function $\mathrm{id}(\mathbf{1}-\mathrm{id})$ and its orthogonal subspace Y given by

$$Y := \left\{ v \in L^2(0,1) \mid \int_0^1 t(1-t)\,v(t)\,dt = 0 \right\}\ .$$

Taking the same sequences as before in this case we obtain the operator $(Q_n)_{n\geq 1}$ of linear operators on $L^2(0,1)$ by setting

$$Q_n f(x) := P_X(K_n(f))(x) + P_Y(M_n(f))(x)\ , \quad f \in L^2(0,1)\ ,\ x \in [0,1]\ . \tag{4.2.20}$$

Similarly to the preceding case, from (4.1.6) we obtain, for every $f \in L^2(0,1)$ and $x \in [0,1]$,

$$P_X(K_n(f))(x) = \frac{\int_0^1 t(1-t) K_n f(t)\,dt}{\int_0^1 t^2(1-t)^2\,dt}\ x(1-x)$$
$$= 30(n+1) \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)!(n-k+1)!}{(n+3)!} \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\ x(1-x)$$
$$= \frac{30}{(n+2)(n+3)} \sum_{k=0}^{n} (k+1)(n-k+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\ x(1-x)$$

and consequently

$$\begin{aligned}
P_Y(M_n(f))(x) &= M_n f(x) - P_X(M_n(f))(x) \\
&= M_n f(x) - 30 \int_0^1 t(1-t) M_n f(t)\, dt \; x(1-x) \\
&= M_n f(x) - \frac{30}{(n+2)(n+3)} \\
&\quad \times \sum_{k=0}^{n} (k+1)(n-k+1) \int_0^1 \binom{n}{k} s^k (1-s)^{n-k} f(s)\, ds \; x(1-x) \; .
\end{aligned}$$

Hence, from (4.2.20),

$$\begin{aligned}
Q_n f(x) = M_n f(x) - \frac{30}{(n+2)(n+3)} \sum_{k=0}^{n} (k+1)(n-k+1) \qquad (4.2.21) \\
\times \left(\int_0^1 \binom{n}{k} s^k (1-s)^{n-k} f(s)\, ds - \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\, ds \right) x(1-x)
\end{aligned}$$

for every $f \in L^2(0,1)$ and $x \in [0,1]$.

Clearly, the sequence $(Q_n)_{n\geq 1}$ converges to the identity operator on $L^2(0,1)$ and a quantitative estimate of the convergence can be obtained as before from (4.2.20) and (4.2.5)–(4.2.6).

A Voronovskaja's formula for the sequence $(Q_n)_{n\geq 1}$ can be also established using the same arguments of Theorem 4.2.1 and yields, for every $f \in C^2([0,1])$

$$\lim_{n\to+\infty} n(L_n f(x) - f(x)) = Af(x) - 15\, x(1-x) \int_0^1 (6t^2 - 6t + 1)\, f(t)\, dt \; . \qquad (4.2.22)$$

uniformly with respect to $x \in [0,1]$.

Finally, the differential operator arising from the (4.2.22) is again a bounded perturbation of the operator A and consequently its closure generates an analytic C_0-semigroup $(Q(t))_{t\geq 0}$ in $L^2(0,1)$ with angle $\pi/2$ on the same domain $D(A)$. The semigroup $(Q(t))_{t\geq 0}$ can be represented as

$$\lim_{n\to+\infty} Q_n^{k(n)} = Q(t) \qquad \text{strongly on } L^2(0,1) \; .$$

whenever $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$.

Hence, even in this case we have the possibility of approximating the solutions of the associated evolution problem using iterates of the operators Q_n evaluated at the initial point.

Remark 4.2.6 It is worthwhile mentioning that if we consider the identity operator in place of one of the preceding sequences we obtain the operators considered in [31] in connection with a best approximation property with respect to a linear operator.

Hence, the problem considered in [31] in the one-dimensional setting can be completely framed in the more general setting considered here.

In the following section we give an example of such situation by considering the case of convolution operators. □

4.3 Best perturbation of Jackson convolution operators

In this section we consider a perturbation of the classical Jackson convolution operators obtained by imposing a best approximation property on the subspace of all trigonometric polynomials having degree less or equal to 2. Since the treatment of this case is very similar to the preceding one, we shall omit several details and we shall only describe the main steps.

We consider the space $L^2_{2\pi}$ of all real 2π-periodic functions which are square summable on the interval $[-\pi, \pi]$ endowed with the usual scalar product

$$\langle f, g\rangle_{2\pi} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx\ , \qquad f, g \in L^2_{2\pi}\ .$$

For every $n \geq 1$, we recall that the n-th Jackson operator $J_n : L^2_{2\pi} \to L^2_{2\pi}$ is defined by setting, for every $f \in L^2_{2\pi}$ and $x \in \mathbb{R}$,

$$J_n f(x) := \frac{3}{2\pi n(2n^2+1)}\int_{-\pi}^{\pi} f(x-t)\,\frac{\sin^4 n\,t/2}{\sin^4 t/2}\,dt\ . \tag{4.3.1}$$

It is well-known that $J_n(f)$ is a trigonometric polynomial of degree $2n-2$ and the following estimate is satisfied for every $f \in L^2_{2\pi}$ (see [63, pp. 79–84], [21, p. 60] and also [9, (5.4.45)])

$$\|J_n(f) - f\|_{2\pi} \leq (1+\pi)\ \omega^{(2)}\left(f, \frac{1}{n+1}\right)\ ,$$

where $\omega^{(2)}(f,\delta) := \sup_{|h|\leq\delta} \|f(\cdot + h) - f\|_{2\pi}$.

We consider the subspace V of $L^2_{2\pi}$ generated by the trigonometric polynomials with degree less or equal to 2 and its orthogonal subspace W. If P_V and P_W denote the orthogonal projection onto the subspaces V and respectively W, we can define the sequence $(H_n)_{n\geq 1}$ of linear operators on $L^2_{2\pi}$ by setting

$$H_n f := P_V(f) + P_W(J_n(f)) = J_n(f) + P_V(f - J_n(f))\ , \quad f \in L^2_{2\pi}\ . \tag{4.3.2}$$

Taking into account that Jackson convolution operators preserve the trigonometric polynomials having degree less or equal to 1, from (4.1.5) we get, for every $f \in L^2_{2\pi}$ and $x \in \mathbb{R}$,

$$\begin{aligned} H_n f(x) \quad &= \quad J_n f(x) - \frac{\cos 2x}{\pi}\int_{-\pi}^{\pi} (J_n f(t) - f(t))\,\cos 2t\,dt \qquad (4.3.3)\\ &\qquad -\frac{\sin 2x}{\pi}\int_{-\pi}^{\pi} (J_n f(t) - f(t))\,\sin 2t\,dt \end{aligned}$$

for every $f \in L^2_{2\pi}$ and $x \in \mathbb{R}$.

It is clear from (4.3.3) that $(H_n)_{n\in\mathbb{N}}$ converges to the identity operator on $L^2_{2\pi}$; moreover, for every $f \in L^2_{2\pi}$,

$$\|H_n f - f\|_{2\pi} \leq (1+\pi)\, \omega^{(2)}\left(f, \frac{1}{n+1}\right)$$

since

$$H_n f - f = P_V(f) + P_W(J_n(f)) - (P_V(f) + P_W(f)) = P_W(J_n(f) - f)\ .$$

Since the Jackson convolution operators preserves the trigonometric polynomials of degree less or equal to 1, the same happens for the operators H_n; moreover, by definition H_n also preserves all trigonometric polynomials having degree less or equal to 2.

Finally, we recall that Jackson convolution operators satisfy the following Voronovskaja-type formula, for every $f \in C^1_{2\pi}$

$$\lim_{n\to+\infty} n(J_n(f) - f) = \frac{\sqrt{3}}{2}\,\pi\, f'\ , \tag{4.3.4}$$

(see also [21] and [9, 365–369 and 357]).

Consequently, the operators H_n satisfy the following Voronovskaja-type formula, for every $f \in C^1_{2\pi}$,

$$\begin{aligned}
&\lim_{n\to+\infty} n(H_n(f) - f) \\
&\quad = \frac{\sqrt{3}}{2}\,\pi\left(f' - \frac{\cos 2\mathrm{id}}{\pi}\int_{-\pi}^{\pi} f'(t)\,\cos 2t\,dt - \frac{\sin 2\mathrm{id}}{\pi}\int_{-\pi}^{\pi} f'(t)\,\sin 2t\,dt\right) \\
&\quad = \frac{\sqrt{3}}{2}\,\pi\, f' - \sqrt{3}\cos 2\mathrm{id}\int_{-\pi}^{\pi} f(t)\sin 2t\,dt + \sqrt{3}\sin 2\mathrm{id}\int_{-\pi}^{\pi} f(t)\cos 2t\,dt
\end{aligned}$$

In this case, if we denote by C the differential operator arising from the preceding Voronovskaja's formula, we can also point out that the closure of $(C^2, C^1(\mathbb{R}))$ generates a cosine function $(C(t))_{t\in\mathbb{R}}$ on $L^2_{2\pi}$ and every $C(t)$ is the strong limit of iterates of the operators H_n (see [35, Theorem 1.2] or Chapter 5 for more details).