Appendix D

A boundedness criterion

Here we give the proof of an improved version of the L^p boundedness criterion mentioned above ([42, Theorem 3.1], Chapter 5) useful to obtain our a-priori estimates in Chapter 5. As nice application we will deduce an alternative proof of the well known a-priori estimates for the heat operator.

In this appendix, as in Chapter 5, we use the following notation.
Given $X_0 = (x_0^1, ..., x_0^N, t_0)$, $R > 0$, with parabolic cylinder of center $X_0 = (x_0, t_0)$ and radius R we mean the set

$$K = K(X_0, R) = \{(x^1, ..., x^N, t) \in R^{N+1} : |x^i - x_0^i| < R,\ |t - t_0| < R^2\}.$$

D.1 Shen's Theorem

The main result of the section is the following Theorem.

Theorem D.1.1. *Let $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a bounded sublinear operator on $L^{p_0}(\mathbb{R}^{N+1})$. Suppose moreover that there exist $\alpha_2 > \alpha_1 > 1,\ C > 0$ such that*

$$\left\{\frac{1}{|K|}\int_K |Tf|^{q_0}\right\}^{\frac{1}{q_0}} \leq C\left\{\left(\frac{1}{|\alpha_1 K|}\int_{\alpha_1 K} |Tf|^{p_0}\right)^{\frac{1}{p_0}} + \sup_{K' \supset K}\left(\frac{1}{|K'|}\int_{K'} |f|^{p_0}\right)^{\frac{1}{p_0}}\right\}$$

for every $K \subset \mathbb{R}^{N+1}$ parabolic cylinder and every function $f \in L_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$. Then T is bounded in $L^p(\mathbb{R}^{N+1})$ for every $p_0 \leq p < q_0$.

We note that in [42, Theorem 3.1] $p_0 = 2$ and the parabolic cylinders are replaced by cubes of $\mathbb{R}^N$. We give a proof of the Theorem inspired by Shen's one.

We recall some auxiliary classical results from harmonic analysis concerning the Maximal Hardy-Littlewood function and the Lebesgue points. The proofs of the results only stated here can be found in [47] for d euclidean distance but it is possible to check that they are also true in the more general setting of the homogeneous spaces (see for example [48, Chapter I]).

Let (Ω, μ) be a measure space and $\mathcal{M}(\Omega)$ be the set of the measurable functions in Ω. Let d be a distance on Ω. Through this section, we denote with $B(x, r)$ the ball of center x and radius r for the metric induced by the distance d.
Let $f \in \mathcal{M}(\Omega)$. For every $\alpha > 0$ we set $\lambda(\alpha) = \lambda_f(\alpha) = \mu\{|f| > \alpha\}$. λ is a decreasing function in $(0, \infty)$. In the next lemma we recall an easy property of λ.

Lemma D.1.2. *Let $f \in \mathcal{M}(\Omega)$. Then*

$$\int_\Omega |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) \, d\alpha.$$

Let $f \in L^p(\Omega)$ with $p < \infty$, we recall the **Chebychev inequality**

$$\lambda(\alpha) = \mu\{|f| > \alpha\} \leq \frac{\|f\|_p^p}{\alpha^p}. \tag{D.1}$$

Definition D.1.3. *We say that μ is a doubling measure if there exists $C_0 > 0$ such that, for every B in Ω*

$$\mu(2B) \leq C_0 \mu(B)$$

where $2B$ is the ball with same center of B and double radius.

Remark D.1.4. By the previous definition it easily follows that, if μ is a doubling measure, for every $\lambda \geq 1$ there exists $C = C(C_0, \lambda)$ such that

$$\mu(\lambda B) \leq C \mu(B).$$

Definition D.1.5. *Let $f \in L^1_{loc}(\Omega)$. The maximal Hardy-Littlewood function $Mf : \Omega \to \overline{\mathbb{R}}$ is so defined*

$$Mf(x) = \sup_{B \ni x, B \subseteq \Omega} \frac{1}{\mu(B)} \int_B |f| d\mu$$

for every $x \in \Omega$.

Remark D.1.6. (1) If $f, \ g \in L^1_{loc}(\Omega)$,

$$M(f+g) \leq Mf + Mg.$$

(2) If $f \in L^\infty(\Omega)$, then $Mf \in L^\infty(\Omega)$ and $\|Mf\|_\infty \leq \|f\|_\infty$.

For every $1 \le p \le \infty$ we can define the operator

$$M : L^p(\Omega) \to \mathcal{M}(\Omega), \qquad f \mapsto Mf.$$

By Remark D.1.6, M is sublinear and bounded from L^∞ in L^∞. The following theorem provides us the so called maximal Hardy-Littlewood inequality, which, with the L^∞ boundedness and the Marcinkiewicz Theorem, gives that $M : L^p(\Omega) \to L^p(\Omega)$ is bounded for every $1 < p \le \infty$.
From now on we suppose that μ is a doubling measure.

Theorem D.1.7 (Maximal Hardy-Littlewood inequality)**.** *Let μ a doubling measure. There exists C positive constant such that for every $f \in L^1(\Omega)$ and for every $\alpha > 0$*

$$\mu(\{Mf > \alpha\}) \le C\frac{\|f\|_1}{\alpha}. \tag{D.2}$$

Corollary D.1.8. *Let $1 < p \le \infty$. Then there exists $A_p > 0$ such that*

$$\|Mf\|_p \le A_p\|f\|_p$$

for every $f \in L^p(\Omega)$.

Remark D.1.9. (**Local maximal function**.) Let $Q \subseteq \Omega$, $f \in L^1(Q)$. We consider the local maximal function so defined

$$M_Q f(x) = \sup_{B\subseteq Q,\ x\in B} \frac{1}{\mu(B)} \int_B |f| d\mu$$

for every $x \in Q$. By considering the space Q equipped with the metric induced by d, we obtain the existence of a positive constant C such that for every $\alpha > 0$ and for every $f \in L^1(Q)$

$$\mu(\{M_Q f > \alpha\}) \le C\frac{\|f\|_{L^1(Q)}}{\alpha} \tag{D.3}$$

and, by the Marcinkiewicz Theorem, it follows that, for every $1 < p \le \infty$, there exists a positive constant A_p such that

$$\|M_Q f\|_{L^p(Q)} \le A_p\|f\|_{L^p(Q)} \tag{D.4}$$

for every $f \in L^p(Q)$.

Definition D.1.10. *Let $f \in L^1_{loc}(\Omega)$. We say that $x \in \Omega$ is a Lebesgue point of f (we write $x \in \mathcal{L}(f)$) if*

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)|\, d\mu = 0.$$

Remark D.1.11. (i) If x is a Lebesgue point of f then

$$f(x) = \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f\, d\mu$$

(ii) If f is continuous in x then $x \in \mathcal{L}(f)$.

Theorem D.1.12 (Lebesgue Theorem). *If $f \in L^1(\Omega)$ then $|\Omega \setminus \mathcal{L}(f)| = 0$*

Proof. Given $r > 0$ we set

$$T_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)|\, d\mu$$

and $Tf(x) = \limsup_{r\to 0^+} T_r f(x)$. We have to prove that $Tf = 0$ almost everywhere in Ω.
By the density of $L^1(\Omega) \cap C(\Omega)$ in $L^1(\Omega)$, given $\varepsilon > 0$ there exists $g \in L^1(\Omega) \cap C(\Omega)$ such that $\|f - g\|_1 < \varepsilon$. By Remark D.1.11(ii)

$$Tg = 0 \text{ in } \Omega. \tag{D.5}$$

Set $h = f - g$,

$$\begin{aligned} T_r h(x) &= \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h - h(x)|\, d\mu \qquad \text{(D.6)} \\ &\le \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h|\, d\mu + |h(x)| \le Mh(x) + |h(x)|, \end{aligned}$$

where Mh is the maximal Hardy-Littlewood function. Obviously T_r is sublinear, therefore $T_r f \le T_r g + T_r h$. Taking the limsup for $r \to 0$, by (D.5) and (D.6) we deduce that

$$Tf \le Tg + Th = Th \le Mh + |h|.$$

By the last inequality it follows that for every $\alpha > 0$

$$\{Tf \ge \alpha\} \subset \left\{Mh \ge \frac{\alpha}{2}\right\} \cup \left\{|h| \ge \frac{\alpha}{2}\right\}$$

and then by Theorem D.1.7 and by the Chebychev inequality

$$\begin{aligned} \mu(\{Tf \ge \alpha\}) &\le \mu\left(\left\{Mh \ge \frac{\alpha}{2}\right\}\right) + \mu\left(\left\{|h| \ge \frac{\alpha}{2}\right\}\right) \\ &\le \frac{2\,C}{\alpha}\|h\|_1 + \frac{2}{\alpha}\|h\|_1 \\ &\le \left(\frac{2\,C}{\alpha} + \frac{2}{\alpha}\right)\varepsilon. \end{aligned}$$

Letting ε to zero we deduce $\mu(\{Tf \ge \alpha\}) = 0$ for every $\alpha > 0$. Therefore the measure of the set $\{Tf > 0\} = \bigcup_{n\in\mathbb{N}}\{Tf > \frac{1}{n}\}$ is zero, this means that $Tf = 0$ a.e. in Ω. □

We finally state a consequence of the Lebesgue Theorem.

Definition D.1.13. *Let $\{E_h\}_{h\ge 0}$ a family of subsets of Ω and let $x \in \Omega$. We say that $\{E_h\}$ converges to x for $h \to 0$ if there exist $\alpha > 0$ and $r_h \to 0$ such that for every $h \ge 0$*

$$E_h \subset B(x, r_h) \quad \textit{and} \quad \mu(E_h) \ge \alpha\mu(B(x, r_h)).$$

Corollary D.1.14. *Let $f \in L^1_{loc}(\Omega)$, $x \in \mathcal{L}(f)$ and $\{E_h\} \to x$, then*

$$\lim_{h\to 0} \frac{1}{\mu(E_h)} \int_{E_h} |f - f(x)|\, d\mu = 0.$$

Proof. We have

$$\frac{1}{\mu(E_h)} \int_{E_h} |f - f(x)|\, d\mu \leq \frac{1}{\alpha\mu(B(x, r_h))} \int_{B(x,r_h)} |f - f(x)|\, d\mu$$

and, since x is a Lebesgue point of f, the right and side of the last inequality goes to zero for $h \to 0$. □

Remark D.1.15. If, given $X,\ X_0 \in \mathbb{R}^{N+1}$, we set

$$d(X, X_0) = \max\{|x^i - x_0^i|,\ 1 \leq i \leq N,\ |t - t_0|^{\frac{1}{2}}\},$$

then the ball of center X_0 and radius R is the parabolic cylinder $K(X_0, R)$. This simple remark allows us to apply the general results about the maximal Hardy-Littlewood function and the Lebesgue points stated before in the case $\Omega = \mathbb{R}^{N+1}$, μ Lebesgue measure and d parabolic distance in $\mathbb{R}^{N+1}$.

We will use the following version of the Calderón-Zygmund decomposition. The proof is similar to that in [9, Lemma 1.1] where cubes of $\mathbb{R}^N$ appear instead of parabolic cylinders.

Proposition D.1.16 (Calderón-Zygmund decomposition)**.** *Let K a parabolic cylinder of $\mathbb{R}^{N+1}$ and $A \subset K$ a measurable set satisfying*

$$0 < |A| < \delta|K| \quad \textit{for some} \quad 0 < \delta < 1.$$

Then there is a sequence of disjoint dyadic parabolic cylinders $\{K_j\}_{j\in\mathbb{N}}$ obtained from K such that

1. $|A \setminus \bigcup_{j\in\mathbb{N}} K_j| = 0$*;*
2. $|A \cap K_j| > \delta|K_j|$ *for every* $j \in \mathbb{N}$*;*
3. $|A \cap \overline{K}_j| \leq \delta|\overline{K}_j|$ *if K_j is a dyadic subdivision of $\overline{K}_j$.*

Proof. Divide K in 2^{N+2} dyadic cylinders $K_{1,1}, \ldots, K_{1,2^{N+2}}$ as follows

$$K_{1,j} = \left\{(x,t) : |x^i - x^i_{1,j}| < \frac{R}{2},\ |t - t_{1,j}| < \frac{R^2}{4}\right\}.$$

Choose those for which $|K_{1,j} \cap A| > \delta|K_{1,j}|$. Divide each cylinder that has not been chosen in 2^{N+2} dyadic cylinders $\{K_{2,j}\}$ and repeat the process above iteratively. In this way we obtain a sequence of disjoint dyadic cylinders which we denote $\{K_j\}$. If $X \notin \bigcup_j K_j$, there exists a sequence of cylinders $C_h = K(X_h, R_h)$ containing X with diameter going to zero for $h \to \infty$ and such that

$$|C_h(X) \cap A| \leq \delta|C_h(X)| < |C_h(X)|. \tag{D.7}$$

Observe that $C_h(X) = K(X_h, R_h) \subset K(X, 2R_h)$ indeed if $Y \in C_h(X) = K(X_h, R_h)$ we have $d(Y, X_h) < R_h$, on the other hand, since $X \in C_h$, we have $d(X, X_h) < R_h$, therefore

$$d(Y, X) < d(Y, X_h) + d(X_h, X) < 2R_h.$$

Moreover

$$|C_h(X)| = R_h^{N+2} = \frac{1}{2^{N+2}}(2R_h)^{N+2} = \frac{1}{2^{N+2}}|K(X, 2R_h)|.$$

Apply Corollary D.1.14 to the family $\{C_h\}$ and $f = \chi_A \in L^1(\mathbb{R}^{N+1})$. By (D.7) we obtain that, if X is a Lebesgue point for χ_A,

$$\chi_A(X) = \lim_{h\to\infty} \frac{1}{|C_h|}\int_{C_h} \chi_A(Y)dY = \frac{|C_h(X)\cap A|}{C_h(X)} < 1.$$

This means that $\chi_A(X) = 0$, that is $X \notin A$. By the Lebesgue Theorem it follows that almost everywhere if $X \notin \cup_j K_j$ then $X \in K \setminus A$. This proves (1) and concludes the proof. □

Proof (Theorem D.1.1). Let $p_0 < p < q_0$. Let $f \in L_c^\infty(\mathbb{R}^{N+1})$. For $\lambda > 0$, we consider the set

$$E(\lambda) = \{(x, t) \in \mathbb{R}^{N+1} : M(|Tf|^{p_0})(x, t) > \lambda\}$$

where M is the maximal operator. Since $Tf \in L^{p_0}$, by the maximal inequality

$$|E(\lambda)| \le C\frac{\|Tf\|_{p_0}^{p_0}}{\lambda} < \infty. \tag{D.8}$$

Let $A = 1/(2\delta^{\frac{p_0}{p}})$ with $0 < \delta < 1/2^{\frac{p}{p_0}}$ small constant to be determined. Observe that $A > 1$. Divide $\mathbb{R}^{N+1}$ in parabolic cylinders $\{K_h\}$ big enough such that

$$|K_h \cap E(A\lambda)| < \delta|K_h|$$

and apply the Calderón-Zygmund decomposition to each K_h. For every $h \in \mathbb{N}$ we obtain a family of parabolic cylinders $\{K_{h,j}\}$ such that

$|(K_h \cap E(A\lambda)) \setminus \bigcup_j K_{h,j}| = 0;$

$|(K_h \cap E(A\lambda)) \cap K_{h,j}| > \delta|K_{h,j}|;$

$|(K_h \cap E(A\lambda)) \cap \overline{K}_{h,j}| \le \delta|\overline{K}_{h,j}|.$

Consider the family of cylinders $\{K_{h,j}\}$ obtained for h and j running in $\mathbb{N}$ and call it $\{K_j\}$ again. In this way we have a family of cylinders $\{K_j\}$ satisfying

1. $|E(A\lambda) \setminus \bigcup_j K_j| = 0;$

2. $|E(A\lambda) \cap K_j| > \delta |K_j|$;
3. $|E(A\lambda)) \cap \overline{K}_j| \leq \delta |\overline{K}_j|$.

We split the proof in three steps.

Step 1

There exist $0 < \delta < 1/2^{\frac{p}{p_0}}$, $0 < \gamma < 1$ such that if

$$\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset$$

then $\overline{K}_j \subseteq E(\lambda)$.

Proof (*Step 1*). Suppose by contradiction that for every $0 < \gamma < 1$, $0 < \delta < 1/2^{\frac{p}{p_0}}$ there exists $\overline{K}_j$ such that $\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset$ and $\overline{K}_j \not\subseteq E(\lambda)$. In particular the previous property holds for δ small enough such that $A \geq 5^{n+2}$. Fixed γ and δ, let $\overline{K}_j$ the corresponding cylinder as above and let $\overline{X} \in \overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$ and $X_0 \in \overline{K}_j \setminus E(\lambda)$. Then

$$M(|Tf|^{p_0})(X_0) = \sup_{K \ni X_0} \frac{1}{|K|} \int_K |Tf|^{p_0}(Y)dY \leq \lambda$$

and

$$M(|f|^{p_0})(\overline{X}) = \sup_{K \ni \overline{X}} \frac{1}{|K|} \int_K |f|^{p_0}(Y)dY \leq \gamma\lambda.$$

In particular, if $K \supseteq \overline{K}_j$, then $X_0,\ \overline{X} \in K$ and, consequently,

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \leq \lambda \quad \text{and} \quad \frac{1}{|K|} \int_K |f|^{p_0} \leq \gamma\lambda. \tag{D.9}$$

Let K_j a parabolic cylinder obtained by the dyadic division of $\overline{K}_j$ and prove that if $X \in K_j$

$$M(|Tf|^{p_0})(X) \leq \max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} \tag{D.10}$$

where $M_{2\overline{K}_j}$ is the local maximal function so defined:

$$M_{2\overline{K}_j}(|Tf|^{p_0})(X) = \sup_{K' \ni X,\ K' \subset 2\overline{K}_j} \frac{1}{|K'|} \int_{K'} |Tf|^{p_0}$$

for $X \in 2\overline{K}_j$.

Let $X \in K_j$ and K a parabolic cylinder containing X. If $K \subset 2\overline{K}_j$

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \leq M_{2\overline{K}_j}(|Tf|^{p_0})(X)$$

and (D.10) holds. Suppose now $K \not\subseteq 2\overline{K}_j$ and let $(\overline{Z}, r)$ and (Z_0, R) center and radius respectively of K and $\overline{K}_j$. We have $r \geq \frac{R}{2}$ indeed, if $r < \frac{R}{2}$ and $Y \in K$, we have

$$\begin{aligned} d(Y, Z_0) &\leq d(Y, \overline{Z}) + d(\overline{Z}, Z_0) < r + d(\overline{Z}, X) + d(X, Z_0) \\ &< r + r + R < \frac{R}{2} + \frac{R}{2} + R = 2R \end{aligned}$$

and then $K \subseteq 2\overline{K}_j$ which is a contradiction. It is easy to check that $\widetilde{K}(\overline{Z}, 5r) \supseteq \overline{K}_j(Z_0, R)$. In fact, let $Y \in \overline{K}_j$, then

$$\begin{aligned} d(Y,Z) &\le d(Y,X) + d(X,\overline{Z}) \le d(Y,Z_0) + d(Z_0,X) + d(X,\overline{Z}) \\ &< R + R + r < 5r, \end{aligned}$$

therefore $Y \in \widetilde{K}(\overline{Z}, 5r)$. By (D.9) we have

$$\frac{1}{|\widetilde{K}|} \int_{\widetilde{K}} |Tf|^{p_0} \le \lambda$$

and, since $(5r)^{n+2} = |\widetilde{K}| = 5^{n+2}|K|$,

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \le \frac{5^{n+2}}{|\widetilde{K}|} \int_{\widetilde{K}} |Tf|^{p_0} \le 5^{n+2}\lambda$$

which ends the proof of (D.10).
Let now $X \in K_j \cap E(A\lambda)$, then

$$\max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} = M_{2\overline{K}_j}(|Tf|^{p_0})(X)$$

because if not, since $A \ge 5^{n+2}$, by (D.10) we have

$$5^{n+2}\lambda = \max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} \ge M(|Tf|^{p_0})(X) > A\lambda \ge 5^{n+2}\lambda$$

and this is a contradiction. Then $M_{2\overline{K}_j}(|Tf|^{p_0}) = M(|Tf|^{p_0})$ in $K_j \cap E(A\lambda)$ and

$$\begin{aligned} |K_j \cap E(A\lambda)| &= |\{X \in K_j : M(|Tf|^{p_0})(X) > A\lambda\}| \\ &= |\{X \in K_j : M_{2\overline{K}_j}(|Tf|^{p_0})(X) > A\lambda\}|. \end{aligned}$$

Let $\eta \in C_c^\infty(\mathbb{R}^{n+1})$ such that $0 \le \eta \le 1$, $\eta = 1$ in $2\alpha_2\overline{K}_j$ e $\eta = 0$ in $\mathbb{R}^{n+1} \backslash 3\alpha_2\overline{K}_j$. Split f as follows:

$$f = \eta f + (1-\eta)f.$$

The support of $(1-\eta)f$ is contained in $\mathbb{R}^{n+1} \setminus 2\alpha_2\overline{K}_j$. Since T is sublinear,

$$|Tf|^{p_0} \le 2^{p_0-1}\left(|T(\eta f)|^{p_0} + |T((1-\eta)f)|^{p_0}\right)$$

and, since the maximal operator is sublinear,

$$M_{2\overline{K}_j}(|Tf|^{p_0}) \le 2^{p_0-1} M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) + 2M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0}).$$

It follows

$$
\begin{aligned}
&|K_j \cap E(A\lambda)| = |\{X \in K_j : M_{2\overline{K}_j}(|Tf|^{p_0})(X) > A\lambda\}| \\
\leq\ & |\{X \in K_j : M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) + M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0} > \frac{A\lambda}{2^{p_0-1}}\}| \\
\leq\ & |\{X \in K_j : M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) > \frac{A\lambda}{2^{p_0}}\}| \\
+\ & |\{X \in K_j : M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0}) > \frac{A\lambda}{2^{p_0}}\}| \\
\leq\ & \frac{C}{A\lambda}\int_{2\overline{K}_j} |T(\eta f)|^{p_0}) + \frac{C}{(A\lambda)^{\frac{q_0}{p_0}}}\int_{2\overline{K}_j} |M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0})|^{\frac{q_0}{p_0}} \\
\leq\ & \frac{C}{A\lambda}\int_{2\overline{K}_j} |T(\eta f)|^{p_0} + \frac{C}{(A\lambda)^{\frac{q_0}{p_0}}}\int_{2\overline{K}_j} |T((1-\eta)f)|^{q_0}
\end{aligned}
$$

with C depending on $n,\ p_0,\ q_0$. The last two addenda have been obtained estimating the previous ones using respectively the local maximal Hardy-Littlewood inequality (D.3) and the Chebychev inequality. Moreover the second addendum has been estimated using the boundedness of the local maximal operator (see (D.4)).
By the boundedness in L^{p_0}, the sublinearity of T and the hypothesis we obtain

$$
\begin{aligned}
&|K_j \cap E(A\lambda)| \\
&\quad \leq \frac{C}{A\lambda}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Big\{ \left(\frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} |T((1-\eta)f)|^{p_0}\right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K'\supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |(1-\eta)f|^{p_0}\right)^{\frac{1}{p_0}} \Big\}^{q_0} \leq \frac{C}{A\lambda}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} \\
&\quad + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Big\{ \left(\frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} (|Tf|^{p_0} + |T(\eta f)|^{p_0})\right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K'\supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |(1-\eta)f|^{p_0}\right)^{\frac{1}{p_0}} \Big\}^{q_0} \leq \frac{C}{A\lambda}\frac{|3\alpha_2\overline{K}_j|}{|3\alpha_2\overline{K}_j|}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} \\
&\quad + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Big\{ \left(\frac{1}{|3\alpha_2\overline{K}_j|}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} + \frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} |Tf|^{p_0}\right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K'\supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |f|^{p_0}\right)^{\frac{1}{p_0}} \Big\}^{q_0}.
\end{aligned}
$$

Observe that, since $\alpha_i > 1$, $\alpha_i \overline{K}_j \supset \overline{K}_j$, then by (D.9)

$$|K_j \cap E(A\lambda)| \leq C|\overline{K}_j| \left\{ \frac{\gamma\lambda}{A\lambda} + \left(\frac{\gamma\lambda + \lambda}{A\lambda} \right)^{\frac{q_0}{p_0}} \right\} \leq C|\overline{K}_j| \left\{ \frac{\gamma}{A} + \left(\frac{1}{A} \right)^{\frac{q_0}{p_0}} \right\}$$
$$= C|\overline{K}_j| \left\{ 2\gamma\delta^{\frac{p_0}{p}} + \left(2\delta^{\frac{p_0}{p}} \right)^{\frac{q_0}{p_0}} \right\} = \delta|K_j|C \left\{ 2\gamma\delta^{\frac{p_0}{p}-1} + 2^{\frac{q_0}{p_0}}\delta^{\frac{q_0}{p}-1} \right\}$$

where $C = C(n, p_0, q_0, \alpha_1, \alpha_2)$. If we choose δ small enough such that

$$C 2^{\frac{q_0}{p_0}} \delta^{\frac{q_0}{p}-1} \leq \frac{1}{2}$$

(this is possible since $\dfrac{q_0}{p} > 1$) and $A = \dfrac{1}{2\delta^{\frac{p_0}{p}}} \geq 5^{n+2}$ and γ such that

$$2C\gamma\delta^{\frac{q_0}{p}-1} \leq \frac{1}{p_0}$$

we obtain

$$|K_j \cap E(A\lambda)| \leq \delta|K_j|.$$

This contradicts the properties of the Calderón-Zygmund decomposition and proves the assertion in Step 1.

Step 2

There exist $0 < \gamma < 1$, $0 < \delta < 1/2^{\frac{p}{p_0}}$ such that

$$|E(A\lambda)| \leq \delta|E(\lambda)| + |\{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) > \gamma\lambda\}| \qquad \text{(D.11)}$$

for every $\lambda > 0$.

Proof (*Step 2*). Let $\{\overline{K}_j\}$ a disjoint subcover of $E(A\lambda) \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$ with the property that

$$\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset.$$

A such subcover exists in fact by property (1) of the Calderön-Zygmund decomposition there exists a family K_j of disjoint cylinders such that tale che

$$|E(A\lambda) \setminus \cup_j K_j| = 0$$

and each K_j is obtained by the dyadic division of a cylinder $\overline{K}_j$. Therefore we can cover $E(A\lambda)$ with the dyadic parents of each K_j. In order to have disjoint cylinders $\overline{K}_j$, if K_r, K_s have the same parent, we include it only one time, if $\overline{K}_r \subset \overline{K}_s$ we take $\overline{K}_s$. Reject finally all the cylinders that don't intersect $\{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$.

By Step 1,

$$\begin{aligned} |E(A\lambda) \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t)| \leq \gamma\lambda\} &\leq \sum_j |E(A\lambda) \cap \overline{K}_j| \\ &\leq \delta \sum_j |\overline{K}_j| \leq \delta|E(\lambda)|. \end{aligned}$$

Hence

$$\begin{aligned}|E(A\lambda)| &\leq |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)|\leq\gamma\lambda\}| \\ &+ |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)|>\gamma\lambda\}| \\ &\leq \delta|E(\lambda)| + |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)|>\gamma\lambda\}|\end{aligned}$$

and the statement in Step 2 is proved.
Step 3
We finally deduce the L^p boundedness of T from the results proved in the previous steps.
For every $\lambda_0 > 0$

$$\begin{aligned}\int_0^{A\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda &\leq \int_0^{A\lambda_0}\lambda^{\frac{p}{p_0}-1}\big[\delta\big|E\big(\frac{\lambda}{A}\big)\big| \\ &+\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)>\frac{\gamma\lambda}{A}\big\}\big|d\lambda\big] \\ &=\delta\int_0^{A\lambda_0}\lambda^{\frac{p}{p_0}-1}\big|E\big(\frac{\lambda}{A}\big)\big|d\lambda \\ &+\int_0^{A\lambda_0}\lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)>\frac{\gamma\lambda}{A}\big\}\big|d\lambda \\ &=\delta A^{\frac{p}{p_0}}\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \\ &+\left(\frac{A}{\gamma}\right)^{\frac{p}{p_0}}\int_0^{\lambda_0\gamma}\lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)>\lambda\big\}\big|d\lambda \\ &\leq\delta A^{\frac{p}{p_0}}\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \\ &+\left(\frac{A}{\gamma}\right)^{\frac{p}{p_0}}\int_0^{\infty}\lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)>\lambda\big\}\big|d\lambda \\ &=\delta A^{\frac{p}{p_0}}\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|M(|f|^{p_0})|^{\frac{p}{p_0}} \\ &\leq\delta A^{\frac{p}{p_0}}\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p\end{aligned}$$

where we used (D.11), Lemma D.1.2 and Corollary D.1.8 (observe that $\frac{p}{p_0} > 1$).
Recall that $A = \dfrac{1}{2\delta^{\frac{p_0}{p}}} > 1$ and $\delta A^{\frac{p}{p_0}} = \dfrac{1}{2^{\frac{p}{p_0}}} < 1$. By the inequalities above

$$\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq \frac{1}{2^{\frac{p}{p_0}}}\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p$$

which implies

$$\left(1-\frac{1}{2^{\frac{p}{p_0}}}\right)\int_0^{\lambda_0}\lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p$$

and, changing the constant C,

$$\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p.$$

Almost everywhere it holds

$$|Tf|^{p_0}(x,t) > \lambda \Rightarrow M(|Tf|^{p_0})(x,t) > \lambda$$

because

$$\begin{aligned} M(|Tf|^{p_0})(x,t) &= \sup_{K\ni(x,t)=X}\frac{1}{|K|}\int_K |Tf|^{p_0}(Y)dY \\ &\geq \frac{1}{|K(X,R)|}\int_K |Tf|^{p_0}(Y)dY \end{aligned}$$

for every $R > 0$ and

$$\frac{1}{|K(X,R)|}\int_K |Tf|^{p_0}(Y)dY \to |Tf|^{p_0}(X)$$

almost everywhere by the Lebesgue Theorem. Therefore we have

$$\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|\{|Tf|^{p_0} > \lambda\}|d\lambda \leq \int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p. \tag{D.12}$$

Moreover $\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda$ is finite indeed, by the maximal Hardy-Littlewood inequality, $B = \sup_{\lambda>0}\lambda|E(\lambda)| < \infty$, this implies $\lambda^{\frac{p}{p_0}-1}|E(\lambda)| \leq B\lambda^{\frac{p}{p_0}-2}$ which is integrable near zero for $2 - \dfrac{p}{p_0} < 1 \Leftrightarrow p > p_0$. Letting λ_0 to $+\infty$ in (D.12) we obtain

$$\int_0^{\infty} \lambda^{\frac{p}{p_0}-1}|\{|Tf|_0^p > \lambda\}|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p$$

and, by Lemma D.1.2,

$$\int_{\mathbb{R}^{n+1}}|Tf|^p \leq C\int_{\mathbb{R}^{n+1}}|f|^p.$$

□

Remark D.1.17. By the proof, it follows that it is sufficient to require that the inequality in the assumption of Theorem D.1.1 is verified for all $f \in C_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$.

D.2 An application of Shen's Theorem

The boundeness result for operators just proved allows us to give an alternative proof of the classical a-priori estimates for the operator $\partial_t - \Delta$.
In this Section we will denote by X the space $(\partial_t - \Delta)C_c^\infty(\mathbb{R}^{N+1})$.

Proposition D.2.1. *Let $1 < p < \infty$. There exist $C_1,\ C_2 > 0$ such that*

$$\|D_{ij}(\partial_t - \Delta)^{-1} g\|_p \le C_1 \|g\|_p$$

and

$$\|\partial_t(\partial_t - \Delta)^{-1} g\|_p \le C_2 \|g\|_p$$

for all $1 \le i, j \le N$ and for all $g \in X$.

Theorem D.2.2. *Let $1 < p < \infty$. Then there exists $C > 0$ such that*

$$\|D^2 u\|_p + \|\partial_t u\|_p \le C \|\partial_t u - \Delta u\|_p \tag{D.13}$$

for all $u \in W_p^{2,1}(\mathbb{R}^{N+1})$.

PROOF. Let $u \in C_c^\infty(\mathbb{R}^{N+1})$, then $u = (\partial_t - \Delta)^{-1}(\partial_t - \Delta)u$ and $g = (\partial_t - \Delta)u \in X$. By proposition D.2.1 we obtain the claimed inequality for test functions. By density the estimate follows for the functions in $W_p^{2,1}(\mathbb{R}^{N+1})$. $\square$

Lemma D.2.3. *The space X is dense in $L^2(\mathbb{R}^{N+1})$.*

PROOF. Denote by $\mathcal{S}(\mathbb{R}^{N+1})$ the Schwartz space and by $\widehat{g}$ the Fourier transform of a function g. First let us prove that $(\partial_t - \Delta)\mathcal{S}(\mathbb{R}^{N+1})$ is dense in $L^2(\mathbb{R}^{N+1})$. Let $v \in L^2(\mathbb{R}^{N+1})$ orthogonal to $(\partial_t - \Delta)u$ for all u in $\mathcal{S}(\mathbb{R}^{N+1})$. We claim that $v \equiv 0$. We have

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)(i\tau + |\xi|^2)\widehat{u}(\xi,\tau) = 0$$

for all $u \in \mathcal{S}(\mathbb{R}^{N+1})$ and then

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2}(1 + i\tau + |\xi|^2)\widehat{u}(\xi,\tau) = 0$$

for all $u \in \mathcal{S}(\mathbb{R}^{N+1})$. The operator $I + \partial_t - \Delta : \mathcal{S}(\mathbb{R}^{N+1}) \to \mathcal{S}(\mathbb{R}^{N+1})$ is surjective, therefore by the previous equality we deduce

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2} w(\xi,\tau) = 0$$

for all $w \in \mathcal{S}(\mathbb{R}^{N+1})$ and then

$$\widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2} \equiv 0$$

almost everywhere in $\mathbb{R}^{N+1}$. This implies $v \equiv 0$. Observe now that X is dense in $(\partial_t - \Delta)\mathcal{S}(\mathbb{R}^{N+1})$ indeed if $f = \partial_t u - \Delta u$ with $u \in \mathcal{S}(\mathbb{R}^{N+1})$ then it can be approximated in the L^2 norm by the sequence $(\partial_t(\eta_n u) - \Delta(\eta_n u))$ where $\eta_n(x,t) = \eta\left(\frac{x}{n}, \frac{t}{n}\right)$ with $\eta \in C_c^\infty(\mathbb{R}^{N+1})$, $0 \le \eta \le 1$, $\eta = 1$ if $|(x,t)| \le 1$ and $\eta = 0$ if $|(x,t)| \ge 2$. $\square$

Proof (Proposition D.2.1). Let $1 \le i, j \le N$. Consider the operators $T_1 = D_{ij}(\partial_t - \Delta)^{-1}$ and $T_2 = \partial_t(\partial_t - \Delta)^{-1}$ from X to $C_c^\infty(\mathbb{R}^{N+1})$. By Lemma D.2.3, T_1 and T_2 extend by density to $L^2(\mathbb{R}^{N+1})$ and in particular they are defined on $C_c^\infty(\mathbb{R}^{N+1})$. By Shen's Theorem, applied in correspondence of $p_0 = 2$, we will deduce the boundedness of these operators in L^p, for $2 \le p < \infty$ and then, by duality, the boundedness for $1 < p \le 2$.
Let us prove now the boundedness in L^2 of T_1 and T_2. Let $f \in X$. We have

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f}$$

and then

$$\|T_1 f\|_2 = \|\widehat{T_1 f}\|_2 \le \|\widehat{f}\|_2 = \|f\|_2.$$

Similarly the T_2 boundedness in L^2 follows. Prove now the inequality in the assumptions of Shen's Theorem.
Let $\alpha_2 > \alpha_1 > 1$, $K \subset \mathbb{R}^{N+1}$ parabolic cylinder and $f \in C_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$. We have

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f}.$$

Set $v = T_1 f$. Since $f \in C_c^\infty(\mathbb{R}^{n+1})$, f and $\widehat{f} \in \mathcal{S}(\mathbb{R}^{N+1})$, it follows that

$$-(1 + |(\xi,\tau)|^2)^k \frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f} = (1 + |(\xi,\tau)|^2)^k \widehat{v} \in L^2(\mathbb{R}^{N+1})$$

for all $k \in \mathbb{N}$ and then $v \in H^k(\mathbb{R}^{N+1})$ for all $k \in \mathbb{N}$. This proves that $v \in C^\infty(\mathbb{R}^{N+1})$. Moreover $\partial_t v - \Delta v = D_{ij} f$ and $\partial_t v - \Delta v = 0$ in $\alpha_2 K$ since $f = 0$ in $\alpha_2 K$. In the same way one can prove that $T_2 f$ satisfies the same equation.
Let K be a parabolic cylinder with center (x_0, t_0) and radius R. We will prove that, for all $p \ge 2$, there exists $C > 0$ such that , if $v \in C^\infty$ solves $\partial_t v - \Delta v = 0$ in $\alpha_2 K$, then

$$\left(\frac{1}{|K|}\int_K |v|^p\right)^{\frac{1}{p}} \le C\left(\frac{1}{|\alpha_1 K|}\int_{\alpha_1 K} |v|^2\right)^{\frac{1}{2}}.$$

Observe that it is sufficient to prove

$$\left(\int_{K_1} |w|^p\right)^{\frac{1}{p}} \le C\left(\int_{\alpha_1 K_1} |w|^2\right)^{\frac{1}{2}}$$

for w smooth solution of $\partial_t w - \Delta w = 0$ in $\alpha_2 K_1$ with $K_1 = K_1((x_0, t_0), 1)$ cylinder with unitary radius. Infact let v such that $\partial_t v - \Delta v = 0$ in $\alpha_2 K$ and set $w(x,t) = v(Rx - (R-1)x_0, R^2 t - (R^2 - 1)t_0)$. Then $\partial_t w - \Delta w = 0$ in $\alpha_2 K_1$. Moreover

$$\left(\int_{K_1} |w(x,t)|^p\right)^{\frac{1}{p}} \le C\left(\int_{\alpha_1 K_1} |w(x,t)|^2\right)^{\frac{1}{2}}$$

implies

$$\left(\int_{K_1}|v(Rx-(R-1)x_0,R^2t-(R^2-1)t_0)|^p\right)^{\frac{1}{p}}\leq$$

$$C\left(\int_{\alpha_1 K_1}|v(Rx-(R-1)x_0,R^2t-(R^2-1)t_0)|^2\right)^{\frac{1}{2}}$$

and, setting $\tau=R^2t-(R^2-1)t_0$, $\xi=Rx-(R-1)x_0$,

$$\left(\frac{1}{R^{n+2}}\int_K|v|^p\right)^{\frac{1}{p}}\leq C\left(\frac{1}{R^{n+2}}\int_{\alpha_1 K}|v|^2\right)^{\frac{1}{2}}$$

which is the estimate for general cylinders.
Let K be a parabolic cylinder of radius 1, w such that $\partial_t w-\Delta w=0$ in $\alpha_2 K$ and $1\leq a<b\leq\alpha_1<\alpha_2$. Let $0\leq\eta\leq 1$ be a smooth function such that $\eta=1$ in aK and $\eta=0$ in $\mathbb{R}^{N+1}\setminus bK$. We write K as $Q\times I$ where Q is the cube in the space $\mathbb{R}^N$ and I the time interval, we multiply the equation satisfied by w times $\eta^2 w$ and we integrate both members with respect to the space variable x on bQ. We obtain

$$\int_{bQ}w_t\eta^2 w+\int_{bQ}\eta^2|\nabla w|^2+2\int_{bQ}w(\nabla w)\eta\nabla\eta=0$$

and, writing the first integral in different way,

$$\frac{1}{2}\frac{d}{dt}\int_{bQ}\eta^2w^2-\int_{bQ}w^2\eta\eta_t+\int_{bQ}\eta^2|\nabla w|^2+2\int_{bQ}w(\nabla w)\eta\nabla\eta=0.$$

Integrate now with respect to the time variable on I. For all $\varepsilon>0$, we hawe

$$\begin{aligned}\int_{bK}\eta^2|\nabla w|^2 &\leq \int_{bK}|w^2\eta\eta_t|+2\left(\int_{bK}\eta^2|\nabla w|^2\right)^{\frac{1}{2}}\left(\int_{bK}w^2|\nabla\eta|^2\right)^{\frac{1}{2}}\\ &\leq C\int_{bK}|w|^2+\varepsilon^2\int_{bK}\eta^2|\nabla w|^2+\frac{1}{\varepsilon^2}\int_{bK}w^2|\nabla\eta|^2.\end{aligned}$$

Choosing ε small enough,

$$\int_{bK}\eta^2|\nabla w|^2\leq C\int_{bK}|w|^2$$

and, since $\eta=1$ on aK,

$$\int_{aK}|\nabla w|^2\leq C\int_{bK}|w|^2.$$

Note that, for every β multi-index,

$$\partial_t(D^\beta w)-\Delta(D^\beta w)=0$$

in $\alpha_2 K$ and, by the previous computations,

$$\int_{aK} |D^\gamma w|^2 \leq C \int_{bK} |D^\beta w|^2 \tag{D.14}$$

for γ multi-index of lenght $|\gamma| = |\beta| + 1$ (with D^γ we mean the derivatives of order γ with respect to the space variable). Choose α multi-index of lenght $m = |\alpha| > N + 1$ and divide the interval $[1, \alpha_1]$ in m intervals $[a_i, b_i]$ with $1 = a_1 < b_1 < a_2 < \ldots < a_m < b_m = \alpha_1$. Applying (D.14) iteratively to $[a_i, b_i]$, we obtain

$$\int_K |D^\alpha w|^2 \leq C \int_{\alpha_1 K} |w|^2$$

and

$$\int_K |D^\mu w|^2 \leq C \int_{\alpha_1 K} |w|^2$$

for all μ multi-index of lenght less than m. Moreover, since

$$\partial_t^{\frac{\alpha}{2}} w = \Delta^\alpha w,$$

$$\int_K |\partial_t^\alpha w|^2 \leq C \int_{\alpha_1 K} |w|^2.$$

We obtained

$$\|w\|_{W_2^{\frac{N+1}{2}}(K)} \leq \|w\|_{L^2(\alpha_1 K)}.$$

By the Sobolev embedding Theorem, $W_2^{\frac{N+1}{2}}(K) \subset L^\infty(K)$, it follows that

$$\|w\|_{L^\infty(K)} \leq \|w\|_{L^2(\alpha_1 K)}$$

and

$$\|w\|_{L^p(K)} \leq \|w\|_{L^\infty(K)} \leq \|w\|_{L^2(\alpha_1 K)}$$

for all $1 \leq p \leq \infty$. By Theorem D.1.1, T_1 and T_2 are bounded in $L^p(\mathbb{R}^{N+1})$ for all $2 \leq p < \infty$.

Let $1 < p \leq 2$ and p' such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Consider

$$T_1 : L^2(\mathbb{R}^{N+1}) \to L^2(\mathbb{R}^{N+1})$$

so defined

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2} \widehat{f}.$$

$T_1 = \mathcal{F}^{-1} M_q \mathcal{F}$ where M_q is the multiplication operator with

$$q(\xi, \tau) = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}$$

and $\mathcal{F}$ is the unitary operator that to $f \in L^2(\mathbb{R}^{N+1})$ associates its Fourier transform. Denoted by T_1^* the adjoint operator of T_1, we have

$$T_1^* = \mathcal{F}^{-1} M_{\overline{q}} \mathcal{F}$$

with $M_{\overline{q}}$ multiplication operator and $\overline{q}(\xi, \tau) = -\dfrac{\xi_i \xi_j}{-i\tau + |\xi|^2}$. Observe that, if $f \in X$, $T_1^* f = D_{ij}(-\partial_t - \Delta)^{-1} f$ and, since we are considering the heat operator all over $\mathbb{R}^{N+1}$, T_1^* enjoies the same properties of T_1. Let $f, g \in C_c^\infty(\mathbb{R}^{N+1})$. Obvioulsy $2 \leq p' < \infty$. By the first part of the proof, there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^{N+1}} (T_1 f) g \right| = \left| \int_{\mathbb{R}^{N+1}} f (T_1^* g) \right| \leq C \|f\|_p \|g\|_{p'}.$$

It follows that $\|T_1 f\|_p \leq \|f\|_p$. In similar way one can prove the same result for T_2. □

If u does not depend on the time variable, the following elliptic version of the Calderón- Zygmund Theorem immediately follows.

Theorem D.2.4. *Let $1 < p < \infty$. There exists C positive constant such that*

$$\|D^2 u\|_p \leq C \|\Delta u\|_p$$

for all $u \in W^{2,p}(\mathbb{R}^N)$.

Anyway, by means of the mean value Theorem for harmonic functions, an alternative direct proof gives the same result.

Proposition D.2.5. *Let $1 < p < \infty$. There exists $C > 0$ such that*

$$\|D_{ij}(\Delta)^{-1} g\|_p \leq C \|g\|_p$$

for all $1 \leq i, j \leq N$ and for all $g \in \Delta(C_c^\infty(\mathbb{R}^N))$.

As before, the following lemma can be proved.

Lemma D.2.6. *The space $\Delta(C_c^\infty(\mathbb{R}^N))$ is dense in $L^2(\mathbb{R}^N)$.*

Proof (Proposition D.2.5). Let $1 \leq i, j \leq N$. Consider the operator $T = D_{ij}(\Delta)^{-1}$ from $\Delta(C_c^\infty(\mathbb{R}^N))$ to $C_c^\infty(\mathbb{R}^N)$. By Lemma D.2.6, T extends by density to all $L^2(\mathbb{R}^N)$.
As in the parabolic case the L^2 boundedness follows by using the Fourier transform. Let us prove the assumption in Shen's Theorem.
Choose $\alpha_2 = 4,\ \alpha_1 = 2$. Let $Q \subset \mathbb{R}^N$ and $f \in C_c^\infty(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N \setminus 4Q$. Set $v = Tf$. As in the parabolic case we have $v \in C^\infty(\mathbb{R}^N)$ and $\Delta v = D_{ij} f$. Since $f = 0$ in $4Q$, $\Delta v = 0$ in $4Q$. Suppose $Q = Q(y, R)$, consider

the ball $B(y,R)$. Obviously $B(y,R) \subset Q(y,R)$ and $\Delta v = 0$ in $4B(y,R)$. By the mean value Theorem for harmonic functions

$$v(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} v(z)dz$$

for all $x \in 4B(y,R)$, $r > 0$ such that $B(x,r) \subset 4B(y,R)$. Note that if $x \in B(y,R)$ then $B(x,R) \subset B(y,2R)$ and

$$\begin{aligned} v(x) &= \frac{1}{|B(x,R)|} \int_{B(x,R)} v(z)dz \leq \frac{C}{|B_R|^{\frac{1}{2}}} \left(\int_{B(x,R)} |v|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C}{|B_R|^{\frac{1}{2}}} \left(\int_{B(y,2R)} |v|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Let $p > 2$. By taking the p-power and integrating over $B(y,R)$,

$$\frac{1}{|B_R|} \int_{B(y,R)} |v|^p \leq \frac{C}{|B_R|^{\frac{p}{2}}} \left(\int_{B(y,2R)} |v|^2 \right)^{\frac{p}{2}}.$$

By Theorem D.1.1 the boundedness of T in L^p for $2 \leq p < \infty$ follows and then by duality we deduce the boundedness in L^p for $1 < p \leq 2$. □