Appendix C

An inequality in Sobolev spaces

The aim of the Appendix is to study the validity of the inequality

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u \leq 0$$

for functions $u \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. Actually a more precise result can be proved, the following equality that one formally obtains integrating by parts holds

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u = -(p-1)\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}. \tag{C.1}$$

If $p \geq 2$ and $u \in W^{2,p}(\mathbb{R}^N)$, then the function $u|u|^{p-2}$ belongs to $W^{2,p'}(\mathbb{R}^N)$, where p' is the conjugated exponent of p. Therefore integration by parts is allowed in the left hand side of (C.1) and the stated equality follows, in particular the inequality which we need is proved too. On the other hand, the situation is more complicated for $1 < p < 2$ due to the presence of the singularity of $|u|^{p-2}$ near the zeros of u. An analogous result remains true for more general elliptic operators in divergence form. Since in our proofs we need only the negativity of the right hand side, here we deduce it by elementary computations.The proof of the equality is more involved and requires a sectional characterization of Sobolev spaces, we refer to [32] for a detailed study of the subject.
We focus our attention on the case $1 < p < 2$ since, as observed, for $p \geq 2$ the equality immediately follows.

Proposition C.0.14. *Let $1 < p < 2$, $u \in C_0^2(\mathbb{R}^N)$, then u satisfies*

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u = -(p-1)\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}.$$

PROOF. Given $\delta > 0$, set

$$u_\delta := u(u^2+\delta)^{\frac{p-2}{2}} \in C_0^2(\mathbb{R}^N).$$

We can apply the integration by parts formula to the functions u_δ to deduce

$$\begin{aligned} & \int_{\mathbb{R}^N} u(u^2+\delta)^{\frac{p-2}{2}}\Delta u = \int_{\mathbb{R}^N} u_\delta \Delta u = -\int_{\mathbb{R}^N} \nabla u \nabla u_\delta \\ = & -\int_{\mathbb{R}^N} |\nabla u|^2 (u^2+\delta)^{\frac{p-4}{2}}((p-1)u^2+\delta). \end{aligned} \tag{C.2}$$

Observe that, for $\delta \to 0$,

$$u_\delta \Delta u \to u|u|^{p-2}\Delta u$$

pointwise and, since $p < 2$,

$$|u_\delta \Delta u| \leq |u|^{p-1}|\Delta u| \in L^1(\mathbb{R}^N).$$

Moreover

$$(u^2+\delta)^{\frac{p-4}{2}}((p-1)u^2+\delta)|\nabla u|^2 \to (p-1)|u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}$$

for $\delta \to 0$ almost everywhere, since $\nabla u = 0$ almost everywhere on $\{u = 0\}$ by Stampacchia's Lemma. By Fatou's Lemma and dominated convergence Theorem, we obtain

$$\begin{aligned} (p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2}\chi_{\{u\neq 0\}} & \leq \liminf_{\delta\to 0} -\int_{\mathbb{R}^N} u(u^2+\delta)^{\frac{p-2}{2}}\Delta u \\ & = -\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u \end{aligned}$$

and then $|\nabla u|^2|u|^{p-2}\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N)$. Recalling that $1 < p < 2$, we have

$$(p-1)u^2(u^2+\delta)^{\frac{p-4}{2}}|\nabla u|^2 \leq (p-1)|u|^{p-2}|\nabla u|^2\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N);$$

$$\delta(u^2+\delta)^{\frac{p-4}{2}}|\nabla u|^2 \leq (u^2+\delta)^{\frac{p-2}{2}}|\nabla u|^2 \leq |u|^{p-2}|\nabla u|^2\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N).$$

Applying the dominated convergence Theorem again in (C.2), the claim follows. □

The desired inequality for functions in $W^{2,p}(\mathbb{R}^N)$ immediately follows by the last proposition.

Corollary C.0.15. *Let $u \in W^{2,p}(\mathbb{R}^N)$, $1 < p < 2$. Then*

$$(p-1)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\chi_{\{u\neq 0\}} \leq -\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u < \infty$$

and, in particular,

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u \leq 0.$$

Proof. Let $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{2,p}(\mathbb{R}^N)$, $u_n \to u$, $\nabla u_n \to \nabla u$ almost everywhere in $\mathbb{R}^N$. Therefore

$$|\nabla u_n|^2 |u_n|^{p-2} \chi_{\{u_n \neq 0\}} \chi_{\{u \neq 0\}} \to |\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}}$$

almost everywhere. By Fatou's Lemma, Proposition (C.0.14) and observing that $u_n|u_n|^{p-2} \to u|u|^{p-2}$ in $L^{p'}$, we deduce

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}} &\leq -\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n |u_n|^{p-2} \Delta u_n \\ &= -\int_{\mathbb{R}^N} u|u|^{p-2} \Delta u.\end{aligned}$$

□