Appendix A

Embedding Theorems and Solvability of Cauchy problems

In this appendix we only state some results about embeddings of parabolic Sobolev spaces and solvability of Cauchy problems in the same spaces useful to prove integrability and regularity of kernels in Section 2.1.2.
For their proofs we refer to [23, Lemma II.3.3, Theorem IV.9.1] and [20, Theorem 9.2.3].

According to notation used in [23], we introduce the norm

$$\|f\|_{r,q_T}^{loc} = \sup_{q_T} \|f\|_{L^r(q_T)}$$

where the supremum is taken over all the cylinders $q_T = \omega \times (0,T)$, the bases ω of which are some domain of unit measure, for examples cubes of $\mathbb{R}^N$. We consider the elliptic operator A given by $Au(x,t) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x,t) + \sum_{i=1}^N F_i(x)D_iu(x,t) - V(x)u(x,t)$ with V positive.

We recall that the parabolic distance between the points $X_1 = (x_1,t_1)$ and $X_2 = (x_2,t_2)$ is defined as

$$d(X_1,X_2) = \max\{|x^i - x_0^i|,\ 1 \le i \le N,\ |t-t_0|^{\frac{1}{2}}\}.$$

If u is a function defined on $\mathbb{R}^{N+1}$, given $\delta \in (0,1)$, we denote

$$[u]_{\delta,\frac{\delta}{2};Q_T} = \sup_{X_1 \neq X_2, X_i \in Q_T} \frac{|u(X_1)-u(X_2)|}{d(X_1,X_2)^\delta};$$

$$|u|_{\delta,\frac{\delta}{2};Q_T} = \sup \|u\|_{\infty,Q_T} + [u]_{\delta,\frac{\delta}{2};Q_T}.$$

By $C^{\delta,\frac{\delta}{2}}(Q_T)$ we denote the space of the functions u for which $|u|_{\delta,\frac{\delta}{2};Q_T}$ is finite. If u is a function depending only on the space variable we use the analogous notation for the classical Hölder spaces.

Theorem A.0.8. *Let $q > 1$. Suppose that the coefficients a_{ij} of the operator A are bounded continuous functions in Q_T, while the coefficients F_i and V have finite norms $\|F_i\|^{loc}_{r,q_T}$ and $\|V\|^{loc}_{s,q_T}$ with*

$$r = \begin{cases} \max(q, N+2) & \text{for} \quad q \neq N+2 \\ N+2+\varepsilon & \text{for} \quad q = N+2. \end{cases}.$$

$$s = \begin{cases} \max(q, \frac{N+2}{2}) & \text{for} \quad q \neq \frac{N+2}{2} \\ \frac{N+2}{2}+\varepsilon & \text{for} \quad q = \frac{N+2}{2}. \end{cases}.$$

and ε arbitrarily small positive number. Suppose moreover that the quantities $\|F_i\|^{loc}_{r,q(t,t+\tau)}$ and $\|V\|^{loc}_{s,q(t,t+\tau)}$ tend to zero for $\tau \to 0$. Then, for any $f \in L^q(Q_T)$, $\phi \in W_q^{2-\frac{2}{q}}(\mathbb{R}^N)$, the problem

$$\begin{cases} \partial_t u - Au = f & \text{in} \quad Q_T \\ u(x,0) = \phi \end{cases}$$

has a unique solution $u \in W^{2,1}_q(Q_T)$. It satisfies the estimate

$$\|u\|_{W^{2,1}_q(Q_T)} \leq C(\|f\|_{L^q(Q_T)} + \|\phi\|_{W_q^{2-\frac{2}{q}}(\mathbb{R}^N)}).$$

Theorem A.0.9. *For any function $u \in W^{2,1}_q(Q_T)$ the inequality*

$$\|\partial_t^r D_x^s u\|_{p,Q_T} \quad \leq \quad C_1(\|\partial_t u\|_{q,Q_T} + \|D_x^2 u\|_{q,Q_T}) + C_2\|u\|_{q,Q_T}$$

is valid under the condition $p \geq q$, $2-2r-s-\left(\frac{1}{q}-\frac{1}{p}\right)(N+2) \geq 0$ and for some constants C_1, C_2 depending on r, s, N, q, p. In addition, if $2-2r-s-\frac{N+2}{q} > 0$, then for any $0 \leq \delta < 2-2r-s-\frac{N+2}{q}$

$$[\partial_t^r D_x^s u]_{\delta,Q_T} \leq C_3(\|\partial_t u\|_{q,Q_T} + \|D_x^2 u\|_{q,Q_T}) + C_4\|u\|_{q,Q_T}$$

for some constants C_3, C_4 depending on r, s, N, q, p.

Finally we state a solvability result in spaces of Hölder functions used in Section 2.1.2. It can be found in [20, Theorem 9.2.3].

Theorem A.0.10. *Let A be the second order elliptic operator above defined and suppose that a, F, V are Hölder continuous for some $\delta \in (0,1)$ and with $|a|_{\delta;\mathbb{R}^N}$, $|F|_{\delta,\mathbb{R}^N}$, $|V|_{\delta,\mathbb{R}^N} \leq K$. Then, for any $f \in C^{\delta,\frac{\delta}{2}}(Q_T)$, $\phi \in C^{2+\delta}(\mathbb{R}^N)$, the problem*

$$\begin{cases} \partial_t u - Au = f & \text{in} \quad Q_T \\ u(x,0) = \phi \end{cases}$$

has a unique solution $u \in C^{2+\delta,1+\frac{\delta}{2}}(Q_T)$. *It satisfies the estimate*

$$|u|_{2+\delta,1+\frac{\delta}{2};Q_T} \leq C(|f|_{\delta,\frac{\delta}{2};Q_T} + |\phi|_{2+\delta;\mathbb{R}^N}$$

for some positive constant C *depending on* $N, \delta, K,\ \lambda$ *and the ellipticity constant of* A.