Chapter 3

Kernel estimates for a class of Schrödinger semigroups

3.1 Introduction

We consider again a Schrödinger operator $A = -\Delta + V$ with a nonnegative potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and we look for some sharp estimates for the kernel p of the semigroup e^{-tA} generated by the operator $-A$ in $L^p(\mathbb{R}^N)$. As previously observed, the kernel is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$.

In the case $V(x) = |x|^\alpha$, $\alpha > 0$, Sikora proves precise on-diagonal bounds of the form $p(x, x, t) \leq h(x, t)$ and then he deduces off-diagonal bounds from the semigroup law, see [45]. Estimates of the same forme have been deduced in the previous chapter and will be improved here.

In Section 2 we prove Sikora-type bounds for radial increasing potentials and we treat also the case of potentials consisting of a radial part and lower order terms.

In Section 3, we report on some upper and lower bounds obtained by Sikora in suitable space-time regions to show the sharpness of our estimates.

In Section 4, we study the asymptotic distribution of eigenvalues of A using the bounds on the heat kernel of e^{-tA} and a Tauberian theorem due to Karamata. When V has a polynomial behaviour, these results have been proved by Titchmarsh (see [51] or [40, Section XIII]) using cube-decomposition methods. Our approach allows us to treat also non polynomial type potential and this seems to be new.

3.2 Pointwise estimates of kernels

Given a positive potential $V \in L^1_{loc}(\mathbb{R}^N)$, for each $s > 0$ we consider the level set

$$E_s = \{x \in \mathbb{R}^N : V(x) \leq s\}.$$

We introduce a new potential V_s

$$V_s(x) = \begin{cases} s & \text{in } E_s \\ V(x) & \text{in } \mathbb{R}^N \setminus E_s \end{cases}$$

and the heat kernel p_s of the Schrödinger operator $A_s = -\Delta + V_s$.
Let us observe that $V_s \geq s$ and $V_s \geq V$. Therefore by Remark 1.3.21 it follows that

$$0 \leq p_s(x, y, t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \exp\{-ts\} \tag{3.1}$$

and

$$0 \leq p(x, y, t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \tag{3.2}$$

for all $x,\ y \in \mathbb{R}^N$ and $t > 0$. To improve the bound for p, as in [45], we estimate the difference between the kernels p and p_s and then we use the triangle inequality. Sikora used the functional calculus to estimate such a difference. Our approach, though more elementary, yields more precise bounds.

Lemma 3.2.1. *Let $p_s,\ E_s$ as above. Then there exists a positive constant $C = C(N)$ such that for all $x \in \mathbb{R}^N,\ t > 0$*

$$|p_s(x, x, t) - p(x, x, t)| \leq \frac{C}{t^{\frac{N}{2}}} \int_{E_s} \frac{\exp\left\{-\frac{|x-y|^2}{4t}\right\}}{|x-y|^N} dy. \tag{3.3}$$

Remark 3.2.2. Let us observe that the integral in the right hand side above is divergent whenever $x \in E_s$. Therefore (3.3) is meaningful only if $x \notin E_s$.

Proof. Let $u,\ w$ respectively the solutions of

$$\begin{cases} u_t = \Delta u - Vu \\ u(0) = f \end{cases}$$

and

$$\begin{cases} w_t = \Delta w - V_s w \\ w(0) = f. \end{cases}$$

Then the difference $z = u - w$ satisfies $z_t = \Delta z - V_s z - (V - V_s)u,\ z(0) = 0$ and, by the variation of constants formula,

$$z(t) = -\int_0^t e^{-(t-r)A_s}(V - V_s)u(r)dr.$$

Representing the semigroup generated by $-A_s$ in the integral form through the kernel p_s we get

$$z(x,t) = -\int_0^t dr \int_{\mathbb{R}^N} p_s(x,y,t-r)(V(y)-V_s(y))u(y,r)\,dy.$$

Representing now u through the kernel p and using (3.1) and (3.2) we obtain

$$\begin{aligned}
|z(x,t)| &\le \int_0^t dr \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} p_s(x,y,t-r)|V(y)-V_s(y)|p(y,l,r)|f(l)|\,dl \\
&\le \frac{1}{(4\pi)^N}\int_0^t dr \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}\times \\
&|V(y)-V_s(y)| \exp\left\{-\frac{|y-l|^2}{4r}\right\} |f(l)|\,dl.
\end{aligned}$$

By definition $V - V_s = 0$ in $\mathbb{R}^N \setminus E_s$ and $|V - V_s| \le s$ in E_s , then

$$\begin{aligned}
|z(x,t)| \le \frac{s}{(4\pi)^N} &\int_0^t dr \int_{E_s} dy \int_{\mathbb{R}^N} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \\
&\times \exp\{-(t-r)s\} \exp\left\{-\frac{|y-l|^2}{4r}\right\} |f(l)|\,dl.
\end{aligned}$$

On the other hand

$$z(x,t) = u(x,t) - w(x,t) = \int_{\mathbb{R}^N} [p(x,l,t) - p_s(x,l,t)]f(l)dl.$$

Comparing this representation and the estimate above we deduce a bound for the difference of the kernels

$$\begin{aligned}
|p(x,x,t) - p_s(x,x,t)| \le \frac{s}{(4\pi)^N} &\int_{E_s} dy \int_0^t \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \\
&\times \exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr.
\end{aligned}$$

We split the integral over $[0,t]$ as the sum of the integrals over $[0,t/2]$ and $[t/2,t]$. Let us consider the first one. In $[0,t/2]$, $(t-r)^{\frac{N}{2}} \ge \left(\frac{t}{2}\right)^{\frac{N}{2}}$ and $t-r \le t$, therefore $\exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \le \exp\left\{-\frac{|x-y|^2}{4t}\right\}$ and

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr \\
&\le \left(\frac{2}{t}\right)^{\frac{N}{2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \int_0^{\frac{t}{2}} \frac{1}{r^{\frac{N}{2}}} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr.
\end{aligned}$$

Similarly

$$\int_{\frac{t}{2}}^{t} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr$$
$$\leq \left(\frac{2}{t}\right)^{\frac{N}{2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \int_{\frac{t}{2}}^{t} \frac{1}{(t-r)^{\frac{N}{2}}} s\exp\{-(t-r)s\}$$
$$\times \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} dr.$$

The function $g(r) = \frac{1}{r^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4r}\right\}$ attaints its maximum at

$$r = \frac{|x-y|^2}{2N},$$

so $g(r) \leq \left(\frac{2N}{e}\right)^{\frac{N}{2}} \frac{1}{|x-y|^N}$. Therefore

$$\frac{s}{(4\pi)^N} \int_{E_s} dy \int_0^{\frac{t}{2}} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}$$
$$\times \exp\left\{-\frac{|x-y|^2}{4r}\right\} \leq \frac{1}{(4\pi)^N} \left(\frac{4N}{e}\frac{1}{t}\right)^{\frac{N}{2}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\}$$
$$\times \frac{1}{|x-y|^N} \int_0^{\frac{t}{2}} s\exp\{-(t-r)s\}\, dr\, dy$$
$$= C(N)\frac{1}{t^{\frac{N}{2}}} \exp\{-ts\} \left(\exp\left\{\frac{t}{2}s\right\} - 1\right) \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy$$
$$\leq C(N)\frac{1}{t^{\frac{N}{2}}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy.$$

Similar computations yield

$$\frac{s}{(4\pi)^N} \int_{E_s} dy \int_{\frac{t}{2}}^{t} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}$$
$$\exp\left\{-\frac{|x-y|^2}{4r}\right\} dr \leq C(N)\frac{1}{t^{\frac{N}{2}}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy$$

and the proof is complete. $\square$

Theorem 3.2.3. *There exists a positive constant $C = C(N)$ such that for all $s > 0$, $x \in \mathbb{R}^N$, $t > 0$*

$$p(x,x,t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-ts\} + \frac{C}{t^{\frac{N}{2}}} \int_{E_s} \frac{\exp\left\{-\frac{|x-y|^2}{4t}\right\}}{|x-y|^N} dy. \tag{3.4}$$

Proof. The proof easily follows from (3.1) and Lemma 3.2.1. □

Assuming that the Lebesgue measure of the level sets E_s is finite, we deduce the following result.

Corollary 3.2.4. *There exists a positive constant $C = C(N)$ such that for all $s > 0$, $x \in \mathbb{R}^N \setminus E_s$ and $t > 0$*

$$p(x,x,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-ts\} + \frac{C}{t^{\frac{N}{2}}} |E_s| \frac{\exp\left\{-\frac{d(x,E_s)^2}{4t}\right\}}{d(x,E_s)^N} dy. \tag{3.5}$$

The estimate just obtained can be more explicitly written if we ask further assumptions on the potential. In particular, for radial, increasing potentials we have the upper bound stated in the following corollary.

Corollary 3.2.5. *If V is radial and increasing ($|x| < |y|$ implies $V(x) < V(y)$), then for all $x \in \mathbb{R}^N$, $t > 0$, $0 < c < 1$*

$$p(x,x,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-tV(cx)\} + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}.$$

Proof. Let $x \in \mathbb{R}^N$. If we choose $s = V(cx)$, from the assumptions on V we deduce that the level set E_s coincides whit the ball $B(0, c|x|)$. Moreover, since $0 < c < 1$, $x \notin E_s$. Then (3.5) holds and the bound easily follows. □

Potentials like $|x|^\alpha$, $\alpha > 0$, belong to the class of radial, increasing potentials, so from Corollary 3.2.5 we deduce the following upper bound which improves that of [45].

Example 3.2.6. Let $V(x) = M|x|^\alpha$ with $\alpha > 0$, then for all $0 < c < 1$, $x \in \mathbb{R}^N$ and $t > 0$

$$p(x,x,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-tMc^\alpha|x|^\alpha\} + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}$$

where ω_N is the measure of the unitary ball in $\mathbb{R}^N$.

Remark 3.2.7. Similar bounds can be obtained for low-order perturbation of the potentials above, that is if $V(x) = |x|^\alpha + o(|x|^\alpha)$, as $|x| \to \infty$. In fact for every $\varepsilon > 0$ there exist $C_\varepsilon,\ C'_\varepsilon > 0$ such that

$$(1-\varepsilon)|x|^\alpha + C_\varepsilon \le V(x) \le (1+\varepsilon)|x|^\alpha + C'_\varepsilon$$

and then, by Corollary 1.3.21,

$$p(x,x,t) \le e^{-C_\varepsilon t} p_\varepsilon(x,x,t),$$

where p_ε is the heat kernel of the Schrödinger operator with potential $(1-\varepsilon)|x|^\alpha$. By Example 3.2.6, for every $0 < c < 1$,

$$\begin{aligned} p(x,x,t) \quad &\le \quad e^{-C_\varepsilon t}\left\{\frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-t(1-\varepsilon)c^\alpha|x|^\alpha\}\right. \\ &+ \quad \left.\frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}\right\}. \end{aligned}$$

Therefore, given $0 < \widetilde{c} < 1$, it is sufficient to choose $\varepsilon > 0$ such that $c = \frac{\widetilde{c}}{(1-\varepsilon)^{\frac{1}{\alpha}}} < 1$ to obtain

$$p(x,x,t) \le e^{-C_\varepsilon t}\left\{ \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-t\widetilde{c}^\alpha |x|^\alpha\} \right.$$
$$\left. + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{ -\frac{(1-c)^2|x|^2}{4t} \right\} \right\}.$$

Remark 3.2.8. Estimate for potentials going to infinity in a different way in different directions can be, sometimes, easily obtained from the previous results. For example, if $V(x,y) = x^2 + y^4$ in $\mathbb{R}^2$, then the heat kernel is the product of the heat kernels of the two one-dimensional operators $-D^2 + x^2$, $-D^2 + y^4$ which follow into the range of application of Example 3.2.6.

Remark 3.2.9. Using the semigroup law it is possible to deduce from the on-diagonal estimates just obtained some off-diagonal estimates. It is sufficient to recall that

$$p(x,y,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})p(z,y,\frac{t}{2})dz. \tag{3.6}$$

In particular

$$p(x,x,t) = \|p(x,\cdot,\frac{t}{2})\|_{L^2}^2.$$

Therefore

$$p(x,y,t) \le p(x,x,t)^{\frac{1}{2}} p(y,y,t)^{\frac{1}{2}},$$

and applying the on-diagonal bounds one can estimate the right hand side.

3.3 Estimates in space-time regions

Considering suitable space-time regions, one can control the gaussian term in Theorem 3.2.3 and its corollaries with the first addendum. In what follows we consider the operator $A = -\Delta + V$ with $V(x) = |x|^\alpha$ but in a similar way bounds in regions can be obtained for other radial, increasing potentials. Moreover it is possible to prove that in these regions similar lower estimates hold and so the estimates are sharp. We refer to [45] for the next results which, however, we recall and prove here for a future discussion in the next section (see Remark 3.4.3).

In the next result, λ_1 is the first eigenvalue of A.

Proposition 3.3.1. *There exist positive constant c_1, c_2, c_3, c_4, C_1, C_2, C_3, C_4 such that, if $t \le (1+|x|)^{1-\frac{\alpha}{2}}$,*

$$\frac{C_1}{t^{\frac{N}{2}}} \exp\{-c_1 t|x|^\alpha\} \le p(x,x,t) \le \frac{C_2}{t^{\frac{N}{2}}} \exp\{-c_2 t|x|^\alpha\}$$

and, if $t > (1+|x|)^{1-\frac{\alpha}{2}}$,

$$C_3 e^{-\lambda_1 t} \exp\{-c_3|x|^{1+\frac{\alpha}{2}}\} \le p(x,x,t) \le C_4 e^{-\lambda_1 t} \exp\{-c_4|x|^{1+\frac{\alpha}{2}}\}.$$

PROOF. Suppose first $t \leq (1+|x|)^{1-\frac{\alpha}{2}}$.
The upper bound easily follows observing that the gaussian term in Theorem 3.2.3 can be controlled with the first addendum. Indeed for $\alpha \leq 2$ we have

$$\begin{aligned} t|x|^\alpha &\leq (1+|x|)^{1-\frac{\alpha}{2}}|x|^\alpha \leq (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^\alpha \\ &= (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^{\alpha-2}(1+|x|)^2 \\ &= \frac{(1+|x|)^2}{(1+|x|)^{1-\frac{\alpha}{2}}} \leq \frac{2}{(1+|x|)^{1-\frac{\alpha}{2}}} + \frac{2|x|^2}{(1+|x|)^{1-\frac{\alpha}{2}}} \leq 2 + \frac{2|x|^2}{t} \end{aligned}$$

and for $\alpha > 2$

$$\begin{aligned} t|x|^\alpha = t|x|^{\alpha-2}|x|^2 &\leq (1+|x|)^{1-\frac{\alpha}{2}}|x|^{\alpha-2}|x|^2 \\ &\leq (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^{\alpha-2}|x|^2 \leq \frac{|x|^2}{t}. \end{aligned}$$

Concerning the lower bound we refer to [45, Proposition 6.1].
If $t > (1+|x|)^{1-\frac{\alpha}{2}}$, the lower bound follows as in Remark 2.2.13 and the upper bound as in the proof of Proposition 2.2.14. □

Let us now consider small times, say $0 < t \leq 1$. We need also to distinguish between the cases $\alpha < 2$ and $\alpha \geq 2$.

Proposition 3.3.2. *If p is the heat kernel corresponding to the operator $-\Delta + |x|^\alpha$ with $\alpha < 2$ then for every $\varepsilon > 0$ there exist positive constants C_ε and C'_ε such that for $t \leq 1$*

$$\frac{C_\varepsilon}{t^{\frac{N}{2}}} \exp\{-(1+\varepsilon)t|x|^\alpha\} \leq p(x,x,t) \leq \frac{C'_\varepsilon}{t^{\frac{N}{2}}} \exp\{-(1-\varepsilon)t|x|^\alpha\}.$$

PROOF. By Remark 2.2.13 we know that

$$p(x,x,t) \geq \frac{C}{t^{\frac{N}{2}}} \exp\{-t(|x|+1)^\alpha\}.$$

Observe that, given $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$(|x|+1)^\alpha = |x|^\alpha + 1 + o(|x|^\alpha) \leq (1+\varepsilon)|x|^\alpha + M_\varepsilon \leq (1+\varepsilon)|x|^\alpha + \frac{M_\varepsilon}{t}$$

and so the lower bound follows. Concerning the upper bound it is sufficient to choose $c_\varepsilon = (1-\varepsilon)^{\frac{1}{\alpha}}$ in Example 3.2.6 and to observe that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\frac{(1-c_\varepsilon)^2}{4}|x|^2 \geq (1-\varepsilon)|x|^\alpha + C_\varepsilon.$$

□

On the other hand, if $\alpha \geq 2$, $1 - \frac{\alpha}{2} \leq 0$ and $(1+|x|)^{1-\frac{\alpha}{2}} \leq 1$. So, by Proposition 3.3.1, for $0 < t \leq (1+|x|)^{1-\frac{\alpha}{2}}$,

$$p(x,x,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\{-ct|x|^\alpha\}$$

and, for $(1+|x|)^{1-\frac{\alpha}{2}} < t \le 1$,

$$p(x,x,t) \le Ce^{-\lambda_1 t} \exp\{-c|x|^{1+\frac{\alpha}{2}}\}.$$

In any case, if $0 < t \le 1$, $\alpha \ge 2$, we have

$$p(x,x,t) \le \frac{C}{t^{\frac{N}{2}}} \exp\{-ct|x|^{1+\frac{\alpha}{2}}\}$$

for suitable positive constants $C,\ c$.
In the next section we will see that a similar lower bound cannot be true.

We observe that the results just proved improve the ones obtained in the previous chapter.

3.4 The asymptotic distribution of the eigenvalues

In this section we investigate the asymptotic distribution of the eigenvalues of $-\Delta+V$, when $V(x)=|x|^\alpha$ or $V(x)=\exp\{|x|^\alpha\}$. Theorem 3.4.2 and Proposition 3.4.4 can be deduced from [51, Section 17.8] or [40, Section XIII], where the proof is different. Instead of using cube decompositions or pointwise estimates on the resolvent we apply the bounds on the heat kernels obtained in the previous sections. This allows us to treat potentials having more than polynomial growth, see Proposition 3.4.5 which seems to be new.
Denote by

$$0 < \lambda_1 \le \lambda_2 \le ...$$

the eigenvalues of A and, for $\lambda > 0$, let $N(\lambda)$ be the number of λ_j such that $\lambda_j \le \lambda$. From the Spectral Theorem it follows that the eigenvalues of e^{-tA} are $e^{-\lambda_n t}$, $n \in \mathbb{N}$. The following well-known Proposition is usually obtained as a corollary of the classical Mercer's Theorem. For completeness, we provide a simple proof based on the semigroup property of the kernel.

Proposition 3.4.1. *Let $t>0$. Then*

$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

Proof. By the estimates in the previous sections it follows $p(x,x,t) \in L^1(\mathbb{R}^N)$. By the semigroup law and the symmetry of p

$$p(x,y,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})p(y,z,\frac{t}{2})\,dz,$$

in particular

$$p(x,x,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})^2\,dz$$

and
$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \int\int_{\mathbb{R}^N\times\mathbb{R}^N} p(x,z,\frac{t}{2})^2\,dx\,dz.$$
Therefore $p(\cdot,\cdot,\frac{t}{2}) \in L^2(\mathbb{R}^N\times\mathbb{R}^N)$ and the operator
$$T(\frac{t}{2})f(x) = e^{-\frac{t}{2}A}f(x) = \int_{\mathbb{R}^N} p(x,y,\frac{t}{2})f(y)\,dy$$
is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^N)$. It follows that
$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \|p(\cdot,\cdot,\frac{t}{2})\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

$\square$

Let us now define the discrete measure μ on $\mathbb{R}_+$ by $\mu(\lambda) = |\{n : \lambda = \lambda_n\}|$. Then $\mu([0,\lambda]) = N(\lambda)$ and
$$\hat{\mu}(t) = \int_0^\infty e^{-\lambda t}d\mu(\lambda) = \sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_{\mathbb{R}^N} p(x,x,t)\,dx.$$

Theorem 3.4.2. *Let $V(x) = |x|^\alpha$ and $N(\lambda)$ as before. Then*
$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{N\omega_N}{(4\pi)^{\frac{N}{2}}} \frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)} \frac{1}{\alpha}\Gamma\left(\frac{N}{\alpha}\right).$$

Proof. By Proposition 3.4.1
$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_{\mathbb{R}^N} p(x,x,t)\,dx.$$
By Example 3.2.6 there exists $C(N)$ such that for all $0 < c < 1$ and $t > 0$
$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\le \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-tc^\alpha|x|^\alpha\}\,dx \\ &\quad + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N}{(1-c)^N} \int_{\mathbb{R}^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\,dx\right\} \\ &= \frac{1}{(4\pi t)^{\frac{N}{2}}} \frac{1}{t^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} \exp\{-c^\alpha|y|^\alpha\}\,dy \\ &\quad + C(N)\frac{c^N}{(1-c)^N} \int_{\mathbb{R}^N} \exp\left\{-(1-c)^2|y|^2\,dy\right\}.\end{aligned}$$
Therefore for all $0 < c < 1$
$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})} \int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-c^\alpha|x|^\alpha\}\,dx$$

and, letting c to 1,

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx. \tag{3.7}$$

In order to obtain a lower bound we proceed as in [13, Lemma 4.5.9].
If A_D is the operator obtained from A by imposing Dirichlet boundary conditions on the surface of the ball B with center x and radius r then

$$p(x,x,t) \ge p_D(x,x,t).$$

Moreover $V(x) \le (|x|+r)^\alpha$ in $B(x,r)$, so

$$p(x,x,t) \ge \exp\{-t(|x|+r)^\alpha\}p_\Delta(x,x,t)$$

where p_Δ is the heat kernel for the Laplacian on B with Dirichlet boundary conditions. By Kac's principle (see [15])

$$p_\Delta(x,x,t) \ge c(r,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\left(1-e^{-\frac{r^2}{4t}}\right)$$

for $t \le \frac{r^2}{2N}$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^N} p(x,x,t)\,dx &\ge c(r,t)\int_{\mathbb{R}^N}\exp\{-t(|x|+r)^\alpha\}\,dx \\
&= c(r,t)|S_{N-1}|\int_0^\infty \exp\{-t(\rho+r)^\alpha\}\rho^{N-1}d\rho \\
&= c(r,t)|S_{N-1}|\int_{rt^{\frac{1}{\alpha}}}^\infty \exp\{-s^\alpha\}\left(\frac{s}{t^{\frac{1}{\alpha}}}-r\right)^{N-1}\frac{ds}{t^{\frac{1}{\alpha}}} \\
&= \frac{1}{(4\pi)^{\frac{N}{2}}}\frac{1}{t^{N(\frac{1}{2}+\frac{1}{\alpha})}}\left(1-e^{-\frac{r^2}{4t}}\right)|S_{N-1}|\int_{rt^{\frac{1}{\alpha}}}^\infty \exp\{-s^\alpha\}\left(s-t^{\frac{1}{\alpha}}r\right)^{N-1}ds
\end{aligned}$$

where $|S_{N-1}|$ is the measure of the unitary sphere in $\mathbb{R}^N$. Finally

$$\begin{aligned}
\liminf_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\ge \frac{1}{(4\pi)^{\frac{N}{2}}}|S_{N-1}|\int_0^\infty \exp\{-s^\alpha\}s^{N-1}ds \qquad (3.8)\\
= \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx.
\end{aligned}$$

From (3.7) and (3.8) it follows that

$$\lim_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx = \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx$$

and so, by Karamata's Theorem (see the Appendix)

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{1}{(4\pi)^{\frac{N}{2}}}\frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx.$$

Finally, observing that

$$\int_{\mathbb{R}^N} \exp\{-|x|^\alpha\}\, dx = \frac{N\omega_N}{\alpha} \int_0^\infty e^{-z} z^{\frac{N}{\alpha}-1}\, dz = \frac{1}{\alpha}\Gamma\left(\frac{N}{\alpha}\right),$$

the proof follows. □

Remark 3.4.3. The last result allows us to deduce some information on the lower bound of the heat kernel relative to the potential $V = |x|^\alpha$, for $t \le 1$ and $\alpha > 2$. We recall that, under these assumptions on t and α, the following upper bound holds

$$p(x,x,t) \le \frac{C}{t^{\frac{N}{2}}} \exp\{-ct|x|^{1+\frac{\alpha}{2}}\}.$$

If a similar lower bound were true, following the proof of Theorem 3.4.2 and applying Proposition B.0.12, we would deduce

$$\liminf_{\lambda\to\infty} \lambda^{-N(\frac{1}{2}+\frac{2}{2+\alpha})} N(\lambda) \ge C$$

for some positive constant C. Since this contradicts Theorem 3.4.2, we conclude that a similar lower bound cannot be true.

Adding a term of the form $o(|x|^\alpha)$ to the previous potential does not affect too much the asymptotic distribution of eigenvalues. In fact the following holds.

Proposition 3.4.4. *Let $V(x) = |x|^\alpha + o(|x|^\alpha)$ (as $|x| \to \infty$). Then*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{N\omega_n}{(4\pi)^{\frac{N}{2}}} \frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)} \frac{1}{\alpha}\Gamma\left(\frac{N}{\alpha}\right).$$

Proof. It is sufficient to observe that, given $\varepsilon > 0$, there exist $C'_\varepsilon,\ C_\varepsilon > 0$ such that

$$(1-\varepsilon)|x|^\alpha + C_\varepsilon \le V(x) \le (1+\varepsilon)|x|^\alpha + C'_\varepsilon$$

and, by the maximum principle,

$$p(x,x,t) \le e^{-C_\varepsilon t} p_\varepsilon(x,x,t)$$

where p_ε is the kernel corresponding to the potential $(1-\varepsilon)|x|^\alpha$. As in the proof of Theorem 3.4.2, it follows that for all $\varepsilon > 0$

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})} \int_{\mathbb{R}^N} p(x,x,t)\, dx \le \frac{1}{(4\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-(1-\varepsilon)|x|^\alpha\}\, dx$$

and, letting ε to 0,

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})} \int_{\mathbb{R}^N} p(x,x,t)\, dx \le \frac{1}{(4\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-|x|^\alpha\}\, dx.$$

In a similar way one obtains the bound for the lim inf and the proof follows. □

From the bound on the kernel proved in the previous section we can deduce the asymptotic behavior of $N(\lambda)$ for other radial potentials.

Proposition 3.4.5. *Let $V(x) = \exp\{|x|^\alpha\}$ with $\alpha > 0$. Then there exist $C_1,\ C_2 > 0$ such that*

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \le C_1$$

and

$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \ge C_2.$$

Lemma 3.4.6. *Let g be measurable and positive in $\mathbb{R}^N$ and let $E_s = \{x \in \mathbb{R}^N : g(x) \le s\}$. Then*

$$\int_{\mathbb{R}^N} e^{-tg(x)}dx = \int_0^\infty |E_{\frac{z}{t}}|e^{-z}dz.$$

Proof. The proof easily follows by observing that

$$\int_{\mathbb{R}^N} \exp\{-tg(x)\}dx = \int_0^\infty |\{x \in \mathbb{R}^N : \exp\{-tg(x)\} > s\}|\,ds. \tag{3.9}$$

$\square$

Proof (Proposition 3.4.5.) By Corollary 3.2.5 there exists $C = C(N)$ such that for all $0 < c < 1$ and $t > 0$

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi t)^{\frac{N}{2}}} &\int_{\mathbb{R}^N} \exp\{-t\exp\{c^\alpha|x|^\alpha\}\}\,dx \\ &+ C\frac{c^N\omega_N}{(1-c)^N}\int_{\mathbb{R}^N} \exp\left\{-(1-c)^2|x|^2\right\}\,dx.\end{aligned}$$

By Lemma 3.4.6

$$\begin{aligned}\int_{\mathbb{R}^N} \exp\{-t\exp\{c^\alpha|x|^\alpha\}\}\,dx &= \int_0^\infty e^{-z}|\{x : \exp\{c^\alpha|x|^\alpha\} \le \frac{z}{t}\}|dz \\ &= \frac{\omega_N}{c^N}\int_t^\infty e^{-z}(\log z - \log t)^{\frac{N}{\alpha}}dz.\end{aligned}$$

Taking the $\limsup$ as $t \to 0$ and letting $c \to 1$ we obtain

$$\limsup_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\omega_N\int_0^\infty e^{-z}dz = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}. \tag{3.10}$$

To prove a lower bound for the $\liminf$ of the same quantity we proceed as in the proof of Theorem 3.4.2. If A_D is the operator obtained from A by imposing Dirichlet boundary conditions on the surface of the ball B with center x and radius r then $p(x,x,t) \ge p_D(x,x,t)$. Moreover $V \le \exp\{(|x|+r)^\alpha\}$ in $B(x,r)$, so $p(x,x,t) \ge \exp\{-t\exp\{(|x|+r)^\alpha\}\}p_\Delta(x,x,t)$ where p_Δ is the heat kernel for the Laplacian on B with Dirichlet boundary conditions. By Kac's principle (see [15])

$$p_\Delta(x,x,t) \ge c(r,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\left(1 - e^{-\frac{r^2}{4t}}\right)$$

for $t \leq \frac{r^2}{2N}$. Therefore, from Lemma 3.4.6,

$$\int_{\mathbb{R}^N} p(x,x,t)\,dx \geq c(r,t)\int_{\mathbb{R}^N} \exp\{-t\exp\{(|x|+r)^\alpha\}\}\,dx$$
$$= c(r,t)\omega_N \int_t^\infty [(\log z - \log t)^{\frac{1}{\alpha}} - r]^N e^{-z} dz.$$

As above

$$\liminf_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} p(x,x,t)\,dx \geq \frac{1}{(4\pi)^{\frac{N}{2}}}\omega_N \int_0^\infty e^{-z}dz = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}. \quad (3.11)$$

From (3.10) and (3.11) it follows that

$$\lim_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} p(x,x,t)\,dx = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}.$$

By Proposition B.0.13, we find $C_1,\ C_2 > 0$ such that

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \leq C_1, \qquad \liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \geq C_2.$$

□