Appendix B

Smooth domains and regularity properties of the distance function

In this Appendix we collect some regularity results of the distance function \( r(x) = \text{dist}(x, \partial \Omega) \), when \( \partial \Omega \) is the boundary of a smooth open subset \( \Omega \) of \( \mathbb{R}^N \). These results are well-known in the case where \( \Omega \) is bounded (see e.g. [26, section 14.6]), but most of them may be extended, without much effort, to the unbounded case, as it is shown below.

First we define open sets with uniformly \( C^{2+\alpha} \) boundaries, for \( 0 \leq \alpha < 1 \).

**Definition B.0.15** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). We say that \( \partial \Omega \) is uniformly of class \( C^{2+\alpha} \) if there exist a covering of \( \partial \Omega \), at most countable, \( \{U_j\}_{j \in \mathbb{N}} \), and a sequence of diffeomorphisms \( \varphi_j : \overline{U}_j \to \mathbb{B}_1 \) of class \( C^{2+\alpha} \) such that

\[
\varphi_j(U_j \cap \Omega) = \{ y \in B_1 | y_N > 0 \}
\]

\[
\varphi_j(U_j \cap \partial \Omega) = \{ y \in B_1 | y_N = 0 \}
\]

and the following properties are satisfied:

(i) there exists \( k \in \mathbb{N} \) such that \( \bigcap_{j \in J} U_j = \emptyset \), if \( |J| > k \);

(ii) there exists \( 0 < \varepsilon < 1 \) such that \( \{ x \in \Omega | r(x) < \varepsilon \} \subseteq \bigcup_{j \in \mathbb{N}} V_j \), where \( V_j = \varphi_j^{-1}(B_{1/2}) \);

(iii) there exists \( C > 0 \) such that

\[
\sup_{j \in \mathbb{N}} \sum_{0 \leq |\beta| \leq 2+\alpha} \| D^\beta \varphi_j \|_{\infty} + \| D^\beta \varphi_j^{-1} \|_{\infty} \leq C.
\]

Now we show that such a set \( \Omega \) satisfies a uniform interior sphere condition, i.e. at each point \( y_0 \in \partial \Omega \) there exists a ball \( B_{y_0} \) depending on \( y_0 \), contained in \( \Omega \) and such that \( \overline{B}_{y_0} \cap \partial \Omega = \{ y_0 \} \); moreover the radii of these balls are bounded from below by a positive constant.

**Proposition B.0.16** If \( \partial \Omega \) is uniformly of class \( C^2 \), then it satisfies a uniform interior sphere condition.

**Proof.** Using condition (iii) and taking into account that \( \varphi_j \) is a diffeomorphism from \( \overline{U}_j \) into \( \mathbb{B}_1 \), it is easy to see that if \( y \in V \) and \( |x - y| < 1/(2C) \), then \( x \in U_j \).

Let \( y_0 \in \partial \Omega \) and let \( \eta(y_0) \) denote the unit inward normal vector to \( \partial \Omega \) at \( y_0 \). For \( 0 \leq t < 1/(2C) \) the point \( x = y_0 + t \eta(y_0) \) belongs to \( U_j \) and \( (\varphi_j) \) denotes the \( N \)-th component of \( \varphi_j \) \( \varphi_j(N)(x) = tD\varphi_j(N)(y_0) \cdot \eta(y_0) + R(t) \)
with $|R(t)| \leq Ct^2/2$. Since $\varphi^{(N)}_j = 0$ on $U_j \cap \partial \Omega$, then $D\varphi^{(N)}_j(y_0) = k\eta(y_0)$, with $k \geq C^{-1}$, by (iii). This yields $\varphi^{(N)}_j(x) \geq tC^{-1} - Ct^2/2 > 0$ for $0 < t < 2/C^3 := \delta$.

Thus, we have proved that

$$y + t\eta(y) \in \Omega, \quad y \in \partial \Omega, \quad t \in [0, \delta].$$

Now, let $y \in \partial \Omega$ and set $B = B(z, \delta/2)$, where $z = y + \eta(y)\delta/2$. Then, it is easy to see that $B \subset \Omega$ and $y \in \partial B$. If $y$ is not the unique point in $\partial \Omega \cap \partial B$, then it suffices to replace the above ball with that of radius $\delta/4$, centered at $z = y + \eta(y)\delta/4$.

We are now ready to prove the properties of the distance function used in this paper.

**Proposition B.0.17** Assume that $\partial \Omega$ is uniformly of class $C^2$ and let $\delta$ be a positive constant such that at each point of $\partial \Omega$ there exists a ball which satisfies the interior sphere condition at $y_0$ with radius greater or equal to $\delta$. Then

(a) for every $x \in \Omega_\delta = \{y \in \overline{\Omega} | r(y) < \delta\}$ there exists a unique $\xi = \xi(x) \in \partial \Omega$ such that $|x - \xi| = r(x)$;

(b) $r \in C^2_b(\Omega_\delta)$;

(c) $Dr(x) = \eta(\xi(x))$, for every $x \in \Omega_\delta$.

**Proof.** (a) The existence part is obvious. For the uniqueness assertion, let $x \in \Omega_\delta$ and $y \in \partial \Omega$ such that $r(x) = |x - y|$. From Proposition B.0.16 there exists a ball $B = B(z, \rho)$ such that $B \subset \Omega$ and $\overline{B} \cap \partial \Omega = \{y\}$. Moreover from the definition of $\delta$, $x \in B$. It is easy to see that $x$ and $z$ lie on the normal direction $\eta(y)$ and that the balls $B(x, r(x))$ and $B(z, \rho)$ are tangent at $y$. Then $B(x, r(x))$ still verifies the interior sphere condition at $y$. It follows that for every $\eta \in \partial \Omega \setminus \{y\}$, one has $\eta \notin B(x, r(x))$, so that $y$ is actually the unique point such that $|x - y| = r(x)$.

The proof of the last two assertions relies on the first statement and the implicit function theorem and it is completely similar to that of the case $\Omega$ bounded. We refer to [26, section 14.6].

142