Chapter 3

Gradient estimates in Dirichlet parabolic problems in regular domains

The aim of the present chapter is to prove global gradient estimates for the bounded classical solution u to the following Dirichlet parabolic problem

$$\begin{cases} u_t(t,x) - Au(t,x) = 0 & t \in (0,T),\ x \in \Omega, \\ u(t,\xi) = 0 & t \in (0,T),\ \xi \in \partial\Omega, \\ u(0,x) = f(x) & x \in \Omega, \end{cases} \tag{3.0.1}$$

where Ω is an unbounded smooth connected open set in $\mathbb{R}^N$, f a continuous and bounded function in Ω and A a second order elliptic operator, with (possibly) unbounded regular coefficients, i.e.,

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i,j=1}^{N} F_i D_i - V = \mathrm{Tr}(qD^2) + \langle F, D \rangle - V. \tag{3.0.2}$$

More precisely, we determine conditions on the coefficients of A yielding the following estimate

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}} \|f\|_\infty, \ \ t \in (0,T). \tag{3.0.3}$$

In Chapter 2 we have already studied gradient estimates for parabolic problems with Neumann boundary conditions. The main tool was Bernstein's method, which consists in applying the maximum principle to the function $u_n^2 + at|Du_n|^2$, where (u_n) approximates the solution. The crucial point was that the convexity assumption on Ω ensured the boundary condition $\frac{\partial |Du_n|^2}{\partial \eta} \le 0$. Here, we cannot proceed exactly in the same way, since for a given function v satisfying $v = 0$ on $\partial\Omega$, it is not possible to establish *a priori* the sign of $|Dv|^2$ on $\partial\Omega$. Hence, after having proved existence and uniqueness of bounded classical solutions u to (3.0.1) (Section 3.2), our first aim is to obtain boundary estimates for Du. This is done by comparison with certain one dimensional operators, which arise by introducing the distance function from the boundary. Then, using Bernstein's method, one shows that the boundary estimates can be extended to the whole Ω (Section 3.3). However, the method works (and gives (3.0.3) with the right dependence of all constants involved), if one already knows that Du is bounded up to the boundary of Ω for positive t, see Proposition 3.3.3. To circumvent this difficulty, we subtract to the operator A a potential εW, where W is big enough to dominate the growth of F and, following ideas in [11], [12], [41], we show that the perturbed operator $A_\varepsilon = A - \varepsilon W$ generates an analytic semigroup in $L^p(\Omega)$ and characterize its domain. Choosing a large p and using Sobolev embedding, it follows that the bounded classical solution u_ε of problem (3.0.1) with A_ε instead of A and a smooth f has a bounded gradient in $[0,T) \times \Omega$. Therefore Proposition 3.3.3 applies and gives (3.0.3) for u_ε with a constant C independent of ε. An approximation argument then completes the proof. This program is carried out in Sections 3.4 and 3.5. In Section 3.6 we present a counterexample.

3.1 Assumptions and main result

Let us collect our hypotheses on Ω and the coefficients of A.

Hypothesis 1.1

(i) Ω is a connected open subset of $\mathbb{R}^N$ with uniformly $C^{2+\alpha}$-boundary for some $0 < \alpha < 1$, see Appendix B.

(ii) $q_{ij}, F_i, V \in C^{1+\alpha}(\Omega \cap B_R)$ for every $i, j = 1, \ldots, N$ and $R > 0$; moreover $V \geq 0$ in Ω.

(iii) $q_{ij} = q_{ji} \in C^1_b(\Omega)$, and there exists $\nu_0 > 0$ such that $\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2$, for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iv) There exist a positive function $\varphi \in C^2(\overline{\Omega})$ and $\lambda_0 > 0$ such that

$$\lim_{|x|\to+\infty,\, x\in\overline{\Omega}} \varphi(x) = +\infty, \qquad A\varphi - \lambda_0\varphi \leq 0.$$

The *Lyapunov* map φ introduced in assumption (iv) ensures that maximum principles hold, see Appendix A. Moreover condition (i) ensures that the *distance function*

$$r(x) = \mathrm{dist}(x, \partial\Omega), \quad x \in \overline{\Omega} \tag{3.1.1}$$

is a C^2-function with bounded second order derivatives in Ω_δ, for some $\delta > 0$, where we set

$$\Omega_\delta = \{x \in \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < \delta\},$$

see [26, Lemma 14.16] and also Appendix B (note that (i) implies that the principal curvatures of $\partial\Omega$, when $\partial\Omega$ is considered as an hypersurface, are bounded). Our main result will be proved assuming also the conditions listed below.

$$\sum_{i,j=1}^N D_iF_j(x)\,\xi_i\,\xi_j \leq (sV(x) + k)\,|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{3.1.2}$$

$$\sum_{i,j=1}^N q_{ij}(x)D_{ij}r(x) + \sum_{i=1}^N F_i(x)\,D_ir(x) \ \leq M, \quad x \in \Omega_\delta \ \text{(for some } \delta > 0), \tag{3.1.3}$$

$$|DV(x)| \leq \beta(1 + V(x)), \quad x \in \Omega, \tag{3.1.4}$$

$$|F(x)| \leq c_1 e^{c_2|x|}, \quad x \in \Omega, \tag{3.1.5}$$

for some constants $k, M, \beta, c_1, c_2 \in \mathbb{R}$, $s < 1/2$.

Observe that, since $q_{ij} \in C^1_b(\Omega)$ and Ω is uniformly C^2, (3.1.3) is only a condition on the component of F along the inner normal to $\partial\Omega$ in a neighborhood of $\partial\Omega$.

Let us explain our main assumptions in the particular case where $A = \Delta + \langle F, D\rangle$. The dissipativity condition on F (3.1.2) is quite natural since a one-dimensional counterexample to gradient estimates has been constructed in Example 2.4.7 when it fails. Observe also that, if $F = D\Phi$, then (3.1.2) is a concavity assumption on Φ.

Condition (3.1.3) means that the component of the drift F along the inner normal is bounded from above in a neighborhood of $\partial\Omega$. Even though its connection with gradient estimates is not evident from an analytic point of view, its necessity is clear if one considers the Markov process governed by the operator A under Dirichlet boundary conditions. In fact the solution $u(t,x)$ to (3.0.1) corresponding to $f = \mathbf{1}$ represents the probability that the process starting from $x \in \Omega$ at time $t = 0$ is not absorbed by the boundary up to time t. If the (inner) normal component of F is unbounded from above in a neighborhood of $\partial\Omega$, one expects that $u(t,x) \to 1$ as $|x| \to \infty$ along the boundary. Since $u(t,\xi) = 0$ for $\xi \in \partial\Omega$, it follows that $u(t,\cdot)$ is even not uniformly continuous, see Example 3.6.1 where this heuristic argument is made rigorous.

Finally, we point out that the growth assumption (3.1.5), even though not very restrictive, seems to be a technical one in order to use our methods, see the proof of Theorem 3.1.2.

We stress the fact that we use mainly analytic tools and we do not need any convexity assumption on Ω. Moreover we note that our operator A may contain a potential term V which is difficult to treat by probabilistic methods.

Remark 3.1.1 Observe that assumption (iv) of Hypothesis 1.1 follows from the positivity of V and the boundedness of q_{ij}, when condition (3.1.2) holds with $s = 0$. In fact (3.1.2) implies, by differentiating the function $t \to \langle F(tx), x\rangle$, that $\langle F(x), x\rangle \le \langle F(0), x\rangle + k|x|^2$, hence the function $\varphi(x) = 1 + |x|^2$ satisfies (iv), for a suitable λ_0.

To specify the dependence of some constants we also introduce the quantity

$$h = \sup_{x\in\Omega}\Big(\sum_{i,j=1}^{N} |Dq_{ij}(x)|^2\Big)^{1/2}$$

which is finite, since $q_{ij} \in C_b^1(\Omega)$.

We will prove the following theorem.

Theorem 3.1.2 *There exists a constant C depending on $\nu_0, k, s, h, N, M, \beta, \delta, T$ such that the bounded classical solution u of (3.0.1) satisfies*

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \quad t \in (0,T), \quad f \in C_b(\Omega).$$

3.2 Existence and uniqueness

In this section we show that (3.0.1) has a unique bounded classical solution, where by *bounded classical solution* of (3.0.1) we mean a function $u \in C^{1,2}(Q)$, such that u is continuous in $\overline{Q}\setminus\partial_{tx}Q$, bounded in Q and solves (3.0.1). To this purpose we use both classical Schauder estimates and a nonstandard maximum principle for discontinuous solutions to (3.0.1), see Theorem A.0.13.

Proposition 3.2.1 *Assume Hypothesis 1.1. If $f \in C^{2+\alpha}(\Omega)$ has compact support in Ω, then problem (3.0.1) has a unique bounded solution u which belongs to $C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))$ for every $R > 0$. Moreover, $\|u\|_\infty \le \|f\|_\infty$ and $u \ge 0$ if $f \ge 0$. Finally, Du belongs to $C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times\Omega'))$ for every $\varepsilon > 0$ and Ω' open bounded set with* $\operatorname{dist}(\Omega', \mathbb{R}^N\setminus\Omega) > 0$. *In particular, $Du \in C^{1,2}(Q)$.*

PROOF. Uniqueness is immediate consequence of a classical maximum principle, see Proposition A.0.12.

To prove the existence part, we consider a sequence of uniformly elliptic operators with coefficients in $C^\alpha(\Omega)$,

$$A^n = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i=1}^N F_i^n D_i - V^n u,$$

such that $F_i^n = F_i$, $V^n = V$ in $\Omega \cap B_n$, $V^n \geq 0$ and let $u_n \in C^{1+\alpha/2, 2+\alpha}(Q)$ be the solution of (3.0.1), with A^n instead of A (see e.g. [30, Theorem IV.5.2]). The classical maximum principle yields $\|u_n\|_\infty \leq \|f\|_\infty$. Let us fix $R > 0$ and observe that, since Ω is unbounded and connected, $\mathrm{dist}(\Omega \setminus B_{R+1}, \Omega \cap B_R) > 0$. Since $A^n = A^m = A$ in $\Omega \cap B_{R+1}$ for $n, m > R+1$, by the local Schauder estimates [30, Theorem IV.10.1], there exists a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))} \leq C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leq 2C\|f\|_\infty.$$

Therefore (u_n) is relatively compact in $C^{1,2}([0,T] \times (\overline{\Omega \cap B_R}))$. Considering an increasing sequence of balls and using a diagonal procedure we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((0,T) \times (\Omega \cap B_R))$ for every $R > 0$ which solves (3.0.1) and satisfies $\|u\|_\infty \leq \|f\|_\infty$. By the maximum principle, $u \geq 0$, whenever $f \geq 0$.

In order to prove the last part of the statement it is sufficient to apply [29, Theorem 8.12.1] directly to the operator $D_t - A$. □

We now introduce linear operators $(P_t)_{t\geq 0}$ via the formula $(P_t f)(x) = u(t,x)$ for $f \in C^{2+\alpha}(\Omega)$, with compact support in Ω, where u is the solution of (3.0.1) given by the above proposition. Each operator P_t is positive and contractive with respect to the sup-norm, by the above proposition.

Now we consider the case where f is only continuous and bounded in Ω and extend the above maps $(P_t)_{t\geq 0}$ to a semigroup in $C_b(\Omega)$.

Proposition 3.2.2 *Assume Hypothesis 1.1. If f belongs to $C_b(\Omega)$, then problem (3.0.1) has a unique bounded classical solution u. Moreover, $u(t,x) \to f(x)$ as $t \to 0$, uniformly on compact sets of Ω.*

Proof. Uniqueness is an immediate consequence of a nonstandard maximum principle, see Theorem A.0.13. To show existence, we consider a sequence $(f_n) \in C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \leq \|f\|_\infty$. Let $u_n \in C^{1+\alpha/2,2+\alpha}((0,T) \times (\Omega \cap B_R))$, for every $R > 0$, be the solution of (3.0.1) with f_n instead of f, given by the previous proposition. Let us fix $\varepsilon > 0$. By the Schauder estimates [30, Theorem IV.10.1], as in the proof of Proposition 3.2.1, we get a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B_R))} \leq C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leq 2C\|f\|_\infty$$

and then, by a compactness argument, we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B_R))$ for every $\varepsilon, R > 0$ which solves the equation $u_t - Au = 0$ in Q and such that $u(t,x) = 0$ for $t \in (0,T), x \in \partial\Omega$. In the following, we write $u = P_t f$, for $f \in C_b(\Omega)$.

It remains to show that $u(t,x) \to f(x)$ as $t \to 0$, uniformly on compact sets of Ω.

Assume first that $f \in C_0(\Omega)$, i.e. f vanishes on $\partial\Omega$ and at infinity. Then we can choose (f_n) as above in such a way that $\|f_n - f\|_\infty \to 0$. The maximum principle implies that (u_n) is a Cauchy sequence in $C([0,T] \times \overline{\Omega})$, hence $u_n \to u$ uniformly in $\overline{Q}$ and $u(0,x) = f(x)$ for every $x \in \overline{\Omega}$.

Let $K \subset \Omega$ be a compact set and $\eta \in C_0(\Omega)$, $0 \leq \eta \leq 1$, be such that $\eta = 1$ in K. Then $P_t\eta \to \eta$ as $t \to 0$, uniformly in Ω, hence $P_t\eta \to 1$ uniformly in K and, since $0 \leq P_t(1-\eta) \leq 1 - P_t\eta$, we get $P_t(1-\eta) \to 0$ uniformly in K. For $f \in C_b(\Omega)$, writing $P_t f = P_t(\eta f) + P_t((1-\eta)f)$ and observing that $P_t(\eta f) \to \eta f$ uniformly in Ω and that $P_t((1-\eta)f) \to 0$ uniformly in K we obtain that $P_t f \to f$, uniformly in K. □

Corollary 3.2.3 *The family $(P_t)_{t\geq 0}$ is a semigroup in $C_b(\Omega)$.*

Proof. The semigroup law $P_{t+s} = P_tP_s$ is immediate consequence of the uniqueness statement in Proposition 3.2.2. □

Observe that the semigroup $(P_t)_{t\geq 0}$ is not strongly continuous. In fact $P_t f \to f$ as $t \to 0$, only uniformly on compact subsets of Ω. However, $P_t f \to f$ uniformly in Ω for every $f \in C_0(\Omega)$.

3.3 Some a-priori estimates

In the following proposition we prove a preliminary boundary gradient estimate for bounded solutions of problem (3.0.1). We need the following lemma on gradient estimates for certain one-dimensional operators.

Lemma 3.3.1 *Let $\delta > 0$ and $g : [0,+\infty) \times [0,\delta] \to \mathbb{R}$ be the solution to*

$$\begin{cases} g_t(t,r) = \nu_0 g_{rr}(t,r) + M g_r(t,r), & t > 0,\ r \in (0,\delta), \\ g(t,0) = 0,\ \ g(t,\delta) = 1 & t > 0, \\ g(0,r) = 1 & r \in (0,\delta). \end{cases} \tag{3.3.1}$$

Then $g_r \geq 0$, $g_{rr} \leq 0$ and for any $T > 0$ there exists $c_T > 0$ such that

$$0 \leq g(t,r) \leq \frac{c_T}{\sqrt{t}} r,\ \ 0 < t \leq T,\ r \in (0,\delta).$$

Proof. We define the operator $(B, D(B))$ in $C([0,\delta])$ by

$$Bu = \nu_0 u'' + Mu' \qquad D(B) = \{u \in C^2([0,\delta]) : u(0) = 0, (Bu)(\delta) = 0\}.$$

Let us show that $(B, D(B))$ generates an analytic semigroup S_t of positive contractions in $C([0,\delta])$ (note that S_t is not strongly continuous since the domain $D(B)$ is not dense in $C([0,\delta])$.

Let $D = \{u \in C^2([0,\delta]) : u(0) = u(\delta) = 0\}$. Then (B, D) generates an analytic semigroup $(T_t)_{t\geq 0}$ in $C([0,\delta])$. Set $\psi(r) = a \int_0^r e^{-M\,s/\nu_0} ds$. Then $B\psi = 0$, $\psi(0) = 0$ and $\psi(\delta) = 1$, if a is suitably chosen. It is easily seen that $S_t f = T_t(f - f(\delta)\psi) + f(\delta)\psi$ is the analytic semigroup generated by $(B, D(B))$ in $C([0,\delta])$. Since the regularity properties of $S_t f$ coincide with those of $T_t f$, it follows that $u(t,r) = S_t f(r)$ is a C^∞ function for $t > 0$, continuous at the points $(0,r)$, with $0 < r < \delta$. The maximum principle, see Theorem A.0.13, now yields positivity and contractivity of S_t.

We can prove the stated properties of g. Since $g = S_t 1$ we have $0 \leq g \leq 1$. Moreover $g(t+s,\cdot) = S_{t+s}1 = S_t S_s 1 \leq S_t 1 = g(t,\cdot)$, hence g is decreasing with respect to t and $g_t \leq 0$. To prove that $g_r \geq 0$ we write

$$g_t = \nu_0 \Big(g_{rr} + \frac{M}{\nu_0} g_r \Big) = \nu_0 e^{-\frac{M}{\nu_0} r} \frac{d}{dr} \Big(e^{\frac{M}{\nu_0} r} g_r \Big) \leq 0,$$

$r \in (0,\delta)$. Then $e^{\frac{M}{\nu_0} r} g_r$ is decreasing. Since $g(t,\delta) = 1$ and $0 \leq g \leq 1$, we have $g_r(t,\delta) \geq 0$, hence $g_r \geq 0$. Now the identity $g_t = \nu_0 g_{rr} + M g_r$ yields $g_{rr} \leq 0$.

Since $(S_t)_{t\geq 0}$ is analytic, for $0 < t \leq T$ we have $\|D^2 g(t,\cdot)\| \leq c_T t^{-1}$, hence $\|Dg(t,\cdot)\| \leq c_T t^{-1/2}$ and the inequality $g(t,r) \leq c_T t^{-1/2} r$ follows, since $g(t,0) = 0$. □

Proposition 3.3.2 *Assume Hypothesis 1.1 and (3.1.3). Then there exists γ depending on ν_0, M, δ, T such that every bounded classical solution u of (3.0.1), differentiable with respect to the space variables on $]0,T[\times\overline{\Omega}$, satisfies the estimate*

$$|Du(t,\xi)| \leq \frac{\gamma}{\sqrt{t}} \|f\|_\infty,\ t \in (0,T),\ \xi \in \partial\Omega. \tag{3.3.2}$$

PROOF. For each $x \in \Omega_\delta$ let $\xi(x)$ be the unique point in $\partial\Omega$ satisfying $|x-\xi| = r(x)$. Note that

$$x = \xi(x) + \eta(\xi(x))r(x),$$

where $\eta(\xi)$ is the unit inner normal to $\partial\Omega$ at $\xi \in \partial\Omega$. Recall also that $Dr(x) = \eta(\xi(x))$, $x \in \Omega_\delta$. See Appendix B for these properties of the distance function r. To proceed we remark that, since $u = 0$ on $\partial\Omega$,

$$Du(t,\xi) = \partial_\eta u(t,\xi), \;\; \xi \in \partial\Omega, \;\; t > 0.$$

In order to prove the claim it is enough to show that

$$|w(t,x)| = w(t,x) \le \frac{\gamma}{\sqrt{t}}\, r(x), \quad t \in (0,T), \;\; x \in \Omega_\delta, \tag{3.3.3}$$

where w is the solution to (3.0.1), corresponding to $f = 1$, and γ depends only on the stated parameters. Indeed, in the general case it is sufficient to observe that, for $x = \xi + r(x)\eta(\xi)$, $\xi \in \partial\Omega$ fixed,

$$|P_t f(x) - P_t f(\xi)| = |P_t f(x)| \le P_t|f|(x) \le \|f\|_\infty P_t 1(x) = \|f\|_\infty w(t,x) \le \frac{\gamma}{\sqrt{t}} r(x)\|f\|_\infty,$$

and (3.3.2) follows easily dividing by r and letting $r \to 0$. To prove (3.3.3) we compare w with an auxiliary function z, using Theorem A.0.13. Let

$$z(t,x) = g(t, r(x)), \;\; x \in \Omega_\delta,$$

where $g : [0,+\infty) \times [0,\delta] \to \mathbb{R}$ is the solution to (3.3.1). Now Lemma 3.3.1 yields

$$|z(t,x)| = g(t,r(x)) \le \frac{\gamma}{\sqrt{t}} r(x), \;\; 0 < t < T, \; x \in \Omega_\delta.$$

Thus we have only to prove that

$$w(t,x) \le z(t,x), \;\; x \in \Omega_\delta, \; t \in (0,T). \tag{3.3.4}$$

To verify (3.3.4), we consider $v = z - w$ in the cylinder $Q_\delta = (0,T) \times \Omega_\delta$. It is clear that v belongs to $C^{1,2}(Q_\delta)$, is continuous in $\overline{Q}_\delta \setminus \partial_{tx} Q_\delta$, bounded on Q_δ and nonnegative on $\partial' Q_\delta \setminus \partial_{tx} Q_\delta$. Moreover

$$\begin{aligned} & v_t - Av = z_t - Az = g_t - \nu_0 g_{rr} - M g_r \\ &+ \Big(\nu_0 g_{rr} + M g_r - g_{rr} \sum_{i,j=1}^N q_{ij} D_i r D_j r - g_r \langle F, Dr\rangle - g_r \sum_{i,j=1}^N q_{ij} D_{ij} r + Vz\Big) \\ &= g_{rr}\Big(\nu_0 - \sum_{i,j=1}^N q_{ij} D_i r D_j r\Big) + g_r\Big(M - \sum_{i,j=1}^N q_{ij} D_{ij} r - \langle F, Dr\rangle\Big) + Vz \ge 0, \end{aligned}$$

since $z, g_r \ge 0$, $g_{rr} \le 0$. The maximum principle Theorem A.0.13 now implies (3.3.4) and concludes the proof. □

The following proposition is an a-priori estimate on Du, where u is the bounded classical solution of (3.0.1). Its importance relies on pointing out the dependence of the constant C below.

Proposition 3.3.3 *Assume Hypothesis 1.1, (3.1.2) and (3.1.4). Then there exists a constant C depending on ν_0, h, k, s, β, T with the following property. Every bounded classical solution u of (3.0.1) such that*

(i) Du belongs to $C^{1,2}(Q)$,

(ii) $\sqrt{t}|Du|$ *is continuous in* $\overline{Q}\setminus \partial_{tx}Q$*, bounded in* Q *and verifies* $\lim_{t\to 0}\sqrt{t}|Du(t,x)| = 0$*,* $x \in \Omega$*,*

(iii) u *satisfies (3.3.2)*

fulfills the estimate

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \;\; t\in(0,T). \tag{3.3.5}$$

Proof. Changing V to $V+1$ (hence u to $e^{-t}u$) we may assume that $|DV| \le \beta V$. We use Bernstein's method and define the function

$$v(t,x) = u^2(t,x) + a\,t\,|Du(t,x)|^2\;, \quad t\in(0,T),\; x\in\Omega,$$

where $a>0$ is a parameter to be chosen later. Then we have $v \in C^{1,2}(Q)$, v is continuous in $\overline{Q}\setminus\partial_{tx}Q$, bounded in Q and $v(0,x) = f^2(x)$. We claim that for a suitable value of $a>0$, depending on ν_0, h, k, s, β, T we have

$$v_t(t,x) - Av(t,x) \le 0, \qquad 0<t<T,\; x\in\Omega. \tag{3.3.6}$$

This, by Theorem A.0.13, implies that

$$v(t,x) \le \sup_{x\in\Omega}|v(0,x)| + \sup_{\xi\in\partial\Omega,\, t\in(0,T)} at\,|Du(t,\xi)|^2 \le (1+a\gamma^2)\|f\|_\infty^2,$$

$0<t\le T$, $x\in\Omega$, and (3.3.5) follows with $C = (a^{-1}+\gamma^2)^{1/2}$.

To verify inequality (3.3.6), note that, by a straightforward computation, v satisfies the equation

$$v_t - Av = a|Du|^2 - 2\sum_{i,j=1}^N q_{ij}\,D_iu\,D_ju + g_1 + g_2,$$

where

$$\begin{aligned} g_1 &= a\,t\left(2\sum_{i,j=1}^N D_iF_j\,D_iu\,D_ju - 2u\langle Du, DV\rangle - V|Du|^2\right) - Vu^2,\\ g_2 &= 2\,a\,t\left(\sum_{i,j,k=1}^N D_kq_{ij}D_kuD_{ij}u - \sum_{i,j,k=1}^N q_{ij}D_{ik}uD_{jk}u\right). \end{aligned}$$

Using the assumptions one has, for all $\varepsilon>0$, $x\in\Omega$, $t\in(0,T)$,

$$\begin{aligned} v_t - Av \;\le\;& \Big(a - 2\nu_0 + 2akt + at(2s-1)V\Big)|Du|^2\\ &+2at\Big(h|Du||D^2u| + \beta V|u||Du| - \nu_0|D^2u|^2\Big) - Vu^2\\ \le\;& \Big(a - 2\nu_0 + 2akt + at(2s-1)V\Big)|Du|^2\\ &+at\Big(h\varepsilon^{-1}|Du|^2 + h\varepsilon|D^2u|^2 + \beta\varepsilon^{-1}Vu^2 + \beta\varepsilon V|Du|^2 - 2\nu_0|D^2u|^2\Big) - Vu^2, \end{aligned}$$

where $|D^2u|^2 = \sum_{i,j=1}^N |D_{ij}u|^2$. Since $2s<1$, choosing ε and a small enough we get immediately (3.3.6). □

3.4 An auxiliary problem

In this section we keep Hypothesis 1.1 and condition (3.1.4) and write our operator in divergence form

$$A = A_0 + \sum_{i=1}^{N} G_i D_i - V,$$

where $A_0 = \sum_{i,j=1}^{N} D_i(q_{ij} D_j)$ and $G_i = F_i - \sum_{j=1}^{N} D_j q_{ij}$.

Moreover, we assume that the potential V and the drift G satisfy the inequality

$$|G(x)| \le \sigma V(x)^{1/2} + c_\sigma, \qquad x \in \Omega, \tag{3.4.1}$$

for some $\sigma > 0$ and show generation of an analytic semigroup in $L^p(\Omega)$, for $\sigma < \min\{2\nu_0(p-1), 2\}$. We follow the ideas of [11], [12] and [41] where the situation $\Omega = \mathbb{R}^N$ is considered.

For simplicity, we assume throughout this section that $2 \le p < \infty$. Observe that, since $q_{ij} \in C_b^1(\Omega)$, condition (3.4.1) holds equivalently for F or G with the same constant σ, possibly with a different choice of c_σ.

We endow A with the domain

$$D_p = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : Vu \in L^p(\Omega)\}$$

which is a Banach space when endowed with the norm

$$\|u\|_{D_p} = \|u\|_{W^{2,p}(\Omega)} + \|Vu\|_{L^p(\Omega)},$$

and remark that the set

$$D = \{u \in C^\infty(\overline{\Omega}) : u_{|\partial\Omega} = 0, \operatorname{supp} u \text{ compact in } \overline{\Omega}\}$$

is dense in D_p.

We need the following interpolative lemma which is analogous to [41, Proposition 2.3].

Lemma 3.4.1 *Assume Hypothesis 1.1 and that condition (3.1.4) hold. Then there exists C depending on N, p, β and the coefficients (q_{ij}) such that for every $0 < \varepsilon < 1$ and $u \in D_p$, $2 \le p < \infty$, the following inequality holds:*

$$\|V^{1/2} Du\|_p \le \varepsilon \|A_0 u\|_p + C\varepsilon^{-1}(\|u\|_p + \|Vu\|_p).$$

PROOF. It suffices to establish the inequality above for functions $u \in D$. Moreover, changing V with $V + 1$, we may assume that $|DV| \le \beta V \le \beta V^{3/2}$.

Integrating by parts and using the fact that $u = 0$ on $\partial\Omega$ and $p \ge 2$ we have

$$\begin{aligned}
\int_\Omega V^{\frac{p}{2}} |D_k u|^p &= \int_\Omega V^{\frac{p}{2}} |D_k u|^{p-2} D_k u D_k u \\
&= -\frac{p}{2} \int_\Omega V^{\frac{p}{2}-1} D_k V u |D_k u|^{p-2} D_k u - (p-1) \int_\Omega V^{\frac{p}{2}} u |D_k u|^{p-2} D_{kk} u \\
&\le \frac{\beta p}{2} \int_\Omega |u||D_k u|^{p-1} V^{\frac{p-1}{2}} V + (p-1) \int_\Omega V^{\frac{p-2}{2}} |D_k u|^{p-2} V |u||D_{kk} u| \\
&\le \frac{\beta p}{2} \Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p \Big)^{1-1/p} \Big(\int_\Omega V^p |u|^p \Big)^{1/p} \\
&+ (p-1) \Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p \Big)^{1-2/p} \Big(\int_\Omega V^p |u|^p \Big)^{1/p} \Big(\int_\Omega |D_{kk} u|^p \Big)^{1/p}.
\end{aligned}$$

Setting $x = \|V^{1/2}D_k u\|_p$, $y = \|Vu\|_p$, $z = \|D_{kk}u\|_p$ we have obtained $x^2 \le (\beta p)/2xy + (p-1)yz$, hence

$$x \le \frac{\beta p}{2} y + \sqrt{(p-1)yz} \le C\varepsilon^{-1} y + \varepsilon z$$

for $\varepsilon < 1$, with C depending on β, p and the statement follows with $\|D^2 u\|_p$ instead of $\|A_0 u\|_p$. To complete the proof it suffices to use the closedness of A_0 on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. $\square$

Proposition 3.4.2 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma < \min\{2\nu_0(p-1), 2\}$. Then (A, D_p) is closed in $L^p(\Omega)$, $2 \le p < \infty$. Moreover, there is a constant λ_0 depending on c_σ with the following property: for every $\lambda > \lambda_0$ there exist C_1, C_2 depending only on $\lambda, N, p, \beta, \sigma, c_\sigma$ and the coefficients (q_{ij}), such that for every $u \in D_p$*

$$\|u\|_{D_p} \le C_1 \|\lambda u - Au\|_p \le C_2 \|u\|_{D_p}.$$

Finally, if $c_\sigma = 0$, then $\lambda_0 = 0$ and the inequality $\lambda \|u\|_p \le \|(\lambda - A)u\|_p$ holds.

Proof. By density we may assume that $u \in D$. The right hand side of the above inequality follows immediately from Lemma 3.4.1, since $|G| \le \sigma V^{1/2} + c_\sigma$.

Changing V with $V + \omega$ for a suitable large ω, we may assume that $c_\sigma = 0$ and that $|DV| \le \beta V$.

Let us multiply the identity $f = \lambda u - Au$ by $u|u|^{p-2}$. Integrating over Ω we get, since $u = 0$ on $\partial\Omega$,

$$\int_\Omega (\lambda + V)|u|^p + (p-1)\int_\Omega q_{ij}|u|^{p-2} D_i u D_j u \le \|f\|_p \|u\|_p^{p-1} + \sigma \int_\Omega V^{1/2}|Du||u|^{p-1}.$$

The last term can be estimated with

$$\sigma\Big(\int_\Omega V|u|^p\Big)^{1/2}\Big(\int_\Omega |u|^{p-2}|Du|^2\Big)^{1/2} \le \frac{\sigma}{2}\Big(\int_\Omega V|u|^p + |u|^{p-2}|Du|^2\Big).$$

Since $\sigma < \min\{2\nu_0(p-1), 2\}$ we easily obtain, for $\lambda > 0$, $\lambda\|u\|_p \le \|f\|_p$. To estimate Vu we observe that

$$\begin{aligned}
\int_\Omega (A_0 u)V^{p-1}u|u|^{p-2} &= -\sum_{i,j=1}^N \int_\Omega q_{ij} D_i u D_j \big(V^{p-1}u|u|^{p-2}\big) \\
&= -(p-1)\int_\Omega \sum_{i,j=1}^N q_{ij} V^{p-1}|u|^{p-2} D_i u D_j u \\
&\quad -(p-1)\int_\Omega \sum_{i,j=1}^N q_{ij} V^{p-2} u|u|^{p-2} D_i u D_j V.
\end{aligned}$$

Multiplying the identity $\lambda u - Au = f$ by $V^{p-1}u|u|^{p-2}$ and integrating over Ω we obtain

$$\begin{aligned}
&\int_\Omega (\lambda V^{p-1} + V^p)|u|^p + \nu_0(p-1)\int_\Omega V^{p-1}|u|^{p-2}|Du|^2 \\
&\quad \le \int_\Omega (\lambda V^{p-1} + V^p)|u|^p + (p-1)\int_\Omega V^{p-1}|u|^{p-2} q(Du, Du) \\
&\quad = -(p-1)\int_\Omega V^{p-2}u|u|^{p-2} q(Du, DV) + \int_\Omega V^{p-1}u|u|^{p-2}\langle G, Du\rangle + \int_\Omega f V^{p-1} u|u|^{p-2},
\end{aligned}$$

where $q(Du, DV) = \sum_{i,j=1}^N q_{ij} D_i u D_j V$ and similarly for $q(Du, Du)$. Next, observe that

$$\begin{aligned}
\Big|\int_\Omega V^{p-1}u|u|^{p-2}\langle G, Du\rangle\Big| &\le \sigma \int_\Omega V^{p-1/2}|u|^{p-1}|Du| \\
&\le \sigma\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2\Big)^{1/2}\Big(\int_\Omega V^p|u|^p\Big)^{1/2} \\
&\le \frac{\sigma}{2}\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2 + \int_\Omega V^p|u|^p\Big)
\end{aligned}$$

and that, for a suitable K depending only on $\|q_{ij}\|_\infty$,

$$\begin{aligned}\int_\Omega |u|^{p-1}V^{p-2}|q(Du,DV)| &\leq K\int_\Omega |u|^{p-1}V^{p-2}|Du||DV| \\ &\leq K\beta\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2\Big)^{1/2}\Big(\int_\Omega |u|^pV^{p-1}\Big)^{1/2} \\ &\leq K\beta\varepsilon\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2+\int_\Omega V^p|u|^p\Big)+C_\varepsilon\int_\Omega |u|^p.\end{aligned}$$

In the last inequality we have used the inequality $t^{p-1}\leq \varepsilon t^p+C_\varepsilon$.

Since $\sigma<\min\{2\nu_0(p-1),2\}$, taking a small ε one concludes that $\|Vu\|_p\leq C\|f\|_p$, with C as in the statement.

We now use Lemma 3.4.1 to estimate the second order derivatives of u. We have

$$\begin{aligned}\|\langle G,Du\rangle\|_p &\leq \sigma\|V^{1/2}Du\|_p\leq \sigma(\varepsilon\|A_0u\|_p+C\varepsilon^{-1}\|u\|_p+C\varepsilon^{-1}\|Vu\|_p) \\ &\leq \sigma(\varepsilon\|f\|_p+\varepsilon\|\langle G,Du\rangle\|_p+\varepsilon\|Vu\|_p+\varepsilon\lambda\|u\|_p+C\varepsilon^{-1}\|u\|_p+C\varepsilon^{-1}\|Vu\|_p)\end{aligned}$$

hence, taking a small ε, $\|\langle G,Du\rangle\|_p\leq C\|f\|_p$ and $\|A_0u\|_p\leq C\|f\|_p$, by difference. Using the closedness of A_0 on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ given by the Calderon-Zygmund estimates, we get $\|D^2u\|_p\leq C\|f\|_p$, with C as in the statement. □

Proposition 3.4.3 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma<\min\{2\nu_0(p-1),2\}$. Then (A,D_p) generates a semigroup in $L^p(\Omega)$, $2\leq p<\infty$.*

Proof. As in the proof of Proposition 3.4.2, we may assume that $c_\sigma=0$, $|DV|\leq\beta V$, so that $\lambda\|u\|_p\leq\|\lambda u-Au\|_p$ for $\lambda>0$. By the Lumer-Phillips Theorem it suffices to show $\lambda-A$ is surjective for $\lambda>0$.

Setting for $\varepsilon>0$

$$V_\varepsilon=\frac{V}{1+\varepsilon V}\qquad G_\varepsilon=\frac{G}{\sqrt{1+\varepsilon V}},$$

it is immediate to check that $V_\varepsilon, G_\varepsilon$ satisfy

$$|DV_\varepsilon|\leq\beta V_\varepsilon\qquad |G_\varepsilon|\leq\sigma V_\varepsilon^{1/2}.$$

Since $V_\varepsilon, G_\varepsilon$ are bounded, the operator $A_\varepsilon=A_0+\langle G_\varepsilon,D\rangle-V_\varepsilon$ with domain $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ generates an analytic semigroup in $L^p(\Omega)$ see [32, Theorem 3.1.3], which is contractive by Proposition 3.4.2.

Given $f\in L^p(\Omega)$, let $u_\varepsilon\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that $(\lambda-A_\varepsilon)u_\varepsilon=f$. By Proposition 3.4.2, $\|u_\varepsilon\|_{2,p},\|V_\varepsilon u_\varepsilon\|_p\leq C\|f\|_p$ with C independent of ε. By weak compactness we find $\varepsilon_n\to0$ such that (u_{ε_n}) converges weakly to a function u in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and strongly in $W^{1,p}_{loc}(\Omega)$. Moreover we may assume that $(u_{\varepsilon_n})\to u$ a.e. in Ω. By Fatou's lemma $\|Vu\|_p\leq C\|f\|_p$, hence $u\in D_p$ and it is easy to check that $(\lambda-A)u=f$. □

Let us show that the above semigroup is analytic.

Theorem 3.4.4 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma<\min\{2\nu_0(p-1),2\}$. Then (A,D_p) generates an analytic semigroup in $L^p(\Omega)$, $2\leq p<\infty$.*

Proof. We keep the same notation of the proof of Proposition 3.4.2. We may assume that $c_\sigma=0$. Let $u\in D$ and set $u^*:=\overline{u}|u|^{p-2}$. Integrating by parts, since $u=0$ on $\partial\Omega$, a lengthy but straightforward computation yields

$$-Re\left(\int_\Omega (Au)u^*\right) = (p-1)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))$$
$$+\int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du)) - \int_\Omega \langle G, Re(\overline{u}Du)|u|^{p-2}\rangle + \int_\Omega V|u|^p$$

and

$$\left|Im\int_\Omega (Au)u^*\right| \leq (p-2)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Im(\overline{u}Du)) + \int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)|.$$

Condition (3.4.1) implies

$$\begin{aligned}\int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)| &\leq \sigma\int_\Omega V^{\frac{1}{2}}|Im(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}\\ &\leq \sigma\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}|Im(\overline{u}Du)|^2\right)^{\frac{1}{2}}\\ &\leq \frac{\sigma}{\sqrt{\nu_0}}\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du))\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |G||u|^{p-2}|Re(\overline{u}Du)| &\leq \sigma\int_\Omega V^{\frac{1}{2}}|Re(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}\\ &\leq \sigma\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}|Re(\overline{u}Du)|^2\right)^{\frac{1}{2}}\\ &\leq \frac{\sigma}{\sqrt{\nu_0}}\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))\right)^{\frac{1}{2}}.\end{aligned}$$

If we put $B^2 := \int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))$, $C^2 := \int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du))$, and $D^2 := \int_\Omega V|u|^p$, then we deduce from the previous estimates

$$-Re\left(\int_\Omega (Au)u^*\right) \geq \left(p-1-\frac{\sigma}{2\nu_0}\right)B^2 + C^2 + \left(1-\frac{\sigma}{2}\right)D^2.$$

Therefore,

$$\left|Im\left(\int_\Omega (Au)u^*\right)\right| \leq (p-2)BC + \frac{\sigma}{\sqrt{\nu_0}}CD$$

and one can find $\kappa > 0$ such that

$$\left|Im\left(\int_\Omega (Au)u^*\right)\right| \leq \kappa\left[-Re\left(\int_\Omega (Au)u^*\right)\right]$$

for every $u \in D$ and, by density, for every $u \in D_p$. Since we already know that (A, D_p) generates a semigroup, by [44, Theorem 3.9, Chapter I] the proof is complete. □

Remark 3.4.5 Observe that all the results proved until now, in this section (but not the next lemma), hold assuming less local regularity on the coefficients. For example $q_{ij} \in C_b^1(\Omega), F \in L^\infty_{loc}(\Omega), V \in C^1(\Omega)$ suffice. Moreover, the existence of the Lyapunov function φ is not necessary.

We call $(T_t)_{t\geq 0}$ the semigroup generated by A in $L^p(\Omega)$. For the proof of our main result we need some regularity results of the function $u(t,x) = (T_t f)(x)$.

Lemma 3.4.6 *Assume that the conditions of Theorem 3.4.4 hold for a fixed $p > N + 1$ and let $f \in C_0^\infty(\Omega)$. Then the function $u(t, x) = (T_t f)(x)$ is the bounded classical solution of problem (3.0.1) and therefore has the regularity properties stated in Proposition 3.2.1. Moreover, Du is continuous and bounded in $\overline{Q}$.*

PROOF. Since $f \in D_p$, the function $t \to T_t f$ is continuous from $[0, T]$ to $W^{2,p}(\Omega)$ and Sobolev embedding implies that u, Du are bounded and continuous in $\overline{Q}$. To complete the proof, we have to show that $u \in C^{1,2}(Q)$.

Let us fix $\varepsilon > 0$ and open bounded sets Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$ and $\overline{\Omega}_2 \subset \Omega$. Since $(T_t)_{t\geq 0}$ is analytic, u is continuously differentiable from $[\varepsilon, T]$ to $W^{2,p}(\Omega)$ and Sobolev embedding yields $u_t \in C(Q)$. Set

$$\kappa = \sup_{\varepsilon \leq t \leq T} \Big(\|u(t, \cdot)\|_{W^{2,p}(\Omega)} + \|u_t(t, \cdot)\|_{W^{2,p}(\Omega)} \Big).$$

For every fixed $t \in [\varepsilon, T]$ the function $u(t, \cdot)$ belongs to $W^{2,p}(\Omega)$ and solves the equation

$$\sum_{i,j=1}^{N} q_{ij} D_{ij} u = -\langle F, Du \rangle + V u - u_t$$

in Ω. Since the right hand side belongs to $W^{1,p}_{loc}(\Omega)$ it follows that $u(t, \cdot) \in W^{3,p}_{loc}(\Omega)$ and that, for a suitable c depending on Ω_1, Ω_2 and the coefficients of A,

$$\sup_{\varepsilon \leq t \leq T} \|u(t, \cdot)\|_{W^{3,p}(\Omega_1)} \leq c\kappa,$$

see [26, Theorem 9.19]. We have proved that for every $i, j = 1, \dots, N$, $D_t D_{ij} u, DD_{ij} u \in L^p([\varepsilon, T] \times \Omega_1)$. By Sobolev embedding, since $p > N+1$, $D_{ij} u \in C(Q)$ and the proof is complete. □

3.5 Proof of Theorem 3.1.2

For $\varepsilon > 0$ let $V_\varepsilon(x) = \varepsilon \exp\{4c_2\sqrt{1 + |x|^2}\}$. Then $|DV_\varepsilon| \leq 4c_2 V_\varepsilon$ and for every $\sigma > 0$ there exists $c_\sigma > 0$ (depending on ε) such that $|F| \leq \sigma(V + V_\varepsilon)^{1/2} + c_\sigma$. Define $A_\varepsilon = A - V_\varepsilon$ and note that the hypotheses of Theorem 3.4.4 are satisfied.

Fix $p > N + 1$, $f \in C_0^\infty(\Omega)$ and let u_ε be the semigroup solution of (3.0.1) with A_ε instead of A, given by Theorem 3.4.4. By Lemma 3.4.6 the function u_ε is the bounded solution of the above problem and Du_ε is continuous and bounded in $\overline{Q}$. By Proposition 3.3.2 we deduce that $|Du_\varepsilon(t, \xi)| \leq (\gamma/\sqrt{t})\|f\|_\infty$, $\xi \in \partial\Omega$, with γ depending on ν_0, M, δ, T and independent of ε.

Since u_ε satisfies the hypotheses of Proposition 3.3.3, we deduce that

$$\|Du_\varepsilon(t, \cdot)\|_\infty \leq (C/\sqrt{t})\|f\|_\infty,$$

with C as in the statement.

Observe that $\|u_\varepsilon\|_\infty \leq \|f\|_\infty$. Let us fix $R > 0$ and note that the C^α-norm of the coefficients of A_ε is bounded, uniformly with respect to $\varepsilon < 1$, in $\Omega \cap B_{R+1}$. By the local Schauder estimates [30, Theorem IV.10.1] applied to the operator $D_t - A_\varepsilon$, there exists a constant C, independent of $\varepsilon < 1$, such that

$$\begin{aligned} \|u_\varepsilon\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))} &\leq C\Big(\|u_\varepsilon\|_{C((0,T)\times(\Omega\cap B_{R+1})}\Big) + \|f\|_{C^{2+\alpha}(\Omega\cap B_{R+1})} \\ &\leq 2C\|f\|_{C^{2+\alpha}(\Omega)}. \end{aligned}$$

By a standard compactness argument we conclude that a subsequence (u_{ε_n}) converges in $C^{1,2}([0,T]\times\overline{(\Omega\cap B_R)})$ for every R to a function u which is the bounded classical solution of (3.0.1) and satisfies $\|Du(t,\cdot)\|_\infty \le (C/\sqrt{t})\|f\|_\infty$.

Finally, to treat the general case of $f\in C_b(\Omega)$ we consider a sequence $(f_n)\in C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty\le\|f\|_\infty$. Let u_n be the bounded classical solution of (3.0.1) relative to f_n. Then $\|Du_n(t,\cdot)\|_\infty\le(C/\sqrt{t})\|f\|_\infty$, by the previous step. Since $(u_n)\to u$ in $C^{1,2}(Q)$, see the proof of Proposition 3.2.2, the estimate for Du follows. □

3.6 Examples and applications

We first show that gradient estimates fail, in general, if condition (3.1.3) is not satisfied. We refer the reader to [8, Example 5.6] for an operator defined on the whole space, for which condition (3.1.2) is violated and gradient estimates fail. The following result refines and generalizes an example in [57].

Example 3.6.1 We consider the following Dirichlet problem in $\Omega=\mathbb{R}^2_+=\{(x,y)\in\mathbb{R}^2,\ x>0\}$

$$\begin{cases} u_t(t,x,y)=u_{xx}(t,x,y)+u_{yy}(t,x,y)+g(y)u_x(t,x,y) & t>0,\ x>0,\\ u(t,0,y)=0 & t>0,\ y\in\mathbb{R},\\ u(0,x,y)=1 & (x,y)\in\Omega,\end{cases}$$

where $g\in C^2(\mathbb{R})$ and

$$\lim_{y\to+\infty} g(y)=+\infty.$$

Observe that (3.1.3) fails. However, Proposition 3.2.2 yields existence and uniqueness of a bounded solution u. Let us show that, for $t>0$, $u(t,\cdot)$ is *not* uniformly continuous in Ω. To this end, it is enough to show that, for every $t,x>0$,

$$\sup_{y>0} u(t,x,y)=1. \tag{3.6.1}$$

Fix $n>0$ and take c_n such that $g(y)\ge n$ for $y\ge c_n$. Define $R_n=(0,+\infty)\times(c_n,+\infty)$ and consider $v=v_n$ which solves

$$\begin{cases} v_t(t,x,y)=v_{xx}(t,x,y)+v_{yy}(t,x,y)+nv_x(t,x,y) & t>0,\ (x,y)\in R_n,\\ v(t,z)=0 & t>0,\ z\in\partial R_n,\\ v(0,x,y)=1 & (x,y)\in R_n,\end{cases}$$

We prove that for $t,x>0$

$$(i)\ \lim_{n\to\infty}\sup_{y>c_n} v_n(t,x,y)=1;\quad (ii)\ u(t,x,y)\ge v_n(t,x,y).$$

Clearly (i) and (ii) give (3.6.1). Let us verify (i). Note that $v_n(t,x,y)=a_n(t,x)b_n(t,y)$, where $a=a_n$, $b=b_n$ solve respectively

$$\begin{cases} a_t(t,x)=a_{xx}(t,x)+na_x(t,x) & t>0,\\ a(t,0)=0 & t>0,\\ a(0,x)=1 & x>0,\end{cases}\qquad \begin{cases} b_t(t,y)=b_{yy}(t,y) & t>0,\\ b(t,c_n)=0 & t>0,\\ b(0,y)=1 & y>c_n.\end{cases}$$

To find an explicit formula for a_n, we first remark that $a_n(t,x)=a_1(n^2t,nx)$. Then, setting $v(t,x)=e^{x/2}e^{\frac14 t}a_1(t,x)$, v solves

$$\begin{cases} v_t(t,x)=v_{xx}(t,x) & t>0,\ x>0,\\ v(t,0)=0 & t>0,\\ v(0,x)=e^{x/2} & x>0;\end{cases}$$

By a reflection argument we get easily an explicit expression for v and finally we obtain for any $t > 0$, $y \geq c_n$, $x \geq 0$,

$$a_n(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} - e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz, \quad b_n(t,y) = \int_0^{y-c_n} \frac{e^{\frac{-z^2}{4t}}}{\sqrt{\pi t}} dz.$$

To we check that (i) holds we write

$$a_n(t,x) = A_n^1(t,x) - A_n^2(t,x),$$
$$A_n^1(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz,$$
$$A_n^2(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz.$$

Let us consider A_n^1. By a change of variables we obtain

$$A_n^1(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\frac{x+n}{2\sqrt{t}}}^{+\infty} e^{-s^2} ds,$$

which is increasing in x and converges to 1 as $n \to +\infty$. In a similar way we get that $A_n^2(t,x)$ is decreasing in x and converges to 0 as $n \to +\infty$. Then (i) easily follows.

To prove (ii) we use Theorem A.0.13. Set $w = u - v_n$ in $(0,T) \times R_n$. We have $w(0,x,y) = 0$, $(x,y) \in R_n$. Moreover $w(t,z) \geq 0$, $z \in \partial R_n$, $t > 0$. To conclude it suffices to verify that

$$w_t(t,x,y) \geq \Delta w(t,x,y) + g(y) w_x(t,x,y), \quad t > 0, \ (x,y) \in R_n. \tag{3.6.2}$$

Since $w_t = \Delta w + g(y) w_x + [g(y) - n](v_n)_x$, $g(y) \geq n$, for $y \geq c_n$ and $(v_n)_x(t,x,y) = a'_n(t,x) b_n(t,y) \geq 0$, $t > 0$, $(x,y) \in R_n$, as verified above, (3.6.2) follows and the proof is complete.

For instance, we can take, in the above example, $g(y) = \sqrt{1+y^2}$. On the other hand, if $g(y) = -\sqrt{1+y^2}$ then all the conditions of Theorem 3.1.2 hold and gradient estimates hold.

Remark 3.6.2 We point out that our main result can be used to prove some boundary gradient estimates for solutions of Dirichlet elliptic problems, involving the operator A. Indeed if $\varphi \in C_b(\Omega) \cap C^2(\Omega)$ solves

$$\begin{cases} A\varphi(x) = 0, & x \in \Omega, \\ \varphi(\xi) = 0, & \xi \in \partial\Omega, \end{cases} \tag{3.6.3}$$

then φ is the bounded classical solution of (3.0.1) with $f = \varphi$. Thus, under the assumptions of Theorem 3.1.2, we get

$$\sup_{x \in \overline{\Omega}} |D\varphi(x)| \leq C \|\varphi\|_\infty.$$

This extends some classical boundary gradient estimates concerning linear and nonlinear second elliptic operators, involving bounded coefficients, see for instance [26, Section 14].

Remark 3.6.3 Theorem 3.1.2 has also some applications to isoperimetric inequalities, see [31] and [57].