Chapter 10

Copulas and Schur–concavity

The notion of Schur–concavity (and the closely related concept of Schur–convexity) has a great importance in the recent applications of statistics; witness of this is the recent monograph of Spizzichino [152] where Schur–concavity is one of the central themes in the Bayesian models of aging. However, the study of Schur–concavity of copulas does not seem to have yet received any attention in the literature, although twenty years ago Alsina studied the same question for t–norms (see [1]). To this topic this chapter is devoted.

In section 10.1 we present some results about the class of Schur–concave copulas and several examples are given in section 10.2. The concept of Schur–concavity, moreover, allows us to discuss an open problem on the classes of copulas and triangular norms (section 10.3).

The presented resulted are also contained in [44, 33].

10.1 The class of Schur–concave copulas

At the beginning of the study on Schur–concavity of copulas, we recall some properties that can be directly derived from section 1.2.

Proposition 10.1.1. *Let $C\colon [0,1]^2 \to [0,1]$ be a semicopula.*

(a) *If C is Schur–concave (or Schur-convex), then it is symmetric.*

(b) *If C is Schur–concave (or Schur–convex) on $\Delta_+ := \{(x,y) \in [0,1]^2 : x \geq y\}$, then C is Schur–concave (or Schur–convex) on $[0,1]^2$.*

Proposition 10.1.2. *A semicopula $C\colon [0,1]^2 \to [0,1]$ is Schur–concave if, and only if, for all x, y and λ in $[0,1]$*

$$C(x,y) \leq C(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y).$$

Proof. It suffices to consider the definition of Schur–concavity and Corollary 1.2.1. □

Example 10.1.1. Consider the copula M. For every $x \geq y$, we have $y \leq \lambda x + (1-\lambda)y$ and $y \leq (1-\lambda)x + \lambda y$, so that

$$M(x, y) \leq M(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y);$$

and, analogously, we have the same result for $x < y$. Therefore M is Schur–concave.

Proposition 10.1.3. *Let C be a continuously differentiable semicopula. Then C is Schur–concave on $[0,1]^2$ if, and only if,*

(i) *C is symmetric;*

(ii) *for all $(x, y) \in \Delta_+$, $\partial_1 C(x, y) \leq \partial_2 C(x, y)$.*

As a consequence, it is easily proved that the copula Π is Schur–concave. Note that not every symmetric copula is Schur–concave, as the following example shows.

Example 10.1.2. Let C be the absolutely continuous copula defined by

$$C(x, y) := \begin{cases} xy/2, & \text{if } (x, y) \in [0, 1/2] \times [0, 1/2]; \\ x\,(3y-1)/2, & \text{if } (x, y) \in [0, 1/2] \times [1/2, 1]; \\ y\,(3x-1)/2, & \text{if } (x, y) \in [1/2, 1] \times [0, 1/2]; \\ (xy + x + y - 1)/2, & \text{if } (x, y) \in [1/2, 1] \times [1/2, 1]. \end{cases}$$

This copula is symmetric and has a density c given by

$$c(x, y) := \begin{cases} 1/2, & \text{if } (x, y) \in [0, 1/2]^2 \cup [1/2, 1]^2; \\ 3/2, & \text{otherwise.} \end{cases}$$

The three points $\mathbf{x} = (6/10, 4/10)$, $\mathbf{y} = (7/10, 3/10)$ and $\mathbf{z} = (8/10, 2/10)$ are such that $\mathbf{x} \prec \mathbf{y} \prec \mathbf{z}$, but

$$C\left(\frac{6}{10}, \frac{4}{10}\right) = \frac{32}{200} < \frac{33}{200} = C\left(\frac{7}{10}, \frac{3}{10}\right),$$
$$C\left(\frac{7}{10}, \frac{3}{10}\right) = \frac{33}{200} > \frac{28}{200} = C\left(\frac{8}{10}, \frac{2}{10}\right).$$

Therefore C is not Schur–concave.

The following result allows us to investigate only on the class of Schur–concave copulas.

Proposition 10.1.4. *The copula W is the only Schur–convex (quasi–)copula.*

Proof. Let C be a Schur–convex copula. Given $x, y \in [0,1]$ such that $x + y \leq 1$, we have $(x, y) \prec (x + y, 0)$, from which

$$C(x, y) \leq C(x + y, 0) = 0.$$

Furthermore, given $x, y \in [0,1]$ such that $x + y > 1$, we have $(x, y) \prec (1, x + y - 1)$, from which

$$C(x, y) \leq C(1, x + y - 1) = x + y - 1.$$

Then, for all $x, y \in [0,1]$

$$C(x, y) \leq \max(x + y - 1, 0) = W(x, y),$$

but, from the Fréchet–Hoeffding bounds inequalities (1.13) it follows that $C = W$. □

Notice that W is also the only Schur–constant (semi–)copula, as showed in Proposition 2.2.2.

Now, we give some results about the class $\mathcal{C}_{SC}$ of Schur–concave copulas.

Proposition 10.1.5. *The class $\mathcal{C}_{SC}$ is a compact subset of $\mathcal{C}$ with respect to the topology of uniform convergence.*

Proof. It is known that $\mathcal{C}$ is compact space with respect to the topology of uniform convergence. But, if $(C_n)_{n \in \mathbf{N}}$ is a sequence in $\mathcal{C}_{SC}$, then the pointwise limit

$$C(x, y) = \lim_{n \to +\infty} C_n(x, y)$$

is Schur–concave. It follows that the set $\mathcal{C}_{SC}$ is a closed subset of $\mathcal{C}$, and therefore it is also compact. □

Proposition 10.1.6. *The class $\mathcal{C}_{SC}$ is a convex subset of $\mathcal{C}$.*

Proof. Let (x_1, x_2) and (y_1, y_2) be two points in $[0,1]^2$ such that $(x_1, x_2) \prec (y_1, y_2)$ and suppose that C_1 and C_2 are Schur–concave copulas. Then, for every $\lambda \in [0,1]$

$$\begin{aligned} C(x_1, x_2) &= \lambda C_1(x_1, x_2) + (1 - \lambda) C_2(x_2, y_2) \\ &\geq \lambda C_1(y_1, y_2) + (1 - \lambda) C_2(y_1, y_2) = C(y_1, y_2), \end{aligned}$$

which concludes the proof. □

Proposition 10.1.7. *A copula C is Schur–concave if, and only if, the survival copula $\hat{C}$ associated with C is Schur–concave.*

Proof. If C is Schur–concave, then, given (x_1, x_2), (y_1, y_2) two points in Δ_+ such that $(x_1, x_2) \prec (y_1, y_2)$, we have

$$(1 - x_1, 1 - x_2) \prec (1 - y_1, 1 - y_2),$$

from which

$$C(1-x_1, 1-x_2) \geq C(1-y_1, 1-y_2),$$

and

$$x_1 + x_2 - 1 + C(1-x_1, 1-x_2) \geq y_1 + y_2 - 1 + C(1-y_1, 1-y_2),$$

Then $\hat{C}$ is Schur-concave. The same argument applies if $\hat{C}$ is assumed to be Schur–concave □

In view of Sklar's Theorem, given a copula C and two univariate d.f.'s F and G, it is possible to construct a bivariate d.f. $H(x,y) := C(F(x), G(y))$ for every $(x,y) \in \mathbb{R}^2$. Now, it is useful to stress the fact that, with suitable marginal d.f.'s, Schur–concave copulas may yield Schur–concave, –convex or constant bivariate d.f.'s (see [115]).

10.2 Families of Schur–concave copulas

Theorem 10.2.1. *Every associative copula is Schur–concave.*

In order to prove this result, first we establish the following two lemmas.

Lemma 10.2.1. *An ordinal sum of Schur–concave copulas is a Schur–concave copula.*

Proof. Let $\{J_i = [a_i, b_i]\}_{i\in\mathfrak{I}}$ be a partition of the unit square and let $\{C_i\}_{i\in\mathfrak{I}}$ be a family of Schur–concave copulas. Let C be the ordinal sum of $\{C_i\}_{i\in\mathfrak{I}}$ with respect to $\{J_i\}_{i\in\mathfrak{I}}$, viz.

$$C(x,y) := \begin{cases} a_i + (b_i - a_i)\ C_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right), & \text{if } (x,y) \in J_i^2; \\ M(x,y), & \text{otherwise.} \end{cases}$$

Notice that C is symmetric and we shall show that, if every C_i is Schur–concave, then C is Schur–concave. Let $(x_1,x_2), (y_1,y_2)$ be two points in Δ_+ such that $(x_1,x_2) \prec (y_1,y_2)$. Suppose that there exists an index $i_0 \in \mathfrak{I}$ such that $(x_1,x_2), (y_1,y_2) \in J_{i_0}^2$. We observe that

$$\left(\frac{x_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{x_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right) \prec \left(\frac{y_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{y_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right),$$

that implies

$$C_{i_0}\left(\frac{x_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{x_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right) \geq C_{i_0}\left(\frac{y_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{y_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right),$$

since C_{i_0} is Schur–concave, and it follows $C(x_1,x_2) \geq C(y_1,y_2)$. Similarly, if (x_1,x_2) and (y_1,y_2) does not belong to J_i^2 for all $i \in \mathfrak{I}$, since M is also Schur–concave, it follows $C(x_1,x_2) \geq C(y_1,y_2)$. Finally, suppose that exists an index i_0 such that $(x_1,x_2) \in J_{i_0}^2$ and $(y_1,y_2) \notin J_i^2$ for all $i \in \mathfrak{I}$. We set $k := x_1 + x_2 = y_1 + y_2$ and we distinguish two cases.

Case 1. If $2a_{i_0} \le k \le a_{i_0} + b_{i_0}$, then $(x_1, x_2) \prec (k - a_{i_0}, a_{i_0})$ and

$$C(x_1, x_2) \ge C(k - a_{i_0}, a_{i_0}) = a_{i_0} \ge M(y_1, y_2) = C(y_1, y_2);$$

hence C is Schur–concave.
Case 2. If $a_{i_0} + b_{i_0} < k < 2b_{i_0}$, then $(x_1, x_2) \prec (b_{i_0}, k - b_{i_0})$ and

$$C(x_1, x_2) \ge C(b_{i_0}, k - b_{i_0}) = k - b_{i_0} \ge M(y_1, y_2) = C(y_1, y_2),$$

from which it follows that C is Schur–concave. □

Lemma 10.2.2. *Every Archimedean copula is Schur–concave.*

Proof. Let (x_1, x_2) and (y_1, y_2) two points in $[0, 1]^2$ such that $(x_1, x_2) \prec (y_1, y_2)$. It follows from Corollary 1.2.1 that there exists $\alpha \in [0, 1]$ such that, if $\overline{\alpha} := 1 - \alpha$, then

$$x_1 = \alpha\, y_1 + \overline{\alpha}\, y_2, \qquad x_2 = \overline{\alpha}\, y_1 + \alpha\, y_2.$$

Let C_φ be an Archimedean copula with additive generator φ. Since φ is convex and strictly decreasing

$$\begin{aligned} C(x_1, x_2) &= C(\alpha\, y_1 + \overline{\alpha}\, y_2, \overline{\alpha} y_1 + \alpha\, y_2) \\ &= \varphi^{[-1]}\left(\varphi(\alpha\, y_1 + \overline{\alpha}\, y_2) + \varphi(\overline{\alpha}\, y_1 + \alpha\, y_2)\right) \\ &\ge \varphi^{[-1]}\left(\alpha\, \varphi(y_1) + \overline{\alpha}\, \varphi(y_2) + \overline{\alpha}\, \varphi(y_1) + \alpha\, \varphi(y_2)\right) \\ &= \varphi^{[-1]}\left(\varphi(y_1) + \varphi(y_2)\right) = C(y_1, y_2), \end{aligned}$$

which concludes the proof. □

Proof. (Theorem 10.2.1) It was shown that M and every Archimedean copula are Schur–concave, moreover the ordinal sum of two Schur–concave copulas is Schur–concave too. In view of Representation Theorem for associative copulas (Theorem 1.6.9), the assertion follows. □

Here we give some other examples of Schur–concave copulas.

Example 10.2.1 (The Fréchet family). Every copula $C_{\alpha,\beta}$ belonging to the Fréchet family (see Example 1.6.2), defined by

$$C_{\alpha,\beta}(x, y) = \alpha\, M(x, y) + (1 - \alpha - \beta)\, \Pi(x, y) + \beta\, W(x, y)$$

is Schur–concave, because it is a convex sum of Schur–concave copulas.

Example 10.2.2 (The FGM family). For all $x, y \in [0, 1]$ and $\theta \in [-1, 1]$

$$C_\theta(x, y) = xy + \theta xy\, (1 - x)\, (1 - y)$$

is a member of the FGM family (see Example 1.6.3). For every $x, y \in [0,1]$ we have

$$\partial_1 C_\theta(x,y) = y + \theta y\,(1-x)\,(1-y) - \theta xy\,(1-y),$$
$$\partial_2 C_\theta(x,y) = x + \theta x\,(1-x)\,(1-y) - \theta xy\,(1-x).$$

As a consequence of the inequality $|1-x-y+2xy| \leq 1$, which holds for all x and y in $[0,1]$, if $x \geq y$ we have

$$\partial_2 C_\theta(x,y) - \partial_1 C_\theta(x,y) = (x-y)[1+\theta(1-x-y+2xy)] \geq 0.$$

Thus, it follows from Proposition 10.1.3 that C_θ is Schur–concave.

Example 10.2.3 (The Plackett family). For all $u, v \in [0,1]$ and $\theta > 0$, $\theta \neq 1$,

$$C_\theta(u,v) = \frac{[1+(\theta-1)\,(u+v)] - \sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}}{2\,(\theta-1)}$$

is a family of copulas, known as *Plackett family* (see [130]). For all $x, y \in [0,1]$, we have

$$\partial_1 C_\theta(u,v) = \frac{1}{2} - \frac{1\,+(\theta-1)\;(u+v)\;-2\theta v}{2\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}},$$
$$\partial_2 C_\theta(u,v) = \frac{1}{2} - \frac{1\,+(\theta-1)\;(u+v)\;-2\theta u}{2\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}}.$$

Moreover, for $u \geq v$, it follows that

$$\partial_2 C_\theta(u,v) - \partial_1 C_\theta(u,v) = \frac{\theta\,(u-v)}{2\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}} \geq 0.$$

Thus C_θ is Schur–concave.

10.3 Solution of an open problem for associative copulas

Recently, E.P. Klement, R. Mesiar and E. Pap ([85]) posed some open problems concerning triangular norms and related operators. In particular, the following problem was formulated:

Problem 10.3.1. *Let T be a continuous Archimedean t–norm. Prove or disprove that:*

$$T\,(\max\{x-a,0\}, \min\{x+a,1\}) \leq T(x,x) \tag{10.1}$$

holds for all $x \in [0,1]$ and for all $a \in\,]0,1/2[$.

In particular, the authors added that "a positive solution of this problem would induce a new characterization of associative copulas". This comment spurs us to investigate inequality (10.1) in the class of copulas: to this end, the notion of Schur–concavity will be useful.

First, notice that inequality (10.1) is not true for every copula.

Example 10.3.1. Let C be the copula given in [114, Example 3.3],

$$C(x,y) := \begin{cases} x, & \text{if } 0 \le x \le \dfrac{y}{2} \le \dfrac{1}{2}; \\ \dfrac{y}{2}, & \text{if } 0 \le \dfrac{y}{2} < x < 1 - \dfrac{y}{2}; \\ x + y - 1, & \text{if } \dfrac{1}{2} \le 1 - \dfrac{y}{2} \le x \le 1. \end{cases}$$

Then

$$C\left(\frac{4}{10}, \frac{6}{10}\right) = \frac{3}{10} > C\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}.$$

Note that C is not associative:

$$C\left(C\left(\frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}\right) = C\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \ne \frac{1}{8} = C\left(\frac{1}{2}, C\left(\frac{1}{2}, \frac{1}{2}\right)\right).$$

But, in general, we have

Lemma 10.3.1. *Let A be a semicopula. If A is Schur–concave, then A satisfies* (10.1).

Proof. Let a be in $]0, 1/2[$. We distinguish three cases. If $x \le a$, then (10.1) follows since A is positive. If $a < x \le 1 - a$, then (10.1) is equivalent to $A(x - a, x + a) \le A(x, x)$, which is a direct consequence of the Schur–concavity. If $x > 1 - a$, then (10.1) is equivalent to $x - a \le A(x, x)$ and this last inequality follows from the fact that

$$A(x, x) \ge A(2x - 1, 1) = 2x - 1 > x - a.$$

□

Lemma 10.3.1 and Theorem 10.2.1 together yield:

Theorem 10.3.1. *If C is an associative copula, then C satisfies* (10.1).

Notice that, if a copula C satisfies (10.1), then it need not be associative.

Example 10.3.2. We consider the FGM family of copulas given, for all $x, y \in [0, 1]$ and $\theta \in [-1, 1]$, by $C_\theta(x, y) = xy + \theta xy\,(1 - x)\,(1 - y)$. From Example 10.2.2, C_θ is Schur–concave, and thus satisfies (10.1), but, if $\theta \ne 0$, C_θ is not associative.

Notice also that, if a copula C satisfies (10.1), then it need not be Schur–concave.

Example 10.3.3. Let C be the copula defined by

$$C(x,y) := \begin{cases} \frac{1}{3}M(3x, 3y-2), & \text{if } (x,y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right]; \\ \frac{1}{3}M(3x-1, 3y-1), & \text{if } (x,y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]; \\ \frac{1}{3}M(3x-2, 3y), & \text{if } (x,y) \in \left[\frac{2}{3}, 1\right] \times \left[0, \frac{1}{3}\right]; \\ W(x,y), & \text{otherwise.} \end{cases}$$

This copula is obtained by using the block–based construction method introduced in [28]. Simple, but tedious, calculations show that C satisfies (10.1), but C is not Schur–concave. In fact, given the points $(2/10, 7/10)$ and $(3/10, 6/10)$, we have

$$C\left(\frac{3}{10}, \frac{6}{10}\right) = 0 < \frac{1}{30} = C\left(\frac{2}{10}, \frac{7}{10}\right),$$

which implies that C is not Schur–concave.

Remark 10.3.1. A geometrical interpretation can be given of the difference between inequality (10.1) and Schur–concavity. If $z = C(s,t)$ is the surface associated with a copula C that satisfies (10.1), the intersections of the surface with all the vertical planes of the form $s + t = 2x$, for all $x \in [0,1]$ and $s \in [0,x]$, are curves that take the maximum value in the point (x,x). But, if C is Schur–concave, we have the stronger condition that such curves are also decreasing from (x,x) to $(2x,0)$ (resp. $(2x-1,1)$).

10.3.1 Discussion in the class of triangular norms

In the class of continuous Archimedean t–norms, inequality (10.1) was characterized in [67] (see also [98, 127]).

Theorem 10.3.2. *Let T be a continuous Archimedean t–norm with additive generator t. Let ξ be defined by $\xi := t^{-1}(t(0)/2)$. Then T satisfies (10.1), for all $a \in]0, 1/2[$ and $x \in [0,1]$, if, and only if, the two following statements hold:*

(a) *for all $z \in]0, \min\{\xi, 1-\xi\}[$, $\quad t(\xi - z) + t(\xi + z) \geq 1$;*

(b) *t is convex on $[\xi, 1]$.*

In particular, if T is strict (viz. $t(0) = +\infty$), then the following statements are equivalent:

(a') *T satisfies (10.1), for all $a \in]0, 1/2[$ and $x \in [0,1]$;*

(b') *t is convex on $[0,1]$.*

On the other hand, we have also the characterization of continuous Archimedean t–norms that are Schur–concave (see [1]).

Theorem 10.3.3. *Let T be a continuous Archimedean t–norm with additive generator t. Then we have:*

(a) *if T is strict, then T is Schur–concave if, and only if, t is convex;*

(b) *if T is nilpotent, then T is Schur–concave if, and only if, t satisfies the following inequality:*

$$t\left(\alpha x + (1-\alpha)y\right) + t\left((1-\alpha)x + \alpha y\right) \le t(x) + t(y)$$

for every α in $[0,1]$ and for all x, y in $[0,1]$ such that $t(x) + t(y) \le 1$.

From the two previous results, we derive

Theorem 10.3.4. *Let T be a strict Archimedean t–norm with additive generator t. The following statements are equivalent:*

(i) *T is a copula;*

(ii) *T is Schur–concave;*

(iii) *T satisfies* (10.1).

Proof. From Theorem 1.6.6, T is a copula if, and only if, the additive generator t is convex and, then, T is Schur–concave (Theorem 10.3.3). Moreover, from Lemma 10.3.1, (ii) implies (iii), which, in its turn, is equivalent to the convexity of t (Theorem 10.3.2), which concludes the proof. □

Remark 10.3.2. The previous result also holds in the case of a continuous t–norm T which is jointly strictly monotone, i.e. $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$ (see [88]).

Looking at Theorem 10.3.4 in the class of nilpotent t–norm, we have $(i) \implies (ii) \implies (iii)$. But, there exists a Schur–concave nilpotent t–norm T, which is not a copula: consider, for example, a t–norm additively generated by $t(x) := \frac{1+\cos(\pi x)}{2}$ (see [1, Example 2.1]). Moreover, in the class of nilpotent t–norms, inequality (10.1) does not imply Schur–concavity as the following example shows.

Example 10.3.4. Consider a t–norm T with additive generator t given by

$$t(x) := \begin{cases} 1 - \frac{x}{10}, & \text{if } x \in \left[0, \frac{1}{10}\right]; \\ -\frac{49\sqrt{2}}{10\left(9\sqrt{2}-10\right)}\left(x - \frac{1}{10}\right) + \frac{99}{100} & \text{if } x \in \left]\frac{1}{10}, 1 - \frac{1}{\sqrt{2}}\right]; \\ (1-x)^2, & \text{otherwise.} \end{cases}$$

Then T satisfies the assumptions of Theorem 10.3.2, and thus the inequality (10.1), but

$$T(5/100, 95/100) = 25/1000 > 0 = T(1/10, 9/10),$$

which implies that T is not Schur–concave.