Chapter 6

A generalization of Archimedean copulas

In this chapter, we introduce and study a class of bivariate copulas that depend on two univariate functions. This new family, which contains the Archimedean copulas (see section 1.6.4), is presented in section 6.1. Several examples are then provided in section 6.2. Section 6.3 is devoted to the study of the concordance order in our class. The same circle of ideas will also enable us to construct and characterize a new family of quasi–copulas (section 6.4).

The contents of this chapter can be also found in the papers [41, 42].

6.1 The new family

We denote by Φ the class of all functions $\varphi : [0,1] \to [0,+\infty]$ that are continuous and strictly decreasing, and by Φ_0 the subset of Φ formed by the functions φ that satisfy $\varphi(1) = 0$. Moreover, we denote by Ψ the class of all functions $\psi : [0,1] \to [0,+\infty]$ that are continuous, decreasing and such that $\psi(1) = 0$. Notice that $\Phi_0 \subset \Psi$.

For all $(\varphi, \psi) \in \Phi \times \Psi$, we introduce the function $C_{\varphi,\psi} : [0,1]^2 \to [0,1]$ defined by

$$C_{\varphi,\psi}(x,y) := \varphi^{[-1]}\left(\varphi(x \wedge y) + \psi(x \vee y)\right). \tag{6.1}$$

Evidently, $C_{\varphi,\psi}$ is symmetric and, by using the properties (1.1) of the pseudo–inverse of a function, it is easily proved that, for all $x \in [0,1]$,

$$C_{\varphi,\psi}(x,1) = \varphi^{[-1]}\left(\varphi(x)\right) = x = C_{\varphi,\psi}(1,x)$$

and

$$0 \le C_{\varphi,\psi}(x,0) = \varphi^{[-1]}\left(\varphi(0) + \psi(x)\right) = C_{\varphi,\psi}(0,x) \le \varphi^{[-1]}\left(\varphi(0)\right) = 0,$$

viz. $C_{\varphi,\psi}$ satisfies the boundary conditions (C1).

Below we shall investigate under which conditions on φ and ψ, the function $C_{\varphi,\psi}$ defined by (6.1) is a copula.

Theorem 6.1.1. *Let φ and ψ belong to Φ and to Ψ, respectively, and let $C = C_{\varphi,\psi}$ be the function defined by* (6.1)*. If φ is convex and $(\psi - \varphi)$ is increasing in $[0,1]$, then C is a copula.*

Proof. Since C satisfies the boundary conditions (C1), it suffices to show that C is 2–increasing. Let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in the unit square. We distinguish three cases. If $R \subset \Delta_+$, then

$$\begin{aligned} V_C(R) =& \varphi^{[-1]}\left(\varphi(y_1) + \psi(x_1)\right) + \varphi^{[-1]}\left(\varphi(y_2) + \psi(x_2)\right) \\ &- \varphi^{[-1]}\left(\varphi(y_2) + \psi(x_1)\right) - \varphi^{[-1]}\left(\varphi(y_1) + \psi(x_2)\right). \end{aligned}$$

Set

$$\begin{aligned} s_1 &:= \varphi(y_1) + \psi(x_1), \qquad s_2 := \varphi(y_2) + \psi(x_2), \\ t_1 &:= \varphi(y_2) + \psi(x_1), \qquad t_2 := \varphi(y_1) + \psi(x_2). \end{aligned}$$

Then $(t_1, t_2) \prec (s_1, s_2)$ and $\varphi^{[-1]}$ is convex, because φ is convex. Thus Theorem 1.2.2 implies $V_C(R) \geq 0$.

Since C is symmetric, the same argument yields $V_C(R) \geq 0$ if the rectangle R is entirely contained in the region Δ_-.

Next, consider the case in which the diagonal of R lies on the diagonal of the unit square, viz. $x_1 = y_1$ and $x_2 = y_2$. If $x_1 = 0$, then $V_C(R) = \varphi^{[-1]}\left(\varphi(x_2) + \psi(x_2)\right) \geq 0$. Assume then that $x_1 > 0$. We obtain

$$\begin{aligned} V_C(R) =& \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_1)\right) + \varphi^{[-1]}\left(\varphi(x_2) + \psi(x_2)\right) \\ &- \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_2)\right). \end{aligned}$$

Now, set

$$s_1 := \varphi(x_1) + \psi(x_1), \quad s_2 := \varphi(x_2) + \psi(x_2), \quad t_1 := \varphi(x_1) + \psi(x_2) =: t_2.$$

Since $t \mapsto (\psi(t) - \varphi(t))$ is increasing in $[0,1]$, we obtain

$$\min\{t_1, t_2\} = \varphi(x_1) + \psi(x_2) \geq \varphi(x_2) + \psi(x_2) = \min\{s_1, s_2\}$$

and

$$t_1 + t_2 \geq s_1 + s_2.$$

Therefore $(t_1, t_2) \prec^w (s_1, s_2)$, and since $\varphi^{[-1]}$ is convex and decreasing, from Tomic's Theorem 1.2.3 we have $V_C(R) \geq 0$. By using the Proposition 1.6.1, we have the desired assertion. □

Remark 6.1.1. Notice that, since $t \mapsto (\psi(t) - \varphi(t))$ is increasing, then $\varphi(t) \geq \psi(t)$ for all $t \in [0,1]$. In fact, if there existed $x_0 \in (0,1)$ such that $\varphi(x_0) < \psi(x_0)$, then

$$0 < \psi(x_0) - \varphi(x_0) \leq \psi(1) - \varphi(1) = -\varphi(1) \leq 0,$$

which is a contradiction.

If (φ, ψ) is a pair of functions that generate a copula C of type (6.1), then, for any $c > 0$, also $(c\varphi, c\psi)$ generates C.

Given a copula C of type (6.1) generated by φ and ψ, let h and k be the two functions given by $h(t) := \exp(-\varphi(t))$ and $k(t) := \exp(-\psi(t))$. Then, we have

$$\begin{aligned} C(x,y) &= \varphi^{[-1]}\left(-\ln(h(x \wedge y)) - \ln(k(x \vee y))\right) \\ &= h^{[-1]}\left(\exp\left[\ln(h(x \wedge y)) + \ln(k(x \vee y))\right]\right) \\ &= h^{[-1]}\left(h(x \wedge y) \cdot k(x \vee y)\right). \end{aligned}$$

In particular, Theorem 6.1.1 can be easily reformulated in a multiplicative form.

Theorem 6.1.2. *Let h, k be two continuous and increasing functions from $[0,1]$ into $[0,1]$ such that $k(1) = 1$. If h is log–concave and $t \mapsto h(t)/k(t)$ is increasing, then*

$$C_{h,k}(x,y) := h^{[-1]}\left(h(x \wedge y) \cdot k(x \vee y)\right) \tag{6.2}$$

is a copula.

6.2 Examples

The most important family of copulas of type (6.1) is the Archimedean one. Specifically, given a function φ in Φ_0, $C_{\varphi,\varphi}$ is an Archimedean copula with additive generator φ. In particular, the copulas Π and W are of this type. On account of this fact, a copula of type (6.1) is called *generalized Archimedean copula* (briefly *GA–copula*).

Notice that also the copula M is of type (6.1): it suffices to take $\psi = 0$ and $\varphi \in \Phi$. As a consequence, the family of copulas of type (6.1) is comprehensive, viz. M, Π and W are GA–copulas.

Example 6.2.1. Given an increasing and differentiable function $f : [0,1] \to [0,1]$, let $\varphi(t) = -\ln t$ and $\psi(t) = -\ln f(t)$ be two functions satisfying the assumptions of Theorem 6.1.1. The corresponding copula of type (6.1) is given by

$$C_{\varphi,\psi}(x,y) := (x \wedge y) f(x \vee y),$$

which is a member of the family of copulas studied in chapter 4. In fact, notice that, if $\psi - \varphi$ is increasing, then we can deduce that $tf'(t) \leq f(t)$ on $[0,1]$ and, therefore, f satisfies the assumptions of Theorem 4.1.1.

Example 6.2.2. Let $\delta : [0,1] \to [0,1]$ be in the class $\mathcal{D}_2$ of the diagonals of a copula. Take $\varphi(t) = 1 - t$ and $\psi(t) = t - \delta(t)$. If ψ is decreasing, then $(\psi - \varphi)$ is increasing and Theorem 6.1.1 ensures that the pair (φ, ψ) generates a copula $C_{\varphi,\psi} = C_\delta$ given by

$$C_\delta(x,y) := \max\{0, \delta(x \vee y) - |x - y|\} \qquad \text{for all } x, y \in [0,1].$$

Thus C_δ is a member of the family of MT–copulas, characterized and studied in chapter 5.

Example 6.2.3. Take $\varphi \in \Phi_0$ and $\psi(t) = \alpha\varphi$ for $\alpha \in [0,1]$. Then $\psi \in \Psi$ and the corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$\begin{aligned} C_\alpha(x,y) &= \varphi^{[-1]}\left(\varphi(x \wedge y) + \alpha\varphi(x \vee y)\right) \\ &= \varphi^{[-1]}\left((\varphi(x) + \varphi(y)) \cdot A\left(\frac{\varphi(x)}{\varphi(x) + \varphi(y)}\right)\right), \end{aligned}$$

where

$$A(t) = \begin{cases} 1 - (1-\alpha)t, & t \in [0, 1/2]\,; \\ \alpha + (1-\alpha)t, & t \in [1/2, 1]\,. \end{cases}$$

Therefore C_α belongs to the Archimax family of copulas presented in Example 1.6.9.

Example 6.2.4. Take $\varphi(t) = -\alpha t + \alpha$ $(\alpha \geq 1)$ and $\psi(t) = 1 - t$. Then the pair (φ, ψ) belongs to $\Phi \times \Psi$ and satisfies the assumptions of Theorem 6.1.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$\begin{aligned} C_\alpha(x,y) &= \max\left\{0, x \wedge y - \frac{1}{\alpha}(1 - x \vee y)\right\} \\ &= \begin{cases} \dfrac{\alpha(x \wedge y) + (x \vee y) - 1}{\alpha}, & \alpha(x \wedge y) + (x \vee y) \geq 1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The copula C_α has a probability mass $\frac{2}{\alpha+1}$ uniformly distributed on the two segments connecting the point $\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$ with $(0,1)$ and $(1,0)$, respectively, and a probability mass $\frac{\alpha-1}{\alpha+1}$ uniformly distributed on the segment joining the point $\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$ to $(1,1)$ (see also page 57 of [114]). In particular, we obtain $C_1 = W$ and $C_\infty = M$. Notice that this class of copulas has been also used in [29].

Example 6.2.5. Take $\varphi(t) = 1 - t$ and, for $\alpha \in [0,1]$,

$$\psi(t) = \begin{cases} \alpha/2, & \text{if } t \in [0, \alpha/2]\,; \\ \alpha - t, & \text{if } t \in [\alpha/2, \alpha]\,; \\ 0, & \text{if } t \in [\alpha, 1]\,. \end{cases}$$

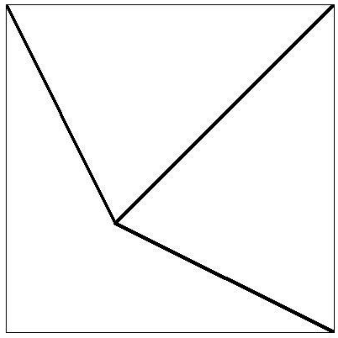

Figure 6.1: Support of the copula C_α ($\alpha = 2$) in Example 6.2.4

Then the pair (φ, ψ) belongs to $\Phi \times \Psi$ and satisfies the assumptions of Theorem 6.1.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$C_\alpha(x,y) := \begin{cases} \max\{0, x+y-\alpha\} & \text{if } (x,y) \in [0,\alpha]^2\,; \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Therefore, C_α spreads uniformly the mass on the two segments connecting, respectively, the points $(1,1)$ with (α,α) and $(\alpha,1)$ with $(1,\alpha)$. Notice that C_α is a member of the Mayor–Torrens family (1.5).

Note that copulas of type (6.1) that are ordinal sums of copulas are characterized in the following

Proposition 6.2.1. *The only (non trivial) ordinal sum of copulas that can be expressed in the form* (6.1) *is the ordinal sum* $(\langle 0, a, C\rangle)$*, where* C *is a suitable copula and* $a \in \,]0,1[$.

Proof. It suffices to observe that the set of idempotent elements of C is given by $\{0\} \cup [a,1]$, where $a := \inf\{t \in [0,1] : \psi(t) = 0\}$. In fact, given the copula C, let δ be its diagonal section given by $\delta(t) := C(t,t) = \varphi^{[-1]}\left(\varphi(t) + \psi(t)\right)$. In particular, for all $t \in \,]0,1[$ we have $\delta(t) < t$ if, and only if, $\min\{\varphi(t) + \psi(t), \varphi(0)\} > \varphi(t)$, which is equivalent to $\psi(t) > 0$. Since ψ is decreasing and $\psi(1) = 0$, we have $\delta(t) < t$ if, and only if, t is in $]0,a[$ where $a := \inf\{t \in [0,1] : \psi(t) = 0\}$. □

Theorem 6.1.1 highlights the importance of finding generators in order to construct GA–copulas. To this purpose the following result provides useful methods (the proofs are the same given in [61]).

Theorem 6.2.1. *Let* (φ, ψ) *be a pair in* $\Phi \times \Psi$*. The following statements hold:*

(a) *if* $f : [0,1] \to [0,1]$ *is an increasing and concave bijection, then* $(\varphi \circ f, \psi \circ f)$ *is in* $\Phi \times \Psi$*;*

(b) *if* $f : [0, +\infty[\to [0, +\infty[$ *is an increasing convex function such that* $f(0) = 0$, *then* $(f \circ \varphi, f \circ \psi)$ *is in* $\Phi \times \Psi$*;*

(c) *if* $0 < \alpha < 1$, *then* $(\varphi(\alpha t) - \varphi(\alpha), \psi(\alpha t) - \psi(\alpha))$ *is in* $\Phi \times \Psi$.

Notice that additive generators of Archimedean copulas can be combined together in order to construct copulas of the type (6.1). In fact, let φ and ψ belong to Φ_0; in view of Theorem 6.1.1, the convexity of φ and the condition that $(\psi - \varphi)$ be increasing ensure that $C_{\varphi,\psi}$ is a copula. Consider, for instance, the functions $\alpha(t) := 1 - t$, $\beta(t) := -\ln t$ and $\gamma(t) := 1/t - 1$, which are, respectively, the additive generators of the Archimedean copulas W, Π and the *Hamacker copula*

$$\frac{\Pi}{\Sigma - \Pi}(x, y) := \frac{xy}{x + y - xy};$$

then we obtain the following copulas:

$$\begin{aligned} C_{\beta,\alpha}(x, y) &= (x \wedge y) \exp\big((x \vee y) - 1\big), \\ C_{\gamma,\alpha}(x, y) &= \frac{x \wedge y}{1 + (x \wedge y) - xy}, \\ C_{\gamma,\beta}(x, y) &= \frac{x \wedge y}{1 - (x \wedge y)\ln(x \vee y)}. \end{aligned}$$

6.3 Concordance order

The concordance order between two GA–copulas is determined only by the properties of their generators.

Theorem 6.3.1. *Let* C *and* D *be two GA–copulas generated, respectively, by the pairs* (φ, ψ) *and* (γ, η). *Let* $\alpha := \varphi \circ \gamma^{[-1]}$ *and* $\beta := \psi \circ \eta^{[-1]}$. *Then* $C \leq D$ *if, and only if,*

$$\alpha(a + b) \leq \alpha(a) + \beta(b) \qquad \textit{for all } a, b \in [0, +\infty]\,. \tag{6.3}$$

Proof. Let x and y be in $[0, 1]$ and suppose, first, that $x \leq y$. Then $C \leq D$ if, and only if,

$$\varphi^{[-1]}\left(\varphi(x) + \psi(y)\right) \leq \gamma^{[-1]}\left(\gamma(x) + \eta(y)\right).$$

Let $\gamma(x) = a$ and $\eta(y) = b$, then the above inequality is equivalent to

$$\varphi^{[-1]}\left(\varphi \circ \gamma^{[-1]}(a) + \psi \circ \eta^{[-1]}(b)\right) \leq \gamma^{[-1]}\left(a + b\right).$$

Applying the function γ to both sides, we obtain

$$\alpha^{[-1]}\left(\alpha(a) + \beta(b)\right) \geq a + b,$$

viz. condition (6.3).

If $x > y$, the proof can be completed by using the same arguments. □

Notice that, if C and D are Archimedean copulas generated, respectively, by φ and γ, then $\alpha = \beta$ and condition (6.3) is equivalent to the subadditivity of α, as stated in Theorem 4.4.2 of [114].

In two particular cases, the concordance order can be expressed in a form simpler than (6.3).

Corollary 6.3.1. *Let C and D be two copulas of type* (6.1) *generated, respectively, by the pairs (φ, ψ) and (γ, η). Let $\alpha := \varphi \circ \gamma^{[-1]}$ and $\beta := \psi \circ \eta^{[-1]}$.*

(a) *If $\varphi = \gamma$ is a strictly decreasing function with $\varphi(0) = +\infty$, then $C \leq D$ if, and only if, $\psi(t) \geq \eta(t)$ for every $t \in [0, 1]$.*

(b) *If $\psi = \eta$ is a strictly decreasing function with $\psi(0) = +\infty$, then $C \leq D$ if, and only if, α is 1–Lipschitz.*

Proof. Since $\varphi = \gamma$ admits an inverse, $\alpha(t) = t$. Therefore condition (6.3) is equivalent to

$$b \leq \beta(b) = \psi \circ \eta^{[-1]}(b) \qquad \text{for all } a, b \in [0, +\infty] .$$

Taking $b := \eta(t)$, we have $C \leq D$ if, and only if, $\psi(t) \geq \eta(t)$ for every $t \in [0, 1]$.

Analogously, for (b), since $\psi = \eta$ admits an inverse, we have $\beta(t) = t$, and (6.3) is equivalent to

$$\alpha(a + b) - \alpha(a) \leq b \qquad \text{for all } a, b \in [0, +\infty] ,$$

as asserted. □

6.4 A similar new class of quasi–copulas

It is interesting to ascertain under which conditions the function $C_{\varphi,\psi}$ defined by (6.1) is a quasi–copula; the following theorem provides a characterization of quasi–copulas in the class $\{C_{\varphi,\psi} : \varphi \in \Phi, \psi \in \Psi\}$.

Theorem 6.4.1. *Let φ and ψ belong to Φ and to Ψ, respectively. Let $C_{\varphi,\psi}$ be the function defined by* (6.1)*. Then $C_{\varphi,\psi}$ is a quasi–copula if, and only if, both the following statements hold:*

(a) *for all $r \leq s$ and $t \in \left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$*

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \varphi^{[-1]}(r) - \varphi^{[-1]}(s);$$

(b) *for all $r \leq s$ and $t \geq \left(\varphi \circ \psi^{[-1]}\right)(r)$*

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \psi^{[-1]}(r) - \psi^{[-1]}(s).$$

Proof. We already know that, when φ and ψ belong to Φ and to Ψ respectively, the function $C := C_{\varphi,\psi}$ given by (6.1) satisfies condition (Q1) for a quasi–copula. That both $x \mapsto C(x, y)$ and $y \mapsto C(x, y)$ are increasing functions for every $y \in [0, 1]$ and for every $x \in [0, 1]$, respectively, is an obvious consequence of the fact that both φ and ψ are decreasing functions. Therefore, in order to complete the proof, it suffices to show that the assumptions are equivalent to the Lipschitz condition (Q3) for C.

Assume, first, that $x_1 < x_2 \leq y$. The inequality

$$C(x_2, y) - C(x_1, y) \leq x_2 - x_1 \tag{6.4}$$

is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(x_2) + \psi(y)\right) - \varphi^{[-1]}\left(\varphi(x_1) + \psi(y)\right)& \\ \leq x_2 - x_1 = \varphi^{[-1]}(\varphi(x_2)) - \varphi^{[-1]}(\varphi(x_1)).&\end{aligned}$$

By setting $r := \varphi(x_2)$, $s := \varphi(x_1)$ and $t := \psi(y)$, we obtain that t belongs to $\left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$ and $r \leq s$; moreover, the last inequality is equivalent to (a).

Next assume $y \leq x_1 < x_2$. The inequality (6.4) is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(y) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(y) + \psi(x_1)\right) &\leq x_2 - x_1 \\ &= \psi^{[-1]}(\psi(x_2)) - \psi^{[-1]}(\psi(x_1)).\end{aligned}$$

By setting $r := \psi(x_2)$, $s := \psi(x_1)$, $t := \varphi(y)$, we have $t \geq \left(\varphi \circ \psi^{[-1]}\right)(s)$ and $r \leq s$. For the arbitrariness of $s \geq r$, it follows that $t \geq \left(\varphi \circ \psi^{[-1]}\right)(r)$ and the last inequality is equivalent to condition (b).

The final case, $x_1 \leq y \leq x_2$, follows from the two previous cases, since

$$\begin{aligned}C(x_2, y) - C(x_1, y) &= C(x_2, y) - C(y, y) + C(y, y) - C(x_1, y) \\ &\leq x_2 - y + y - x_1 = x_2 - x_1,\end{aligned}$$

which concludes the proof. □

Although Theorem 6.4.1 characterizes quasi–copulas of the type (6.1), conditions (a) and (b) may be somewhat impractical. However, these conditions are equivalent to the convexity of φ, when $\varphi = \psi$, as is shown in the following

Corollary 6.4.1. *Let φ belong to Φ_0 and let $C_{\varphi,\varphi}$ be a function of the type* (6.1). *Then $C_{\varphi,\varphi}$ is a quasi–copula if, and only if, φ is convex.*

Proof. By Theorem 6.4.1, $C_{\varphi,\varphi}$ is a quasi–copula if, and only if, for every $r \leq s$ and for every $t \geq 0$, we have

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \varphi^{[-1]}(r) - \varphi^{[-1]}(s),$$

which can be written in the form

$$\varphi^{[-1]}(r + t) + \varphi^{[-1]}(s) \leq \varphi^{[-1]}(s + t) + \varphi^{[-1]}(r). \tag{6.5}$$

If φ is convex, so is $\varphi^{[-1]}$, and therefore (6.5) follows directly from Theorem 1.2.2, observing that $(r+t, s) \prec (s+t, r)$. Conversely, if (6.5) holds, for all $a, b \geq 0$ we can put

$$r = a, \qquad t = \frac{b-a}{2} \qquad s = \frac{a+b}{2}.$$

Then, we have

$$2\varphi^{[-1]}\left(\frac{a+b}{2}\right) \leq \varphi^{[-1]}(a) + \varphi^{[-1]}(b),$$

viz. $\varphi^{[-1]}$ is mid–convex and, because $\varphi^{[-1]}$ is continuous, it follows that $\varphi^{[-1]}$ is convex, and hence so is φ. □

The following result provides a sufficient condition for $C_{\varphi,\psi}$ to be a quasi–copula.

Proposition 6.4.1. *Let φ and ψ belong to Φ and to Ψ, respectively. If φ is convex, then, for the function $C_{\varphi,\psi}$ defined by* (6.1)*, the following statements are equivalent:*

(a) *$C_{\varphi,\psi}$ is a quasi–copula;*

(b) *for every $\lambda \in [\varphi(1), \varphi(0)]$ the function $\rho_\lambda : \left[\varphi^{[-1]}(\lambda), 1\right] \to \mathbb{R}$ given by*

$$\rho_\lambda(t) := \varphi^{[-1]}\left(\lambda + \psi(t)\right) - t$$

is decreasing.

Proof. From Theorem 6.4.1, it suffices to show that C satisfy the 1–Lipschitz condition (Q3). Assume, first, that $x_1 \leq x_2 \leq y$. The inequality

$$C(x_2, y) - C(x_1, y) \leq x_2 - x_1 \tag{6.6}$$

is equivalent to

$$\varphi^{[-1]}\left(\varphi(x_2) + \psi(y)\right) + \varphi^{[-1]}(\varphi(x_1)) \leq \varphi^{[-1]}\left(\varphi(x_1) + \psi(y)\right) + \varphi^{[-1]}(\varphi(x_2)).$$

By setting $s_1 := \varphi(x_2) + \psi(y)$, $s_2 := \varphi(x_1)$, $t_1 := \varphi(x_1) + \psi(y)$ and $t_2 := \varphi(x_2)$, we have $(s_1, s_2) \prec (t_1, t_2)$ and therefore, since $\varphi^{[-1]}$ is convex, Theorem 1.2.2 ensures that (6.6) is satisfied. In this case, the Lipschitz condition is a consequence of the convexity of φ alone.

Next assume $y \leq x_1 < x_2$. The inequality (6.6) is equivalent to

$$\varphi^{[-1]}\left(\varphi(y) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(y) + \psi(x_1)\right) \leq x_2 - x_1;$$

viz. condition (b).

The final case, $x_1 \leq y \leq x_2$, follows from the two previous cases. □

Example 6.4.1. Take the functions

$$\varphi(t) := -\ln t \quad \text{and} \quad \psi(t) := -\ln\left(t + t^2 - t^3\right).$$

For every $\lambda \in [0, +\infty]$ the function $\rho_\lambda : [\exp(-\lambda), 1] \to \mathbb{R}$ is given by

$$\rho_\lambda(t) := \exp(-\lambda)\left(t + t^2 - t^3\right) - t.$$

Now, (φ, ψ) is in $(\Phi \times \Psi)$ and ρ_λ is decreasing, therefore Theorem 6.4.1 ensures that the function $C_{\varphi,\psi}$, given by (6.1) is a quasi–copula. Notice that $C_{\varphi,\psi}$ is not a copula, as shown in Example 4.3.1. This implies that the family $\{C_{\varphi,\psi} : \varphi \in \Phi, \psi \in \Psi\}$, where φ and ψ satisfy conditions (a) and (b) of Theorem 6.4.1, contains proper quasi–copulas.