Chapter 5

A family of copulas with given diagonal section

Given a copula C, its diagonal δ satisfies the following properties:

(D1) $\delta(1) = 1$;

(D2) $\delta(t) \leq t$ for all $t \in [0,1]$;

(D3) δ is increasing;

(D4) $|\delta(t) - \delta(s)| \leq 2|t - s|$ for all $t, s \in [0,1]$.

We recall that $\mathcal{D}$ denotes the set of functions $\delta : [0,1] \to [0,1]$ satisfying (D1)–(D3) and $\mathcal{D}_2$ denotes the subset of $\mathcal{D}$ of the functions satisfying also (D4). In literature the question of determining a copula beginning from a function $\delta \in \mathcal{D}_2$ has been already studied, as showed in subsection 1.6.3.

In this chapter, we give another class of copulas that can be derived from the diagonal section. Specifically, we are interested on copulas C satisfying the functional equation:

$$C(x,y) + |x - y| = C(x \vee y, x \vee y) \qquad \text{whenever } C(x,y) > 0.$$

In other words, we analyse under which conditions on $\delta \in \mathcal{D}_2$ the function

$$D_\delta(x,y) := \max\{0, \delta(x \vee y) - |x - y|\} \qquad \text{for all } x, y \in [0,1], \tag{5.1}$$

is a copula. Notice that t–norms of type (5.1) were already studied by G. Mayor and J. Torrens ([105]), who obtained the following characterization.

Theorem 5.0.2. *Let T be a continuous t–norm with diagonal section δ. Then T satisfies the functional equation*

$$T(x, y) + |x - y| = \delta(x \vee y),$$

whenever $T(x, y) > 0$, if, and only if, T belongs to the Mayor–Torrens family of t–norms presented in Example 1.4.1.

For these reasons, we shall use the prefix MT to indicate a function of type (5.1) (e.g. MT–copula, MT–quasicopula, MT–semicopula), where "MT" stands for "à la Mayor and Torrens".

MT–copulas are characterized in section 5.1 and their properties are studied in section 5.2. Section 5.3 is devoted to the study of a simple procedure to generate an aggregation operator with additional properties (Lipschitz, 2–increasing, etc.) beginning from two aggregation operators of the same type and with the same diagonal section.

The results of this chapter are also contained in [39, 40].

5.1 Characterization of MT–copulas

In order to characterize MT–copulas, first, we establish an analogous characterization for semicopulas of the same type.

Lemma 5.1.1. *The following statements are equivalent:*

(a) *$\delta \in \mathcal{D}$ and there exists $a \in [0, 1[$ such that $\delta(x) = 0$ on $[0, a]$ and the function $x \mapsto (\delta(x) - x)$ is increasing on $[a, 1]$.*

(b) *D_δ is an MT–semicopula;*

Proof. $(a) \Longrightarrow (b)$: For all $t \in [0, 1]$

$$D_\delta(t, 1) = \max\{0, \delta(1) - |t - 1|\} = t = D_\delta(1, t).$$

In order to ensure that D_δ is increasing in each variable, consider $x, x', y \in [0, 1]$ with $x \leq x'$ such that $D_\delta(x, y) > 0$ and $D_\delta(x', y) > 0$. If $y \geq x'$, then

$$D_\delta(x, y) = \delta(y) - y + x \leq \delta(y) - y + x' = D_\delta(x', y).$$

If $y \leq x$, then, since $t \mapsto (\delta(t) - t)$ is increasing,

$$D_\delta(x, y) = \delta(x) - x + y \leq \delta(x') - x' + y = D_\delta(x', y). \tag{5.2}$$

Finally, if $x < y \leq x'$, then, again since $t \mapsto (\delta(t) - t)$ is increasing,

$$D_\delta(x, y) = \delta(y) - y + x \leq \delta(x') - x' + y = D_\delta(x', y). \tag{5.3}$$

$(b) \Longrightarrow (a)$: Set $a := \sup\{t \in [0,1] : D_\delta(t,t) = 0\}$ that satisfies the required conditions. The isotony of $(\delta(t)-t)$ is established in the same way of the proof of $(a) \Rightarrow (b)$ (see inequalities (5.2) and (5.3)). □

Theorem 5.1.1. *The following statements are equivalent:*

(a') $\delta \in \mathcal{D}_2$ *and there exists* $a \in [0,1/2]$ *such that* $\delta(x) = 0$ *on* $[0,a]$ *and the function* $x \mapsto (\delta(x) - x)$ *is increasing on* $[a,1]$*;*

(b') D_δ *is a copula.*

Proof. $(a') \Longrightarrow (b')$: In view of Proposition 1.6.1, it suffices to prove that

$$D_\delta(x',y') + D_\delta(x,y) - D_\delta(x',y) - D_\delta(x,y') \geq 0 \tag{5.4}$$

in three cases: on a rectangle $R := [x,x'] \times [y,y']$ contained in Δ_- or in Δ_+, and on a rectangle $R := [x,y] \times [x,y]$.

In the first case, put $F(x,y) := \delta(y) - y + x$. Then

$$F(x',y') + F(x,y) = F(x,y') + F(x',y)$$

and $D_\delta(x,y) = \max\{0, F(x,y)\}$. If two terms on the left hand side of (5.4) are equal to 0, then inequality (5.4) follows from the monotony of δ. If one of the terms in the left hand side of (5.4) is 0, then it is necessarily the value on the left–lower corner of the rectangle R and, on the remaining three corners, the values of D_δ are equal to those of F. Then $F(x,y) \leq D_\delta(x,y)$ implies

$$0 = V_F(R) \leq V_{D_\delta}(R).$$

If R is contained on Δ_+, the proof follows from the commutativity of D_δ.

In the third case, it suffices to prove that, for every $x \leq y$, $\delta(x) + \delta(y) \geq 2(\delta(y) - y + x)$, i.e. $(\delta(y) - y) - (\delta(x) - x) \leq y - x$. However, this inequality follows from (D4) because, if $\delta \in \mathcal{D}_2$, then $(\delta(t) - t)$ is 1-Lipschitz.

$(b') \Longrightarrow (a')$: It follows directly from Lemma 5.1.1, by observing that, because of the Fréchet–Hoeffding bounds (1.13), we have $a \in [0,1/2]$. □

Corollary 5.1.1. *The following statements are equivalent:*

(a') $\delta \in \mathcal{D}_2$ *and there exists* $a \in [0,1/2]$ *such that* $\delta(x) = 0$ *on* $[0,a]$ *and the function* $x \mapsto (\delta(x) - x)$ *is increasing on* $[a,1]$*;*

(c') D_δ *is a quasi–copula.*

In other words, no proper MT–quasi–copula exists.

Proof. As in Theorem 5.1.1, we prove that $(c') \Longrightarrow (a')$. The assertion follows directly, since every copula is a quasi–copula. □

5.2 Properties of MT–copulas

In this section, we denote by D an MT–copula and by δ its diagonal satisfying the assumptions of Theorem 5.1.1.

Proposition 5.2.1. *Every MT–copula D is a simple Bertino copula.*

Proof. First, observe that $D(x, y) = 0$ if, and only if, $x \vee y \leq a$. In fact, if there exist $x, y \in [0, 1]$ such that $D(x, y) = 0$ with $x \vee y > a$, we have $\delta(x \vee y) - |x - y| \leq 0$, from which, for all $x > a \vee y$, $\delta(x) - x \leq -y \leq \delta(y) - y$: a contradiction, because $(\delta(x) - x)$ is increasing on $[a, 1]$.

Let x, y be in $[0, 1]$ such that $D(x, y) > 0$ so that x and y both belong to $[a, 1]$. By Theorem 5.1.1, $x \mapsto (x - \delta(x))$ is decreasing on $[a, 1]$. If $x \geq y$, we have

$$D(x, y) = \delta(x) - x + y = \min\{x, y\} - \min\{x - \delta(x), y - \delta(y)\}.$$

In the other case $x < y$, the proof is analogous. □

As a consequence, the following statistical characterization of MT–copulas can be formulated ([57, Corollary 3.2])

Corollary 5.2.1. *Let U and V be r.v.'s uniformly distributed on $[0, 1]$ whose joint distribution function is the copula D. Then, for each $(x, y) \in [0, 1]^2$, either*

$$P(U \leq x, V \leq y) = P(\max\{U, V\} \leq \min\{x, y\})$$

or

$$P(U > x, V > y) = P(\min\{U, V\} > \max\{x, y\}).$$

Moreover, since $t \mapsto (t - \delta(t))$ has slope 1 in the interval $[0, a]$ on which it is strictly increasing, in view of [57, Theorem 4.1], it follows

Proposition 5.2.2. *Every MT–copula D is extremal, in the sense that, if there exist two copulas A and B such that $D = \alpha A + (1 - \alpha)B$, with $\alpha \in]0, 1[$, then $D = A = B$.*

Remark 5.2.1. The support of D contains the part of the main diagonal of the unit square corresponding to the union of the intervals on which $\delta > 0$ and $\delta' < 2$ and a line which is the boundary of its zero region (see also [57, Theorem 2.2]).

Remark 5.2.2. Observing that the family T_α of Theorem 5.0.2 is an ordinal sum of W, $T_\alpha = (\langle 0, \alpha, W \rangle)$, and thus it is a copula for every α in $[0, 1]$, we have that, as a consequence of Theorem 5.0.2, the only associative MT–copulas are of this type.

Now, we present a result on symmetries.

Proposition 5.2.3. *Let X and Y be continuous r.v.'s with copula D. If X and Y are symmetric about α and β, respectively ($\alpha, \beta \in \mathbb{R}$), then (X, Y) is radially symmetric about (α, β) if, and only if, there exists $a \in [0, 1/2]$ such that D is a member of the family of copulas given by*

$$C_a(x, y) = \max\{W(x, y), M(x, y) - a\}. \tag{5.5}$$

Proof. Let D be an MT–copula with diagonal δ. From Proposition 1.6.3, it suffices to show that $D = \widehat{D}$, viz. for every $(x, y) \in [0, 1]^2$

$$\max\{0, \delta(x \vee y) - |x - y|\} = x + y - 1 + \max\{0, \delta(1 - x \wedge y) - |x - y|\}, \tag{5.6}$$

which is equivalent to

$$\delta(t) = 2t - 1 + \delta(1 - t) \qquad \text{for every } t \in [0, 1].$$

For some $a \in [0, 1/2]$, $\delta(t) = 0$ on $[0, a]$ and $D = \widehat{D}$ implies that $\delta(t) = 2t - 1$ on $[1 - a, 1]$. Since $\delta(a) - a = -a = \delta(1 - a) - (1 - a)$ and, from Theorem 5.1.1, $(\delta(x) - x)$ is increasing on $[a, 1]$, this latter function must necessarily be a constant, which can only be equal to $-a$ on $[a, 1 - a]$, so that $\delta(t) = t - a$ on $[a, 1 - a]$. Thus we have that there exists $a \in [0, 1/2]$ such that

$$\delta(t) = \begin{cases} 0, & \text{if } t \in [0, a]; \\ t - a, & \text{if } t \in [a, 1 - a]; \\ 2t - 1, & \text{if } t \in [1 - a, 1]; \end{cases}$$

and D coincides with C_a. □

Notice that the copula C_a is a shuffle of Min, as showed in Example 1.6.8.

It is known from [57] that the Bertino copulas are the weakest (in the pointwise ordering) copulas with given diagonal section. Moreover, the following result is easily derived.

Proposition 5.2.4. *Let D_δ and D_γ be two MT–copulas with diagonals δ and γ, respectively. Then $D_\delta \le D_\gamma$ if, and only if, $\delta(t) \le \gamma(t)$ for all $t \in [0, 1]$.*

Thus, the concordance order on MT–copulas depends on the pointwise ordering of their diagonals. In the same way, the diagonal of an MT–copula describes the most common non–parametric measures of association between random variables.

Theorem 5.2.1. *Let D be the MT–copula associated with the random pair (X, Y). The values of the measures of association between X and Y are given, respecively, by*

$$\tau_D = 8 \int_0^1 \delta(x)\, dx\ - 3, \qquad \rho_D = 12 \cdot \int_0^1 \delta^2(x)\, dx\ - 3,$$

$$\gamma_D = 4 \cdot \left[3 \int_{1/2}^1 \delta(x) dx + \int_0^{1/2} \delta(x) dx - 1 \right],$$

$$\beta_D = 4 \cdot \delta(1/2) - 1, \qquad \varphi_D = 6 \int_0^1 \delta(x)\, dx - 2.$$

Proof. Let D be an MT–copula and let Ω, Ω_+ and Ω_- be the three subsets of the unit square defined by:

$$\Omega := \{(x,y) \in [0,1]^2 : D(x,y) > 0\};$$
$$\Omega_+ := \Delta_+ \cap \Omega; \qquad \Omega_- := \Delta_- \cap \Omega.$$

In view of Theorem 1.8.1, we have

$$\tau_D = 1 - 4 \int\int_{[0,1]^2} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy,$$

where

$$\partial_x D(x,y) \cdot \partial_y D(x,y) = \begin{cases} \delta'(x) - 1, & \text{if } (x,y) \in \Omega_+; \\ \delta'(y) - 1, & \text{if } (x,y) \in \Omega_-; \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} &\int\int_{[0,1]^2} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy \\ &= \int\int_{\Omega_+} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy + \int\int_{\Omega_-} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy \\ &= 2 \cdot \int_0^1 (\delta'(x) - 1)\, dx \int_{x-\delta(x)}^{x} dy = 1 - 2\int_0^1 \delta(x) dx. \end{aligned}$$

Simple calculations lead to the value of τ_D.

By using Theorem 1.8.2, ρ_D is given by:

$$\begin{aligned} \rho_D &= 12 \int\int_{[0,1]^2} D(x,y) dx dy\ - 3 = 24 \int\int_{\Omega_+} (\delta(x) - x + y)\, dx dy\ - 3 \\ &= 24 \cdot \int_0^1 (\delta(x) - x)\, dx \int_{x-\delta(x)}^{x} dy + 24 \cdot \int_0^1 dx \int_{x-\delta(x)}^{x} y\, dy\ - 3 \\ &= 24 \cdot \int_0^1 \left(\delta^2(x) - x\delta(x)\right) dx + 12 \cdot \int_0^1 \left(-\delta^2(x) + 2x\delta(x)\right)\, dx\ - 3 \\ &= 12 \cdot \int_0^1 \delta^2(x)\, dx\ - 3. \end{aligned}$$

In the same manner, from Theorem 1.8.3

$$\gamma_D = 4 \left[\int_0^1 D(x, 1-x) dx - \int_0^1 (x - D(x,x))\, dx \right].$$

For all $x \in [0,1]$

$$D(x, 1-x) = \begin{cases} \max\left(0, \delta(1-x) - 1 + 2x\right) & \text{if } x \le 1/2; \\ \max\left(0, \delta(x) - 2x + 1\right) & \text{if } x > 1/2. \end{cases}$$

It is easy to show that

$$\int_0^1 D(x,1-x)dx = 2\int_{1/2}^1 \delta(x)dx - 1/2$$

and

$$\int_0^1 (x-\delta(x))\,dx = 1/2 - \int_0^1 \delta(x)dx,$$

from which we have the asserted value of γ_D.

The expressions of β_D and φ_D follow directly from Theorems 1.8.4 and 1.8.5. □

5.3 A construction method

Let Δ_+ and Δ_- be the two subsets of the unit square given in (1.12). For two binary aggregation operators A and B, we introduce the function $F_{A,B} : [0,1]^2 \to [0,1]$ given, for all x, y in $[0,1]$, by

$$F_{A,B}(x,y) := A(x,y)\,1_{\Delta_+}(x,y) + B(x,y)\,1_{\Delta_-}(x,y).$$

In other words, if we divide the unit square by means of its diagonal, then $F_{A,B}$ is equal to A in the lower triangle and equal to B in the upper one.

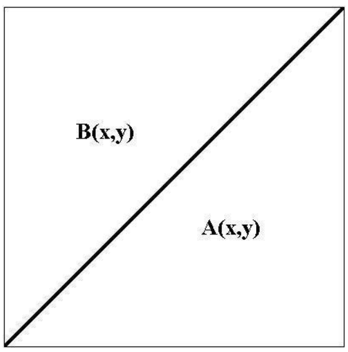

Figure 5.1: The function $F_{A,B}$

Proposition 5.3.1. *If A and B are agops with the same diagonal section, then $F_{A,B}$ is an agop. Moreover, if A and B are semicopulas, so is $F_{A,B}$.*

Proof. It is obvious that $F_{A,B}(0,0) = 0$ and $F_{A,B}(1,1) = 1$. Moreover, $F_{A,B}$ is increasing in each place because $\delta_A = \delta_B$. Finally, notice that, if A and B are semicopulas, then

$$F_{A,B}(x,1) = B(x,1) = x, \quad F_{A,B}(1,x) = A(1,x) = x,$$

for every $x \in [0,1]$. Therefere $F_{A,B}$ has neutral element 1. □

Notice that, if A and B are agops such that $\delta_A \neq \delta_B$, then $F_{A,B}$ need not be increasing. For example, if $A = M$ and $B = \Pi$, then

$$F_{A,B}(0.5, 0.4) = 0.4 > 0.3 = F_{A,B}(0.5, 0.6).$$

In the following results we consider the case in which A and B are copulas or quasi–copulas.

Proposition 5.3.2. *If A and B are quasi–copulas with the same diagonal section, then $F_{A,B}$ is a quasi–copula.*

Proof. In view of Proposition 5.3.1 we have to prove only that $F_{A,B}$ is 1–Lipschitz. Let x, x', y, y' be points in $[0, 1]$. If (x, y) and (x', y') are both in Δ_+ (or Δ_-), then $F_{A,B}$ is obviously 1–Lipschitz. Therefore, suppose that, for example, $(x, y) \in \Delta_+$ and $(x', y') \in \Delta_-$ and, without loss of generality, $x > x'$ and $y < y'$.

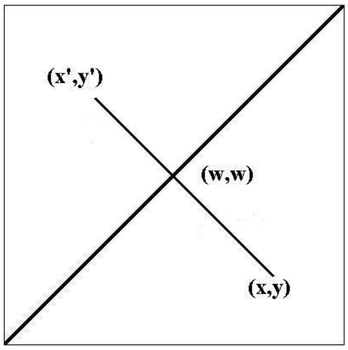

Figure 5.2: Proof of Proposition 5.3.2

Let (w, w) be the point of intersection between the segment line joining (x, y) and (x', y') and the diagonal section of the unit square. We have

$$\begin{aligned}
|F_{A,B}(x, y) - F_{A,B}(x', y')| &\leq |F_{A,B}(x, y) - F_{A,B}(w, w)| + |F_{A,B}(w, w) - F_{A,B}(x', y')| \\
&\leq |A(x, y) - A(w, w)| + |B(w, w) - B(x', y')| \\
&\leq (x - w) + (w - y) + (w - x') + (y' - w) \\
&\leq |x - x'| + |y - y'|.
\end{aligned}$$

The other cases can be proved in an analogous manner. □

Corollary 5.3.1. *If A and B are 1–Lipschitz agops with the same diagonal section, then $F_{A,B}$ is a 1–Lipschitz agop.*

Proposition 5.3.3. *Let A and B be copulas with the same diagonal section. If A and B are symmetric, then $F_{A,B}$ is a copula.*

Proof. In view of Proposition 5.3.1, we have to proof only the 2–increasing property for $F_{A,B}$. On the rectangles entirely contained in either Δ_+ or Δ_-, the rectangular inequality (C2) follows directly from the 2–increasing property of A and B. Therefore, in view of Proposition 1.6.1, it suffices to show that, for all $s, t \in [0,1]$ with $s < t$,

$$V_{F_{A,B}}([s,t]^2) := A(s,s) + A(t,t) - B(s,t) - A(t,s) \geq 0.$$

Because A and B are both symmetric, we have

$$V_{F_{A,B}}([s,t]^2) := \frac{1}{2}\left(V_A([s,t]^2) + V_B([s,t]^2)\right) \geq 0,$$

which concludes the proof. □

Remark 5.3.1. In Proposition 5.3.3, the assumption of the symmetry of the copulas A and B is essential. If, for example, A is a non–symmetric copula, then $F_{A,B}$ need not be a copula. We consider, for example, the copula A given by

$$A(x,y) = \begin{cases} \max\left(x + \frac{1}{2}(y-1), 0\right), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \\ \min\left(x + \frac{1}{2}(y-1), y\right), & \text{if } x \in \left]\frac{1}{2}, 1\right]; \end{cases}$$

and the copula B given by

$$B(x,y) := \min\left\{x, y, \frac{\delta_A(x) + \delta_A(y)}{2}\right\}.$$

If $R := [1/3, 2/3]^2$, we have

$$V_{F_{A,B}}(R) = -1/12 < 0,$$

viz. $F_{A,B}$ is not a copula. Specifically, because of Proposition 5.3.2, $F_{A,B}$ is a proper quasi–copula.

Remark 5.3.2. In [108], a general method was described to symmetrize an agop. Specifically, let A be an agop (generally, non-symmetric), for every $x, y \in [0,1]$, the *symmetrized version of* A is defined by

$$\widetilde{A}(x,y) = \begin{cases} A(x,y), & \text{if } x \geq y; \\ A(y,x), & \text{if } x < y. \end{cases} \tag{5.7}$$

Since it is a clear that, if A is an agop (quasi–copula), then the transpose A^T is an agop (quasi–copula), it follows from Proposition 5.3.1 (Proposition 5.3.2) that $\widetilde{A}$ is an agop (quasi–copula). Notice that, given a copula C, $\widetilde{C}$ need not be a copula. We consider, for example, the copula C_λ ($\lambda \in]0,1[$) defined by

$$C_\lambda(x,y) = \begin{cases} y, & \text{if } y \le \lambda x; \\ \lambda x, & \text{if } \lambda x < y \le 1-(1-\lambda)x; \\ x+y-1, & \text{if } 1-(1-\lambda)x < y \le 1. \end{cases}$$

For a fixed $\epsilon \in \left]0, \frac{1}{2}\right[$, we have

$$V_{\widetilde{C_\lambda}}\left(\left[\frac{1}{2}, \frac{1}{2}+\epsilon\right] \times \left[\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon\right]\right) = \frac{\lambda}{2} - \lambda(\frac{1}{2}+\epsilon) < 0.$$

A similar construction method can also be introduced for agops that have the same values in some fixed linear sections ([80]), for example, with the same opposite diagonal sections. In this case, let Γ_+ and Γ_- be the two subsets of the unit square defined by

$$\Gamma_+ := \{(x,y) \in [0,1]^2 : x+y \le 1\}, \qquad \Gamma_- := \{(x,y) \in [0,1]^2 : x+y > 1\}.$$

Given the agops A and B, we introduce the function $F^{A,B} : [0,1]^2 \to [0,1]$ given, for all $x, y \in [0,1]$, by

$$F^{A,B}(x,y) := A(x,y)\, 1_{\Gamma_+}(x,y) + B(x,y)\, 1_{\Gamma_-}(x,y).$$

As above, we have

Proposition 5.3.4. *If A and B are agops with the same opposite diagonal section, then $F^{A,B}$ is an agop. Moreover, if A and B are quasi–copulas, then $F^{A,B}$ is a quasi–copula too.*

Theorem 5.3.1. *Let A and B be copulas with the same opposite diagonal section. If $B(x,y) \ge A(x,y)$ for every $(x,y) \in \Gamma_-$, then $F^{A,B}$ is a copula.*