Chapter 4

A new family of PQD copulas

In this chapter we introduce a new class of bivariate copulas, depending on a univariate function, that includes some already known families. This class is characterized in section 4.1, where a probabilistic interpretation is given, and its properties (dependence, measures of association, symmetries, associativity, absolute continuity) are studied in detail in section 4.2. Section 4.3 is devoted to the introduction of a similar class in the set of quasi–copulas.

The contents of this chapter can be also found in [36, 42, 43].

4.1 Characterization of the new class

Let f be a mapping from $[0,1]$ into $[0,1]$. Consider the function C_f given, for every $x, y \in [0,1]$, by

$$C_f(x,y) := (x \wedge y)\, f(x \vee y). \tag{4.1}$$

It is obvious that every C_f is symmetric and the copulas Π and M are of this type: it suffices to take, respectively, $f(t) = t$ and $f(t) = 1$ for all $t \in [0,1]$. Our aim is to study under which conditions on f, C_f is a copula. Notice that, in view of the properties (1.9) and (1.10) of a copula, it is quite natural to require that f is increasing and continuous and, then, simple considerations of real analysis imply that f is differentiable almost everywhere on $[0,1]$ and the left and right derivatives of f exist for every $x \in [0,1]$ and assume finite values. We aim to characterize the copulas of type (4.1).

Lemma 4.1.1. *Let $f : [0,1] \to [0,1]$ be a continuous and increasing function, differentiable except at finitely many points. The following statements are equivalent:*

(a) *for every $s, t \in \,]0,1]$, with $s \le t$, $sf(s) + tf(t) - 2sf(t) \ge 0$;*

(b) *the function $t \mapsto f(t)/t$ is decreasing on $]0,1]$.*

Proof. $(a) \Rightarrow (b)$: Let s_i $(i = 1, 2, \dots, n)$ be the points in $[0, 1]$ such that $f'(s_i^+) \neq f'(s_i^-)$. Set $s_0 := 0$ and $s_{n+1} := 1$. For every $i \in \{0, 1, \dots, n\}$, let s and t be in $]s_i, s_{i+1}[$, $s < t$. The inequality

$$sf(s) + tf(t) - 2sf(t) \geq 0$$

is equivalent to

$$\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s}.$$

In the limit $t \downarrow s$, we have $f(s) \geq sf'(s)$. It follows that

$$\left(\frac{f(s)}{s}\right)' = \frac{sf'(s) - f(s)}{s^2} \leq 0,$$

viz. $t \mapsto f(t)/t$ is decreasing in each interval $]s_i, s_{i+1}[$, $(i = 0, 1, \dots, n)$. But $f(t)/t$ is continuous and, therefore, it is decreasing on the whole $]0, 1]$.
$(b) \Rightarrow (a)$: Let s, t be in $]0, 1]$, with $s < t$. Then

$$\frac{f(s)}{s} \geq \frac{f(t)}{t}$$

is equivalent to

$$\frac{f(s)}{s} \geq \frac{f(t) - f(s)}{t - s},$$

and, because f is increasing,

$$\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s},$$

viz. condition (a). □

Theorem 4.1.1. *Let $f : [0, 1] \to [0, 1]$ be a differentiable function (except at finitely many points). Let C_f be the function defined by* (4.1). *Then C_f is a copula if, and only if, the following statements hold:*

(i) $f(1) = 1$*;*

(ii) f *is increasing;*

(iii) *the function $t \mapsto f(t)/t$ is decreasing on* $]0, 1]$.

Proof. It is immediate that C_f satisfies the boundary conditions (C1) if, and only if, $f(1) = 1$. We now prove that C_f is 2–increasing if, and only if, (ii) and (iii) hold. Let x, x', y, y' be in $[0, 1]$ with $x \leq x'$ and $y \leq y'$. First, we suppose that the rectangle $[x, x'] \times [y, y']$ is a subset of Δ_+ (see notations (1.12)). Then

$$V_C([x, x'] \times [y, y']) = (y' - y)\,(f(x') - f(x)) \geq 0$$

if, and only if, f is increasing. Analogously, the 2–increasing property is equivalent to (ii) for rectangles contained in Δ_-. If, instead, the diagonal of $[x, x'] \times [y, y']$ lies on the diagonal $\{(x, y) \in [0, 1]^2 : y = x\}$ of the unit square, then $x = y$ and $x' = y'$ and, in view of Lemma 4.1.1,

$$V_C([x, x'] \times [x, x']) = xf(x) + x'f(x') - 2xf(x') \geq 0$$

if, and only if, (iii) holds. Now, the assertion follows from Proposition 1.6.1. □

A function f that satisfies the assumptions of Theorem 4.1.1 is called *generator* of a copula of type (4.1). In particular, the class of generators is convex and, because of condition (iii), it has minimal element $\mathrm{id}_{[0,1]}$ and maximal element the constant function equal to 1. Note that $f : [0, 1] \to [0, 1]$ satisfies condition (iii) of Theorem 4.1.1 if, and only if, f is *star–shaped*, i.e., $f(\alpha x) \geq \alpha f(x)$ for all $\alpha \in [0, 1]$. Moreover, every concave function satisfies (iii) (these results can also be found in [103, Chap. 16]). Now, we give a probabilistic interpretation of the generators.

Proposition 4.1.1. *Let U and V be r.v.'s uniformly distributed on $[0, 1]$ with copula C_f of type (4.1). Then*

$$f(t) = P\left(\max\{U, V\} \leq t \mid U \leq t\right).$$

Proof. For every t in $[0, 1]$, we have

$$C(t, t) = tf(t) = P\left(U \leq t, V \leq t\right),$$

and

$$P\left(\max\{U, V\} \leq t \mid U \leq t\right) = \frac{P\left(U \leq t, V \leq t\right)}{P(U \leq t)} = f(t),$$

namely the assertion. □

In the sequel we give some sub–classes of copulas $\{C_\alpha\}$ of type (4.1) generated by a one–parameter family $\{f_\alpha\}$.

Example 4.1.1 (Fréchet copulas)**.** Given $f_\alpha(t) := \alpha t + (1 - \alpha)$ ($\alpha \in [0, 1]$), we obtain $C_\alpha = \alpha\Pi + (1 - \alpha)M$, which is a convex sum of Π and M and, therefore, is a member of the Fréchet family of copulas (see Example 1.6.2) (see, also, family (B11) in [74]). Notice that $C_0 = M$ and $C_1 = \Pi$.

Example 4.1.2 (Cuadras–Augé copulas)**.** Given $f_\alpha(t) := t^\alpha$ ($\alpha \in [0, 1]$), C_α is defined by

$$C_\alpha(x, y) = (x \wedge y)(x \vee y)^\alpha = \begin{cases} xy^\alpha, & \text{if } x \leq y; \\ x^\alpha y, & \text{if } x > y. \end{cases}$$

Then C_α describes the Cuadras–Augé family of copulas (see Example 1.6.4). Notice that $C_0 = M$ and $C_1 = \Pi$.

Example 4.1.3. Given $f_\alpha(t) := \min(\alpha t, 1)$ $(\alpha \geq 1)$, C_α is defined by

$$C_\alpha(x,y) = (x \wedge y)\min\{\alpha(x \vee y), 1\} = \begin{cases} \alpha xy, & \text{if } (x,y) \in [0,1/\alpha]^2\,; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

viz. C_α is the ordinal sum $(\langle 0, 1/\alpha, \Pi\rangle)$. Notice that $C_1 = \Pi$ and $C_\infty = M$, where, if $g(x) = \lim f_\alpha(x)$ as $\alpha \to +\infty$ and $x \in]0,1]$, $C_\infty := C_g$.

Example 4.1.4. Given the function $f_\alpha(t) := c\exp(t^\alpha/\alpha)$, where $\alpha > 0$ and $c = \exp(-1/\alpha)$, we obtain the following family

$$C_\alpha(x,y) = \begin{cases} cx\exp(y^\alpha/\alpha), & \text{if } x \leq y; \\ cy\exp(x^\alpha/\alpha), & \text{if } x > y. \end{cases}$$

Example 4.1.5. The function $f_\alpha(t) := \frac{1}{\sin\alpha}\sin(\alpha t)$ $(\alpha \in]0, \pi/2])$ is increasing with $f_\alpha(t)/t$ decreasing on $]0,1]$, as is easily proved. Therefore, Theorem 4.1.1 ensures that

$$C_\alpha(x,y) = \begin{cases} \frac{x}{\sin\alpha}\sin(\alpha y)\,, & \text{if } x \leq y; \\ \frac{y}{\sin\alpha}\sin(\alpha x)\,, & \text{if } x > y. \end{cases}$$

is a copula.

For a copula C_f of type (4.1) the following result holds (see [100] for details).

Theorem 4.1.2. *If C_f is the copula given by* (4.1) *and $H(x,y) = C_f(F_1(x), F_2(y))$ for univariate d.f.'s F_1 and F_2, then the following statements are equivalent:*

(a) *random variables X and Y with joint d.f. H have a representation of the form*

$$X = \max\{R, W\} \qquad \text{and} \qquad Y = \max\{S, W\}$$

where R, S and W are independent r.v.'s;

(b) *H has the form $H(x,y) = F_R(x)F_S(y)F_W(x \wedge y)$, where F_R, F_S and F_W are univariate d.f.'s.*

4.2 Properties of the new class

In this section we give the most important properties of a copula C_f of type (4.1).

4.2.1 Concordance order

Proposition 4.2.1. *Let C_f and C_g be two copulas of type* (4.1)*. Then $C_f \leq C_g$ if, and only if, $f(t) \leq g(t)$ for all $t \in [0,1]$.*

In particular, for every copula C_f, $\Pi \le C_f \le M$ and, therefore, every C_f is positively quadrant dependent.

Example 4.2.1. Consider the family $\{f_\alpha\}$ ($\alpha \ge 1$), given by $f_\alpha(t) := 1-(1-t)^\alpha$. It is easily proved by differentation that every f_α is increasing with $f_\alpha(t)/t$ decreasing on $]0,1]$. Therefore, this family generates a family of copulas C_α, that is positively ordered, with $C_1 = \Pi$ and $C_\infty = M$.

Example 4.2.2. Consider the family of copulas generated by the function $f_\alpha(t) := (1+\alpha)t/(\alpha t+1)$ for every $\alpha \ge 0$. This family is positively ordered with $C_0 = \Pi$ and $C_\infty = M$.

4.2.2 Dependence concepts

Theorem 4.2.1. *Let (X,Y) be a continuous random pair with copula C_f. Then*

(a) *Y is left tail decreasing in X;*

(b) *Y is stochastically increasing in X if, and only if, f' is decreasing a.e. on $[0,1]$;*

(c) *X and Y are left corner set decreasing.*

Proof. In order to prove $LTD(Y|X)$, according to Proposition 1.7.2 it suffices to notice that, for every $(x,y) \in [0,1]^2$

$$\frac{C_f(x,y)}{x} = \begin{cases} f(y), & \text{if } x \le y; \\ \dfrac{yf(x)}{x}, & \text{if } x > y; \end{cases}$$

is decreasing in x.

Property $SI(Y|X)$ follows from Proposition 1.7.3, observing that $\partial_x C_f$ is decreasing in the first place if, and only if, f' is decreasing a.e. on $[0,1]$.

In order to prove (c), because of Proposition 1.7.4, it suffices to prove that, for all x, x', y, y' in $[0,1]$, with $x \le x'$ and $y \le y'$,

$$C_f(x,y)C_f(x',y') \ge C_f(x,y')C_f(x',y) \ge 0. \tag{4.2}$$

Because $f(t)/t$ is decreasing and C_f is symmetric, inequality (4.2) follows easily from simple calculations on rectangles $[x,x'] \times [y,y']$ that have 4, 3 or 2 vertices in the set Δ_+. For instance, if $[x,x'] \times [y,y']$ has only two vertices, say (x,y) and (x',y) in Δ_+, then (4.2) holds if, and only if, $x'f(x) \ge xf(x')$, viz. $f(t)/t$ is decreasing. □

The following result for the tail dependence holds.

Proposition 4.2.2. *Let C_f be a copula of type (4.1). Then, the lower tail dependence of C_f is $f(0^+)$ and the upper tail dependence of C_f is $1 - f'(1^-)$.*

Proof. The diagonal section of C_f is $\delta_{C_f}(t) = tf(t)$. Therefore, from Proposition 1.7.5, we have $\lambda_L = \delta'_C(0^+) = f(0^+)$ and $\lambda_U = 2 - \delta'_C(1^-) = 1 - f'(1^-)$. □

Remark 4.2.1. As noted, a copula of type (4.1) is PQD and, therefore, it is suitable to describe positive dependence of a random vector (X, Y). However, it is very simple to introduce a copula to describing, for example, the (negative) dependence of the random vector $(X, -Y)$. It suffices to consider the copula $C^f_{0,1}$ given by

$$C^f_{0,1}(x,y) := x - C(x, 1-y) = \begin{cases} x(1 - f(1-y)), & \text{if } x + y \le 1; \\ x - (1-y)f(x), & \text{otherwise.} \end{cases}$$

4.2.3 Measures of association

Theorem 4.2.2. *The values of several measures of association of C_f are, respectively, given by*

$$\tau_C = 4\int_0^1 xf^2(x)\,dx - 1, \qquad \rho_C = 12\int_0^1 x^2 f(x)\,dx - 3,$$
$$\gamma_C = 4\left(\int_0^{1/2} x\,[f(x) + f(1-x)]\,dx + \int_{1/2}^1 f(x)\,dx\right) - 2,$$
$$\beta_C = 2f(1/2) - 1, \qquad \varphi_C = 6\int_0^1 xf(x)\,dx - 2.$$

Proof. In view of Theorem 1.8.1, the Kendall's tau of C_f is given by

$$\tau_C = 1 - 4\int_0^1\int_0^1 \partial_x C(x,y)\partial_y C(x,y)\,dx\,dy.$$

Now, we have

$$\begin{aligned} &\int_0^1\int_0^1 \partial_x C(x,y)\partial_y C(x,y)\,dx\,dy \\ =\ & \int_0^1 dy \int_0^y xf(y)f'(y)\,dx + \int_0^1 dx\int_0^x yf(x)f'(x)\,dy \\ =\ & \int_0^1 x^2 f(x)f'(x)\,dx = \frac{1}{2} - \int_0^1 xf^2(x)\,dx, \end{aligned}$$

where the last equality is obtained through integration by parts. Then

$$\tau_C = 4\int_0^1 xf^2(x)\,dx - 1.$$

From Theorem 1.8.2, Spearman's rho is given by:

$$\begin{aligned} \rho_C &= 12\int_0^1\int_0^1 C(x,y)\,dx\,dy - 3 \\ &= 12\int_0^1 dy\int_0^y xf(y)\,dx + \int_0^1 dx\int_0^x yf(x)\,dy - 3 \\ &= 12\int_0^1 x^2 f(x)\,dx - 3. \end{aligned}$$

Following Theorem 1.8.3, we have

$$\begin{aligned}
\gamma_C &= 4\left(\int_0^1 C(x,1-x)\,dx - \int_0^1 (x - C(x,x))\ dx\right) \\
&= 4\left(\int_0^{1/2} xf(1-x)\,dx - \int_0^{1/2} [x - xf(x)]\ dx\right. \\
&\qquad \left. + \int_{1/2}^1 (1-x)f(x) - \int_{1/2}^1 [x - xf(x)]\ dx\right) \\
&= 4\left(\int_0^{1/2} x\,[f(x) + f(1-x)]\ dx + \int_{1/2}^1 f(x)\,dx\ - \frac{1}{2}\right) \\
&= 4\left(\int_0^{1/2} x\,[f(x) + f(1-x)]\ dx + \int_{1/2}^1 f(x)\,dx\right) - 2.
\end{aligned}$$

The expressions of β_C and φ_C follow easily from Theorems 1.8.4 and 1.8.5. □

As an application of Theorem 4.2.2, the measures of association for the copulas in Examples 1.6.2 and 1.6.4 can be easily given:

– If C is a copula of the Fréchet family, then

$$\tau_C = \frac{(\alpha-1)(\alpha-3)}{3}, \qquad \rho_C = 1 - \alpha = \gamma_C = \varphi_C.$$

– If C is a Cuadras–Augé copula, then

$$\tau_C = \frac{1-\alpha}{1+\alpha}, \qquad \rho_C = \frac{3-3\alpha}{3+\alpha}, \qquad \varphi_C = \frac{2-2\alpha}{2+\alpha}.$$

4.2.4 Symmetry properties

Theorem 4.2.3. *Let (X,Y) be continuous r.v.'s with copula C_f.*

(a) *If X and Y are identically distributed, then X and Y are exchangeable.*

(b) *If X and Y are symmetric about a and b, respectively $(a, b \in \mathbb{R})$, then (X,Y) is radially symmetric about (a,b) if, and only if, $C_f = \alpha\Pi + (1-\alpha)M$ for some $\alpha \in [0,1]$.*

(c) *If X and Y are symmetric about a and b, respectively $(a, b \in \mathbb{R})$, then (X,Y) is jointly symmetric about (a,b) if, and only if, $C_f = \Pi$.*

Proof. Statement (a) is a consequence of the symmetry of C_f. From Proposition 1.6.3, statement (b) holds if, and only if, C_f satisfies the following functional equation:

$$\forall x, y \in [0,1] \qquad C_f(x,y) = x + y - 1 + C_f(1-x, 1-y). \tag{4.3}$$

But, equality (4.3) is equivalent to

$$(x \wedge y) f(x \vee y) = x + y - 1 + [1 - (x \vee y)]\, f\,[1 - (x \wedge y)]\,;$$

in particular, for all $y \in [x, 1[$, we have

$$\begin{aligned} &xf(y) = x + y - 1 + (1-y)f(1-x) \\ &\Longrightarrow x\,(1 - f(y)) + (1-y)f(1-x) = 1 - y \\ &\Longrightarrow x \cdot \frac{1 - f(y)}{1-y} + f(1-x) = 1 \Longrightarrow f(1-x) = 1 - x \cdot \frac{f(y)-1}{y-1}. \end{aligned}$$

In the limit $y \uparrow 1$, we can derive

$$\frac{1 - f(y)}{1 - y} \longrightarrow f'(1^-),$$

where $f'(1^-)$ is a real number in $[0, 1]$. Thus $f(1-x) = 1 - cx$, i.e. $f(x) = cx + (1-c)$, which corresponds to the family $C_f = c\Pi + (1-c)M$.

From Proposition 1.6.3, (X, Y) is jointly symmetric about (a, b) if, and only if, for all $(x, y) \in [0, 1]^2$

$$C_f(x, y) = x - C_f(x, 1-y) \qquad \text{and} \qquad C_f(x, y) = y - C_f(1-x, y). \tag{4.4}$$

In particular, for $x = y$, we obtain

$$\forall x \in [0, 1] \qquad xf(x) = x - [x \wedge (1-x)]\, f\,[x \vee (1-x)]\,,$$

which implies

$$\begin{aligned} &\forall x \in [1/2, 1] \qquad xf(x) = x - (1-x)f(x), \\ &\forall x \in [0, 1/2] \qquad xf(x) = x - xf(1-x), \end{aligned}$$

viz. $f(x) = x$ on $[0, 1]$, which corresponds to $C_f = \Pi$. □

4.2.5 Associativity

Lemma 4.2.1. *Let C_f be a copula of type* (4.1). *Then C_f is Archimedean if, and only if, $C_f = \Pi$.*

Proof. If C_f is an Archimedean copula, then, there exists a convex function $\varphi : [0, 1] \to [0, +\infty]$, which is continuous and strictly increasing, $\varphi(1) = 0$, such that $C_f(x, y) = \varphi^{[-1]}\left(\varphi(x) + \varphi(y)\right)$. In view of Theorem 1.6.8,

$$\varphi'(x)\, \frac{\partial C_f(x, y)}{\partial y} = \varphi'(y)\, \frac{\partial C_f(x, y)}{\partial x} \qquad \text{a.e. on } [0, 1]^2.$$

In particular, if $x = y$, we obtain $\varphi'(x) \cdot xf'(x) = \varphi'(x) \cdot f(x)$, which leads to $xf'(x) = f(x)$. In the class of the generators of a copula of type (4.1), this differential equation has as unique solution the function $f(x) = x$, viz. $C_f = \Pi$. □

Theorem 4.2.4. *Let C_f be a copula of type* (4.1). *Then C_f is associative if, and only if, C_f is an ordinal sum of type $(\langle 0, a, \Pi \rangle)$ with $a \in [0, 1]$.*

Proof. First, notice that every ordinal sum of type $(\langle 0, a, \Pi\rangle)$ is associative and it is generated by the function $f(t) = \min\{t/a, 1\}$.

Conversely, let C_f be an associative copula. As asserted in Theorem 1.6.9, the representation of C_f depends on the set I_D of idempotent elements of C_f, given by $I_D := \{0\} \cup [a, 1]$, where $a := \inf\{t \in [0, 1] : f(t) = 1\}$. If $I_D = \{0, 1\}$, then C_f is Archimedean and, therefore, Lemma 4.2.1 ensures that $C_f = \Pi = (\langle 0, 1, \Pi\rangle)$. If $I_D = [0, 1]$, then $C_f = M = (\langle 0, 0, \Pi\rangle)$. Otherwise, C_f is an ordinal sum of type $(\langle 0, a, D\rangle)$ for a suitable Archimedean copula D. Therefore, if φ is a generator of D, for all x, y in $[0, a]$,

$$C_f\,(x,y) = a\,\varphi^{[-1]}\left(\varphi\left(\frac{x}{a}\right) + \varphi\left(\frac{y}{a}\right)\right).$$

Hence, applying the chain rule to $\varphi(C_f(x, y)/a) = \varphi(x/a) + \varphi(y/a)$, we obtain

$$\varphi'\left(\frac{C_f(x,y)}{a}\right)\frac{\partial C_f(x,y)}{\partial x} = \varphi'\left(\frac{x}{a}\right), \quad \varphi'\left(\frac{C_f(x,y)}{a}\right)\frac{\partial C_f(x,y)}{\partial y} = \varphi'\left(\frac{y}{a}\right).$$

Therefore, a.e. on $[0, 1]^2$, we have

$$\varphi'\left(\frac{x}{a}\right)\,\frac{\partial C_f(x,y)}{\partial y} = \varphi'\left(\frac{y}{a}\right)\,\frac{\partial C_f(x,y)}{\partial x}.$$

An argument similar to the proof of Lemma 4.2.1 gives $D = \Pi$, as asserted. □

4.2.6 Absolute continuity

Proposition 4.2.3. *The only absolutely continuous copula of type* (4.1) *is* Π.

Proof. Let C_f be a copula of type (4.1). If C_f is absolutely continuous, then

$$1 = C_f(1,1) = \int_0^1 \int_0^1 \frac{\partial^2 C}{\partial x \partial y}\,dx\,dy = \int_0^1 \int_0^1 f'(x \vee y)\,dx\,dy.$$

It follows that

$$\frac{1}{2} = \int_0^1 ds \int_0^s f'(s)\,dt = \int_0^1 s f'(s)\,ds;$$

integrating by parts, we have

$$\int_0^1 f(x)\,dx = \frac{1}{2}.$$

The function $f(x) = x$ is a solution of the above equation and, because all functions generating a copula of type (4.1) are greater than $\mathrm{id}_{[0,1]}$, it follows that $\mathrm{id}_{[0,1]}$ is the only solution in this class. □

Remark 4.2.2. Let C_f be a copula of type (4.1), $C \neq \Pi$. Consider the first derivative of C_f

$$\partial_1 C_f(x,y) = \begin{cases} f(y), & \text{if } x < y; \\ y \cdot f'(x), & \text{otherwise.} \end{cases}$$

For a fixed y_0, the mapping $t \mapsto \partial_1 C_f(t, y_0)$ has a jump discontinuity in y_0, and, thus, C_f has a singular component along the main diagonal of the unit square. By using [74, Theorem 1.1], the mass of this singular component is given by

$$m = \int_0^1 (f(x) - xf'(x))\, dx = 2 \cdot \int_0^1 f(x)\, dx - 1.$$

This m has a graphical interpretation if f admits an inverse: in fact, m is the area of the region of the unit square between the graph of f and the graph of f^{-1}.

4.3 A similar new class of quasi–copulas

Given a function $f : [0,1] \to [0,1]$, we are also interested in studying under which conditions on f, the following function

$$Q_f(x, y) := (x \wedge y)\, f(x \vee y), \quad \text{for all } (x, y) \in [0,1]^2, \tag{4.5}$$

is a quasi–copula. The following result provides a characterization.

Theorem 4.3.1. *Let $f : [0,1] \to [0,1]$ be a continuous function and let Q_f be defined by (4.5). Then Q_f is a quasi–copula if, and only if, the three following statements hold:*

(i) $f(1) = 1$;

(ii) f *is increasing;*

(iii) $x_1 \cdot \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \le 1$ *for every* $x_1, x_2 \in [0,1]$, *with* $x_1 < x_2$.

Proof. First, observe that Q_f satisfies (Q1) if, and only if, $f(1) = 1$ and Q_f satisfies (Q2) if, and only if, (ii) holds. In order to prove that Q_f satisfies (Q3), let x_1, x_2 and y be three points in $[0,1]$ with $x_1 < x_2$. We distinguish three cases. If $x_1 < x_2 \le y$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = x_2 f(y) - x_1 f(y) \le x_2 - x_1$$

because $f \le 1$. If $y \le x_1 < x_2$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = y \cdot (f(x_2) - f(x_1)) \le \frac{y}{x_1} \cdot (x_2 - x_1) \le x_2 - x_1$$

if, and only if, (iii) holds. Finally, if $x_1 \le y \le x_2$, in view of the two above cases we obtain

$$\begin{aligned} Q_f(x_2, y) - Q_f(x_1, y) &= (Q_f(x_2, y) - Q_f(y, y)) + (Q_f(y, y) - Q_f(x_1, y)) \\ &\le (x_2 - x_1) \end{aligned}$$

if, and only if, (iii) holds. In every case, (iii) is a necessary and sufficient condition that ensures that Q_f satisfies (1.10). □

Corollary 4.3.1. *Let* $f : [0,1] \to [0,1]$ *be a differentiable function and let* Q_f *be defined by* (4.5)*. Then* Q_f *is a quasi–copula if, and only if, the three following statements hold:*

(i) $f(1) = 1$*;*

(ii) f *is increasing;*

(iii) $xf'(x) \leq 1$ *for every* $x \in [0,1]$.

Notice that if Q_f is a copula, then $t \mapsto f(t)/t$ is decreasing and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_1)}{x_1}$$

for every $x_1, x_2 \in [0,1]$, with $x_1 < x_2$, from which the condition (iii) of Theorem 4.3.1 follows, viz. Q_f is a quasi–copula. The converse implication need not be true, as the following example shows.

Example 4.3.1. Consider the function $f(t) := t + t^2 - t^3$ on $[0,1]$. So, f satisfies the assumptions of Theorem 4.3.1, viz. $f'(t) \leq 1/t$ on $[0,1]$, but $f(t)/t$ is increasing on $[0,1/2]$. So Q_f is a proper quasi–copula. Another (not everywhere) differentiable function g, which leads to a proper quasi–copula, is given by

$$g(x) = \begin{cases} x, & \text{if } x \in [0,1/4]\,; \\ 2x - 1/4, & \text{if } x \in]1/4,1/2[\,; \\ (x+1)/2, & \text{if } x \in [1/2,1]\,. \end{cases}$$

We have $g'(x) \leq 1/x$ and thus Q_g is a quasi-copula; however, $h(x) := g(x)/x$ is not decreasing (e.g. $h(1/4) = 1$ but $h(1/2) = 3/2$).