Chapter 2

The new concept of semicopula

The focus of this chapter is on the notion of *semicopula*. To the best of our knowledge, this term was used for the first time by B. Bassan and F. Spizzichino ([7]) and arises from a statistical application: the study of multivariate aging through the analysis of the Schur–concavity of the survival distribution function. Specifically, in order to define some notions of aging from the univariate case to the bivariate case, B. Bassan and F. Spizzichino introduced the so–called *bivariate aging function*, which *"has all the formal properties of a copula, except possibly for the rectangle inequlity"* (see [6]). Therefore, they call "semicopula" a function of this type. As it will be seen shortly, this function generalizes the concept both of copula and of triangular norm.

However, this concept was already known, in different contexts, as *conjunctor*, a monotone extension of the Boolean conjunction with neutral element 1 ([26, 27]), *t–seminorm* ([154]), or *generalized copula* ([136]). Moreover, the class of semicopulas appeared also in [140, Definition 2], where it is used in order to characterize some operations on d.f.'s that are not derivable from any operation on r.v.'s.

In section 2.1, we give the basic properties and examples of semicopulas. Some characterizations of the semicopulas M, Π and W are given in section 2.2, where super– and sub– harmonic semicopulas are studied and their statistical interpretation is presented. The study of the class of semicopulas is the object of section 2.3. The extension of semicopulas to the multivariate case is presented in section 2.4, where an interesting connection to the theory of fuzzy measures is also given.

These results can be also found in [47, 42, 34, 45].

2.1 Definition and basic properties

Definition 2.1.1. A function $S : [0,1]^2 \to [0,1]$ is said to be a *semicopula* if, and only if, it satisfies the two following conditions:

(S1) $S(x,1) = S(1,x) = x$ for all x in $[0,1]$;

(S2) $S(x,y) \le S(x',y')$ for all $x,x',y,y' \in [0,1]$, $x \le x'$ and $y \le y'$.

The class of semicopulas will be denoted by $\mathcal{S}$.

In other words, a semicopula is a binary aggregation operator with neutral element 1 and, consequently, annihilator 0, because

$$0 \le S(x,0) \le S(1,0) = 0,$$

and, analogously, $S(0,x) = 0$ for all $x \in [0,1]$.

The class $\mathcal{S}$ strictly includes the class $\mathcal{Q}$ of quasi–copulas and, if we denote by $\mathcal{S}_C$ the set of continuous semicopulas, $\mathcal{S}_C \subset Q$. Moreover, the set $\mathcal{S}_S$ of symmetric semicopulas is a proper subset of $\mathcal{S}$ and it strictly includes the set $\mathcal{T}$ of t–norms.

Example 2.1.1.

- ▷ The drastic t–norm Z is a semicopula, but it is not a quasi–copula, because it is not continuous.
- ▷ $S_1(x,y) = xy \max\{x,y\}$ is a continuous semicopula, but, because it is not associative, it is not a t–norm. Moreover, S_1 is not a quasi–copula, because

$$S_1(8/10, 9/10) - S_1(8/10, 8/10) = 136/1000 > 1/10.$$

- ▷ The following mapping S_2 is an associative semicopula that is not commutative

$$S_2(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1/2] \times [0,1[; \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Proposition 2.1.1. *If $S : [0,1]^2 \to [0,1]$ is a semicopula, then*

$$Z(x,y) \le S(x,y) \le M(x,y) \qquad \textit{for all } x \textit{ and } y \textit{ in } [0,1]. \tag{2.1}$$

Proof. If S is a semicopula, then, for all $x, y \in [0,1[$, we obtain

$$0 = S(x,0) \le S(x,y) \le S(x,1) = x.$$

Analogously,

$$0 = S(x,0) \le S(x,y) \le S(1,y) = y,$$

so that $S(x,y) \le \min\{x,y\}$. □

It must be noticed that no assumption on the (left– or right–) continuity of a semicopula has hitherto been made and different types of continuity can be also considered in the class of semicopulas in the spirit of [88]; but, the next result can be useful (see, e.g., [95]).

Proposition 2.1.2. *Let* $H : [0,1]^2 \to [0,1]$ *be increasing in each variable. The following statements are equivalent:*

(a) *H is jointly (left–) continuous, in the sense that if $\{s_n\}$ and $\{t_n\}$ are two increasing sequences of points of $[0,1]$ that tend to s and t respectively, then*

$$\lim_{n\to+\infty} H(s_n, t_n) = H(s,t);$$

(b) *H is (left–) continuous in each place.*

Because of (S2), every semicopula has derivatives almost everywhere on $[0,1]^2$. In particular, some conditions on derivatives allow us to characterize the semicopulas that are also quasi–copulas. But, first, we give two technical lemmata (see, respectively, page 333 and 337 of [153]).

Lemma 2.1.1. *Let $f : [a,b] \to \mathbb{R}$ be given. If f is continuous on $[a,b]$ and differentiable except at countably many points of $[a,b]$, and f' is Lebesgue integrable on $[a,b]$, then f is absolutely continuous on $[a,b]$.*

Lemma 2.1.2. *Let $f : [a,b] \to \mathbb{R}$ be given. The following statements are equivalent:*

(a) *for some $k > 0$, we have*

$$|f(x) - f(y)| \le k|x - y| \quad \textit{for all } x, y \in [a,b];$$

(b) *f is absolutely continuous on $[a,b]$ and $|f'(t)| \le k$ on $[a,b]$ for some $k > 0$.*

Proposition 2.1.3. *Let S be a semicopula such that all the horizontal and vertical sections of S are differentiable on $[0,1]$ except at countably many points. The following statements are equivalent:*

(a) *S is a quasi–copula;*

(b) *S satisfies the following two conditions:*

  (b1) *S is continuous;*

  (b2) *for every (x,y) in $[0,1]^2$ that admits first–order partial derivatives of S*

$$0 \le \partial_x S(x,y) \le 1 \quad \textit{and} \quad 0 \le \partial_y S(x,y) \le 1.$$

Proof. Implication $(a) \implies (b)$ is trivial. In order to prove $(b) \implies (a)$, let $S_y(t)$ be the horizontal section of S at $y \in [0,1]$ and $S_x(t)$ be the vertical section of S at $x \in [0,1]$. The functions S_x and S_y are continuous and differentiable on $[0,1]$ except at countably many points and their derivatives are bounded. Therefore, from Lemma 2.1.1 it follows that they are absolutely continuous. But, again, if S_x and S_y are absolutely continuous and their derivatives are bounded from above by 1, then Lemma 2.1.2 ensures that S_x and S_y are Lipschitz with constant 1. Therefore, for every (x,y) and (x',y') in $[0,1]^2$, we have

$$\begin{aligned} |S(x,y) - S(x',y')| &\leq |S(x,y) - S(x',y)| + |S(x',y) - S(x',y')| \\ &\leq |S_y(x) - S_y(x')| + |S_{x'}(y) - S_{x'}(y')| \\ &\leq |x - x'| + |y - y'|, \end{aligned}$$

which is the desired assertion. □

Notice that there exists also a semicopula which is not Lebesgue measurable.

Example 2.1.2. Let J be a subset of $[0,1]$ that is not Lebesgue measurable. Define the function

$$S(x,y) = \begin{cases} 0, & (x+y<1) \quad \text{or} \quad (x+y=1 \text{ and } x \in J); \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Then S is a semicopula that is not Lebesgue measurable. In [79] there is an analogous example of a t–norm which is not Lebesgue measurable.

Given a semicopula S, its diagonal section δ satisfies the following properties:

(a) $\delta(1) = 1$;

(b) $\delta(t) \leq t$ for all $t \in [0,1]$;

(c) δ is increasing.

Conversely, given a function δ satisfying properties (a), (b) and (c), it is always possible to construct a semicopula whose diagonal section is δ; for instance:

$$S_\delta(x,y) := \begin{cases} \delta(x) \wedge \delta(y), & \text{if } (x,y) \in [0,1[^2; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

A semicopula need not be uniquely determined by its diagonal. For example, if $\delta(t) = t^2$ for all $t \in [0,1]$, there are two different semicopulas, Π and S_δ with diagonal section equal to δ. The only semicopulas uniquely determined by their diagonal sections are M and Z, as asserted in the following

Proposition 2.1.4. *The only semicopula with diagonal section equal to* $id_{[0,1]}$ *is* M.

Proof. Suppose that $\delta(t) = t$ for all t in $[0,1]$. For all $x, y \in [0,1]$, if $x \geq y$, then

$$S(y,y) = y \leq S(x,y) \leq S(1,y) = y;$$

whereas if $x < y$, then

$$S(x,x) = x \leq S(x,y) \leq S(x,1) = x;$$

that is $S(x,y) = \min\{x,y\}$. □

Analogously, we can prove

Proposition 2.1.5. *The only semicopula with diagonal* $\delta(t) = 0$ *on* $[0,1[$ *is* Z*.*

The proof of the following result is immediate and will not be given.

Proposition 2.1.6. *Let* $S = (\langle a_i, b_i, S_i \rangle)_{i \in I}$ *be an ordinal sum of semicopulas. Then* S *is a semicopula.*

Another simple construction method for semicopulas is presented here.

Example 2.1.3 (Frame semicopula)**.** Let the points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

partition the unit interval $[0,1]$ and let

$$0 = v_0 \leq v_1 \cdots \leq v_n < 1$$

be points in $[0,1]$ such that $v_i \leq t_i$ $(i \in \{1,2,\ldots,n\})$. The *frame semicopula* S_f corresponding to $(t_0, t_1, \ldots, t_n)$ and $(v_0, v_1, \ldots, v_n)$ is defined by

$$S_f(x,y) := \begin{cases} v_{i-1}, & \text{if } (x,y) \in [t_{i-1},1[^2 \backslash [t_i,1[^2; \\ x \wedge y, & \text{if } x \vee y = 1. \end{cases}$$

Moreover, if continuity questions arise, we may choose as the value taken on the side of each frame the value taken on the frame below.

2.2 Characterizations of some semicopulas

At a first glance, the definition of semicopula might appear somewhat more general than actually is. In this sense, it will be shown in this section that condition (S1) is quite restrictive and that it allows to characterize some basic semicopulas.

Proposition 2.2.1. *Let* S *be a semicopula. The following statements are equivalent:*

(a) S *is concave;*

(b) *S is super–homogeneous, viz.* $S(\lambda x, \lambda y) \geq \lambda\, S(x, y)$ *for all x, y and λ in* $[0, 1]$*;*

(c) *S is idempotent, viz.* $S(x, x) = x$ *for every* $x \in [0, 1]$*;*

(d) $S = M$.

Proof. If S is concave, then $S(\lambda x, \lambda y) = S(\lambda(x, y) + (1 - \lambda)(0, 0)) \geq \lambda S(x, y)$, and (b) holds. If S is super–homogeneous, then $S(x, x) \geq xS(1, 1) = x$, which together with $S(x, x) \leq S(x, 1) = x$, leads to (c). If S is idempotent, then Proposition 2.1.4 ensures that $S = M$. Finally, it is clear that M is concave. □

Proposition 2.2.2. *Let S be a semicopula. The following statements are equivalent:*

(a) *S is convex and 1–Lipschitz;*

(b) *S is a function of the sum of its arguments, i.e.* $S(x, y) = F(x + y)$ *for some function F from* $[0, 2]$ *into* $[0, 1]$*;*

(c) $S = W$.

Proof. (a) ⇒ (c): Suppose that S is convex and 1–Lipschitz. If $x + y \in\,]0, 1]$, define $\lambda := y/(x + y)$, which is in $[0, 1]$; then $(x, y) = \lambda(0, x + y) + (1 - \lambda)(x + y, 0)$. Now, since S is convex,

$$0 \leq S(x, y) \leq \lambda S(0, x + y) + (1 - \lambda)S(x + y, 0) = 0;$$

therefore, $S(x, y) = 0$. If $x + y \geq 1$, define $\lambda := (1 - y)/[2 - (x + y)]$, which is in $[0, 1]$, in order to obtain $(x, y) = \lambda(1, x + y - 1) + (1 - \lambda)(x + y - 1, 1)$. Again, since S is convex,

$$S(x, y) \leq \lambda S(1, x + y - 1) + (1 - \lambda)S(x + y - 1, 1) = x + y - 1,$$

and, since it is 1–Lipschitz,

$$S(1, 1) - S(x, y) \leq 1 - x + 1 - y.$$

Therefore $S(x, y) = x + y - 1$, and (c) holds.

(b) ⇒ (c): Suppose that there exists a function F from $[0, 2]$ into $[0, 1]$ such that $S(x, y) = F(x + y)$. If t is in $[0, 1]$, then $F(t) = S\,(0, t) = 0$, and if t is in $[1, 2]$, then $F(t) = S(1, t - 1) = t - 1$. Therefore, $F(t) = \max\{0, t - 1\}$, and $S(x, y) = F(x + y) = \max\{x + y - 1, 0\} = W(x, y)$.

Parts "(c)⇒(a)" and "(c)⇒(b)" can be easily proved. □

In particular, part (b) is equivalent to the fact that S is Schur–constant.

Proposition 2.2.3. *The following properties are equivalent for a semicopula S:*

(a) *S is positively homogeneous with respect to one variable, viz. for every x, y, λ in $[0,1]$, either $S(x,\lambda y)=\lambda S(x,y)$ or $S(\lambda x,y)=\lambda S(x,y)$;*

(b) *S has separate variables, viz. there exist two functions F_1 and F_2 defined from $[0,1]$ into $[0,1]$ such that $S(x,y)=F_1(x)\cdot F_2(y)$;*

(c) *S has linear section in both the variables;*

(d) $S=\Pi$.

Proof. Without loss of generality assume that S is homogeneous with respect to the first variable; then $S(x,y)=x\,S(1,y)=xy$; therefore (a) implies (b).

Now, suppose that (b) holds and let $S(x,y)=F_1(x)\cdot F_2(y)$ be a semicopula. It follows that $S(x,1)=F_1(x)\cdot F_2(1)=x$ and $S(1,x)=F_1(1)\cdot F_2(x)=x$. Therefore, for every $a\in[0,1]$, we have $S(x,a)=F_1(x)\cdot F_2(a)=(F_2(a)/F_2(1))\cdot x$, viz. the horizontal section of S at the point a is linear. The same result holds for the vertical section of S.

Finally, if S has linear sections in both the variables, then, fixed $a\in[0,1]$, we have $S(x,a)=\lambda_a x$ for a suitable $\lambda_a\in[0,1]$. But $S(1,a)=a$ and, hence, $\lambda_a=a$ and $S=\Pi$. Obviously, (d) implies (a). □

2.2.1 Harmonic semicopulas

Let Ω be an open subset of $\mathbb{R}^2$. A twice continuously differentiable function $F\colon\Omega\to\mathbb{R}$ is said to be *harmonic* if

$$\Delta F(x,y):=\frac{\partial^2F(x,y)}{\partial x^2}+\frac{\partial^2F(x,y)}{\partial y^2}=0\quad\text{for all }(x,y)\in\Omega.$$

Moreover, such F is said to be *superharmonic* (resp. *subharmonic*) if $\Delta F\le 0$ (resp. $\Delta F\ge 0$). For more details on harmonic function theory, we refer to [5]. Here we recall two important results for harmonic functions.

Theorem 2.2.1 (Maximum–minimum principle for harmonic functions)**.** *Let Ω be a connected open subset of $\mathbb{R}^2$ and let F be a harmonic function on Ω. If F has either a maximum or a minimum on Ω, then F is constant on Ω.*

Theorem 2.2.2. *Let Ω be a connected open subset of $\mathbb{R}^2$ and let F be a superharmonic (respectively, subharmonic) function on Ω. If F has a minimum (respectively, a maximum) on Ω, then it is constant on Ω.*

Proposition 2.2.4. *The only harmonic semicopula is* Π.

Proof. It is easily shown that Π is harmonic. Suppose that there exists another harmonic semicopula F and let (x_0,y_0) be a point in $]0,1[^2$ such that $\Pi(x_0,y_0)\neq F(x_0,y_0)$. Now, $G:=F-\Pi$ is a harmonic function that vanishes on the boundary of $[0,1]^2$. Therefore, G has either a maximum or a minimum on $]0,1[^2$, and, in view of the maximum–minimum principle for harmonic functions, G is constant, and this constant is equal to zero, viz. $F = \Pi$. □

Proposition 2.2.5. *If S is a superharmonic (resp. subharmonic) semicopula, then $S \geq \Pi$ (resp. $S \leq \Pi$).*

Proof. If S is a superharmonic semicopula, then $G := S - \Pi$ is also superharmonic and it vanishes on the boundary of $[0,1]^2$. Therefore, $S(x,y) - \Pi(x,y) \geq 0$ for every (x,y) in $[0,1]^2$, because, otherwise, Theorem 2.2.2 would imply $S = \Pi$. A similar argument holds for subharmonic semicopulas. □

In the case of copulas, the following result holds.

Proposition 2.2.6. *Let (X,Y) be a continuous random pair with copula C. If C is superharmonic, then (X,Y) is positively quadrant dependent. Analogously, if C is subharmonic, then (X,Y) is negatively quadrant dependent.*

Proposition 2.2.7. *Let the copula C of a pair (X,Y) of continuous random variables be twice–differentiable.*

(a) *If Y is stochastically increasing in X and if X is stochastically increasing in Y, then C is superharmonic.*

(b) *If Y is stochastically decreasing in X and if X is stochastically decreasing in Y, then C is subharmonic.*

Proof. In view of Proposition 1.7.3, the property $SI(Y|X)$ is equivalent to the concavity of the function $x \mapsto C(x,y)$ for every $y \in [0,1]$, and $SI(X|Y)$ is equivalent to the concavity of the function $y \mapsto C(x,y)$ for every $x \in [0,1]$. Because C is twice differentiable, it follows that $\partial^2_{xx} C(x,y) \leq 0$ and $\partial^2_{yy} C(x,y) \leq 0$, from which $\Delta C(x,y) \leq 0$. The proof of part (b) is analogous. □

Therefore we can insert the concept of super– and sub– harmonicity in the scheme of dependence concepts (note that the converse implications in Table 2.1 are, in general, false).

SI(Y|X) & SI(X|Y)	$\Longrightarrow$	Superharmonicity	$\Longrightarrow$	PQD(X,Y)
SD(Y|X) & SD(X|Y)	$\Longrightarrow$	Subharmonicity	$\Longrightarrow$	NQD(X,Y)

Table 2.1: Superharmonicity and dependence concepts

Example 2.2.1. Let consider the class of copulas given by $C_{fg}(x,y) = xy + \lambda f(x)\, g(y)$, where f and g are suitable functions and $\lambda > 0$ (see [132]). We have

$$\Delta C_{fg}(x,y) = \lambda(f''(x)\, g(y) + f(x)\, g''(y)).$$

If $f(t) = t(1-t)^2$ and $g(t) = t(1-t)$, then C_{fg} is a PQD copula, but

$$\Delta C_{fg}(x,y) = \lambda\left[(6x-4)y(1-y) - 2x(1-x)^2\right]$$

is (strictly) positive on the set $\{(x,y) \in [0,1]^2 : x = 1\}$ and it is (strictly) negative on the set $\{(x,y) \in [0,1]^2 : 0 \le x < 2/3\}$; thus C_{fg} is neither superharmonic nor subharmonic.

Analogously, we can find two functions f and g such that C_{fg} is superharmonic, but f and g are not both concave and, thus, C_{fg} is not $SI(Y|X)$ and $SI(X|Y)$.

2.3 The class of semicopulas

Proposition 2.3.1. *If S_1 and S_2 are semicopulas, then for all $\theta \in [0,1]$ both the weighted arithmetic mean $(1-\theta)S_1 + \theta S_2$ and the weighted geometric mean $S_1^{\theta}\, S_2^{1-\theta}$ are semicopulas. In other words, the set $\mathcal{S}$ is convex and log–convex.*

Let $\mathcal{X}$ denote the set of all functions from $[0,1]^2$ to $[0,1]$ equipped with the product topology (which corresponds to pointwise convergence).

Theorem 2.3.1. *The class $\mathcal{S}$ of semicopulas is a compact subset of $\mathcal{X}$ (under the topology of pointwise convergence).*

Proof. Since $\mathcal{X}$ is a product of compact spaces, it is well known from Tychonoff Theorem (see, e.g., [76]) that $\mathcal{X}$ is compact. The proof is completed by showing that $\mathcal{S}$ is a closed subset of $\mathcal{X}$, viz. given a sequence $\{S_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$, if S_n converges pointwise to S, then S belongs to $\mathcal{S}$. In fact, for all $x, x', y \in [0,1]$ and $n \in \mathbb{N}$,

$$S_n(x,1) = x \xrightarrow[n\to+\infty]{} x = S(x,1),$$

and, if $x \le x'$, $S_n(x,y) \le S_n(x',y)$ implies $S(x,y) \le S(x',y)$, which is the desired conclusion. □

A sequence $\{S_n\}_{n\in\mathbb{N}}$ of semicopulas is a Cauchy sequence with respect to pointwise convergence if, for every $\epsilon > 0$ and for every point (x,y) in $[0,1]^2$, there exists a natural number $n_0 = n_0(\epsilon, x, y)$ such that

$$|S_n(x,y) - S_m(x,y)| < \epsilon,$$

whenever $n, m \ge n_0$. As an immediate consequence, each Cauchy sequence of semicopulas converges pointwise to some semicopula; in other words $\mathcal{S}$ is complete. Notice that it is known that the class $\mathcal{T}$ of t–norms is neither a complete nor a compact subset of $\mathcal{S}$ ([83]).

Now, consider the set $\mathcal{S}$ equipped with the pointwise ordering. Obviously, $(\mathcal{S}, \leq)$ is partially ordered, and not all pairs of semicopulas are comparable: it is sufficient to consider the following example.

Example 2.3.1. Let S_1 and S_2 be, respectively, the two ordinal sums given by

$$S_1(x,y) = (\langle 0, 1/2, Z\rangle) = \begin{cases} 0, & \text{if } (x,y) \in [0,1/2[^2, \\ \min\{x,y\}, & \text{otherwise;} \end{cases}$$

and by

$$S_2(x,y) = (\langle 1/2, 1, Z\rangle) = \begin{cases} 1/2, & \text{if } (x,y) \in [1/2,1[^2; \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Then

$$0 = S_1(1/4,1/4) < S_2(1/4,1/4) = 1/4,$$

but

$$3/4 = S_1(3/4,3/4) > S_2(3/4,3/4) = 1/2.$$

Proposition 2.3.2. *The set $\mathcal{S}$, equipped with the classical pointwise ordering, is a complete lattice.*

Proof. Let $\mathcal{B}$ be a nonempty subset of $\mathcal{S}$. For all $x, x', y \in [0,1]$ such that $x \leq x'$,

$$\vee\mathcal{B}(x,1) = \sup\{S(x,1) \colon S \in \mathcal{B}\} = x,$$

that is $\vee\mathcal{B}$ satisfies the condition $(S1)$ of Definition 2.1.1; moreover,

$$\vee\mathcal{B}(x,y) = \sup\{S(x,y) \colon S \in \mathcal{B}\} \leq \sup\{S(x',y) \colon S \in \mathcal{B}\} = \vee\mathcal{B}(x',y),$$

that is $\vee\mathcal{B}$ satisfies the condition $(S2)$ of Definition 2.1.1, and hence $\vee\mathcal{B}$ is a semicopula. Analogously $\wedge\mathcal{B}$ is a semicopula. □

In particular, the minimum (and the maximum) of two semicopulas is a semicopula. This result holds also for quasi–copulas, but neither for copulas nor for t-norms, as the following examples show (see, also, [123]).

Example 2.3.2. Consider the two copulas defined, for α and β in $]0,1[$ by

$$A_\alpha(x,y) := \begin{cases} \alpha \vee (x+y-1), & \text{if } (x,y) \in [\alpha,1]^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

(this is the ordinal sum $(\langle \alpha, 1, W\rangle)$) and

$$B_\beta(x,y) := \begin{cases} \dfrac{xy}{\beta}, & \text{if } (x,y) \in [0,\beta]^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

(this is the ordinal sum $(\langle 0, \beta, \Pi \rangle)$). Now, for $\alpha = 1/3$ and $\beta = 1/2$, the function $F : [0,1]^2 \to [0,1]$ defined by $F(x,y) := A_{(1/3)}(x,y) \wedge B_{(1/2)}(x,y)$ is not a copula. In fact, choose $s = t = 1/3$ and $s' = t' = 1/2$,

$$F(s',t') - F(s',t) - F(s,t') + F(s,t) = -1/9 < 0.$$

Moreover, $A_{(1/3)}$ and $B_{(1/2)}$ are t–norms, but the function F is not associative, because $F(F(1/2,1/2),1/3) = 2/9$, while $F(1/2, F(1/2,1/3)) = 1/3$.

Example 2.3.3. Consider the two copulas:

$$A_\lambda(x,y) = \begin{cases} y, & 0 \le y < \lambda x; \\ \lambda x, & \lambda x \le y < 1 - (1-\lambda)x; \\ x + y - 1, & \text{otherwise}; \end{cases}$$

and $B_\lambda = A^T$ the transpose of A. Then, for $\lambda = 1/2$, we have

$$\max\left\{A_{(1/2)}, B_{(1/2)}\right\}\left(\left[\frac{1}{3}, \frac{2}{3}\right]^2\right) = -\frac{1}{6} < 0.$$

Example 2.3.4. Consider the two t–norms:

$$T_1(x,y) = \begin{cases} x \wedge y, & x + y > 1; \\ 0, & \text{otherwise}; \end{cases}$$

and $T_2(x,y) = \Pi$. Then

$$T = \max\{T_1(x,y), T_2(x,y)\} = \begin{cases} x \wedge y, & x + y > 1; \\ xy, & \text{otherwise}; \end{cases}$$

is not associative. In fact,

$$T\left(T\left(\frac{4}{10}, \frac{5}{10}\right), \frac{7}{10}\right) = T\left(\frac{20}{100}, \frac{7}{10}\right) = \frac{14}{100},$$

but

$$T\left(\frac{4}{10}, T\left(\frac{5}{10}, \frac{7}{10}\right)\right) = T\left(\frac{4}{10}, \frac{5}{10}\right) = \frac{20}{100}.$$

2.3.1 Extremal semicopulas

Definition 2.3.1. A semicopula S is said to be *extremal* if it can not be expressed as a non–trivial convex sum of two semicopulas; in the sense that, if S admits the representation $S = \lambda A + (1-\lambda) B$ for A and B in $\mathcal{S}$ and $\lambda \in \,]0,1[$, then $S = A = B$.

By connecting Proposition 2.3.1 and Theorem 2.3.1, it follows that $\mathcal{S}$ is a compact and convex subset of $\mathcal{X}$; therefore, in view of the Krein–Millman Theorem (see, e.g., [32]), we have:

Proposition 2.3.3. *The class $\mathcal{S}$ of semicopulas is the convex hull of the set formed by extremal semicopulas.*

Next we show that the semicopulas Z and M are extremal.

Given the semicopula Z, suppose that there exist B and C in $\mathcal{S}$ and $\lambda \in \,]0,1[$ such that $Z(x,y) = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$ on $[0,1]^2$. For all $x, y \in [0,1[$, the equality

$$Z(x,y) = 0 = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$$

implies

$$B(x,y) = 0 = C(x,y),$$

so that $B = Z = C$ on $[0,1]^2$.

Using the same notations, we consider the semicopula M and suppose

$$M(x,y) = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$$

on $[0,1]^2$. In particular, for every $x \in [0,1]$ the equality

$$M(x,x) = x = \lambda\, B(x,x) + (1-\lambda)\, C(x,x)$$

implies

$$\delta_B(x) = \delta_C(x) = x,$$

which, in view of Proposition 2.1.4, yields $B = C = M$.

Extremal semicopulas can be easily constructed beginning from root sets. We recall that a *root set* $A \subset [0,1]^2$ is defined by the property:

$$(x,y) \in A \quad \text{implies} \quad (x',y') \in A \quad \text{for every } 0 \le x' \le x \text{ and } 0 \le y' \le y.$$

Thus, given a root set A, the semicopula S_A defined by

$$S_A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

is extremal, and this can be proved by the same arguments of the cases M and Z. Such S_A are called 1–*internal* semicopulas. Notice that M and Z are 1–internal semicopulas with root sets, respectively, $A_M = \emptyset$ and $A_Z = [0,1[^2$. Moreover, S_A is a t–norm if the set A is symmetric with respect to the main diagonal of the unit square.

Remark 2.3.1. For every semicopula S and for every $u \in [0,1]$, we can define the root set

$$A_u := \{(x,y) \in [0,1[^2 : S(x,y) < u\},$$

and we have

$$S(x,y) = \bigvee_{u \in [0,1]} S_{A_u}(x,y).$$

Thus every semicopula is the supremum of a set formed by 1–internal semicopulas.

Notice that the semicopula W is not extremal in $\mathcal{S}$. In fact, it suffices to consider the two semicopulas

$$S_1(x,y) = W(x,y)\,(2 - \max\{x,y\}) \text{ and } S_2(x,y) = W(x,y)\cdot \max\{x,y\}.$$

Then $W = (S_1 + S_2)/2$.

Analogously, Π is not extremal in $\mathcal{S}$ (and also in the class of copulas). In fact, $\Pi = (C_1 + C_2)/2$, where

$$C_1(x,y) = \begin{cases} \frac{xy}{2}, & (x,y) \in \left[0,\frac{1}{2}\right]^2; \\ \frac{3xy-x}{2}, & (x,y) \in \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right]; \\ \frac{3xy-y}{2}, & (x,y) \in \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right]; \\ \frac{xy+x+y-1}{2}, & (x,y) \in \left[\frac{1}{2},1\right]^2; \end{cases}$$

and

$$C_2(x,y) = \begin{cases} \frac{3xy}{2}, & (x,y) \in \left[0,\frac{1}{2}\right]^2; \\ \frac{xy+x}{2}, & (x,y) \in \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right]; \\ \frac{xy+y}{2}, & (x,y) \in \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right]; \\ \frac{3xy-x-y+1}{2}, & (x,y) \in \left[\frac{1}{2},1\right]^2; \end{cases}$$

and C_1 and C_2 are copulas.

2.4 Multivariate semicopulas

The notion of semicopula can be extended in a natural way to the n–dimensional case ($n \geq 3$).

Definition 2.4.1. A function $S : [0,1]^n \to [0,1]$ is said to be an *n–semicopula* if it satisfies the two following conditions:

(S1') $S(\mathbf{x}) = x_i$ if all coordinates of $\mathbf{x}$ are 1 except at most the i–th one;

(S2') S is increasing in each place.

Higher dimensional semicopulas are easily constructed from lower dimensional ones, in view of the following results, whose easy proofs will not be reproduced here.

Proposition 2.4.1. *Let H be a 2–semicopula and let S_m and S_n be, respectively, an m–semicopula and an n–semicopula ($m, n \in \mathbb{N}$). Then the function $S : [0,1]^{m+n} \to [0,1]$ defined by*

$$S(x_1,\dots,x_{m+n}) := H\left(S_m(x_1,\dots,x_m), S_n(x_{m+1},\dots,x_{m+n})\right) \tag{2.2}$$

is an $(m+n)$–semicopula.

Aggregation operators of type (2.2) are called *double aggregation operators*; they allow to combine two input lists of information coming from different sources into a single output (see [13] for more details).

In the opposite direction we can construct lower dimensional semicopulas from higher dimensional ones.

Proposition 2.4.2. *Any m–marginal of an n–semicopula S_n, $2 \leq m < n$, is an m–semicopula, viz., if S_n is an n–semicopula, then the function $S_m : [0,1]^m \to [0,1]$ defined by*

$$S_m(x_1, x_2, \ldots, x_m) = S_n(x_1, x_2, \ldots, x_m, 1, 1, \ldots, 1)$$

is an m–semicopula, and so any function obtained from it by permuting its arguments.

From Definition 2.4.1, it follows that all n–quasi–copulas are n–semicopulas. On the other hand, it is clear that an n–semicopula is a special n–ary aggregation operator.

In particular, a family of semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is, obviously, a global aggregation operator, but it need not have the neutral element property (in the sense of global agop), because, in general, $S_n(x_1, \ldots, x_{n-1}, 1) \neq S_{n-1}(x_1, \ldots, x_{n-1})$. Here we propose a possible definition of *global semicopula*.

Definition 2.4.2. A family of commutative semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is called a *global semicopula* if $S_1 = \mathrm{id}_{[0,1]}$ and, for every $n \geq 2$,

$$S_{n-1}(x_1, \ldots, x_{n-1}) = S_n(x_1, \ldots, x_{n-1}, 1).$$

Notice that, in this way, a global semicopula is a global aggregation operator with neutral element 1.

Analogously, we can define the concepts of *global quasi–copula* and *global copula*.

In practice, it is not difficult to construct a global semicopula. It suffices to take a commutative 2–semicopula S and construct the family $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ in such a way that $S_1 = \mathrm{id}_{[0,1]}$, and, for every $n \geq 2$,

$$S_n(x_1, \ldots, x_n) := S(S_{n-1}(x_1, \ldots, x_{n-1}), x_n).$$

This method can be used also for quasi–copulas, but not for copulas, where it is not immediate to construct a copula beginning from his margins (see [141] for more details).

Finally, we present a few comments on a possible use of global copulas in a probabilistic context.

Consider a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ in which all the random variables (=r.v.'s) are continuous. In view of Sklar's Theorem, a (unique) k–dimensional copula C_k can be associated with any choice of k r.v.'s $X_{i_1}, \ldots, X_{i_k}$. In particular, if the r.v.'s of the process are *exchangeable*, the copula C_k is commutative and it does not depend on the choice of the r.v.'s. Moreover, C_{k-1} is the $(k-1)$–margin copula of C_k.

Conversely, if $\{C_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is a global copula, in view of the Kolmogorov compatibility Theorem (see [94]), we can construct an exchangeable stochastic process $\{X_n\}_{n\in\mathbb{N}}$ (where each r.v. X_n is uniformly distributed on $[0,1]$) such that, for every $n \in \mathbb{N}$, C_n is the copula associated with any choice of n r.v.'s of the process.

Thus we have established a one–to–one correspondence between global copulas and exchangeable stochastic processes.

2.4.1 Multivariate semicopulas and fuzzy measures

Here, we reformulate a result of M. Scarsini (see [136]) through the concept of multivariate semicopula. To this end, some basic notations will be useful (see [30, 16]).

For every $n \geq 2$, let $\mathcal{B}(\overline{\mathbb{R}}^n)$ be the class of Borel sets in $\overline{\mathbb{R}}^n$. A set function $\nu : \mathcal{B}(\overline{\mathbb{R}}^n) \to [0,1]$ is called *fuzzy measure* (or *capacity*) if it satisfies:

(a) $\nu(\emptyset) = 0$ and $\nu(\overline{\mathbb{R}}^n) = 1$;

(b) $\nu(A) \leq \nu(B)$ for all Borel sets A and B, $A \subseteq B$.

In particular, a fuzzy measure ν is called *supermodular* (or *convex*) if, for all Borel sets A and B

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

Given a fuzzy measure ν, the *distribution function associated with* ν is the function $F_\nu : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ given by

$$F_\nu(x_1, \dots, x_n) = \nu\left([-\infty, x_1] \times \cdots \times [-\infty, x_1]\right).$$

Moreover, we denote by F_{ν_i} the marginal d.f. associated with ν_i, where ν_i is the i–th projection of ν $(i = 1, 2, \dots, n)$. Notice that, due to lack of additivity, a fuzzy measure is not completely characterized by its distribution function.

Theorem 2.4.1 ([136])**.** *Let ν be a supermodular fuzzy measure on $(\overline{\mathbb{R}}^n, \mathcal{B}(\overline{\mathbb{R}}^n))$, F_ν its associated d.f., and F_{ν_i}, $(i = 1, 2, \dots, n)$, the marginal d.f.'s associated with the projections $\nu_1, \nu_2, \dots, \nu_n$ of ν. Then there exists a semicopula $S_\nu : [0,1]^n \to [0,1]$ such that*

$$\forall (x_1, \dots, x_n) \in \overline{\mathbb{R}}^n \qquad F_\nu(x_1, \dots, x_n) = S_\nu\left(F_{\nu_1}(x_1), \dots, F_{\nu_n}(x_n)\right).$$

The above result is a direct generalization of Sklar's Theorem to fuzzy measures; in fact, if ν is a probability measure, we obtain Thereom 1.9.1. Moreover, we stress the fact that as a copula links a joint d.f. to its margins so a semicopula joins the d.f. of a fuzzy measure to its one–dimensional marginal d.f.'s.