Chapter 6

Euler buckling for compressible cylinders

One of the first, and most important, problems to be tackled by the theory of linear elasticity is that of the buckling of a column under an axial load. Using Bernoulli's beam equations, Euler found the critical load of compression N_{cr} leading to the buckling of a slender cylindrical column of radius B and length L. As recalled by Timoshenko and Gere [124], Euler looked at the case of an ideal column with (built in)-(free) end conditions. What is now called "Euler buckling", or the "*fundamental case* of buckling of a prismatic bar" [124] corresponds to the case of a bar with hinged-hinged end conditions. The corresponding critical load is given by

$$\frac{N_{\text{cr}}}{\pi^3 B^2} = \frac{E}{4} \left(\frac{B}{L}\right)^2,\tag{6.1}$$

where E is the Young modulus. The extension of this formula to the case of a thick column is a non-trivial, even sophisticated, problem of non-linear three-dimensional elasticity. In general, progress can be only made by using reductive (rod, shells, etc.) theories. However, there is another choice of boundary conditions for which the criterion (6.1) is valid: namely, the case where both ends are "guided" or "sliding" (the difference between the two cases lies in the shape of the deflected bar, which is according to the half-period of a sine in one case and of a cosine in the other case). In exact non-linear elasticity, there exists a remarkable three-dimensional analytical solution to this problem (due to Wilkes [129]) which describes a small-amplitude (incremental) deflection superimposed upon the large homogeneous deformation of a cylinder compressed between two lubricated platens. In this case, the Euler formula can be extended to the case of a column with finite dimensions, for arbitrary constitutive law.

The exact incremental solution allows for an explicit derivation of Euler's formula at the *onset of non-linearity*, which combines third-order elastic constants with a term in $(B/L)^4$. Goriely et al. [47], showed that for an incompressible cylinder,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4} \left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96} \left(\frac{20}{3} E + 9\mathcal{A}\right) \left(\frac{B}{L}\right)^4,\tag{6.2}$$

where $\mathcal{A}$ is Landau's third-order elasticity constant. This formula clearly shows

that *geometrical non-linearities* (term in $(B/L)^4$) are intrinsically coupled to *physical non-linearities* (term in $\mathcal{A}$) for this problem. (For the connection between Euler's theory of buckling and incremental incompressible nonlinear elasticity, see the early works of Wilkes [129], Biot [13], Fosdick and Shield [41], and the references collected in [47].)

Now, in third-order incompressible elasticity, there are two elastic constants: the shear modulus μ ($= E/3$) and $\mathcal{A}$ (see (2.45)). In third-order compressible elasticity, there are five elastic constants: λ and μ, the (second-order) Lamé constants (or equivalently, E and ν, Young's modulus and Poisson's ratio), and $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the (third-order) Landau constants (see (2.43)). Euler's formula at order $(B/L)^2$, equation (6.1), involves only one elastic constant, E. It is thus natural to ask whether Poisson's ratio, ν, plays a role in the non-linear correction to Euler formula of $(B/L)^4$, the next-order term. The final answer is found in our recent work [26] as formula (6.32) below, which shows that the non-linear correction involves all five elastic constants.

6.1 Finite compression and buckling

In this section, we recall the equations governing the homogeneous compression of a cylinder in the theory of exact (finite) elasticity and we also use the form of some incremental solutions that is, of some small-amplitude static deformations which may be superimposed upon the large compression and which indicate the onset of instability for the cylinder.

The mathematical method used to determine the solutions for incremental solution is another example for the semi-inverse method. In fact our ansatz for the components of the mechanical displacement will be in looking for incremental static solutions that are periodic along the circumferential and axial directions, and have yet unknown radial variations (see formula (6.10)).

6.1.1 Large deformation

We take a cylinder made of a hyperelastic, compressible, isotropic solid with strain energy function $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ say, with radius B and length L in its undeformed configuration. We denote by (R, Θ, Z) the coordinates of a particle in the cylinder at rest, in a cylindrical system.

Then we consider that the cylinder is subject to the following deformation,

$$r = \lambda_1 R, \qquad \theta = \Theta, \qquad z = \lambda_3 Z, \tag{6.3}$$

where (r, θ, z) are the current coordinates of the particle initially at (R, Θ, Z), λ_1 is the radial stretch ratio and λ_3 is the axial stretch ratio. Explicitly, $\lambda_1 = b/B$ and $\lambda_3 = l/L$, where b and l are the radius and length of the deformed cylinder, respectively. The physical components of $\boldsymbol{F}$, the corresponding deformation gradient, are:

$$\boldsymbol{F} = \mathrm{Diag}\,(\lambda_1, \lambda_1, \lambda_3), \tag{6.4}$$

showing that the deformation is equi-biaxial and homogeneous (and thus, universal).

The (constant) Cauchy stresses required to maintain the large homogeneous compression are given by formula (1.42),

$$T_i = J^{-1}\lambda_i W_i, \qquad i = 1, 2, 3 \quad \text{(no sum)}, \tag{6.5}$$

where $W_i \equiv \partial W / \partial \lambda_i$. In our case, $T_1 = T_2$ because the deformation is equi-biaxial, and $T_1 = T_2 = 0$ because the outer face of the cylinder is free of traction. Hence

$$T_1 = T_2 = \lambda_1^{-1}\lambda_3^{-1}W_1 = 0, \qquad T_3 = \lambda_1^{-2}W_3. \tag{6.6}$$

Note that we may use the first equality to express one principal stretch in terms of the other (provided, of course, that inverses can be performed).

6.1.2 Incremental equations

Now we recall the equations governing the equilibrium of incremental solutions, in the neighborhood of the finite compression. They read in general as (1.80)

$$\operatorname{div} \mathbf{\Sigma}^T = \mathbf{0}, \tag{6.7}$$

where $\mathbf{\Sigma}$ is the incremental nominal stress tensor. It is related to $\boldsymbol{u}$, the incremental mechanical displacement, through the incremental constitutive law (1.81), that in component form can be written as

$$\Sigma_{ji} = \mathcal{A}_{0jilk}u_{k,l}, \tag{6.8}$$

where the comma denotes partial differentiation with respect to the current coordinates and $\mathcal{A}_0$ is the fourth-order tensor of incremental elastic moduli. Its non-zero components, in a coordinate system aligned with the principal axes of strain, are given by (1.84). Note that here, some of these components are not independent one from another because $\lambda_1 = \lambda_2$ and $T_1 = T_2 = 0$. In particular, we find that

$$\begin{aligned} &\mathcal{A}_{01212} = \mathcal{A}_{02121} = \mathcal{A}_{01221}, \qquad \mathcal{A}_{02323} = \mathcal{A}_{01313} = \mathcal{A}_{01331} = \mathcal{A}_{02332}, \\ &\mathcal{A}_{02222} = \mathcal{A}_{01111}, \qquad \mathcal{A}_{02233} = \mathcal{A}_{01133}, \qquad \mathcal{A}_{03232} = \mathcal{A}_{03131}, \\ &\mathcal{A}_{01122} = \mathcal{A}_{01111} - 2\mathcal{A}_{01212}. \end{aligned} \tag{6.9}$$

6.1.3 Incremental solutions

We look for incremental static solutions that are periodic along the circumferential and axial directions, and have yet unknown radial variations. Thus our ansatz for the components of the mechanical displacement is the same as Wilkes's [129]:

$$\begin{aligned} u_r &= U_r(r) \cos n\theta \cos kz, \\ u_\theta &= U_\theta(r) \sin n\theta \cos kz, \\ u_z &= U_z(r) \cos n\theta \sin kz, \end{aligned} \tag{6.10}$$

where $n = 0, 1, 2, \ldots$ is the *circumferential mode number*; k is the *axial wavenumber*; the subscripts (r, θ, z) refer to components within the cylindrical coordinates (r, θ, z); and all upper-case functions are functions of r alone.

Dorfmann and Haughton [31] show that the following displacements $\boldsymbol{U}^{(1)}$, $\boldsymbol{U}^{(2)}$, and $\boldsymbol{U}^{(3)}$ are solutions to the incremental equations (6.7),

$$\boldsymbol{U}^{(1)}(r),\, \boldsymbol{U}^{(2)}(r) = \left[I_n'(qkr), -\frac{n}{qkr} I_n(qkr), -\frac{(\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131})}{q(\mathcal{A}_{01313} + \mathcal{A}_{01133})} I_n(qkr) \right]^T,$$

(6.11)

and

$$\boldsymbol{U}^{(3)}(r) = \left[\frac{1}{r} I_n(q_3 kr), -\frac{q_3 k}{n} I_n'(q_3 kr), 0 \right]^T,$$

(6.12)

where $q = q_1, q_2$ in turn and I_n is the modified Bessel function of order n. Here q_1, q_2, and q_3 are the square roots of the roots q_1^2, q_2^2 of the following quadratic in q^2:

$$\mathcal{A}_{01313}\mathcal{A}_{01111}q^4 + [(\mathcal{A}_{01133} + \mathcal{A}_{01313})^2 - \mathcal{A}_{01313}\mathcal{A}_{03131}$$
$$- \mathcal{A}_{03333}\mathcal{A}_{01111}]q^2 + \mathcal{A}_{03333}\mathcal{A}_{03131} = 0, \quad (6.13)$$

and of the root of the following linear equation in q^2

$$\mathcal{A}_{01212}q^2 - \mathcal{A}_{03131} = 0,$$

(6.14)

respectively.

From (6.8) we find that the incremental traction on planes normal to the axial direction has components of the same form as that of the displacements, namely

$$\Sigma_{rr} = S_{rr}(r) \cos n\theta \cos kz,$$
$$\Sigma_{r\theta} = S_{r\theta}(r) \sin n\theta \cos kz,$$
$$\Sigma_{rz} = S_{rz}(r) \cos n\theta \sin kz,$$

(6.15)

say, where again the functions $S_{rr}, S_{r\theta}, S_{rz}$ are functions of r alone. Then we find that the traction solutions corresponding to the solutions (6.11) and (6.12) are given by

$$r\boldsymbol{S}^{(1)}(r),\quad r\boldsymbol{S}^{(2)}(r) = \left[2\mathcal{A}_{01212} I_n'(qkr) \right.$$
$$- \left(2\mathcal{A}_{01212}\frac{n^2}{qkr} + qkr\mathcal{A}_{01111} - \frac{kr\mathcal{A}_{01133}(\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131})}{q(\mathcal{A}_{01313} + \mathcal{A}_{01133})} \right) I_n(qkr),$$
$$\left. 2n\mathcal{A}_{01212}\left(\frac{I_n(qkr)}{qkr} - I_n'(qkr) \right), -kr\mathcal{A}_{01313}\left(\frac{\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131}}{\mathcal{A}_{01313} + \mathcal{A}_{01133}} + 1 \right) I_n'(qkr) \right]^T,$$

(6.16)

and

$$r\boldsymbol{S}^{(3)}(r) = \left[2\mathcal{A}_{01212}\left(\frac{I_n(q_3 kr)}{r} - q_3 k I_n'(q_3 kr) \right), \right.$$
$$\left. \mathcal{A}_{01212}\left(\frac{2q_3 k}{n} I_n'(q_3 kr) - \left(\frac{2n}{r} + \frac{q_3^2 k^2 r}{n} \right) I_n(q_3 kr) \right), -k\mathcal{A}_{01313} I_n(q_3 kr) \right]^T. \quad (6.17)$$

The general solution to the incremental equations of equilibrium is thus of the form

$$rS(r) = \left[rS^{(1)}(r) \middle| rS^{(2)}(r) \middle| rS^{(3)}(r) \right] c, \tag{6.18}$$

where $S \equiv [S_{rr}, S_{r\theta}, S_{rz}]^T$, and c is a constant three-component vector. Note that we use the quantity rS for the traction (instead of S), because it is the Hamiltonian conjugate to the displacement in cylindrical coordinates [117].

Now when the cylinder is compressed (by platens say), its end faces should stay in full contact with the platens so that the first incremental boundary condition is

$$u_z = 0, \qquad \text{on} \quad z = 0, l, \tag{6.19}$$

which leads to [31, 47]

$$k = m\pi/l, \tag{6.20}$$

for some integer m, the *axial mode number*. From (6.15), we now see that on the thrust faces, we have

$$\Sigma_{rz} = 0, \qquad \text{on} \quad z = 0, l, \tag{6.21}$$

which means that the end faces of the column are in sliding contact with the thrusting platens. In other words, in the limit of a slender column, we recover the Euler strut with *sliding-sliding*, or *guided-guided* end conditions. In Figure 6.1, we show the first two axi-symmetric and two asymmetric modes of incremental buckling.

The other boundary condition is that the cylindrical face is free of incremental traction: $S(b) = 0$. This gives

$$\Delta \equiv \det \left[bS^{(1)}(b) \middle| bS^{(2)}(b) \middle| bS^{(3)}(b) \right] = 0. \tag{6.22}$$

6.2 Euler buckling

6.2.1 Asymptotic expansions

We now specialize the analysis to the asymmetric buckling mode $n = 1$, $m = 1$, corresponding to the Euler buckling with guided-guided end conditions, in the limit where the axial compressive stretch λ_3 is close to 1 (the other modes are not reached for slender enough cylinders). To this end, we only need to consider the third-order elasticity expansion of the strain energy density (2.43). (Note that there are other, equivalent, expansions based on other invariants, such as the ones proposed by Murnaghan [86], Toupin and Bernstein [125], Bland [17], or Eringen and Suhubi [37], see Norris [91] for the connections.)

To measure how close λ_3 is to 1, we introduce ϵ, a small parameter proportional to the slenderness of the deformed cylinder,
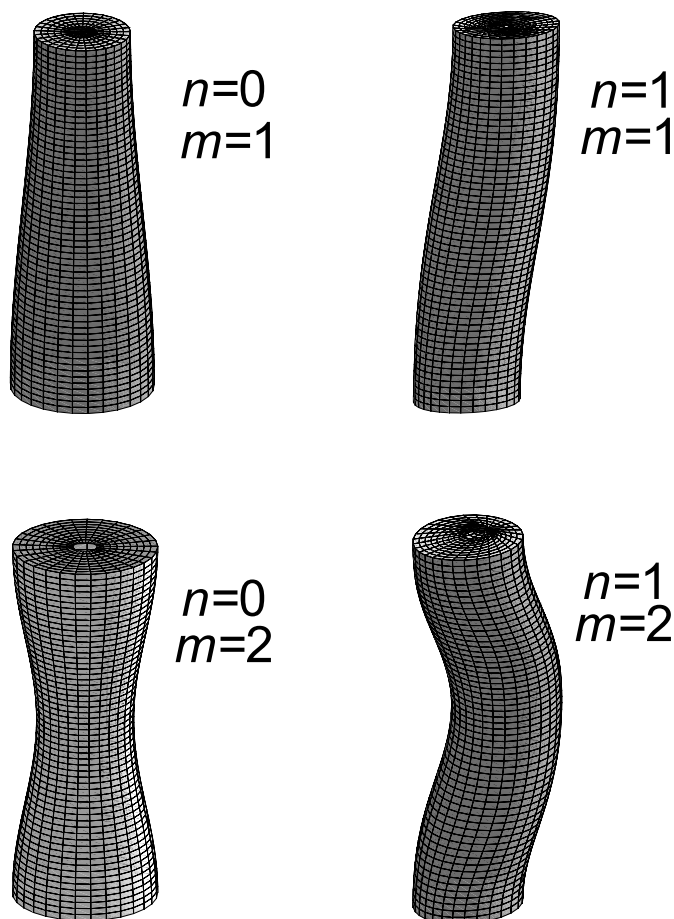
$$\epsilon = kb = \pi b/l. \tag{6.23}$$

Figure 6.1: First two axi-symmetric and two asymmetric modes of buckling for a compressed strut with guided-guided end conditions: n is the circumferential mode number and m the axial mode number. For slender enough cylinders, the $n = 1$, $m = 1$ mode is the first mode of buckling.

Then we expand the radial stretch λ_1 and the critical buckling stretch λ_3 in terms of ϵ up to order M,

$$\lambda_1 = \lambda_1(\epsilon) = 1 + \sum_{p=1}^{M} \alpha_p \epsilon^p + \mathcal{O}(\epsilon^{M+1}), \qquad \lambda_3 = \lambda_3(\epsilon) = 1 + \sum_{p=1}^{M} \beta_p \epsilon^p + \mathcal{O}(\epsilon^{M+1}),$$

$$(6.24)$$

say, where the α's and β's are to be determined shortly. Similarly, we expand Δ in powers of ϵ,

$$\Delta = \sum_{p=1}^{M_d} d_p \epsilon^p + \mathcal{O}(\epsilon^{M_d+1}),$$

and solve each order $d_p = 0$ for the coefficients α_p and β_p, making use of the condition $T_1 = 0$. We find that α_p and β_p vanish identically for all odd values of p, and that λ_1 and λ_3, up to the fourth-order in ϵ, are given by

$$\lambda_1 = 1 + \alpha_2 \epsilon^2 + \alpha_4 \epsilon^4 + O(\epsilon^6), \qquad \lambda_3 = 1 + \beta_2 \epsilon^2 + \beta_4 \epsilon^4 + O(\epsilon^6), \qquad (6.25)$$

with α_2 and α_4 given by

$$\alpha_2 = \frac{\nu}{4}, \tag{6.26}$$

and

$$\alpha_4 = -\frac{\nu(1+\nu)}{32} - \frac{(1+\nu)(1-2\nu)}{16E}\left[\nu^2\mathcal{A} + (1-2\nu+6\nu^2)\mathcal{B} + (1-2\nu)^2\mathcal{C}\right] - \nu\beta_4, \tag{6.27}$$

wherein

$$\beta_2 = -\frac{1}{4}, \tag{6.28}$$

and

$$\beta_4 = \frac{29 + 39\nu + 8\nu^2}{96(1+\nu)} + \frac{1}{16E}\left[(1-2\nu^3)\mathcal{A} + 3(1-2\nu)(1+2\nu^2)\mathcal{B} + (1-2\nu)^3\mathcal{C}\right].$$

Note that we switched from Lamé constants to Poisson's ratio and Young's modulus for these expressions, using the connections (1.53).

6.2.2   Onset of nonlinear Euler buckling

The analytical results presented above are formulated in terms of the current geometrical parameter ϵ, defined in (6.23). In order to relate these results to the classical form of Euler buckling, we introduce the initial geometric slenderness B/L. Recalling that $\epsilon = \pi b/l$, $\lambda_3 = l/L$, and $b = \lambda_1 B$, we find that

$$\epsilon\lambda_3 = \pi\lambda_1(B/L). \tag{6.29}$$

We expand ϵ in powers of B/L, and solve (6.29) to obtain

$$\begin{aligned}\epsilon &= \pi(B/L) + (\alpha_2 - \beta_2)\pi^3(B/L)^3 + \mathcal{O}\left((B/L)^4\right)\\ &= \pi(B/L) + (1+\nu)(\pi^3/4)(B/L)^3 + \mathcal{O}\left((B/L)^4\right).\end{aligned} \tag{6.30}$$

Second, we wish to relate the axial compression to the current axial load N. To do so, we integrate the axial stress over the faces of the cylinder,

$$N = -2\pi\int_0^b rT_3\mathrm{d}r = -\pi b^2 T_3 = -\pi\lambda_1^2 B^2 T_3, \tag{6.31}$$

because T_3 is constant, given by $(6.6)_2$.

Finally, in order to write the nonlinear buckling formula, we expand λ_1 and λ_3 in (6.31), using (6.25), and then expand ϵ in powers of the slenderness (B/L),

using (6.30). It gives the desired expression for the first *non-linear correction to Euler formula*,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96}\delta_{\text{NL}}\left(\frac{B}{L}\right)^4, \tag{6.32}$$

where

$$\delta_{\text{NL}} = 2\frac{13 + 12\nu - 2\nu^2}{(1+\nu)}E +$$
$$12\left[(1-2\nu^3)\mathcal{A} + 3(1-2\nu)(1+2\nu^2)\mathcal{B} + (1-2\nu)^3\mathcal{C}\right]. \tag{6.33}$$

We now check this equation against its incompressible counterpart (6.2). Theoretical considerations and experimental measurements [22, 29, 94, 131], show that in the incompressible limit, E and $\mathcal{A}$ remain finite,

$$\nu \to 1/2, \quad (1-2\nu)\mathcal{B} \to -E/3, \quad (1-2\nu)^3\mathcal{C} \to 0. \tag{6.34}$$

It is then a simple exercise to verify that (6.32) is indeed consistent with (6.2) in those limits.

6.2.3 Examples

Table 6.1: Lamé constants and Landau third-order elastic moduli for five solids (10^9 N·m^{-2})

material	λ	μ	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Polystyrene	1.71	0.95	-10	-8.3	-10.6
Steel Hecla 37	111	82.1	-358	-282	-177
Aluminium 2S	57	27.6	-228	-197	-102
Pyrex glass	13.5	27.5	420	-118	132
SiO$_2$ melted	15.9	31.3	-44	93	36

To evaluate the importance of the non-linear correction, we computed the critical axial stretch ratio of column buckling for two solids. In Table 6.1, we list the second- and third-order elastic constants of five compressible solids, as collected by Porubov [101] (in the Table we converted the "Murnaghan constants" given by Porubov to Landau constants $\mathcal{A}, \mathcal{B}, \mathcal{C}$). Figure 6.2 shows the variations of λ_3 with the squared slenderness $(B/L)^2$, for pyrex and silica (two last lines of Table 1).

Notes

The present analysis can be considered an extension of the article [47] from incompressible solids to compressible solids. It provides an asymptotic formula for the critical value of the load for the Euler buckling problem, with guided-guided (sliding-sliding) end conditions. This formula was checked both in the incompressible limit and on particular cases against the exact value of the buckling
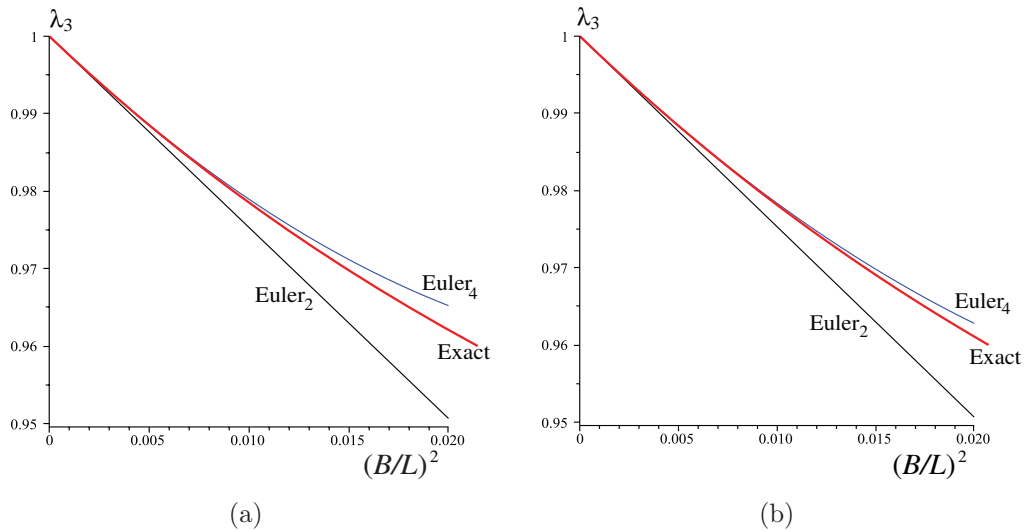
Figure 6.2: Comparison of the different Euler formulas obtained by expanding the exact solution to order 2 (classical Euler buckling formula, plot labeled "Euler$_2$") and to order 4 (plot labeled "Euler$_4$"), for pyrex (a) and for silica (b).

obtained from the exact solutions. Not surprisingly it reinforces the universal and generic nature of the Euler buckling formula, as the correction is small for most systems even when nonlinear elastic effects and nonlinear geometric effects are taken into account. It would be of great interest to see if these effects could be observed experimentally.

We note that Goriely et al. [47] write the equations for the linearized problem of instability using the *Stroh formalism* [122]. They adapt the work of Shuvalov [117] on waves in anisotropic cylinders to develop a Stroh-like formulation of the problem. From a computational perspective, the Stroh formalism is particularly well suited and well behaved (Biryukov [14]; Fu [42]) and if numerical integration was required it would provide an ideal representation of the governing equation.

Here we could have also presented the linearized problem of instability for the compressible cylinder within the Stroh formalism. We have omitted this formalism because it was not essential for our discussion.