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The Semi-Inverse Method in solid mechanics: Theoretical underpinnings and novel applications

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Il metodo semi-inverso in meccanica dei solidi: Basi teoriche e nuove applicazioni

Tesi diretta da Michel DESTRADE e Giuseppe SACCOMANDI

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présentée par
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La méthode semi-inverse en mécanique des solides: Fondements théoriques et applications nouvelles

Thèse dirigée par Michel DESTRADE et Giuseppe SACCOMANDI

Soutenance prévue le 7 décembre 2010 devant le jury composé de:

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Introduction

In the framework of the theory of Continuum Mechanics, exact solutions play a fundamental role for several reasons. They allow to investigate in a direct way the physics of various constitutive models (for example, in suggesting specific experimental tests); to understand in depth the qualitative characteristics of the differential equations under investigation (for example, giving explicit appreciation on the well-posedness of these equations); and they provide benchmark solutions of complex problems.

The Mathematical method used to determine these solutions is usually called the semi-inverse method. This is essentially a heuristic method that consists in formulating a priori a special ansatz on the geometric and/or kinematical fields of interest, and then introducing this ansatz into the field equations. Luck permitting, these field equations reduce to a simple set of equations and then some special boundary value problems may be solved.

Although the semi-inverse method has been used in a systematic way during the whole history of Continuum Mechanics (for example the celebrated Saint Venant solutions in linear elasticity have been found by this method), it is still not known how to generate meaningful ansatzes to determine exact solutions for sure. In this direction, the only step forward has been a partial confirmation of the conjecture by Ericksen [36] on the connection between group analysis and semi-inverse methods [96].

Another important aspect in the use of the semi-inverse method is associated in fluid dynamics with the emergence of secondary flows and in solid mechanics with latent deformations. It is clear that "Navier-Stokes fluid" and an "isotropic incompressible hyperelastic material" are intellectual constructions. No real fluid is exactly a Navier-Stokes fluid and no-real world elastomer can be characterized from a specific elastic potential, such as for example the "neo-Hookean" or "Mooney-Rivlin" models. The experimental data associated with the extension of a rubber band can be approximated by several different models, but we still do not know of a fully satisfying mathematical model. This observation is fundamental in order to understand that the results obtained by a semi-inverse method could be dangerous and misleading.

We know that a Navier-Stokes fluid can move by parallels flows in a cylindrical tube of arbitrary section. We obtain that solution by considering that the kinematic field is a function of the section variables only. In this way, the Navier-Stokes equations are reduced to linear parabolic equations which we solve by considering the usual no-slip boundary conditions. This picture is peculiar to Navier-Stokes fluids. In fact, if the relation between the stress and the stretching is not linear, a fluid can flow in a tube by parallel flows if and only if the tube possesses cylindrical symmetry (see [40]). If the tube is not perfectly cylindrical, then what is going on? Clearly any real fluid may flow in a tube, whether or not it is a Navier-Stokes fluid. In the real world, what is different from what it is predicted by the Navier-Stokes theory is the presence of secondary flows, i.e. flows in the section of the cylinder. This means that a pure parallel flow in a tube is a strong idealization of reality. A classic example illustrating such an approach in solid mechanics is obtained by considering deformations of anti-plane shear type. Knowles [72] shows that a non-trivial (non-homogeneous) equilibrium state of anti-plane shear is not always (universally) admissible, not only for compressible solids (as expected from Ericksen's result [34]) but also for incompressible solids. Only for a special class of incompressible materials (inclusive of the so-called "generalized neo-Hookean materials") is an anti-plane shear deformation controllable. Let us consider, for example, the case of an elastic material filling the annular region between two coaxial cylinders, with the following boundary-value problem: hold fixed the outer cylinder and pull the inner cylinder by applying a tension in the axial direction. It is known that the deformation field of pure axial shear is a solution to this problem valid for every incompressible isotropic elastic solid. In the assumption of non-coaxial cylinders, thereby losing the axial symmetry, we cannot expect the material to deform as prescribed by a pure axial shear deformation. Knowles's result [72] tells us that now the boundary-value problem can be solved with a general anti-plane deformation (not axially symmetric) only for a certain subclass of incompressible isotropic elastic materials. Of course, this restriction does not mean that, for a generic material, it is not possible to deform the annular material as prescribed by our boundary conditions, but rather that, in general, these lead to a deformation field that is more complex than an anti-plane shear.

Hence, we also expect secondary in-plane deformations. The true problem is therefore to understand when these secondary fields can be or cannot be neglected; it is not to determine the special theory for which secondary flows disappears in our mathematical world. These issues are relevant to many stability issues.

The present Thesis originates from the desire to understand in greater detail the analogy between secondary flows and latent deformations (i.e. deformations that are awoken from particular boundary conditions) in solid mechanics. We would also like to question those boundary conditions that allow a semi-inverse simple solution for special materials, but pose very difficult problem for general materials. In some sense we are criticizing all studies that characterize the special strain energy functions for which particular classes of deformations turn out to be possible (or using a standard terminology, turn out to be controllable).

We wish to point out that our criticism is not directed at the mathematical results obtained by these studies. Those results can and do lead to useful exact solutions if the correct subclass of materials is picked. However, with regard to the whole class of materials that are identified in the literature, one has to exercise a great deal of caution, because models that are obtained on the basis of purely mathematical arguments may exhibit highly questionable physical behavior. For example, some authors have determined which elastic compressible isotropic materials support simple isochoric torsion. In fact, it is not of any utility to understand which materials possess this property, because these materials do not exist. It is far more important to understand which complex geometrical deformation accompanies the action of a moment twisting a cylinder. That is why universal solutions are so precious (see [113]). These results may also have important repercussions in biomechanics. In the study of the hemo-dynamics, the hypothesis that the arterial wall deforms according to simple geometric fields does not account for several fundamental factors. A specific example of a missing factor is the effect of torsion on microvenous anastomic patency and early thrombolytic phenomenon (see for example [116]). Nonetheless, we do acknowledge the value of simple exact solutions obtained by inverse or semi-inverse investigations for understanding directly the nonlinear behavior of solids.

The plan of the Thesis is the following: in the first two chapters, we develop an introduction to nonlinear elasticity, essential to the subsequent chapters. The third chapter is entirely devoted to the inverse procedures of Continuum Mechanics and we illustrate some of the most important results obtained by their use, including the "universal solutions". While the inverse procedures have been truly important to obtain exact solutions, on the other hand some of them may misguide and miss real and interesting real phenomena. Here we also begin to expose our criticism of some uses of the semi-inverse method and we describe in detail the "anti-plane shear problem". The core of these considerations is presented in the fourth chapter (see also [28]). Here we illustrate some possible dangers inherent to the use of special solutions to determine classes of constitutive equations. We consider some specific solutions obtained for isochoric deformations but for compressible nonlinear elastic materials: "pure torsion" deformation, "pure axial shear" deformation and the "propagation of transverse waves". We use a perturbation tecnique to predict some risks that they may lead to when they are considered. Mathematical arguments are therefore important when they determine general constitutive arguments, not very special strain energies as the compressible potential that admits isochoric deformations. In the fifth chapter (see also [27]), we give an elegant and analytic example of secondary (or latent) deformations in the framework of nonlinear elasticity. We consider a complex deformation field for an isotropic incompressible nonlinear elastic cylinder and we show that this deformation field provides an insight into the possible appearance of secondary deformation fields for special classes of materials. We also find that these latent deformation fields are woken up by normal stress differences. Then we present some more general and universal results in the sixth chapter, where we use incremental solutions of nonlinear elasticity and we provide an exact solution for buckling instability of a nonlinear elastic cylinder and an explicit derivation for the first nonlinear correction of Euler's celebrated buckling formula (see also [26]).

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Contents

Abstract

Recently, the biomechanics of soft tissues has become an important topic of research in several engineering, biomedical and mathematical fields. Soft tissues are biological materials that can undergo important deformations (both within physiological and pathological fields) and they clearly display a nonlinear mechanical behaviour. In this case the analysis of the deformations by computational methods (e.g. finite elements) can be complex. Indeed, it is not easy to know exactly the "right" constitutive equations to describe the behaviour of the material, and often the commercial software turns out to be unsuited for dealing with trust the solutions for the corresponding balance equations. The geometrical nonlinearity of the model under investigation makes it very difficult to grasp the true physics of the problem and often the intuition of the engineer can do very little if it is not guided by careful and exact mathematical analysis. To this end the possibility of obtaining easy exact solutions for the field equations is an important and privileged tool, helping us to gain a better understanding of several biomechanics phenomena.

The semi-inverse method is one of few known methods available to obtain exact solutions in the mathematical theory of Continuum Mechanics. The semi-inverse method has been used in a systematic way during the whole history of Continuum Mechanics (for example to derive the celebrated Saint Venant solutions [5, 6]), but unfortunately this use has always happened essentially in a heuristic way, completely disconnected from a general method.

Essentially, the purpose of the semi-inverse method consists in formulating a priori a special ansatz for the unknown fields in a certain theory and in reducing the general balance equations to a simplified subset of equations. Here, by simplifying action, one often means that the balance equations are reduced to an easier system of differential equations (for example passing from a system of partial differential equations to an ordinary differential system, see [90]).

The following Thesis, developed in six chapters, studies several points of view of this method and other connected methodologies. The first chapters are essentially introductory while the others collect the results of research obtained during my PhD ([26, 27, 28]).

The First Chapter is devoted to the definitions, symbols and basic concepts of the theory of nonlinear elasticity. In that chapter we define the kinematics of finite deformation, introducing the concept of material body and of deformation. We introduce the balance laws, the stress and the equations of motion. We also propose constitutive concepts, such as those of frame indifference, material isotropy and hyperelasticity. We analyse the restrictions imposed on the mathematical models, such as the empirical inequalities of Truesdell and Noll, to ensure a reasonable

mechanical behaviour.

The Second Chapter exhibits some special constitutive laws for hyperelastic materials. One of the problems encountered in Continuum Mechanics concerns the choice of models for the strain energy function for a good description of the mechanical behaviour of “real” materials. Here we describe some models (both for compressible and incompressible materials) that are commonly used in the literature, including: the neo-Hookean model, the Mooney-Rivlin model, the generalized neo-Hookean model, the Hadamard model, the Blatz-Ko model, and finally an expansion of the strain energy function with respect to the Green Lagrange strain tensor, used to study small-but-finite deformations.

The Third Chapter introduces a small overview of the use of the semi-inverse method in elasticity. We show some examples which may be considered the most representative and/or meaningful and highlight their strengths and weaknesses. We apply the inverse method by searching universal solutions both in the compressible (where the only admissible deformations are homogeneous [34]) and in the incompressible case (where in addition to homogeneous, five other inhomogeneous “families” have been found in the literature [33, 119]).

The Ericksen result [34] shows that there are no other finite deformations beyond those homogeneous that are controllable for all compressible materials. The impact of that result on the theory of nonlinear elasticity was quite important. For many years there has been “the false impression that the only deformations possible in an elastic body are the universal deformations” [25]. In the same time as the publication of Ericksen's result, there was considerable activity in trying to find solutions for nonlinear elastic materials using the semi-inverse method. And the search of the exact solutions for nonlinear isotropic elastic incompressible materials, thanks to the constraint of incompressibility, has been easier than for the compressible ones. In other words it has been possible to find exact solutions which are not universal.

In recent years, there has been a great interest in the possibility to determine classes of exact solutions for compressible materials as well. One of the strategies used is to take inspiration from the inhomogeneous solutions for nonlinear elastic incompressible materials and to seek similar solutions in compressible materials. The Fourth Chapter focusses on the results obtained for compressible materials using this line of research. The object is to determine which compressible materials can sustain isochoric deformations such as, for example, “pure torsion”, “axial pure shear” and “azimuthal pure shear”. We believe that these lines of research can be misleading. To illustrate our thesis we have considered small perturbations on some classes of compressible materials capable to sustain a certain isochoric deformation. As a result, although the perturbation is “small”, the corresponding volume variation is not negligible. We emphasize that it does not turn out to be of any utility to understand which materials can sustain a simple isochoric torsion, because these materials do not exist, but it is far more important to understand which complex geometrical deformation accompanies the action of a moment twisting for a cylinder. Only in this way, can the results obtained with the semi-inverse method be meaningful.

Among the examples of application of the semi-inverse method, we report the search of solutions for the “anti-plane shear” and “radial” deformation. In the

incompressible case we know that, for a general elastic solid, the balance equations are consistent with the anti-plane shear assumption only in the cylindrical symmetry case. We can say nothing when the body geometry is more general, since in that case the equilibrium equations for a generic elastic solid reduce to an overdetermined system that is not always consistent. This means that for general bodies, the anti-plane shear deformation must be coupled with secondary deformations. A complex tensional state is automatically produced in the body.

The Fifth Chapter presents a short overview of the results already obtained in literature on the latent deformations (see [39, 63, 83]). Then we give a new analytical example for the above issue (see also [27]). We consider a complex deformation field for an isotropic incompressible nonlinear elastic cylinder, namely a combination of an axial shear, a torsion and an azimuthal shear. After fixing some boundary conditions, one can show that for the neo-Hookean material, the azimuthal shear is not essential regardless of whether the torsion is present or not. When the material is idealized as a Mooney-Rivlin material, the azimuthal shear cannot vanish when a non-zero amount of twist is considered. Applying the stress field, obtained from the neo-Hookean case, in order to extrude a cork from a bottle of wine, then we conjecture that is more advantageous to accompany the usual vertical axial force by a twisting moment.

The Thesis ends with a Sixth Chapter giving a new application of the semi-inverse method (see also [26]). The celebrated Euler buckling formula gives the critical load for the axial force for the buckling of a slender cylindrical column. Its derivation relies on the assumptions that linear elasticity applies to this problem, and that the slenderness of the cylinder is an infinitesimal quantity. Considering the next order for the slenderness term, we find a first nonlinear correction to the Euler formula. To this end, we specialize the exact solution of non-linear elasticity for the homogeneous compression of a thick cylinder with lubricated ends to the theory of third-order elasticity. This example is especially important because it supposes a general method, even if it is approximated, and it may be applied to several contexts.

These results show again the true complexity of nonlinear elasticity where it is difficult to choose the reasonable reductions. Moreover the results obtained have an important applications in biomechanic, a topic that will be the subject of future research.

Sunto

La Biomeccanica dei tessuti molli è recentemente diventata un importante argomento di ricerca in molti ambiti ingegneristici, bio-medici e anche matematici. I tessuti molli sono materiali biologici che possono subire deformazioni importanti (sia in ambito fisiologico che patologico) ed esibiscono un comportamento meccanico chiaramente nonlineare. In questo frangente lo studio delle deformazioni con metodi computazionali, come gli elementi finiti, può essere molto complesso. Infatti, risulta difficile conoscere con sicurezza le equazioni costitutive "giuste" per descrivere il comportamento del materiale e il software commerciale risulta spesso inadeguato per affrontare con sicurezza la risoluzione delle equazioni di bilancio corrispondenti. La nonlinearità geometrica dei modelli in questione complica di molto la realtà fisica del problema e spesso l'intuito dell'ingegnere può ben poco se non viene accompagnato da dettagliate e rigorose analisi matematiche. In questo frangente la possibilità di avere semplici soluzioni esatte delle equazioni di campo è uno strumento importante e privilegiato per aiutare la nostra comprensione dei vari fenomeni biomeccanici.

Il metodo semi-inverso è uno dei pochi strumenti a nostra disposizione per ottenere soluzioni esatte nell'ambito della teoria matematica della meccanica dei continui. Il metodo semi-inverso è stato utilizzato in modo sistematico già dai fondatori della teoria dell'elasticità lineare (si pensi alle famose soluzioni di Saint Venant [5, 6]), ma purtroppo questo uso è sempre avvenuto in modo euristico e completamente sganciato da una metodologia generale.

Sostanzialmente lo scopo del metodo semi-inverso è quello di fissare a priori una serie di assunzioni sui campi incogniti in una data teoria e di ridurre le equazioni di bilancio generali a sottoinsiemi semplificati di equazioni. Qui per azione semplificativa solitamente si intende che le equazioni di bilancio vengano ridotte ad un sistema di equazioni differenziali più semplici (per esempio da un sistema di equazioni alle derivate parziali si può passare ad un sistema differenziale ordinario, vedi [90]).

La presente Tesi, nei sei capitoli in cui si sviluppa, studia diversi aspetti di questo metodo ed altre metodologie ad esso, in un certo senso, correlate. I primi capitoli sono di carattere introduttivo mentre i rimanenti riportano i risultati ottenuti durante il mio dottorato ([26, 27, 28]).

Il Primo Capitolo è dedicato alle definizioni, ai simboli e ai concetti base della teoria dell'elasticità nonlineare. In questo capitolo si definisce la cinematica delle deformazioni finite, introducendo il concetto di corpo materiale deformabile e di deformazione. Si passa poi alle leggi di bilancio, alla definizione di sforzo (stress) e alla formulazione delle equazioni del moto. Vengono quindi affrontati i concetti

costitutivi come il concetto frame indifference, di isotropia materiale ed il concetto di iperelasticità. Si analizzano le restrizioni imposte ai modelli matematici per assicurare un comportamento meccanico ragionevole come le diseguaglianze empiriche di Truesdell e Noll.

Il Secondo Capitolo espone alcune specifiche leggi costitutive di materiali iperelastici. Uno dei problemi maggiormente incontrati nelle applicazioni in meccanica dei continui riguarda la scelta di modelli per la funzione energia potenziale per poter descrivere al meglio un comportamento meccanico dei materiali "reali". Qui descriviamo alcuni modelli (sia per materiali comprimibili che incomprimibili) che sono maggiormente utilizzati in letteratura, tra cui: il modello neo-Hookeano, il modello di Mooney-Rivlin, il mdello neo-Hookeano generalizzato, il modello di Hadamard, il modello di Blatz-Ko ed infine una funzione energia potenziale ottenuta come espansione in termini del tensore di Lagrange, utile quest'ultima per "piccole" ma finite deformazioni.

Il Terzo Capitolo presenta una piccola overview dell'uso del metodo semi-inverso in elasticità. Si riportano solo alcuni esempi che possono essere considerati tra i più rappresentativi e/o significativi, sottolineandone i punti di forza e di debolezza. Applichiamo il metodo inverso nella ricerca di soluzioni universali sia nel caso comprimibile (dove le sole deformazioni possibili sono quelle omogenee, [34]) sia nel caso incomprimibile (dove oltre alle deformazioni omogenee nella versione isocorica in letteratura sono state trovate altre "cinque famiglie" non omogenee [33, 119]).

Il risultato di Ericksen [34] dimostra che non ci sono altre deformazioni finite oltre quelle omogenee che sono controllabili per tutti i materiali comprimibili. L'impatto di tale risultato sulla teoria dell'elasticità nonlineare è stato fondamentale. Per molti anni c'è stata "la falsa impressione che le uniche deformazioni possibili per un corpo elastico sono quelle universali" (vedi [25]). Nello stesso tempo della pubblicazione del risultato di Ericksen, una considerevole attività di ricerca cercava di trovare soluzioni usando il metodo semi-inverso. Per i materiali elastici nonlineari isotropi ed incomprimibili il vincolo di incomprimibilità ha facilitato la ricerca delle soluzioni esatte rispetto ai materiali comprimibili. Ovvero è stato possibile trovare soluzioni esatte che non sono universali.

Negli anni più recenti ci si è molto interessati della possibilità di determinare classi di soluzioni esatte anche per i mezzi comprimibili. Una delle strategie adottate per trovare soluzioni esatte anche in quest'ultimo caso consiste nel prendere ispirazione dalle soluzioni non omogenee per materiali elastici nonlineari incomprimibili e cercare simili soluzioni per materiali comprimibili. Nel Quarto Capitolo ci si interessa proprio ai risultati ottenuti per materiali comprimibili in questo filone di ricerca. Si tratta di determinare quali materaili comprimibili possono sostenere deformazioni isocoriche quali ad esempio la "torsione pura", lo "shear puro assiale" e lo "shear rotazionale puro". Questi filoni di ricerca a nostro avviso possono essere molto fuorvianti. Per illustrare i nostri argomenti abbiamo considerato delle piccole perturbazioni su alcune classi di materiali comprimibili capaci di sostenere una particolare deformazione isocorica. Ne risulta che seppur la perturbazione può considerarsi "piccola" la variazione di volume che ne corrisponde può non essere trascurabile. Sottolineiamo quindi come non sia importante capire quali materiali elastici ed isotropi comprimibili possono subire ad esempio una torsione semplice ed isocorica, in quanto questi materiali in ogni caso sono inesistenti, ma piuttosto capire quale geometria accompagna l'azione di un momento torcente in un cilindro che viene idealizzato come elastico ed isotropo. Solo in questo modo i risultati ottenuti con il metodo semi-inverso possono essere capiti in modo profondo.

Tra gli esempi di applicazione del metodo semi-inverso riportiamo la ricerca di soluzioni per la deformazione di "anti-plane shear" e per la deformazione "radiale". Nel caso incomprimibile sappiamo che le equazioni di bilancio per un qualunque solido elastico sono compatibili con l'assunzione di antiplane shear solo nel caso di simmetria cilindrica. Non sappiamo dire nulla quando la geometria del corpo è più generale, in quanto in questo caso le equazioni di equilibrio si riducono ad un sistema sovradeterminato che non sempre risulta compatibile. Questo significa che in corpi generali la deformazione di anti-plane shear deve essere accoppiata a deformazioni secondarie. Ovvero anche se le condizioni al contorno risultano compatibili con una deformazione di antiplane shear, questa per essere ammissibile non può essere pura. Automaticamente nel corpo si crea uno stato tensionale complesso. Cercare modelli speciali per cui questo stato tensionale viene meno non permette di capire veramente cosa succede nella realtà.

Nel Quinto Capitolo dopo aver brevemente esposto i risultati già ottenuti in letteratura sulle deformazioni latenti (vedi [39, 63, 83]), presentiamo un nuovo esempio analitico e non approssimato della questione (vedi anche [27]). Consideriamo infatti un campo di deformazioni complesso per un cilindro elastico isotropo nonlineare ed incomprimibile: una combinazione di uno shear assiale, di una torsione e di uno shear rotazionale. Sotto la scelta di alcune condizioni al bordo, si dimostra come nel caso neo-Hookeano lo shear rotazionale è inessenziale indipendentemente se la torsione è presente. Se il materiale invece è idealizzato essere un materiale di Mooney-Rivlin, lo shear rotazionale nel caso di torsione non nulla è strettamente necessario. Applicando il campo di stress, trovato nel caso neo-Hookeano, all'estrazione di un tappo di una bottiglia di vino, congetturiamo infine che è richiesta più forza a "tirare" solamente che "tirare e torcere".

La tesi termina con un Sesto Capitolo nel quale una nuova applicazione del metodo semi-inverso è discussa (vedi anche [26]). La celebre formula di Eulero sull'instabilità in "buckling" trova il valore critico della forza assiale per un cilindro "snello" che diviene instabile. La sua derivazione poggia sull'assunzione di elasticità lineare e che la "snellezza" del cilindro sia infinitesima. Considerando un ordine in più per il paremetro che misura la "snellezza" del cilindro, troviamo la prima correzione non lineare alla formula di Eulero. Per fare questo, specializziamo le soluzioni esatte dell'elasticità nonlineare per la compressione omogenea di un cilindro "spesso" con estremi lubrificati all'interno della teoria dell'elasticità del terzo ordine. Questo esempio è particolarmente interessante perchè prevede l'utilizzo di una metodologia generale, anche se in un certo senso approssimata, che può essere applicata in diversi contesti.

Questi risultati dimostrano ancora una volta come la teoria dell'elasticità sia un argomento complesso dove è difficile scegliere le semplificazioni ragionevoli. I risultati ottenuti hanno inoltre un loro significato applicativo in ambito biomeccanico che sarà argomento delle nostre prossime ricerche.

Résumé

La biomécanique des tissus mous est récemment devenue un sujet de recherche important dans nombreux domaines de l'ingénierie, y compris en bio-médicine et en mathématique. Les tissus mous sont des matériaux biologiques qui peuvent subir des déformations importantes (dans les régimes physiologiques et pathologiques) et qui présentent clairement un comportement mécanique nonlinéaire. Dans ce contexte, l'étude des déformations en s'appuyant sur des méthodes de calcul numérique, comme les éléments finis, peut être s'avérer compliquée. En effet, il est difficile de connaître avec certitude les équations constitutives “exactes” capables de décrire le comportement du matériau et les logiciels commerciaux sont souvent insuffisants pour aborder avec certitude la résolution des équations nonlinéaires correspondantes. La nonlinéarité géométrique de ces modèles complique grandement la réalité physique du problème et l'intuition de l'ingénieur est souvent peu utile si elle n'est pas accompagnée par l'analyse mathématique détaillée et rigoureuse. Dans ce contexte, la possibilité d'avoir des solutions exactes simples pour les équations du champ est un outil important et privilégié pour nous aider à comprendre plusieurs phénomènes biomécaniques.

La méthode semi-inverse est un des rares outils à notre disposition pour obtenir des solutions exactes dans la théorie mathématique de la mécanique des milieux continus. La méthode semi-inverse a déjà été utilisée de manière systématique par les fondateurs de la théorie de l'élasticité linéaire (on pense aux célèbres solutions de Saint Venant [5, 6]); malheureusement, cette utilisation a toujours été employée d'une manière heuristique et complètement détachée d'une méthodologie générale.

Essentiellement, le but de la méthode semi-inverse est d'établir a priori un certain nombre d'hypothèses concernant les champs inconnus dans une théorie donnée et de réduire les équations génerales de l'équilibre à des sous-ensembles simplifiés d'équations. Ici, simplifier signifie généralement que les équations de l'équilibre sont réduites à un système d'équations différentielles plus faciles (par exemple en partant d'un système d'équations différentielles aux dérivées partielles, on peut obtenir un système d'équations différentielles ordinaires, voir [90]).

Cette thèse, qui se développe en six chapitres, étudie divers aspects de cette méthode et aussi d'autres méthodes, dans un certain sens, connexes. Les premiers chapitres sont introductifs et généraux, alors que les suivants présentent les résultats nouveaux obtenus pendant mon doctorat ([26, 27, 28]).

Le Premier Chapitre est consacré aux définitions, symboles et concepts de base de la thorie non-linéaire de l'élasticité. Ce chapitre définit la cinématique des déformations finies par l'introduction des notions de corps déformable et de déformation. Nous passons ensuite aux équations de bilan, à la définition des contraintes et à la formulation des équations du mouvement. Puis nous abordons les concepts constitutifs comme la notion d'isotropie matérielle et le concept d' hyperélasticité. Nous analysons les restrictions imposées sur des modèles mathématiques pour assurer un comportement mécanique raisonnable, comme les inégalités de Truesdell et Noll.

Le Deuxième Chapitre expose certaines lois constitutives pour les matériaux hyperélastiques. Un des principaux problèmes rencontrés dans les applications en mécanique des milieux continus concerne le choix de modèles pour la fonction d'énergie potentielle, permettant de mieux décrire un comportement mécanique des materiaux "réels". Nous décrivons ici certains modèles (pour materiaux compressibles comme incompressibles) qui sont souvent utilisés dans la littérature, y compris: le modèle néo-Hookéen, le modèle Mooney-Rivlin, le modèle néo-Hookéen généralisé, le modèle d'Hadamard, le modèle de Blatz-ko, et finalement une fonction d'energie potentielle obtenue comme expansion en termes d'invariants du tenseur de Green-Lagrange, et utile pour des déformations finies mais modérées.

Le Troisième Chapitre présente un aperçu de l'utilisation de la méthode semi-inverse en élasticité. Nous exposons des exemples qui pourraient être considérés comme les plus représentatifs et/ou importants, et nous mettons en évidence leurs forces et leurs faiblesses. Nous appliquons la méthode inverse dans la recherche de solutions universelles dans le cas compressible (où les seules déformations possibles sont homogènees, [34]) comme dans le cas incompressible (où, en plus des déformations homogènes, existent cinq autres "familles" de solutions universelles).

Le résultat de Ericksen [34] montre qu'il n'y a pas d'autres déformations finies autres qu' homogènes qui soient contrôlables pour tous les matériaux compressibles. L'impact de ce résultat sur la théorie de l'élasticité non-linéaire a été fondamental. Pendant de nombreuses années, on a eu "la fausse impression que les seules déformations possibles pour un corps élastique sont celles qui sont universelles" (voir [25]). À la mème époque que celle de la publication des résultats de Ericksen, une activité considérable de recherche était en cours pour essayer de trouver des solutions en utilisant la méthode semi-inverse. La contrainte d'incompressibilité a facilité la recherche de solutions exactes par rapport aux matériaux compressibles, où il a été possible de trouver des solutions exactes qui ne soient pas universelles.

Ces dernières années, s'est développé un grand intérèt pour la possibilité de trouver des classes de solutions exactes pour les solides compressibles. Une des stratégies utilisées pour trouver des solutions exactes dans ce dernier cas est de s'inspirer des solutions non-homogènes pour matériaux élastiques incompressibles et de rechercher des solutions similaires pour les matériaux compressibles. Dans le Chapitre Quatre nous nous intéressons précisément aux résultats obtenus pour les matériaux compressibles dans cette ligne de recherche. Il s'agit de déterminer les matériaux compressibles qui peuvent soutenir des déformations isochores comme la "torsion pure", le "cisaillement axial pur" et le "cisaillement de rotation pur". Nous pensons que ces lignes de recherche peuvent être très trompeuses. Pour illustrer nos arguments, nous avons considéré des petites perturbations sur certaines classes de matériaux compressibles capables de supporter une certain deformation isochore particulière. Il s'ensuit que même si la perturbation peut être considérée comme étant petite, le changement de volume ne peut cependant pas être négligeable. Nous soulignons par conséquent qu'il n'est pas important de comprendre quels materiaux isotropes élastiques et compressibles peuvent subir par exemple une torsion pure et isochore, parce que dans de tels matériaux n'existent pas, mais plutôt de comprendre la géométrie qui accompagne l'action d'un couple dans un cylindre qui est idéalisé comme élastique et isotrope. C'est uniquement de cette façon que les résultats obtenus avec la méthode semi-inverse peuvent être compris d'une manière approfondie.

Parmi les exemples d'application de la méthode semi-inverse nous rapportons la recherche de solutions à la déformation de "cisaillement anti-plan" et à la déformation "radiale". Dans le cas incompressible nous savons que les équations de bilan, pour n'importe quel solide élastique, sont compatibles avec l'hypothèse de cisaillement anti-plan seulement dans le cas de symmétrie cylindrique. Nous ne pouvons pas progresser lorsque la géométrie du corps est plus générale, parce qu'alors, les équations d'équilibre sont réduites à un système surdéterminé qui n'est pas toujours compatible. Cela signifie qu'en général, la déformation de cisaillement anti-plan doit être couplée avec une déformation secondaire. Donc même si les conditions aux limites sont compatibles avec une deformation de cisaillement anti-plan, celle-ci ne peut pas être pure pour être admissible. Automatiquement dans le corps on a créé un état de contrainte complexe. Rechercher des modèles spéciaux pour lesquels cet état de contraintes est absent, ne peut pas vraiment nous aider comprendre ce qui se passe dans la réalité.

Dans le Cinquième Chapitre, après avoir brièvement présenté les résultats déjà obtenus dans la littérature sur les déformations latentes (voir [39, 63, 83]), nous présentons un nouvel exemple analytique de la question (voir aussi [27]). En fait nous considérons un champ de déformation complexe pour un cylindre elastique non-linéaire isotrope et incompressible: une combinaison d'une inflation, d'une torsion, et d'un cisaillement hélicoïdal. Avec le choix de certaines conditions aux limites, nous montrons que dans le cas néo-Hookéen le cisaillement de rotation est inessentiel, peu importe si la torsion est présente. Si le matériau est idealisé comme un modèle de Mooney-Rivlin, alors il faut avoir nécessairement le cisaillement de rotation avec la torsion non nulle. Avec l'application à la mécanique de l'extraction d'un bouchon d'une bouteille de vin, enfin, nous conjecturons qu' il faut nécessite plus de force pour "tirer" seulement que "tirer et tordre".

La thèse se termine par un Sixième Chapitre dans lequel une nouvelle application de la méthode semi-inverse est discutée (voir aussi [26]). La célèbre formule d'Euler sur l'instabilité en "flambage" trouve la valeur critique de la force axiale d'un cylindre "svelte" instable. Ce calcul est basé sur l'hypothèse d'une élasticité linéaire, où la finesse du cylindre est infinitésimale. Considérant un ordre supérieur pour la "minceur", nous trouvons une première correction non-linéaire à la formule d'Euler. Á cette fin, nous spécialisons les solutions exactes de l'élasticité non-linéaire pour la compression homogène d'un cylindre "épais" avec extrémités lubrifiées à la théorie de l'élasticité de troisième ordre. Cet exemple est particulièrement intéressant car il implique l'utilisation d'une méthodologie générale, bien que dans un certain sens approximative, qui peut être appliquée dans différents contextes.

Ces résultats démontrent une fois de plus que la théorie de l'élasticité est un sujet complexe, où est difficile choisir des simplifications raisonnable. Les résultats obtenus ont aussi une leur importance dans la biomécanique, qui sera l'objet de

notre prochaine recherche.

Chapter 1

Introduction to Elasticity

This introductory chapter presents some basic concepts of continuum mechanics, symbols and notations for future reference.

1.1 Kinematics of finite deformations

We call $\mathcal{B}$ a *material body*, defined to be a three-dimensional differentiable manifold, the elements of which are called *particles* (or *material points*) P. This manifold is referred to a system of co-ordinates which establishes a one-to-one correspondence between particles and a region B (called a *configuration* of $\mathcal{B}$) in three-dimensional Euclidean space by its position vector $\boldsymbol{X}(P)$. As the body deforms, its configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, and associate a unique B_t, the configuration at time t of $\mathcal{B}$; then the one-parameter family of all configurations $\{B_t : t \in I\}$ is called a *motion* of $\mathcal{B}$.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration at some prescribed time r. Then we label by $\boldsymbol{X}$ any particle P of $\mathcal{B}$ in B_r and by $\boldsymbol{x}$ the position vector of P in the configuration B_t (called *current configuration*) at time t. Since B_r and B_t are configurations of $\mathcal{B}$, there exists a bijection mapping $\boldsymbol{\chi} : B_r \to B_t$ such that

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}). \tag{1.1}$$

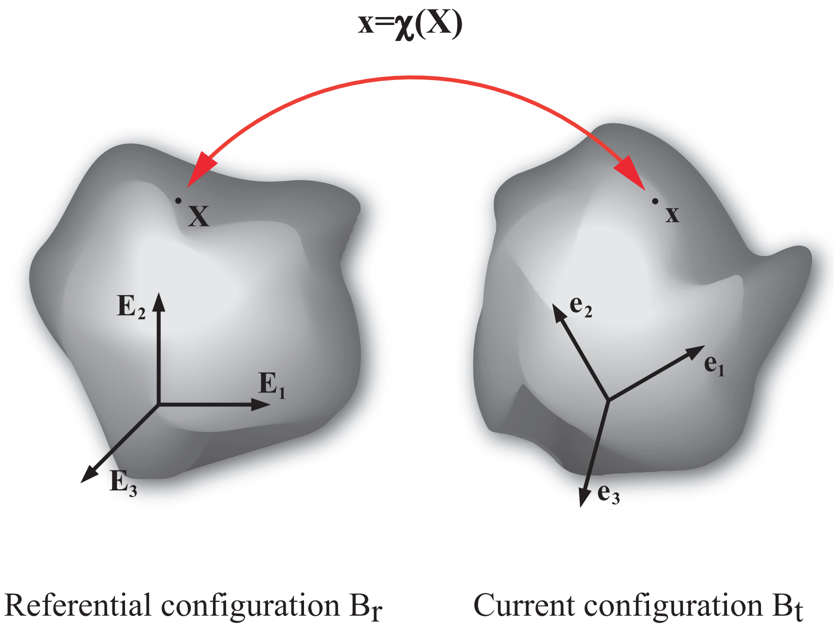
The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body from B_r to B_t and since the latter depends on t, we write

$$\boldsymbol{x} = \boldsymbol{\chi}_t(\boldsymbol{X}) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}_t^{-1}(\boldsymbol{x}), \tag{1.2}$$

instead of (1.1), or equivalently,

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), \tag{1.3}$$

for all $t \in I$. For each particle P (with label $\boldsymbol{X}$), $\boldsymbol{\chi}_t$ describes the motion of P with t as parameter, and hence the motion of $\mathcal{B}$. We assume that a sufficient number of derivatives of $\boldsymbol{\chi}_t$ (with respect to position and time) exists and that they are continuous.



The *velocity* $\boldsymbol{v}$ and the *acceleration* $\boldsymbol{a}$ of a particle P are defined as

$$\boldsymbol{v} \equiv \dot{\boldsymbol{x}} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\boldsymbol{X}, t) \tag{1.4}$$

and

$$\boldsymbol{a} \equiv \dot{\boldsymbol{v}} \equiv \ddot{\boldsymbol{x}} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\boldsymbol{X}, t), \tag{1.5}$$

respectively, where the superposed dot indicates differentiation with respect to t at fixed $\boldsymbol{X}$, i.e. the material time derivative.

We assume that the body is a contiguous collection of particles; we call this body a *continuum* and we define the *deformation gradient tensor* $\boldsymbol{F}$ as a second-order tensor,

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \text{Grad } \boldsymbol{x} \equiv \text{Grad } \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{1.6}$$

Here and henceforth, we use the notation Grad, Div, Curl (respectively grad, div, curl) to denote the gradient, divergence and curl operators in the reference (respectively, current) configuration, i.e with respect to $\boldsymbol{X}$ (respectively, $\boldsymbol{x}$).

We introduce the quantity

$$J = \det \boldsymbol{F} \tag{1.7}$$

and assume that $J \neq 0$, in order to have $\boldsymbol{F}$ invertible, with inverse

$$\boldsymbol{F}^{-1} = \text{grad } \boldsymbol{X}. \tag{1.8}$$

In general the deformation gradient $\boldsymbol{F}$ depends on X, i.e. varies from point to point and such deformation is said to be *inhomogeneous*. If, on the other hand, $\boldsymbol{F}$ is independent of $\boldsymbol{X}$ for the body in question then the deformation is said to be *homogeneous*. If the deformation is such that there is no change in volume, then the deformation is said to be *isochoric*, and

$$J \equiv 1. \tag{1.9}$$

A material for which (1.9) holds for all deformations is called an *incompressible material.*

The polar decomposition theorem of linear algebra applied to the nonsingular tensor $\boldsymbol{F}$ gives two unique multiplicative decompositions:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \quad \text{and} \quad \boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}, \tag{1.10}$$

where $\boldsymbol{R}$ is the *rotation tensor* (and characterizes the local rigid body rotation of a material element), $\boldsymbol{U}$ is the *right stretch tensor*, and $\boldsymbol{V}$ is the *left stretch tensor* of the deformation ($\boldsymbol{U}$ and $\boldsymbol{V}$ describe the local deformation of the element). Using this decomposition for $\boldsymbol{F}$, we define two tensor measures of deformation called the *left* and *right Cauchy-Green strain tensors*, respectively, by

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}^2, \qquad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2. \tag{1.11}$$

The couples ($\boldsymbol{U}$,$\boldsymbol{V}$) and ($\boldsymbol{B}$,$\boldsymbol{C}$) are *similar* tensors, that is, they are such that

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T, \quad \boldsymbol{B} = \boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T, \tag{1.12}$$

and therefore $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same principal values $\lambda_1, \lambda_2, \lambda_3$, say, and $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same principal values $\lambda_1^2, \lambda_2^2, \lambda_3^2$. Their respective principal directions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are related by the rotation $\boldsymbol{R}$,

$$\boldsymbol{\nu} = \boldsymbol{R}\boldsymbol{\mu}. \tag{1.13}$$

The λ's are the stretches of the three principal material lines; they are called *principal stretches.*

1.2 Balance laws, stress and equations of motion

Let A_r, in the reference configuration, be a set of points occupied by a subset $\mathcal{A}$ of a body $\mathcal{B}$. We define a function m called a *mass function* in the following way

$$m(A_r) = \int_{A_r} \rho_r \, \mathrm{d}V, \tag{1.14}$$

where ρ_r is the density of mass per unit volume V. In the current configuration, the mass of A_t is calculated as

$$m(A_t) = \int_{A_t} \rho \, \mathrm{d}v, \tag{1.15}$$

where in this case ρ is the density of mass per unit volume v. The *local mass conservation law* is expressed by

$$\rho = J^{-1}\rho_r, \tag{1.16}$$

or equivalentely in the form

$$\dot{\rho} + \rho \mathrm{div}\boldsymbol{v} = 0. \tag{1.17}$$

This last form of mass conservation equation is also known as the *continuity equation.*

The forces that act on any part $A_t \subset B_t$ of a continuum $\mathcal{B}$ are of two kinds: a distribution of contact forces, which we denote $\boldsymbol{t_n}$ per unit area of the boundary ∂A_t of A_t, and a distribution of body forces, denoted $\boldsymbol{b}$ per unit volume of A_t. Applying the *Cauchy theorem*, we know that there exists a second-order tensor called the *Cauchy stress tensor*, which we denote $\boldsymbol{T}$, such that

(i) for each unit vector $\boldsymbol{n}$,

$$\boldsymbol{t_n} = \boldsymbol{T}\boldsymbol{n}, \tag{1.18}$$

where $\boldsymbol{T}$ is independent of $\boldsymbol{n}$,

(ii)

$$\boldsymbol{T}^T = \boldsymbol{T}, \tag{1.19}$$

and

(iii) $\boldsymbol{T}$ satisfies the *equation of motion*,

$$\text{div}\boldsymbol{T} + \rho\boldsymbol{b} = \rho\boldsymbol{a}. \tag{1.20}$$

Often, the Cauchy stress tensor is inconvenient in solid mechanics because the deformed configuration generally is not known *a priori.* Conversely, it is convenient to use the material description. To this end, we introduce the *engineering stress tensor* $\boldsymbol{T_R}$, also known as the *first Piola-Kirchhoff stress tensor*, in order to define the contact force distribution $\boldsymbol{t_N} \equiv \boldsymbol{T_R}\boldsymbol{N}$ in the reference configuration

$$\boldsymbol{T_R} = J\boldsymbol{T}\boldsymbol{F}^{-T}. \tag{1.21}$$

It is then possible to rewrite the balance laws corresponding to (1.18), (1.19) and (1.20), in the following form

$$\boldsymbol{t_N} = \boldsymbol{T_R}\boldsymbol{N}, \tag{1.22}$$

$$\boldsymbol{T_R}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{T_R}^T, \tag{1.23}$$

$$\text{Div}\boldsymbol{T_R} + \rho_r\boldsymbol{b}_r = \rho_r\ddot{\boldsymbol{x}}, \tag{1.24}$$

where $\boldsymbol{b}_r$ denotes the body force per unit volume in the reference configuration.

1.3 Isotropy and hyperelasticity: constitutive laws

We call *nominal stress tensor* the transpose of $\boldsymbol{T_R}$ that we denote by

$$\boldsymbol{S} = \boldsymbol{T_R}^T \tag{1.25}$$

and we call *hyperelastic* a solid whose elastic potential energy is given by the strain energy function $W(\boldsymbol{F})$ and such that

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}), \tag{1.26}$$

holds, relating the nominal stress and the deformation, or equivalently, such that

$$\boldsymbol{T} = J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}^T \boldsymbol{F}^T, \tag{1.27}$$

relating the Cauchy stress and the deformation. In component form (1.26) and (1.27) read, respectively,

$$S_{ji} = \left(\frac{\partial W}{\partial F_{ij}}\right), \qquad T_{ij} = J^{-1} \frac{\partial W}{\partial F_{i\alpha}} F_{j\alpha}. \tag{1.28}$$

A material having the property that at a point $\boldsymbol{X}$ of undistorted state, every direction is an axis of material symmetry, is called *isotropic at* $\boldsymbol{X}$. A hyperelastic material which is isotropic at every material point in a global undistorted material is called an *isotropic hyperelastic material*; in this case, the strain energy density function can be expressed uniquely as a symmetric function of the principal stretches or in terms of the principal invariants I_1, I_2, I_3 of $\boldsymbol{B}$ (or equivalently, the principal invariants of $\boldsymbol{C}$, because in the isotropic case they coincide for every deformation $\boldsymbol{F}$), or in terms of the principal invariants i_1, i_2, i_3 of $\boldsymbol{V}$. Thus,

$$W = \hat{W}(\lambda_1, \lambda_2, \lambda_3) = \bar{W}(I_1, I_2, I_3) = \tilde{W}(i_1, i_2, i_3), \tag{1.29}$$

say, where

$$I_1 = \mathrm{tr}\boldsymbol{B}, \quad I_2 = \tfrac{1}{2}[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}\boldsymbol{B}^2], \quad I_3 = \det \boldsymbol{B}. \tag{1.30}$$

The principal invariants I_1, I_2, I_3 of $\boldsymbol{B}$ are given in terms of the principal stretches by

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{1.31}$$

The principal invariants of $\boldsymbol{V}$ (and hence of $\boldsymbol{U}$), i_1, i_2, i_3, are given by:

$$\begin{aligned} i_1 &= \mathrm{tr}\boldsymbol{V} = \lambda_1 + \lambda_2 + \lambda_3, \\ i_2 &= \tfrac{1}{2}[i_1^2 - \mathrm{tr}\boldsymbol{V}] = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \\ i_3 &= \det\boldsymbol{V} = \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{1.32}$$

The principal invariants of $\boldsymbol{B}$, given in (1.31), are connected with the principal invariants of $\boldsymbol{V}$ given in (1.32) by the relations

$$I_1 = i_1^2 - 2i_2, \qquad I_2 = i_2^2 - 2i_1 i_3, \qquad I_3 = i_3^2. \tag{1.33}$$

It is usual to require (for convenience) that the strain-energy function W should vanish in the reference configuration, where $\boldsymbol{F} = \boldsymbol{I}$, $I_1 = I_2 = 3$, $I_3 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Thus,

$$\bar{W}(3, 3, 1) = 0, \qquad \hat{W}(1, 1, 1) = 0. \tag{1.34}$$

After some algebraic manipulations, follow two useful forms for the general constitutive equation, which we write as

$$\boldsymbol{T} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^2, \tag{1.35}$$

or, using the Cayley-Hamilton theorem, as

$$\boldsymbol{T} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_{-1} \boldsymbol{B}^{-1}, \tag{1.36}$$

where

$$\alpha_i = \alpha_i(I_1, I_2, I_3), \qquad \beta_j = \beta_j(I_1, I_2, I_3), \tag{1.37}$$

$i = 0, 1, 2$; $j = 0, 1, -1$, are called *the material* or *elastic response functions*. In terms of the strain energy function they are given by

$$\begin{aligned} \beta_0(I_1, I_2, I_3) &= \alpha_0 - I_2\alpha_2 = \frac{2}{\sqrt{I_3}}\left[I_2\frac{\partial W}{\partial I_2} + I_3\frac{\partial W}{\partial I_3}\right], \\ \beta_1(I_1, I_2, I_3) &= \alpha_1 + I_1\alpha_2 = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1}, \\ \beta_{-1}(I_1, I_2, I_3) &= I_3\alpha_2 = -2\sqrt{I_3}\frac{\partial W}{\partial I_2}. \end{aligned} \tag{1.38}$$

When the hyperelastic isotropic material is also incompressible, it is possible to rewrite (1.35) and (1.36) as

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^2, \tag{1.39}$$

and

$$\boldsymbol{T} = -p\boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_{-1} \boldsymbol{B}^{-1}, \tag{1.40}$$

respectively, where p is an undetermined scalar function of $\boldsymbol{x}$ and t (p is a Lagrange multiplier). The undetermined parameter p differs in (1.39) and (1.40) by a $2I_2(\partial W/\partial I_2)$ term. Then the material response coefficients $\alpha_i = \alpha_i(I_1, I_2)$ and $\beta_j = \beta_j(I_1, I_2)$ with $i = 1, 2$ and $j = 1, -1$ are defined respectively by

$$\beta_1 = \alpha_1 + I_1\alpha_2 = 2\frac{\partial W}{\partial I_1}, \qquad \beta_{-1} = \alpha_2 = -2\frac{\partial W}{\partial I_2}. \tag{1.41}$$

We say that a body $\mathcal{B}$ is *homogeneous* if it is possible to choose a single reference configuration B_r of the whole body so that the response functions are the same for all particle.

The formulae (1.35), (1.36), (1.39) and (1.40), may be replaced by any other set of three independent symmetric invariants, for example by i_1, i_2, i_3, the principal invariants of $\boldsymbol{V}$. When the strain energy function W depends by the principal stretches, the principal Cauchy stress components (that we denote by T_i, $i = 1, 2, 3$) are given by

$$T_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i} \tag{1.42}$$

for compressible materials, and by

$$T_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \tag{1.43}$$

for incompressible materials.

1.4 Restrictions and empirical inequalities

The response functions β_j are not completely arbitrary but must meet some requirements. First of all, if we ask our compressible (incompressible) model to be stress free in the reference configuration, then they must satisfy

$$\bar{\beta}_0 + \bar{\beta}_1 + \bar{\beta}_{-1} = 0, \quad (-\bar{p} + \bar{\beta}_1 + \bar{\beta}_{-1} = 0), \tag{1.44}$$

where $\bar{\beta}_j = \beta(3,3,1)$ (and $\bar{p} = p(3,3,1)$) are the values of the material functions (1.37) in the reference configuration. In general (to have hydrostatic stress $\boldsymbol{T}_0$) they must satisfy

$$\boldsymbol{T}_0 = (\bar{\beta}_0 + \bar{\beta}_1 + \bar{\beta}_{-1})\boldsymbol{I}, \quad (\boldsymbol{T}_0 = (-\bar{p}_0 + \bar{\beta}_1 + \bar{\beta}_{-1})\boldsymbol{I}). \tag{1.45}$$

The question of what other restrictions should be imposed in general on the strain energy functions of hyperelasticity theory, in order to capture the actual physical behavior of isotropic materials in finite deformation is of no less importance, and forms the substance of *Truesdell's problem*. To model *real* material behavior, we assume that the response functions β_j are compatible with fairly general empirical descriptions of mechanical response, derived from carefully controlled large deformation tests of isotropic materials. To this end we assume that the empirical inequalities imposed by Truesdell and Noll hold (see [127]). They are, in the compressible case,

$$\beta_0 \leq 0, \quad \beta_1 > 0, \quad \beta_{-1} \leq 0, \tag{1.46}$$

and in the incompressible case,

$$\beta_1 > 0, \quad \beta_{-1} \leq 0. \tag{1.47}$$

1.5 Linear elasticity and other specializations

In the special case of *linear* (linearized) elasticity, some constitutive restrictions must be considered also in order to reflect the real behavior of the material, and these restrictions lead to some important assumptions on the physical constants. Hence, let $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ be the mechanical displacement. In the case of small strains, the linear theory of elasticity is based on the following equations

$$\boldsymbol{T} = \boldsymbol{\mathcal{C}}[\boldsymbol{\epsilon}], \tag{1.48}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right), \tag{1.49}$$

$$\mathrm{Div}\boldsymbol{T} + \boldsymbol{b}_r = \rho\ddot{\boldsymbol{u}}, \tag{1.50}$$

where $\boldsymbol{\epsilon}$ denotes the *infinitesimal strain tensor* and $\boldsymbol{\mathcal{C}}$ the fourth-order tensor of elastic stiffness. These three equations represent the *stress-strain law*, *strain-displacement relation*, and the *equation of motion*, respectively. When the body is homogeneus and isotropic, the constitutive equation (1.48) reduces to

$$\boldsymbol{T} = 2\mu\boldsymbol{\epsilon} + \lambda(\mathrm{tr}\boldsymbol{\epsilon})\boldsymbol{I}, \tag{1.51}$$

where μ and λ are the so-called *Lamé constants* or, in the inverted form,

$$\boldsymbol{\epsilon} = \frac{1}{E}\left[(1+\nu)\boldsymbol{T} - \nu(\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\right], \tag{1.52}$$

where

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \quad \nu = \frac{\lambda}{2(\mu+\lambda)}. \tag{1.53}$$

The second Lamé constant μ determines the response of the body in shear, at least within the linear theory, and for this reason is called the *shear modulus*. The constant E is known as *Young's modulus*, the constant ν as Poisson's ratio, and the quantity $\kappa = (2/3)\mu + \lambda$ as the *modulus of compression* or *bulk modulus*.

A linearly elastic solid should increase its length when pulled, should decrease its volume when acted on by a pure pressure, and should respond to a positive shearing strain by a positive shearing stress. These restrictions are equivalent to either sets of inequalities

$$\mu > 0, \quad \kappa > 0; \tag{1.54}$$

$$E > 0, \quad -1 < \nu \leq 1/2. \tag{1.55}$$

In the incompressible case, the constitutive equation (1.51) is replaced by

$$\boldsymbol{T} = 2\mu\boldsymbol{\epsilon} - p\boldsymbol{I}, \tag{1.56}$$

in which p is an arbitrary scalar function of $\boldsymbol{x}$ and t, independent of the strain $\boldsymbol{\epsilon}$. In the limit of incompressibility ($\mathrm{tr}\boldsymbol{\epsilon} \to 0$)

$$\kappa \to \infty, \quad \lambda \to \infty, \quad \mu = \frac{E}{3}, \quad \nu \to \frac{1}{2} \tag{1.57}$$

so that the strain-stress relation (1.52) becomes

$$\boldsymbol{\epsilon} = \frac{1}{2E}\left[3\boldsymbol{T} - (\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\right]. \tag{1.58}$$

The components of the strain tensor (1.49) must satisfy the *compatibility conditions* of Saint Venant, which can be written in terms of the strain components as

$$\epsilon_{ij,hk} + \epsilon_{hk,ij} - \epsilon_{ik,jh} + \epsilon_{jh,ik} = 0, \tag{1.59}$$

where $i, j, h, k = 1, 2, 3$ and $\epsilon_{ij,hk} = \partial^2\epsilon_{ij}/(\partial x_h \partial x_k)$. Writing (1.59) in full, the 81 possible equations reduce to six essential equations, which are

$$2\epsilon_{12,12} = \epsilon_{11,22} + \epsilon_{22,11}, \tag{1.60}$$

and a further two by cyclic exchanges of indices, and

$$\epsilon_{11,23} = \left(\epsilon_{12,3} + \epsilon_{31,2} - \epsilon_{23,1}\right)_1, \tag{1.61}$$

and a further two by cyclic exchanges of indices. Introducing the equations (1.52) and (1.50) into the compatibility conditions (1.59) in the isotropic and homogeneous case, we obtain *Michell's equations*

$$T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{i,j}b_{k,k} - (b_{i,j} + b_{j,i}), \tag{1.62}$$

or *Beltrami's* simpler *equations*, in the case of no or constant body forces,

$$T_{ij,kk} + \frac{1}{1+\nu} T_{kk,ij} = 0. \tag{1.63}$$

Let us consider the shear modulus $\mu > 0$ and the bulk modulus $\kappa > 0$ and go back to the hyperelastic case. For consistency with the linearized isotropic elasticity theory, the strain-energy function must satisfy

$$\begin{aligned} &\bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = 0, \\ &\bar{W}_{11} + 4\bar{W}_{12} + 4\bar{W}_{22} + 2\bar{W}_{13} + 4\bar{W}_{23} + \bar{W}_{33} = \frac{\kappa}{4} + \frac{\mu}{3}, \end{aligned} \tag{1.64}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i$, $\bar{W}_{ij} = \partial^2\bar{W}/(\partial I_i \partial I_j)$ $(i,j = 1,2,3)$ and the derivatives are evaluated for $I_1 = I_2 = 3$, and $I_3 = 1$. We can observe that $(1.64)_1$ is equivalent to (1.44). The analogues of (1.64) for $\tilde{W}(i_1, i_2, i_3)$ are

$$\begin{aligned} &\tilde{W}_1 + 2\tilde{W}_2 + \tilde{W}_3 = 0, \\ &\tilde{W}_{11} + 4\tilde{W}_{12} + 4\tilde{W}_{22} + 2\tilde{W}_{13} + 4\tilde{W}_{23} + \tilde{W}_{33} = \kappa + \frac{4}{3}\mu, \end{aligned} \tag{1.65}$$

where $\tilde{W}_i = \partial\tilde{W}/\partial i_i$, $\tilde{W}_{ij} = \partial^2\tilde{W}/(\partial i_i \partial i_j)$ $(i,j = 1,2,3)$ and the derivatives are evaluated for $i_1 = i_2 = 3$, and $i_3 = 1$. If instead of (1.64) and (1.65), the form $\hat{W}(\lambda_1, \lambda_2, \lambda_3)$ of the strain energy function is considered, then it must satisfy

$$\begin{aligned} &\hat{W}_i(1,1,1) = 0 \\ &\hat{W}_{ij}(1,1,1) = \kappa - \frac{2}{3}\mu \ (i \neq j), \quad \hat{W}_{ii} = \kappa + \frac{4}{3}\mu, \end{aligned} \tag{1.66}$$

where, in the latter, no summation is implied by the repetition of the index i, the notation $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$, $\hat{W}_{ij} = \partial^2\hat{W}/\partial(\lambda_i\partial\lambda_j)$ $(i,j = 1,2,3)$ is adopted, and the derivatives are evaluated for $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

1.6 Incremental elastic deformations

Let us consider the deformation of a body $\mathcal{B}$ relative to a given reference configuration $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ and then suppose that the deformation is changed to $\boldsymbol{x}' = \boldsymbol{\chi}'(\boldsymbol{X})$. The displacement of a material particle due to this change is $\dot{\boldsymbol{x}}$ say, defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{\chi}'(\boldsymbol{X}) - \boldsymbol{\chi}(\boldsymbol{X}) \equiv \dot{\boldsymbol{\chi}}(\boldsymbol{X}), \tag{1.67}$$

and its gradient is

$$\mathrm{Grad}\dot{\boldsymbol{\chi}} = \mathrm{Grad}\boldsymbol{\chi}' - \mathrm{Grad}\boldsymbol{\chi} \equiv \dot{\boldsymbol{F}}. \tag{1.68}$$

When $\dot{\boldsymbol{x}}$ is expressed as a function of $\boldsymbol{x}$ we call it the *incremental mechanical displacement*, $\boldsymbol{u} = \dot{\boldsymbol{x}}(\boldsymbol{x})$. For a compressible hyperelastic material (1.26), the associated nominal stress difference is

$$\dot{\boldsymbol{S}} = \boldsymbol{S}' - \boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}') - \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}), \tag{1.69}$$

which has the linear approximation

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}}\dot{\boldsymbol{F}}, \tag{1.70}$$

where $\boldsymbol{\mathcal{A}}$ is the fourth-order tensor of elastic moduli, with components

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}. \tag{1.71}$$

The component form of (1.70) is

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \tag{1.72}$$

which provides the convention for defining the product appearing in (1.70). The corresponding form of (1.70) for incompressible materials is

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}}\dot{\boldsymbol{F}} - \dot{p}\boldsymbol{F}^{-1} + p\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \tag{1.73}$$

where $\dot{p}$ is the increment of p and $\boldsymbol{\mathcal{A}}$ has the same form as in (1.71). Equation (1.73) is coupled with the incremental form of the incompressibility constraint (1.9),

$$\mathrm{tr}(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) = 0. \tag{1.74}$$

From the equilibrium equation (1.24) and its counterpart for $\boldsymbol{\chi}'$, we obtain by subtraction the equations of static equilibrium in absence of body forces,

$$\mathrm{Div}\dot{\boldsymbol{S}}^T = \boldsymbol{0}, \tag{1.75}$$

which does not involve approximation. In its linear approximation, $\dot{\boldsymbol{S}}$ is replaced by (1.70) or (1.73) with (1.74). When the displacement boundary conditions on ∂B_r are prescribed, the incremental version is written as

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{\xi}} \quad \text{on } \partial B_r \tag{1.76}$$

or in the case of tractions boundary conditions (1.22), as

$$\dot{\boldsymbol{S}}^T\boldsymbol{N} = \dot{\boldsymbol{\tau}} \quad \text{on } \partial B_r, \tag{1.77}$$

where $\dot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\tau}}$ are the prescribed data for the incremental deformation $\dot{\boldsymbol{\chi}}$. It is often convenient to use the deformed configuration B_t as the reference configuration instead of the initial configuration B_r and one needs therefore to treat all incremental quantities as functions of $\boldsymbol{x}$ instead of $\boldsymbol{X}$. Making use of the following definitions

$$\boldsymbol{u}(\boldsymbol{x}) = \dot{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}(\boldsymbol{x})), \quad \boldsymbol{\Gamma} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \quad \boldsymbol{\Sigma} = J^{-1}\boldsymbol{F}\dot{\boldsymbol{S}}, \tag{1.78}$$

and of the fourth-order (Eulerian) tensor $\boldsymbol{\mathcal{A}}_0$ of *instantaneous elastic moduli*, whose components are given in terms of those of $\boldsymbol{\mathcal{A}}$ by

$$\mathcal{A}_{0piqi} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}, \tag{1.79}$$

it follows that $\boldsymbol{\Gamma} = \text{grad}\boldsymbol{u}$ and the equilibrium equations (1.75) become

$$\text{div}\boldsymbol{\Sigma}^T = \mathbf{0}, \tag{1.80}$$

where for compressible materials

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma}, \tag{1.81}$$

and for incompressible materials

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma} + p\boldsymbol{\Gamma} - \dot{p}\boldsymbol{I}, \tag{1.82}$$

where now $J = 1$ in (1.79). The incompressibility constraint (1.74) takes the form

$$\text{tr}\boldsymbol{\Gamma} \equiv \text{div}\boldsymbol{u} = 0. \tag{1.83}$$

When the strain energy function W is given as a symmetrical function of the principal strains $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$, the non-zero components, in a coordinate system aligned with the principal axes of strain, are given in general by [95]

$$\begin{aligned}
&J\mathcal{A}_{0iijj} = \lambda_i\lambda_j\hat{W}_{ij}, && \\
&J\mathcal{A}_{0ijij} = (\lambda_i\hat{W}_i - \lambda_j\hat{W}_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2), && i \neq j,\ \lambda_i \neq \lambda_j, \\
&J\mathcal{A}_{0ijji} = (\lambda_j\hat{W}_i - \lambda_i\hat{W}_j)\lambda_i\lambda_j/(\lambda_i^2 - \lambda_j^2), && i \neq j,\ \lambda_i \neq \lambda_j, \\
&J\mathcal{A}_{0ijij} = (\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \lambda_i\hat{W}_i)/2, && i \neq j,\ \lambda_i = \lambda_j, \\
&J\mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - \lambda_i\hat{W}_i, && i \neq j,\ \lambda_i = \lambda_j,
\end{aligned} \tag{1.84}$$

(no sums), where $\hat{W}_{ij} \equiv \partial^2\hat{W}/(\partial\lambda_i\partial\lambda_j)$.

Notes

In this chapter we have only introduced some basic concepts, definitions, symbols and basic relationships of continuum mechanics in the field of elasticity. Although there is an extensive literature on the thermomechanics of elastomers, our setting here is purely isothermal and no reference is made to thermodynamics.

For literature on this introductory part, we refer mainly to: Atkin and Fox [4], Beatty [9], Gurtin [50], Holzapfel [57], Landau and Lifshitz [76], Leipholz [77], Ogden [95], Spencer [121] and Truesdell and Noll [127]. These books are an excellent survey of some selected topics in elasticity with an updated list of references.

In Truesdell and Noll [127] (see Section 43), a material is called *elastic* if it is *simple*[1] and if the stress at time t depends only on the local configuration at time t, and not on the entire past history of the motion. This means that the constitutive equation must be expressed as

$$\boldsymbol{T} = \mathcal{G}(\boldsymbol{F}), \tag{1.85}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{F}$ is the deformation gradient at the present time, taken with respect to a fixed but arbitrary, local reference configuration and

[1] A material is simple if and only if its response to any deformation history is known as soon as its response to all homogeneous irrotational histories is specified (see Section 29 in [127]).

$\mathcal{G}$ is the response function of the elastic material. It is important to point out that in recent years, Rajagopal [103, 104] asserted that this interpretation is much too restrictive and he illustrated his thesis by introducing implicit constitutive theories that can describe the *non-dissipative* response of solids. Hence, Rajagopal gives the constitutive equation for the mathematical model of an elastic material in the form

$$\mathcal{F}(\boldsymbol{F}, \boldsymbol{T}) = \boldsymbol{0}, \tag{1.86}$$

and in [104] gives some interesting conceptual and theoretical reasons to adopt implicit constitutive equations. In [103], Rajagopal and Srinivasa show that the class of solids that are incapable of dissipation is far richer than the class of bodies that is usually understood as being elastic.

In the last section of this chapter, we introduced the linearized equations for incremental deformations. They constitute the first-order terms associated with a formal perturbation expansion in the incremental deformation. The higher-order (nonlinear) terms are for example required for weakly nonlinear analysis of the stability of finitely deformed configurations, see Chapter 10 in [43]. For a discussion of the mathematical structure of the incremental equations, see [54]. Applications of the linearized incremental equations for interface waves in pre-stressed solids can be found in Chapter 3 of [30].

Chapter 2

Strain energy functions

The aims of *constitutive theories* are to develop mathematical models for representing the real behavior of matter, to determine the material response and in general, to distinguish one material from another. As described in the preceding chapter, constitutive equations for hyperelastic materials postulate the existence of a strain energy function W. There are several theoretical frameworks for the analysis and derivation of constitutive equations, for example the *Rivlin-Signorini method* where the governing idea is to expand the strain energy function in a power series of the invariants, or the *Valanis-Landel approach* expressing the strain energy directly in terms of the principal stretches [115].

In this chapter, we make no attempt at presenting these methods but instead, we present some classical explicit forms of strain-energy functions used in the literature for some isotropic hyperelastic materials. Many other models have been proposed (for example, a collection of constitutive models for rubber can be found in [32]).

2.1 Strain energy functions for incompressible materials

2.1.1 Neo-Hookean model

The *neo-Hookean model* is one of the simplest strain energy functions. It involves a single parameter and provides a mathematically simple and reliable constitutive model for the non-linear deformation behavior of isotropic rubber-like materials. Its strain energy function is

$$W = \frac{\mu}{2}(I_1 - 3), \tag{2.1}$$

where $\mu > 0$ is the shear modulus for infinitesimal deformations. The neo-Hookean model comes out of the molecular theory, in which vulcanized rubber is regarded as a three-dimensional network of long-chain molecules that are connected at a few points. The elementary molecular theory of networks is based on the postulate that the elastic free energy of a network is equal to the sum of the elastic free energies of the individual chains. In order to derive (2.1), a Gaussian distribution for the probability of the end-to-end vector of the single chain is also assumed. While in a phenomenological theory the constitutive parameters are dictated only by the functional form considered, in a molecular theory the parameters are introduced on the basis of the modeled phenomena and consequently, are related *ex ante* to physical quantities. In this framework the constitutive parameter μ is determined by micromechanics parameters, as

$$\mu = nkT, \tag{2.2}$$

where n is the chain density, k is the Boltzmann constant and T is the absolute temperature. Although it poorly captures the basic features of rubber behaviour, the neo-Hookean model is much used in finite elasticity theory because of its "good" mathematical properties (for example a huge number of exact solutions to boundary value problems may be found using this model).

2.1.2 Mooney-Rivlin model

To improve the fitting to data, Rivlin introduced a dependence of the strain energy function on both the first and second invariants. A slightly more general model than neo-Hookean is therefore a simple, or two-term, *Mooney-Rivlin model*, for which the strain energy function is assumed to be linear in the first and second invariant of the Cauchy-Green strain tensor. This model is of purely phenomenological origin, and was originally derived by Mooney [84]. The strain energy may be written as

$$W = \tfrac{1}{2}\left(\tfrac{1}{2} + \gamma\right)\mu(I_1 - 3) + \tfrac{1}{2}\left(\tfrac{1}{2} - \gamma\right)\mu(I_2 - 3), \tag{2.3}$$

where γ is a dimensionless constant in the range $-1/2 \leq \gamma \leq 1/2$ and $\mu > 0$ is the shear modulus for infinitesimal deformations. When $\gamma = 1/2$, we recover the neo-Hookean model (2.1). Mooney [84] showed that the form (2.3) is the most general one which is valid for large deformations of an incompressible hyperelastic material, isotropic in its undeformed state, for the relation between the shearing force and amount of simple shear to be linear. Hence the constant μ is also the shear modulus for large shears.

By considering the expansion of the strain energy function in power series of $(I_1 - 3)$ and $(I_2 - 3)$ terms, it can be shown that for small deformations, the quantities $(I_1 - 3)$ and $(I_2 - 3)$ are, in general, small quantities of the same order, so that (2.3) represents an approximation valid for sufficiently small ranges of deformations, extending slightly the range of deformations described by the neo-Hookean model. This is pointed out in the figures (2.1 - 2.4) where the classical experimental data of Treloar [126] for simple tension and of Jones and Treloar [69] for equibiaxial tension are plotted (their numerical values having been obtained from the original experimental tables), and compared with the predictions of neo-Hookean and Mooney-Rivlin models.

In the first case, *simple tension*, the principal stresses are

$$t_1 = t, \qquad t_2 = t_3 = 0, \tag{2.4}$$

and requiring for the principal stretches

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda_3 = \lambda^{-1/2}, \tag{2.5}$$

we obtain from the relation (1.40)

$$t = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(W_1 + \frac{1}{\lambda}W_2\right). \tag{2.6}$$

In Figure (2.1) we report the classical data of Treloar, by plotting the Biot stress $f = t/\lambda$ defined per unit reference cross-sectional area against the stretch λ. In Figure (2.2), we used the so-called *Mooney plot* (widely used in the experiment literature to compare the different models) because it is sensitive to relative errors. It represents the Biot stress $f = t/\lambda$ divided by the universal geometrical factor $2(\lambda - 1/\lambda^2)$, plotted against $1/\lambda$:

$$\frac{f}{2(\lambda - 1/\lambda^2)} = W_1 + \frac{1}{\lambda}W_2. \tag{2.7}$$

The Mooney-Rivlin model, fitting to data, improves the neo-Hookean model for small and moderate stretches. In fact, in the case of simple extension, the curves in (2.1) and (2.2) for the models under examination are obtained considering only the early part of the data. For large extensions, the Mooney-Rivlin curve gives a bad fitting. This fact may be emphasized by the Mooney plot (2.2), where the Mooney-Rivlin curve is a straight line, and is seen to fit only a reduced range of data.

For the *equibiaxial tension* test we let

$$t_1 = t_2 = t, \quad t_3 = 0, \tag{2.8}$$

and require the principal stretches to be

$$\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = \lambda^{-2}, \tag{2.9}$$

so that we obtain by (1.40) the following relation for the principal stress,

$$t = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(W_1 + \lambda^2 W_2\right). \tag{2.10}$$

In Figure (2.3), we report the classical data of Jones and Treloar [69] by plotting the Biot stress $f = t/\lambda$ against the stretch λ. In Figure (2.4) we represent the Mooney plot for the Biot stress divided by $2(\lambda - 1/\lambda^5)$, plotted against λ^2:

$$\frac{f}{2(\lambda - 1/\lambda^5)} = W_1 + \lambda^2 W_2. \tag{2.11}$$

The Mooney plot (2.4) reveals how the Mooney-Rivlin model extends slightly the range of data approximation compared to the neo-Hookean model, but cannot fit all of them.

The Mooney-Rivilin model has been studied extensively even though no rubber-like material seems to be described by it to within errors of experiment. It is used as the first illustration for every general result for isotropic incompressible materials for which several analytical solutions have been found.

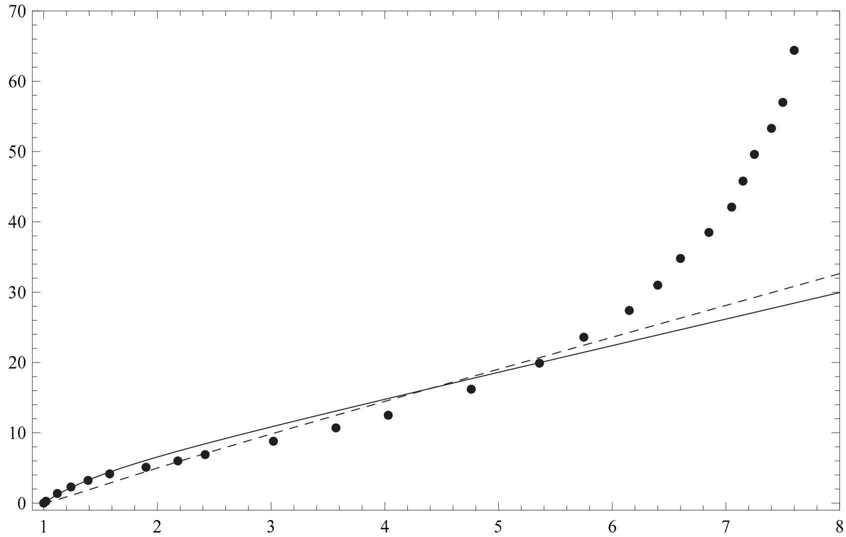


Figure 2.1: Plot of the simple tension data (circles) of Treloar [126] against the stretch λ, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, both models were optimized to fit the first 16 points, i.e. data for which $\lambda \in (1, 6.15)$).

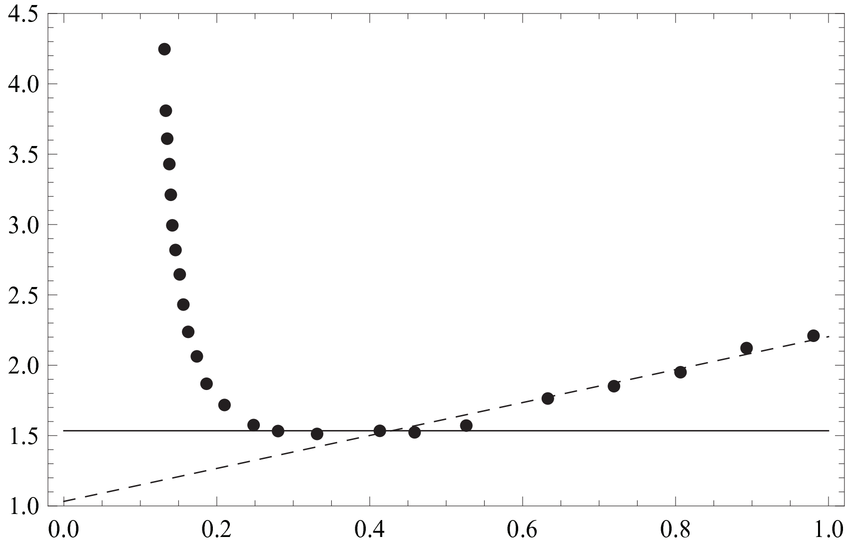


Figure 2.2: Plot of the simple tension data (circles) of Treloar [126] normalized by $2(\lambda - 1/\lambda^2)$ (λ is the stretch), against $1/\lambda$, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, the Mooney-Rivlin model has been optimized to fit the nine points for which $1/\lambda \in (0.33, 0.99)$ and the neo-Hookean model has been optimized to the five points for which $1/\lambda \in (0.28, 0.53)$).

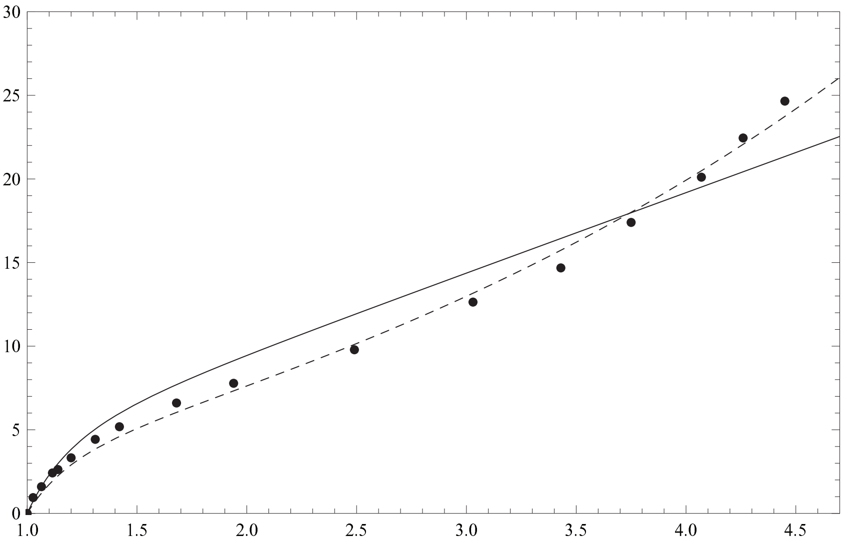


Figure 2.3: Plot of the equibiaxial tension data (circles) of Jones and Treloar [69], against the stretch λ, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, both models have been optimized to fit all seventeen points.)

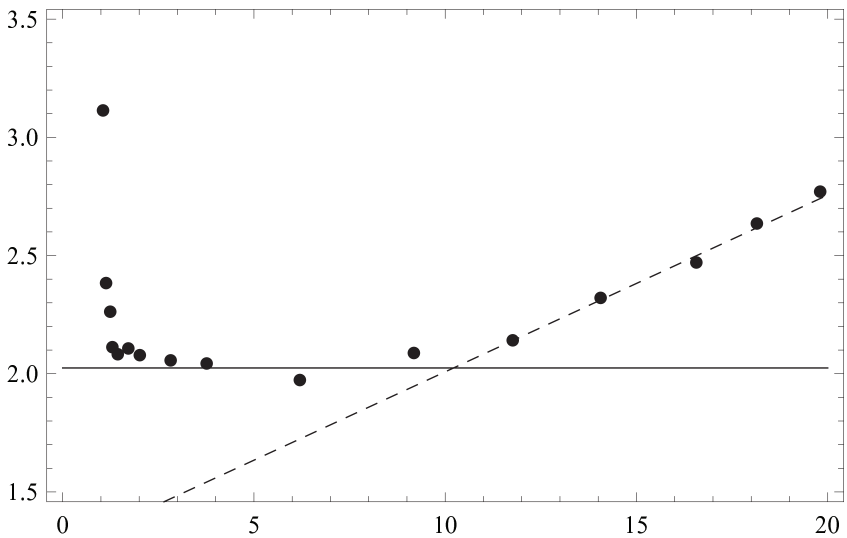


Figure 2.4: Plot of the equibiaxial tension data (circles) of Jones and Treloar [69] normalized by $2(\lambda - 1/\lambda^5)$ (λ is the stretch), against λ^2, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, the Mooney-Rivlin model has been optimized to fit the five data for which $\lambda^2 \in (11.76, 19.81)$ and the neo-Hookean model has been optimized to fit the three points for which $\lambda^2 \in (2.8, 6.2)$).

2.1.3 Generalized neo-Hookean model

Despite the idea of Rivlin to introduce the dependence of W of the second invariant I_2, there are several models of strain energy functions depending on the first invariant I_1 only. In the molecular theory, I_1 is connected to the mean squared end-to-end distance of the chains, but in general the chains cannot assume a completly arbitrary form and length. To overcome this constraint, the second invariant I_2, which is connected instead with the surface extension of material, is needed. Often the introduction of this invariant renders the calculations cumbersome, and from there follows the wide use of strain energies functions depending in a nonlinear manner on the first invariant only. A function of this form is called *generalized neo-Hookean model*,

$$W = W(I_1). \tag{2.12}$$

To account for the finite extensibility of the polymeric chains composing the elastomer network (since Gaussian statistics give rise to a probability density function without *compact support*), some models of the form (2.12) introduce a distribution function for the end-to-end distance of the polymeric chain which is not Gaussian. These models are usually called *non-Gaussian models*. From the phenomenological point of view these models can be divided into two classes: models with *limiting chain extensibility*, and *power-law* models. An example of the first class is due to Gent [45], who proposed the following strain energy density

$$W = -\frac{\mu}{2b} \ln\left[1 - b(\,I_1 - 3\,)\right], \tag{2.13}$$

where $b > 0$ is a limiting parameter value constant for I_1, accounting for limiting polymeric chain extensibility and $\mu > 0$ is the shear modulus for infinitesimal deformations. An example of the second class, widely used in biomechanics, was proposed by Fung [44] as follows

$$W = \frac{\mu}{2b}\exp\left[b\,(I_1 - 3) - 1\right], \tag{2.14}$$

where the dimensionless constant $b > 0$ is a stiffining parameter, and $\mu > 0$ is the shear modulus for infinitesimal deformations. Both classes behave as neo-Hookean solids in the small b/small-deformation limit, since they both obey

$$W(I_1, b) = \frac{\mu}{2}\,(I_1 - 3) + \frac{\mu b}{4}\,(I_1 - 3)^2 + O\left(b^2\,(I_1 - 3)^3\right) \tag{2.15}$$

as $b\,(I_1 - 3) \to 0$. Another power-law constitutive model was proposed by Knowles [73]. It can be written as

$$W = \begin{cases} \dfrac{\mu}{2\alpha\beta}\left[(1 + \beta\,(I_1 - 3))^{\alpha} - 1\right], & \text{if } \alpha \neq 0 \text{ and } \beta \neq 0, \\[2ex] \dfrac{\mu}{2\beta}\log\left(1 + \beta\,(I_1 - 3)\right), & \text{if } \alpha = 0 \text{ and } \beta \neq 0, \\[2ex] \dfrac{\mu}{2}\,(I_1 - 3)\,, & \text{if } \beta = 0,\ (\forall\alpha), \end{cases} \tag{2.16}$$

where α and β are constants; when $\alpha = 1$ the neo-Hookean model (2.1) is recovered. Knowles introduced this model to describe both strain-stiffening and strain-softening effects in elastomeric materials and biological soft tissues. For a careful study of the analytical properties of the Knowles potential, see [15].

Even though some classical experimental data suggest that constitutive equations of the form (2.12) may have limited applicability, they nevertheless often lead to closed-form analytical solution for many interesting problems. Such solutions are useful for a better understanding of the mechanical properties of the matter and also as benchmarks for more complex numerical computations.

2.1.4 Other models

Rivlin and Saunders [112] showed that both neo-Hookean and Money-Rivlin models are not adequate to describe accurately the experimental properties of rubber. Their conclusion was that $\partial W/\partial I_1$ is independent of both I_1 and I_2, and that $\partial W/\partial I_2$ is independent of I_1 and decreases with increasing I_2. They thus deduced the strain energy function in the form

$$W = C(I_1 - 3) + f(I_2 - 3), \tag{2.17}$$

where C is a constant and f is a function whose slope diminishes continuously with increasing I_2. In the more recent work of Obata [92], it is found that neither $\partial W/\partial I_1$ nor $\partial W/\partial I_2$ can be regarded as constant, and that each should depend on both I_1 and I_2.

Valanis and Landel [128] proposed that the strain energy function W may be expressible as the sum of three functions of the principal stretches,

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3), \tag{2.18}$$

in which the function $w(\lambda)$ is, by symmetry, the same for each of the extension ratios. Equivalent to (2.18) is the expansion due to Ogden [93],

$$W = \sum_{m=1}^{\infty} \mu_m(\lambda_1^{\alpha_m} + \lambda_2^{\alpha_m} + \lambda_3^{\alpha_m} - 3)/\alpha_m \tag{2.19}$$

in terms of powers of the principal stretches, where each μ_m and α_m are material constants, not necessarily integers [93]. Jones and Treloar [69] and Ogden [115] show how the biaxial strain experiments are consistent with the Valanis-Landel model (2.18) and the Ogden expansion (2.19).

2.2 Strain energy functions for compressible materials

In the compressible case, as well as (1.34), a further assumption is required for W: it should approach infinity as I_3 tends to infinity or zero$^+$. In other words, an infinite amount of energy is required in order to expand the body to infinite volume or to compress it to a point with vanishing volume, so that

$$\lim_{I_3 \to +\infty} W = +\infty, \qquad \lim_{I_3 \to 0^+} W = +\infty. \tag{2.20}$$

2.2.1 Hadamard model

Hadamard [51] introduced a class of elastic materials characterised by the property that infinitesimal longitudinal waves may propagate in every direction, when they are maintained in an arbitrary state of finite static homogeneous deformation. This constitutive model, called *Hadamard model* by John [68], describes also the only compressible isotropic homogeneous elastic material for which three linearly-polarized *finite* amplitude plane waves, one longitudinal and two transverse, may propagate in every direction when it is homogeneously deformed [24, 68]. The strain energy function is defined by

$$W = c_1(I_1 - 3) + c_2(I_2 - 3) + H(I_3), \tag{2.21}$$

where $c_1,\ c_2$ are material constants such that $c_1 > 0,\ c_2 \geq 0$, or $c_1 \geq 0,\ c_2 > 0$ and $H(I_3)$ is an arbitrary function to be specified on the basis of constitutive arguments. The connection with the Lamé constants of the linear theory is made through the relations

$$c_1 = \mu + H'(1), \quad c_2 = -\frac{\mu}{2} - H'(1), \quad 4H''(1) = \lambda + 2\mu. \tag{2.22}$$

An example for the function $H(I_3)$, accounting for the effects of compressibility, is given by Levinson and Burgess[1] [79]. They propose the following explicit form for the material function $H(I_3)$,

$$H(I_3) = (\lambda + \mu)\,(I_3 - 1) - (\lambda + 2\mu)\,(\sqrt{I_3} - 1). \tag{2.23}$$

2.2.2 Blatz-Ko model

The *Blatz-Ko model* is one of much used models describing the behavior of rubber in the compressible case. Replacing the principal invariants I_k by another set of independent invariants of $\boldsymbol{B}$, J_k defined by

$$J_1 \equiv I_1 = \mathrm{tr}\boldsymbol{B}, \quad J_2 \equiv I_2/I_3 = \mathrm{tr}\boldsymbol{B}^{-1}, \quad J_3 \equiv I_3^{1/2} = \det \boldsymbol{F}, \tag{2.24}$$

the strain energy function may be written as $W(J_1, J_2, J_3)$. Introducing (2.24) into (1.38), we find that

$$\beta_0 = \frac{\partial W}{\partial J_3}, \quad \beta_1 = \frac{2}{J_3}\frac{\partial W}{\partial J_1}, \quad \beta_{-1} = -\frac{2}{J_3}\frac{\partial W}{\partial J_2}. \tag{2.25}$$

Let us now consider a special class of materials whose response functions in (2.25) depend on J_3 alone. This is possible if and only if

$$\beta_0 = W_3(J_3), \quad \beta_1 = \frac{\alpha}{J_3}, \quad \beta_{-1} = -\frac{\beta}{J_3}, \tag{2.26}$$

where $W_3 \equiv \partial W/\partial J_3$ and α and β are constants. It can be shown that

$$\beta_1(1) - \beta_{-1}(1) = \alpha + \beta = \mu, \tag{2.27}$$

[1]We observe that Levinson and Burgess give an explicit form of $H(I_3)$ that does not verify $(2.20)_2$.

and introducing another constant f such that

$$\alpha = \mu f, \quad \beta = \mu(1-f), \tag{2.28}$$

the equation for the Cauchy stress for this special class of material is derived from (1.40) in the form

$$\boldsymbol{T} = W_3(J_3) + \frac{\mu f}{J_3}\boldsymbol{B} - \frac{\mu(1-f)}{J_3}\boldsymbol{B}^{-1}. \tag{2.29}$$

Considering a simple tensile loading

$$T_1 = t, \qquad T_2 = 0, \qquad T_3 = 0, \tag{2.30}$$

with principal stretches $(\lambda, \lambda_2, \lambda_3)$, Blatz and Ko [18] assumed (since in their experiment with $f = 0$ they found $J_3 = \lambda^{1/2}$) the following general constitutive assumption of volume control

$$J_3 = \lambda^n. \tag{2.31}$$

It follows from Batra's theorem [7] that

$$\lambda_2 = \lambda_3, \tag{2.32}$$

and from (2.31), that

$$\lambda_2(\lambda) = \lambda^{(n-1)/2}. \tag{2.33}$$

From (1.49) the infinitesimal strains are of the form $\epsilon_k = \lambda_k - 1$. Following [12] we define the *Poisson function* $\nu(\lambda)$ as

$$\nu(\lambda) = -\frac{\epsilon_3}{\epsilon_1} = \frac{1-\lambda_2(\lambda)}{\lambda - 1}, \tag{2.34}$$

from which the infinitesimal Poisson ratio is deduced in the limit

$$\nu = \lim_{\lambda \to 1} \nu(\lambda) = -\frac{(n-1)}{2}. \tag{2.35}$$

Therefore a Blatz-Ko material must verify

$$\lambda_2(\lambda) = \lambda^{-\nu}, \tag{2.36}$$

and consequently

$$\lambda = J_3^{1/(1-2\nu)}. \tag{2.37}$$

Blatz and Ko integrated the expression W_3 by making use of condition (2.37) and the condition $W(3,3,1) = 0$ in the natural state. They thus obtained the following general expression for the strain energy

$$\begin{aligned} W(J_1, J_2, J_3) = {} & \frac{\mu f}{2}[(J_1 - 3) - \frac{2}{q}(J_3^q - 1)] \\ & + \frac{\mu(1-f)}{2}[(J_2 - 3) - \frac{2}{q}(J_3^{-q} - 1)], \end{aligned} \tag{2.38}$$

where

$$q = \frac{n-1}{n} = \frac{-2\nu}{1-2\nu}. \qquad (2.39)$$

Two special models of this expression (2.38), $f = 0$ and $f = 1$, are often used in applications. The former characterizes the class of *foamed, polyurethane elastomers* and the latter describes the class of *solid, polyurethane rubbers* studied in the Blatz-Ko experiments. We note that in the limit $I_3 \to 1$ it is possible to obtain the Mooney-Rivlin strain energy density for incompressible materials from (2.38). Thus (2.38) may be viewed as a generalization of the Mooney-Rivlin model to compressible materials. In the literature, a special compressible material of the first case ($f = 0$) is often used at $q = -1$, for which the strain energy, rewritten in terms of invariants I_k, is given by

$$W(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right). \qquad (2.40)$$

2.3 Weakly non-linear elasticity

To study small-but-finite elastic effects, the *weakly non-linear elasticity* theory [76], considers an expansion for the strain energy function in the following form

$$W = \frac{1}{2!}C_{ijkl}E_{ij}E_{kl} + \frac{1}{3!}C_{ijklmn}E_{ij}E_{kl}E_{mn} + \dots, \qquad (2.41)$$

where $C_{ijk\dots}$ are constant moduli and $\boldsymbol{E} = \boldsymbol{E}^T$ is the Lagrange, or Green, strain tensor, defined as $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$. In the isotropic case, the strain energy (2.41) has the following expansion to the second order (*second-order elasticity*) as

$$W = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{E})^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2), \qquad (2.42)$$

where λ and μ are the Lamé constants. At the third order (*third-order elasticity*), the expansion is (see [101] for example)

$$W = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{E})^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{A}}{3}\mathrm{tr}(\boldsymbol{E}^3) + \mathcal{B}\,(\mathrm{tr}\boldsymbol{E})\,\mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{C}}{3}(\mathrm{tr}\boldsymbol{E})^3, \qquad (2.43)$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the *Landau third-order elastic constants.*

For incompressible solids the second-order expansion involves only one material constant: μ, and the third-order expansion involves only two material constants: μ and $\mathcal{A}$. They are written respectively as

$$W = \mu\ \mathrm{tr}(\boldsymbol{E}^2), \qquad (2.44)$$

and

$$W = \mu\ \mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{A}}{3}\mathrm{tr}(\boldsymbol{E}^3). \qquad (2.45)$$

Rivlin and Saunders [112] showed that the Mooney-Rivlin strain-energy function (2.3) of exact non-linear incompressible elasticity coincides, at the same order of approximation, with the general weakly nonlinear third-order elasticity expansion (2.45). Introducing the following constants

$$C_1 = \frac{1}{2}\left(\frac{1}{2} + \gamma\right)\mu, \quad C_2 = \frac{1}{2}\left(\frac{1}{2} - \gamma\right)\mu, \tag{2.46}$$

in (2.3), the connections between the material constants are

$$\mu = 2(C_1 + C_2), \quad \mathcal{A} = -8(C_1 + 2C_2). \tag{2.47}$$

Notes

This presentation of theoretical framework for the constitutive equations includes many but not all models proposed in literature. One of the main problems encountered in the applications of mechanics of continua is the complete and accurate determination of the constitutive relations necessary for the mathematical description of the behavior of real materials. Indeed people working with rubber know very well that the mechanical behavior of this material is very complex and outside of the forecast possibilities of nonlinear elasticity (see Saccomandi [115]).

One of the omissions, in this chapter is the so-called Rivlin-Signorini method. First Murnaghan [85] and then Rivlin [110] and Signorini [118] approximated the material response functions by polynomials in the appropriate invariants. In this way, a particular material is then characterized by the constant coefficients of the polynomial rather than by functions. Applications of the Rivlin-Signorini method can be found in [81, 120]. Although from a theoretical point of view, any complete set of invariants is equivalent to another, it has been observed by several authors that the approach used by Rivlin considering the principal invariants it is not very practical in fitting experimental data, because of the possible propagation of experimental errors (see for example [128]). Therefore it may be interesting to consider the possibility of expressing the strain energy directly in terms of the principal stretches and to overcome some difficulties related to the symmetry. That is why Valanis and Landel [128] postulated that the strain energy function be a sum of functions each depending on a single stretch (see (2.18)).

Chapter 3

Inverse methods

A general *boundary value problem* of elastostatics for a body $\mathcal{B}$ consists in finding a motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ that satisfies $\boldsymbol{a}(\boldsymbol{X}, t) = \boldsymbol{0}$ for all particles $\boldsymbol{X}$ of B_r and for all times t. Recalling equation (1.24), this means that the motion must satisfy the equilibrium equation

$$\mathrm{Div}\boldsymbol{T_R} + \rho_r \boldsymbol{b}_r = \boldsymbol{0}, \tag{3.1}$$

everywhere in B_r, and the boundary conditions of surface tractions (1.22) and place,

$$\boldsymbol{t_N} = \boldsymbol{T_R N}, \quad \text{prescribed on } \partial B_r^1, \tag{3.2}$$
$$\boldsymbol{X} = \bar{\boldsymbol{X}}, \quad \text{prescribed on } \partial B_r^2, \tag{3.3}$$

respectively, where ∂B_r^1 and ∂B_r^2 are disjoint parts of ∂B_r such that $\partial B_r = \partial B_r^1 \cup \partial B_r^2$.

The boundary value problem is expressed in terms of material description because, as we have just emphasized in the first chapter, the geometry of the deformed body generally is unknown a priori (otherwise equations (1.18) - (1.20) may be used). From a theoretical point of view, a given rubberlike material can be characterized by an appropriate constitutive equation that will enable us to predict its response to specified loading and displacement boundary conditions. We assume, as a first approximation, that a certain rubber material may be modeled as either a compressible or an incompressible, homogeneous isotropic hyperelastic material such that (1.36) or (1.40) applies. These represention formulae are useful to understand how the given material may be distinguished from another one on grounds of the response functions β_i only. But very little can be said *a priori* about these response functions unless some helpful experiment is made. Then, because measurements can be done only on the boundary of the test specimen, it would better to know *a priori* the kind of deformation that we want to reproduce experimentally in order to know what quantities can be effectively measured. To this end, Beatty [9] says

> *It is clear, in particular, that the experimenter must know a priori the class of deformations that actually may be produced in every compressible or incompressible, homogeneous and isotropic, hyperelastic material by the application of surface loading alone. Also, the surface loads*

> *needed to effect them must be known in order to select the kinds of loading devices that may be used.*

Theoretical results which fit this program are so-called *universal* results. A deformation, or a motion, which satisfies the balance equations with zero body force and which, in equilibrium, is supported by suitable surface tractions alone, is called a *controllable solution*. A controllable solution which is the same for all materials in a given class is a *universal solution* (when the solution is controllable for a specific subclass of material, it is called a *relative universal solution*). Besides universal solutions, other kind of universal results exist, which involve not only the strain but also the stress. For a given deformation or motion, *a local universal relation* is an equation relating the stress components and the position vector which holds at any point of the body and which is the same for any material in a given class.

From the mathematical point of view, the analytical solution for the (3.1) - (3.3) problem may be very hard to attain, even in the simplest boundary value problem, because the set of equations forms a non-trivial system of nonlinear, partial differential equations generating often nonunique solutions. To solve the resulting boundary-value problems, *inverse techniques* can be used to provide simple solutions and to suggest experimental programs for the determination of response functions. Two powerful methods for inverse investigations are the so-called *inverse method* and the *semi-inverse method*. They have been used in elasticity theory as well as in all fields of the mechanics of continua. For example, it is quoted in the book [77] that

> *In the inverse method, a known solution of the displacement is assumed with the aid of which strain and stress states are determined. Finally, using the boundary conditions, the body itself and its load and reactions are determined.*
>
> *In the semi-inverse method, part of unknowns is given, and the missing quantities are determined in such a way that the differential equations and boundary conditions are being satisfied.*

Similarly, Carlson [19] states

> *In the inverse method, we start with a given deformation (i.e., guess an $\boldsymbol{F}$), calculate the corresponding stress from the constitutive equation, and check to see if the stress satisfies equilibrium (generally for zero body force). If equilibrium is not satisfied, then the deformation is discarderd. However, if equilibrium is met, then we attempt to interpret the deformation and stress in a physically meaningful setting. I. e., we consider various shapes for the (deformed) body, calculate the corresponding surface tractions, anh hope to get something of physical interest.*
>
> *The semi-inverse method is just the same, except that in the deformation one includes some arbitrary parameters of functions that can be adjusted so that equilibrium is met or the boundary data comes out to be more interesting.*

Here is what Neményi [90] says about such methods in a general framework of continuum mechanics:

> *We shall call inverse an investigation of a partial differential equation of physics if in it the boundary conditions (or certain other supplementary conditions) are not prescribed at the outset. Instead, the solution is defined by the differential equation, and certain additional analytical, geometrical, kinematical, or physical properties of the field. In the semi-inverse method some of the boundary conditions are prescribed at the outset, whereas others are left open and obtained indirectly through certain simplifying assumption concerning the properties of the fields.*

The true power of the inverse methods is that they can reduce in most cases a system of differential equations in three independent variables to a system having only two, or even one, independent variable(s) which may, or may not, admit an exact solution in closed form. If this reduced system can be solved in closed form, then it is possible to obtain some exact solutions to boundary value problems, that hopefully are meaningful within the framework of the theory that is being employed[1]. Of course, even if it cannot be solved exactly, the semi-inverse method leads to a simpler set of equations that can be resolved numerically. When the use of inverse methods does not lead to new solutions, it may nonetheless yield a negative result in certain cases; that is, the nonexistence of certain types of solutions may be established. Inverse and semi-inverse procedures have been implemented in all fields of the mechanics of continua, and the number of results obtained is very large indeed (see [90] to have an idea of their applications).

3.1 Inverse Method

In order to find exact solutions to the problem (3.1) - (3.3) by the inverse method, the starting point is to assume a suitable form for the deformation, then find the stress fields associated to this deformation by making use of the constitutive equations, and finally verify whether the equilibrium equations are satisfied. In the positive case, one may deduce the surface tractions necessary to maintain the deformation, some of which are of considerable importance experimentally. Let us consider some examples, starting with some *homogeneous deformations*, with zero body forces.

3.1.1 Homogeneous deformations

The most general homogeneous deformation is described by the following form

$$\boldsymbol{x} = \boldsymbol{F}\boldsymbol{X} + \boldsymbol{c}, \tag{3.4}$$

where $\boldsymbol{X}$ and $\boldsymbol{x}$ are the the Cartesian coordinates in the reference and in the current configurations, respectively, $\boldsymbol{F}$ is a constant tensor and $\boldsymbol{c}$ is a constant vector.

[1]This is not always the case, as it is well known in the framework of the Navier-Stokes theory where the exact solutions found by the semi-inverse method are often not compatible with the canonical no-slip boundary conditions.

From (1.35) we deduce that the Cauchy stress $\boldsymbol{T}$ is also constant throughout a compressible material. It follows that the equilibrium equations are satisfied only when the body force $\boldsymbol{b}$ is zero and these deformations therefore may be produced by surface tractions alone[2]. For incompressible materials we deduce from (1.39) that if the hydrostatic pressure p is constant, then the above results also apply.

Let us consider *pure homogeneous deformations*, described by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.5}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are the the Cartesian coordinates in the reference and in the current configurations, respectively, and $\lambda_1, \lambda_2, \lambda_3$, are positive constants. The physical components of $\boldsymbol{B}$ and of his inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix}, \tag{3.6}$$

respectively, and the first three principal invariants are given by

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{3.7}$$

By formula (1.36) the stress components for a compressible material are

$$\begin{aligned} T_{11} &= \beta_0 + \beta_1\lambda_1^2 + \beta_{-1}\lambda_1^{-2}, & T_{22} &= \beta_0 + \beta_1\lambda_2^2 + \beta_{-1}\lambda_2^{-2}, \\ T_{33} &= \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2}, & T_{ij} &= 0 \ (i \neq j). \end{aligned} \tag{3.8}$$

In the incompressible case the deformation (3.5) must satisfy the constraint $I_3 = 1$, that is

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{3.9}$$

so that, in contrast to the compressible case, only two of the constants $\lambda_1, \lambda_2, \lambda_3$ are independent. By formula (1.40), and because we are considering zero body force, the equilibrium equations are satisfied only if $p = p_0$ with p_0 constant. The stress components for an incompressible material are

$$\begin{aligned} T_{11} &= -p_0 + 2W_1\lambda_1^2 - 2W_2\lambda_1^{-2}, & T_{22} &= -p_0 + 2W_1\lambda_2^2 - 2W_2\lambda_2^{-2}, \\ T_{33} &= -p_0 + 2W_1\lambda_3^2 - 2W_2\lambda_3^{-2}, & T_{ij} &= 0 \ (i \neq j). \end{aligned} \tag{3.10}$$

In both cases only normal stresses are present on surfaces parallel to the coordinate planes. The incompressible case (3.10) differs from (3.8) by an arbitrary constant p_0. The appearance of this term is one of the reasons why constrained materials are easier to deal with mathematically than unconstrained ones.

[2]This result is important physically since it is relatively easy to apply forces to a boundary, see Beatty [9].

Dilatation

In the special case where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (with $\lambda > 0$ because $J > 0$), the deformation (3.5) is called *uniform dilatation.* The stress components (3.8) then become

$$T_{ij} = (\beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2})\delta_{ij}, \tag{3.11}$$

where δ is the Kronecker symbol. The term $-(\beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2})$ corresponds therefore to a hydrostatic pressure that we denote by $P(\lambda^2)$. When $P > 0$ the body is subjected to a hydrostatic pressure, while for $P < 0$ it is subjected to a hydrostatic tension. By (1.18) we obtain the stress vector $\boldsymbol{t}$ in the form

$$\boldsymbol{t_n} = -P(\lambda^2)\boldsymbol{n}, \tag{3.12}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface. Hence, to maintain this deformation, the stress vector must be normal to the surface at each point of the boundary. In general one would expect that the volume of a compressible material held in equilibrium under the action of a uniform pressure should be less than its volume before deformation, that under the action of a uniform tension, the volume should be greater than its initial volume, and that when no traction is applied on the boundary, the volume remains unchanged. Since the variation of volume is $J = \lambda^3$, an equivalent statement is that

$$\begin{aligned} &\lambda < 1 \text{ when } P > 0,\\ &\lambda > 1 \text{ when } P < 0,\\ &\lambda = 1 \text{ when } P = 0. \end{aligned} \tag{3.13}$$

Furthermore, P should be a monotonic decreasing function of λ in order to increase the volume when the applied pressure is increased and viceversa, so that

$$\frac{\mathrm{d}P}{\mathrm{d}\lambda} < 0. \tag{3.14}$$

In linear elasticity this constraint is equivalent to require that the bulk modulus κ is positive (see $(1.54)_2$). In view of the definition of P, the relation (3.14) places some restrictions on the response functions β_i. In the incompressible case, the constraint (3.9) requires that $\lambda = 1$ so that there is no deformation.

Simple extension

When $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = \bar{\lambda}$, the deformation (3.5) is called *uniform extension* (when $\lambda > 1$) or *contraction* (when $\lambda < 1$) in the X_1-direction, together with equal extension or contraction in the lateral X_2- and X_3-directions. For compressible materials we deduce from (3.8) that the stress components are

$$\begin{aligned} &T_{11} = \beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2}, \qquad T_{22} = T_{33} = \beta_0 + \beta_1\bar{\lambda}^2 + \beta_{-1}\bar{\lambda}^{-2},\\ &T_{ij} = 0 \ (i \neq j). \end{aligned} \tag{3.15}$$

The simplest stress system arises when (if possible), $T_{22} = T_{33} = 0$ in order to have traction free lateral sides. This gives

$$\beta_0 + \beta_1 \bar{\lambda}^2 + \beta_{-1} \bar{\lambda}^{-2} = 0. \tag{3.16}$$

Equation (3.16) depends on λ^2 and $\bar{\lambda}^2$ and for a given λ, it is not obvious that it should have a single positive root $\bar{\lambda}^2$. If (3.16) has no root then uniform extension cannot be effected by applying a tension T_{11} alone, i.e. others surface tractions are necessary. If (3.16) has more than one root, then there are more than one tensile sress which produce a given extension with the remaining faces traction-free. When it is possible to apply a tension in the X_1-direction with the other stresses being zero, the extension is called *simple*, and we expect the specimen to increase in length in this direction, whereas when we apply a pressure the length should decrease. Also, if no tension is applied on the boundary, then the length should remain unchanged. Finally when T_{11} is increased, the extension should increase and vice-versa. Hence T_{11} should verify

$$\frac{\mathrm{d}T_{11}}{\mathrm{d}\lambda} > 0. \tag{3.17}$$

This inequality places a further restriction on the response functions β_i. For incompressible materials, the constraint (3.9) implies $\bar{\lambda} = 1/\sqrt{\lambda}$. Here the stress components (3.10) become

$$\begin{aligned} T_{11} &= -p_0 + 2W_1\lambda^2 - 2W_2\lambda^{-2}, \qquad\qquad T_{ij} = 0 \ (i \neq j), \\ T_{22} = T_{33} &= -p_0 + 2W_1\lambda^{-1} - 2W_2\lambda. \end{aligned} \tag{3.18}$$

The principal difference with the compressible case is that the boundary conditions $T_{22} = T_{33} = 0$ appropriate to the block subject to a tension T_{11} can always be satisfied on setting

$$p_0 = 2W_1\lambda^{-1} - 2W_2\lambda. \tag{3.19}$$

The uniaxial tension necessary to maintain this deformation (see also (2.6)) is

$$T_{11} = 2(\lambda^2 - \lambda^{-1})W_1 + 2(\lambda - \lambda^{-2})W_2. \tag{3.20}$$

Simple shear

Let us consider the homogeneous deformation of *simple shear*,

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{3.21}$$

where k is a constant parameter representing the *amount of shear*. We consider the shearing by applied surface tractions alone of a block with faces initially parallel to the coordinates planes. This deformation is quite difficult to produce experimentally because of the complex surface tractions needed to maintain it (as we see below). However, it is probably the simplest example illustrating that large deformations are different from infinitesimal deformations described by linear elasticity, not only in magnitude but also in the novel effects they produce.

It easy to determine the physical components of the left Cauchy-Green deformation tensor $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ as

$$\begin{pmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -k & 0 \\ -k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.22}$$

respectively, so that the first three principal invariants of $\boldsymbol{B}$ are $I_1 = I_2 = 3 + k^2$, $I_3 = 1$. The stress components for this deformation for compressible materials (see (1.36)) are given by

$$\begin{aligned} T_{11} &= \beta_0 + \beta_1(1+k^2) + \beta_{-1}, & T_{12} &= k(\beta_1 - \beta_{-1}), \\ T_{22} &= \beta_0 + \beta_1 + \beta_{-1}(1+k^2), & T_{13} &= 0, \\ T_{33} &= \beta_0 + \beta_1 + \beta_{-1}, & T_{23} &= 0. \end{aligned} \tag{3.23}$$

Thus we see that *both* normal and shear stresses are present on surfaces parallel to the coordinate planes and that, as in the previous deformation, the stress components are constants.

To consider the relation between the shear stress and the amount of shear, we define

$$\mu(k^2) = \beta_1(3+k^2, 3+k^2, 1) - \beta_{-1}(3+k^2, 3+k^2, 1), \tag{3.24}$$

so that

$$T_{12} = \mu(k^2)k. \tag{3.25}$$

The quantity $\mu(k^2)$ is called the *generalized shear modulus*. Its value $\mu \equiv \mu(0)$ in the natural state is the initial shear modulus. Physically we would expect the shear stress acting on a surface with normal in the $2-$direction to be in the direction in which the surface has been displaced, so that we expect

$$\mu(k^2) > 0. \tag{3.26}$$

By (3.24) we can see how the empirical inequalities (1.46) are sufficient to establish (3.26). Because the shear stress T_{12} is an odd function of the amount of shear, the shear stress is therefore in the direction of the amount of shear (since the normal stresses are even functions of the amount of shear, they do not depend on its direction). In linear elasticity (making use of (1.51)), simple shear can be maintained by applying only shear stresses on the faces of specimen. A similar situation does not arise with a finite deformation, unless a degenerate material is considered for which a simple shear can be produced in the absence of all stress. In fact, imposing $T_{11} = T_{22} = T_{33} = 0$ gives the conditions

$$\begin{aligned} &\beta_0 + \beta_1(1+k^2) + \beta_{-1} = 0, \\ &\beta_0 + \beta_1 + \beta_{-1}(1+k^2) = 0, \\ &\beta_0 + \beta_1 + \beta_{-1} = 0, \end{aligned} \tag{3.27}$$

from which it follows that $\beta_0 = \beta_1 = \beta_{-1} = 0$ and in this case, T_{12} is also zero. We therefore conclude that, for all materials exhibiting physically reasonable response, the simple shear (3.21) cannot be produced by applying only shear stresses on surfaces parallel to the cordinate planes: normal stresses are also necessary[3].

From (3.23) we obtain

$$\begin{aligned} &\beta_1 k^2 = T_{11} - T_{33}, \quad \beta_{-1} k^2 = T_{22} - T_{33}, \\ &\beta_0 k^2 = (2 + k^2) T_{33} - (T_{11} + T_{22}). \end{aligned} \tag{3.28}$$

This is an example of how the inverse method can be applied to find universal solutions and of how it is possible to use this kind of solutions to design an experimental test to determine the β_i's. However, we note that an experiment based on simple shear only is too restrictive to determine completely the response functions. In fact this deformation allows exploration of what happens only along the line $I_3 = 1,\ I_1 = I_2$ in the space of invariants, made of $I_1 > 0,\ I_2 > 0$ and $I_3 > 0$.

From (3.23) it is also possible to derive the relations

$$T_{13} = T_{23} = 0, \quad kT_{12} = T_{11} - T_{22}. \tag{3.29}$$

These relations provide links between the stress components and the amount of shear k which do not depend on the particular elastic isotropic material. They are *universal relations*. They are important because for example if one finds experimentally that $(3.29)_2$ is not satisfied then one may conclude that the material under investigation is not an isotropic elastic material. Also, except in the case of a degenerate material, T_{11} cannot be equal to[4] T_{22}. By $(3.29)_2$ we deduce also that the knowledge of the behavior of the shear stress in simple shear gives no information about the normal stresses. This intuition is present in linear elasticity, where simple shear alone cannot determine the normal stresses while by $(3.29)_2$ the normal stresses characterize the simple shear.

The unit normal $\boldsymbol{n}$ and the unit tangent $\boldsymbol{\tau}$ on the inclined faces have the components

$$\boldsymbol{n} = (1, -k, 0) \, / \sqrt{(1 + k^2)}, \qquad \boldsymbol{\tau} = (k, 1, 0) \, / \sqrt{(1 + k^2)}, \tag{3.30}$$

so that we may calculate the normal stress N and the shear stress T which have to be applied to the inclined faces of the deformed specimen in order to maintain the simple shear deformation. By use of (1.18), they are

$$N = \boldsymbol{t_n} \cdot \boldsymbol{n}, \quad T = \boldsymbol{t_n} \cdot \boldsymbol{\tau}. \tag{3.31}$$

and we therefore obtain the following relationships

$$\begin{aligned} &(1 + k^2) N = T_{11} + k^2 T_{22} - 2k T_{12}, \\ &(1 + k^2) T = k(T_{11} - T_{22}) + (1 - k^2) T_{12}. \end{aligned} \tag{3.32}$$

[3] When normal stresses are not applied, the material tends to contract or expand. This result was apparently conjectured by Kelvin and is often called the *Kelvin effect* [123].

[4] In 1909 Poynting noticed a similar phenomenon and performed a series of torsion experiments to illustrate the lengthening of a metal wire when no normal force was applied. The existence of unequal normal stresses is often referred to as the *Poynting effect* [102].

Using the universal relation $(3.29)_2$ we deduce

$$\begin{aligned} T_{12} &= (1+k^2)T, \\ kT_{12} &= (1+k^2)(T_{22}-N), \\ N &= T_{22}-kT. \end{aligned} \tag{3.33}$$

We deduce some interesting consequences from these relationships. First, we can see that $|T| < |T_{12}|$ and $|N| < |T_{22}|$. Hence if T_{22} is negative, so is N, i.e. if the normal traction on the shearing planes is a pressure, then so is the normal traction on the inclined faces. Since N is different from T_{22} (otherwise this would again imply that $\mu(k^2) = 0$), the Poynting effect still holds when referred to the current faces of the sheared block. Finally by $(3.33)_2$ it follows that there may be special elastic materials such that $N = 0$ for all shears k.

Many of the results for compressible materials are still valid for incompressible bodies. In this case, by (1.40) the stress components are given as

$$\begin{aligned} T_{11} &= -p_0 + 2(1+k^2)W_1 - 2W_2, \\ T_{22} &= -p_0 + 2W_1 - 2(1+k^2)W_2, \\ T_{33} &= -p_0 + 2W_1 - 2W_2, \\ T_{12} &= k\mu(k^2), \quad T_{13} = T_{23} = 0, \end{aligned} \tag{3.34}$$

where p_0 is a constant to be determined by the prescribed boundary conditions, and $\mu(k^2) = 2(W_1+W_2)$ is obtained from (3.24) by replacing β_1 with $2W_1$ and β_{-1} with $-2W_2$. As in the compressible case, the remarks concerning the behaviour of normal stress and shear stresses when the direction of shear is reversed are still valid, as are the results (3.29), (3.32) and (3.33) from which we deduce that T_{11} and T_{22} cannot be equal, and that the Poynting effect is still present. In constrast with the compressible case, it is possible to make any one of the normal stresses vanish by an appropriate choice of p_0. For example we may choose p_0 such that $T_{33} = 0$ and in this case we see that $T_{11} > 0$ and $T_{22} < 0$ if and only if the empirical inequalities hold. In this particular case ($T_{33} = 0$) we obtain

$$p_0 = 2(W_1 - W_2), \tag{3.35}$$

and

$$T_{11} = 2k^2W_1, \quad T_{22} = -2k^2W_2, \tag{3.36}$$

$$T_{12} = k\mu(k^2), \quad T_{33} = T_{13} = T_{23} = 0, \tag{3.37}$$

showing that the normal stress on the shearing planes is always a pressure since $T_{22} < 0$, from which we deduce (as in the compressible case) that $T < 0$. If these pressures are not applied in addition to the shear forces, then we would expect the material to stretch in the 1− and 2−directions and hence to contract in the 3−direction (because of the incompressibility constraint). In other words, one form of the Poynting effect is observed.

Rivlin was one of the first authors to use inverse procedures to construct some examples of exact solutions of physical interest to both analysts and experimenters.

His work is very interesting also because it marked the birth in 1948 of the modern theory of finite elasticity (see Rivlin [111] for the collected works). Later, Ericksen [33, 34] introduced a different and more general approach to the investigation of inverse solutions, and such results provide the kinds of tools requested by experimenters.

3.1.2 Universal solutions

We know from the previous section that homogeneous deformations, which play a fundamental role in the theory of finite elastic deformations, can be maintained in all homogeneous bodies under the action of surface forces alone, because the stress corresponding to (3.4) is a constant tensor, and the balance equations are then trivially satisfied in absence of body forces. They therefore represent a set of universal solutions for all homogeneous materials. Ericksen [34] proved in 1955 that they are the only controllable deformations possible in every *compressible*, homogeneous and isotropic hyperelastic material. This result is known as *Ericksen's theorem.*

For *incompressible* materials, the definite answer is still lacking in the search for all universal solutions. So far, five families of universal solutions have been found in addition to homogeneous deformations. All solutions are such that appropriate physical components of stress are constants on each member of a family of parallel planes, coaxial cylinders, or concentric spheres. Let us start by looking at how some restrictions on the physical components of the stress can help to simplify the problem, in general by reducing a partial differential system to an ordinary one with less unknowns. To this end we consider the case of cylindrical coordinate (r, θ, z) only. A similar discussion can be conducted for Cartesian and spherical coordinates.

The equilibrium equations, in the absence of body force (1.20), read

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0,\\
&\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta} = 0,\\
&\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} = 0.
\end{aligned} \tag{3.38}$$

If we assume that $\boldsymbol{T} + p\boldsymbol{I}$ depends on r only (such assumption is often made when the problem has cylindrical symmetry), the partial differential system (3.38) simplifies as

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0,\\
&\frac{1}{r^2}\frac{\partial \left(r^2 T_{r\theta}\right)}{\partial r} - \frac{1}{r}\frac{\partial p}{\partial \theta} = 0,\\
&\frac{1}{r}\frac{\partial \left(r T_{rz}\right)}{\partial r} - \frac{\partial p}{\partial z} = 0.
\end{aligned} \tag{3.39}$$

By the further assumptions that

$$T_{r\theta} = 0, \quad T_{rz} = 0, \tag{3.40}$$

it follows from (3.39) that p depends only on r and consequently that the T_{rr} component depends on r only. Under these strong assumptions, the partial differential system (3.38) is reduced to an ordinary differential system that is easier to solve. The T_{rr} and $T_{\theta\theta}$ components are given by

$$T_{rr} = -\int \frac{T_{rr} - T_{\theta\theta}}{r}\,\mathrm{d}r, \quad T_{\theta\theta} = \frac{\mathrm{d}\,(rT_{rr})}{\mathrm{d}r}. \tag{3.41}$$

Let us consider the first family of universal solutions (in the literature, these solutions are classified in "families"). It is given by the following deformation

Family 1:

$$r = \sqrt{2AX}, \quad \theta = BY, \quad z = \frac{Z}{AB} - BCY, \tag{3.42}$$

which describes *bending, stretching and shearing of a rectangular block.* Here (X, Y, Z) and (r, θ, z) are the cartesian and cylindrical coordinates in the reference and in the current configuration, respectively, and A, B, C are constants with $AB \neq 0$. If $C = 0$ the deformation describes *pure bending* and carries the parallepipedic block bounded by the planes $X = X_1,\ X = X_2,\ Y = \pm Y_0,\ Z = \pm Z_0$ into the circular annular wedge bounded by the cylinders $r = r_1 = \sqrt{2AX_1}$, $r = r_2 = \sqrt{2AX_2}$, and the planes $\theta = \pm\theta_0 = \pm BY_0,\ z = \pm z_0 = \pm Z_0/(AB)$. When B is prescribed, then the arbitrary axial stretch $1/(AB)$ is allowed, and the radial stretch is so adjusted as to render the deformation isochoric[5]. The physical components of $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} \dfrac{A^2}{r^2} & 0 & 0 \\ 0 & B^2r^2 & -B^2Cr \\ 0 & -B^2Cr & B^2C^2 + \dfrac{1}{A^2B^2} \end{bmatrix}, \tag{3.43}$$

and

$$\begin{bmatrix} \dfrac{r^2}{A^2} & 0 & 0 \\ 0 & \dfrac{1}{B^2r^2} + \dfrac{A^2B^2C^2}{r^2} & \dfrac{A^2B^2C}{r} \\ 0 & \dfrac{A^2B^2C}{r} & A^2B^2 \end{bmatrix}, \tag{3.44}$$

respectively. The first two principal strain invariants are

$$\begin{aligned} I_1 &= \frac{A^2}{r^2} + B^2r^2 + B^2C^2 + \frac{1}{A^2B^2}, \\ I_2 &= \frac{r^2}{A^2} + \frac{1}{r^2}\left(\frac{1}{B^2} + A^2B^2C^2\right) + A^2B^2, \end{aligned} \tag{3.45}$$

[5] In the general case, the deformation may be effected in two steps, the first of which is the bending and axial stretch, while the second is a homogeneous strain which carries the body into the solid bounded by the cylindrical surfaces $r = r_1$ and $r = r_2$, the planes $\theta = \pm\theta_0$, and the helicoidal surfaces $z + C\theta = \pm z_0$.

respectively, and $I_3 = 1$ in agreement with incompressibility. From (1.40) we see that the physical components of $\boldsymbol{T} + p\boldsymbol{I}$ are functions of r only and that (3.40) is satisfied. By $(3.41)_1$

$$
\begin{aligned}
T_{rr} &= -\int \left[2\frac{\partial W}{\partial I_1}\left(\frac{A^2}{r^3} - B^2 r\right) - 2\frac{\partial W}{\partial I_2}\left(\frac{r}{A^2} - \frac{1}{r^3}\left(\frac{1}{B^2} + A^2B^2C^2\right)\right)\right] \mathrm{d}r \\
&= \int \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r, \qquad (3.46)
\end{aligned}
$$

from which we obtain the other components of the stress.

$$
\begin{aligned}
T_{\theta\theta} &= T_{rr} + 2\left[B^2r^2 - \frac{A^2}{r^2}\right]\frac{\partial W}{\partial I_1} - 2\left[\frac{1}{r^2}\left(\frac{1}{B^2} + A^2B^2C^2\right) - \frac{r^2}{A^2}\right]\frac{\partial W}{\partial I_2}, \\
T_{zz} &= T_{rr} + 2\left[B^2C^2 + \frac{1}{A^2B^2} - \frac{A^2}{r^2}\right]\frac{\partial W}{\partial I_1} - 2\left[A^2B^2 - \frac{r^2}{A^2}\right]\frac{\partial W}{\partial I_2}, \qquad (3.47) \\
T_{\theta z} &= -2B^2Cr\frac{\partial W}{\partial I_1} - 2\frac{A^2B^2C}{r}\frac{\partial W}{\partial I_2}.
\end{aligned}
$$

To obtain the unknown p we make use of (1.40),

$$
p = -T_{rr} + 2W_1(\boldsymbol{B})_{11} - 2W_2(\boldsymbol{B}^{-1})_{11}, \qquad (3.48)
$$

and therefore by (3.43), (3.44) and (3.46),

$$
p = -\int \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r + 2\left(\frac{A^2}{r^2}W_1 - \frac{r^2}{A^2}W_2\right). \qquad (3.49)
$$

It is possible to choose the constants in (3.46) in order to have the cylinder $r = r_1$ free of traction. To have the cylinder $r = r_2$ also free of traction, it is then necessary that

$$
\int_{r_1}^{r_2} \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r = 0, \qquad (3.50)
$$

and when this is verified, a particular relation among the constants A, B, C applies. Independently of whether or not (3.50) can be satisfied, the helicoidal faces $z+C\theta = \pm z_0$ (with unit normal $\boldsymbol{n} = (0, C/r, 1)/\sqrt{1 + C^2/r^2}$) cannot be free of traction in order to maintain the deformation. The normal and tangential tractions N and T, respectively, are

$$
\begin{aligned}
N &= \frac{1}{1 + C^2/r^2}\left[T_{zz} + 2\frac{C}{r}T_{\theta z} + \frac{C^2}{r^2}T_{\theta\theta}\right], \qquad (3.51) \\
T &= \frac{1}{1 + C^2/r^2}\left[\left(1 - \frac{C^2}{r^2}\right)T_{\theta z} + \frac{C}{r}\left(T_{\theta\theta} - T_{zz}\right)\right],
\end{aligned}
$$

respectively. Only when pure bending is considered (i.e. $C = 0$) we can deduce from (3.47) and (3.51) that $T = 0$ and $N = T_{zz}$. In general, both normal and tangential tractions must be applied. The presence of these tractions gives rise to the Poynting effect for bending, similar to the Poynting effect discussed for the simple shear deformation. We thus underline how the Poynting effet is in general inevitable in nonlinear elasticity.

A similar discussion can be made for the other four families, which are described by the following deformations.

Family 2: Straightening, stretching and shearing of a sector of a hollow cylinder,

$$x = \frac{1}{2}AB^2R^2, \quad y = \frac{\Theta}{AB}, \quad z = \frac{Z}{B} + \frac{C\Theta}{AB}. \tag{3.52}$$

Family 3: Inflation, bending, torsion, extension and shearing of an annular wedge,

$$r = \sqrt{AR^2 + B}, \quad \theta = C\Theta + DZ, \quad z = E\Theta + FZ, \tag{3.53}$$

with $A(CF - DE) = 1$.

Family 4: Inflation or eversion of a sector of a spherical shell,

$$r = \left(\pm R^3 + A\right)^{1/3}, \quad \theta = \pm\Theta, \quad \varphi = \Phi. \tag{3.54}$$

Family 5: Inflation, bending, extension, and azimuthal shearing, of an annular wedge,

$$r = \sqrt{A}R, \quad \theta = D\ln(BR) + C\Theta, \quad z = FZ, \tag{3.55}$$

with $ACF = 1$.

Here A, B, C, D, E, F are constants. It seems that the class of static deformations that are possible in all homogeneous, isotropic, incompressible elastic bodies subject to surface tractions only is likely to be exhausted by these cases. Some progress toward determining other deformations may be made if we replace the purely inverse method by a semi-inverse one, considering a family of deformations involving one or more arbitrary functions which may be determined so as to render the deformation possible for a particular material.

3.2 Semi-inverse method

In elasticity the first application of the semi-inverse method is due to Saint-Venant [5, 6] in 1855. He was the first to study the problem of linear elastostatics for a right long cylinder free from volume forces and loaded only at the bases by unspecified tractions. This problem was later on called the problem of Saint-Venant (*Saint-Venantsche Problem*) by Clebsch [23]. The starting point of the application of the semi-inverse method in order to solve the problem is that some components of the stress vanish. In particular, it is assumed that the normal tension on every section parallel to X_3, the axis of the cylinder, be zero:

$$T_{11} = T_{12} = T_{22} = 0. \tag{3.56}$$

When this assumption is made, it is possible to find a closed-form solution of the problem by the use of the linear equilibrium equations (1.50), of the linear constitutive equations (1.48), of the compatibility Beltrami conditions (1.63), and of the prescribed boundary conditions. The displacement field for the points of the cylinder turns out to depend linearly on four constants; these represent kinematic parameters to be specified at one base of the cylinder. Each of them characterizes a simple mode of deformation of the cylinder: extension, bending, torsion, and flexure, and it may be shown that the four kinematic parameters are linear functions of the resultant actions on the bases.

The semi-inverse assumption on the field of stress is of fundamental importance to find the approximate analytical solution. Although this assumption is suggested by the geometrical and boundary surface tractions, it is justified afterwards by the existence of the solution found. The Kirchhoff principle shows then the uniquess of the solution.

> KIRCHHOFF, 1859 *If either the surface displacement or the surface tractions are given, the solution of the problem of equilibrium of an elastic body is unique in the sense that the state of stress (and strain) is determinate without ambiguity, provided that the magnitude of the stress (and strain) is so small that the strain energy function exists and remains positive definite.*

Several applications of the semi-inverse method can be found in the literature on nonlinear elasticity. Of course it is not possible to list all such results because a survey aiming at completeness would require a whole book. Here we present some representative examples in order to underline some aspects of the semi-inverse method, and other useful examples for our discussion are given in the next chapters. First, we recall something just discussed on *simple extension*, but here we modify the problem a little bit. Then we discuss a problem of *anti-plane shear* (see [59, 61, 63, 113] for more details). Finally, some others remarks are discussed for the *radial deformation* problem.

3.2.1 Simple uniaxial extension

Let us consider the uniaxial extension of a rod by prescribing some boundary conditions at the outset: we ask that our model gives traction-free lateral surfaces. In this case, coming back to (3.5), we set $\lambda_1 = \lambda$ to denote the uniaxial stretch, while λ_2 and λ_3 denote the lateral stretches.

In the compressible case, making use of the (3.8) relations and the fact that the stress components are constants, we have to set

$$\begin{aligned} \beta_0 + \beta_1\lambda_2^2 + \beta_{-1}\lambda_2^{-2} = 0, \\ \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2} = 0. \end{aligned} \tag{3.57}$$

Forming the difference, we obtain

$$(\lambda_2^2 - \lambda_3^2)\left(\beta_1 - \frac{1}{\lambda_2^2\lambda_3^2}\beta_{-1}\right) = 0. \tag{3.58}$$

Applying the empirical inequalities (1.46) to (3.58), we obtain a necessary condition: $\lambda_2 = \lambda_3$ to be satisfied. The same condition is also discussed in the previous chapter by using Batra's Theorem [7]. But now we see how the arbitrary parameters λ_2 and λ_3 are adjusted a posteriori to meet the boundary conditions. In this case, by equations (3.57), we may solve uniquely as

$$\lambda_3 = \lambda_2 = \lambda_2(\lambda), \tag{3.59}$$

and obtain a simple extension under a tensile stress

$$T_{11}(\lambda) = (\lambda^2 - \lambda_2^2)\left(\beta_1 - \frac{1}{\lambda^2\lambda_2^2}\beta_{-1}\right). \tag{3.60}$$

In the incompressible case, equation (3.60) is remplaced by (3.20). The Poisson function $\nu(\lambda)$ given by the expression (2.34) for an incompressible material reads here as

$$\nu(\lambda) = \frac{1}{\sqrt{\lambda}(\sqrt{\lambda}+1)}. \tag{3.61}$$

In the natural state of an incompressible material, the Poisson ratio has the value $\nu = \nu(1) = 1/2$, otherwise (3.61) is a monotone decreasing function of the amount of uniaxial stretch. In a simple tension experiment, we see that (3.61) can be used to evaluate whether the material is incompressible[6]. Indeed equation (3.61) is universal for any isotropic uniform elastic material which is incompressible. In the case of compressibility, equation (2.34) is not universal since $\lambda_2(\lambda)$ depends on the special material we are considering. For example, for a general Blatz-Ko material (2.38), we know from (2.36) that

$$\lambda_2(\lambda) = \lambda^{(n-1)/2}, \tag{3.62}$$

where n is parameter characterizing a particular Blatz-Ko model. By formula (2.34) it is clear that the Poisson function,

$$\nu(\lambda) = \frac{1-\lambda^{(n-1)/2}}{\lambda - 1}, \tag{3.63}$$

depends now on the Blatz-Ko model used.

3.2.2 Anti-plane shear deformation

Let us consider the following deformation written in Cartesian coordinates

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1, X_2), \tag{3.64}$$

representing an *anti-plane shear deformation*, where $\boldsymbol{X}$ denote the reference coordinates and $\boldsymbol{x}$ the current coordinates of the body. The displacement is therefore described by a single smooth scalar function (the *out-of-plane* displacement) $w \equiv w(X_1, X_2)$. By the semi-inverse procedure, in the search of static solutions with zero body force, we must verify if the balance equations $\text{div}\boldsymbol{T} = \boldsymbol{0}$ are satisfied for some specified w and/or for some specific class of materials.

Let us consider the incompressible case. Before analysing the general case (3.64) we suppose that our body has axial symmetry (cylindrical body) and we assume that the anti-plane shear problem may be solved by considering an axisymmetric deformation of the form

$$w(X_1, X_2) = w(X_1^2 + X_2^2). \tag{3.65}$$

[6]Beatty and Stalnaker [12] show that although the Poisson function of every incompressible material has the universal constant, natural limit value 1/2, the converse is generally false.

In cylindrical coordinates, the deformation (3.64) can be rewritten as

$$r = R, \quad \theta = \Theta, \quad z = Z + w(R), \tag{3.66}$$

where $w(R)$ is the axial displacement. Such a deformation is also called *telescopic shear*. The physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} 1 & 0 & w' \\ 0 & 1 & 0 \\ w' & 0 & 1+w'^2 \end{bmatrix}, \quad \begin{bmatrix} 1+w'^2 & 0 & -w' \\ 0 & 1 & 0 \\ -w' & 0 & 1 \end{bmatrix}, \tag{3.67}$$

respectively, where the prime denotes differentation respect to R, and the first three principal strain invariants are

$$I_1 = I_2 = 3 + w'(R)^2, \tag{3.68}$$

and $I_3 = 1$ in agreement with the incompressibility constraint. By formula (1.40) we obtain the physical components of the Cauchy stress tensor as

$$\begin{aligned} T_{rr} &= -p + 2W_1 - 2(1+w'^2)W_2, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= -p + 2W_1 - 2W_2, & T_{rz} &= 2(W_1 + W_2)w', \\ T_{zz} &= -p + 2(1+w'^2)W_1 - 2W_2, & T_{\theta z} &= 0. \end{aligned} \tag{3.69}$$

Finally, the equilibrium equations reduce to equations (3.39) but now $T_{r\theta}$ only is zero, showing that $p = p(r, z)$. On using the expressions of T_{rr} and $T_{\theta\theta}$ in $(3.39)_1$ we obtain

$$p(r, z) = 2W_1 - 2W_2(1+w'^2) - \int \frac{2}{r} W_2 w'^2 \, \mathrm{d}r + g(z), \tag{3.70}$$

where g is an arbitrary function of z. By virtue of (3.70) and the expression of T_{rz}, we can rewrite $(3.39)_3$ as

$$\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rz}) = \lambda r, \tag{3.71}$$

where $\lambda = \mathrm{d}g(z)/\mathrm{d}z$. This equation is a second-order nonlinear ordinary differential equation for $w(R)$, with an immediate first integral in the form of a first-order differential equation for $w(R)$, namely

$$2(W_1 + W_2)w' = \frac{\lambda R}{2} + \frac{C_1}{R}, \tag{3.72}$$

where C_1 is a constant of integration. The problem may be completely solved once the strain energy function W is specified.

The important issue to emphasize here is that the system of partial differential equations div$\boldsymbol{T} = \boldsymbol{0}$ is a compatible system that may have an analytical solution when W is given. The telescopic shear is a special anti-plane shear problem and even though a solution to this problem may be found, in general we are not able to get any information on the general anti-plane problem (3.64). In fact, reconsidering (3.64), we show that the previous favourable situation is not verified now. This means that in the search for solutions of the balance equations by a semi-inverse method we are not always lucky; in some cases, the semi-inverse method may be used in a negative sense, by showing the nonexistence of solutions. For example, as in the general antiplane shear (3.64), it may happen that the balance equations reduce to an overdetermined set of differential equations which are not compatible, showing therefore that a pure antiplane shear is not always possible.

The physical components of $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ for the general anti-plane shear deformation (3.64), are given by

$$\begin{bmatrix} 1 & 0 & w_1 \\ 0 & 1 & w_2 \\ w_1 & w_2 & 1+k^2 \end{bmatrix}, \quad \begin{bmatrix} 1+w_1^2 & w_1 w_2 & -w_1 \\ w_1 w_2 & 1+w_2^2 & -w_2 \\ -w_1 & -w_2 & 1 \end{bmatrix}, \tag{3.73}$$

respectively, and the first three principal invariants are $I_1 = I_2 = 3 + k^2,\ I_3 = 1$, where $k = |\nabla w|$ and w_i $(i = 1, 2)$ are the derivatives of w with respect to $X_i,\ (i = 1, 2)$. Following (1.40), the Cauchy stress components are given by

$$\begin{aligned} T_{11} &= -p + 2\left(W_1 - (1+w_1^2)W_2\right), & T_{12} &= -2w_1 w_2 W_2, \\ T_{22} &= -p + 2\left(W_1 - (1+w_2^2)W_2\right), & T_{13} &= 2(W_1 + W_2)w_1, \\ T_{33} &= -p + 2W_1(1+k^2) - 2W_2, & T_{23} &= 2(W_1 + W_2)w_2. \end{aligned} \tag{3.74}$$

It is easy to check that now the balance equations form a system of three differential equations in the two unknowns $p(X_1, X_2, X_3)$ and $w(X_1, X_2)$, i.e.

$$\begin{aligned} &p_{,1} - 2[W_1 - (1+w_1^2)W_2]_{,1} + 2[w_1 w_2 W_2]_{,2} = 0, \\ &p_{,2} - 2[W_1 - (1+w_2^2)W_2]_{,2} + 2[w_1 w_2 W_2]_{,1} = 0, \\ &p_{,3} - 2[(W_1 + W_2)w_1]_{,1} - 2[(W_1 + W_2)w_2]_{,2} = 0, \end{aligned} \tag{3.75}$$

where the subscripts 1 and 2 stand for differentiation with respect to X_1 and X_2, respectively, and where

$$W_i = \left.\frac{\partial W}{\partial I_i}\right|_{I_1 = I_2 = 3+k^2, I_3 = 1}. \tag{3.76}$$

Since w_1 and w_2 are independent of X_3, so are I_1 and I_2. From $(3.75)_3$, we deduce that $p_{,3}$ has the same property. Thus p is linear in X_3:

$$p(X_1, X_2, X_3) = cX_3 + \bar{p}(X_1, X_2), \tag{3.77}$$

where c is a constant (called here the *axial pressure gradient*) and $\bar{p} = \bar{p}(X_1, X_2)$ is an undetermined function. A further reduction of the first two equilibrium equations $(3.75)_1$ and $(3.75)_2$ may be obtained by eliminating $p_{,12}$ by appropriate cross-differentation. In the end, we obtain an overdetermined differential system in which the unknown w must satisfy simultaneously the following two nonlinear ordinary differential equations,

$$\begin{aligned} &\left[\left(w_1^2 - w_2^2\right) W_2\right]_{,12} = [w_1 w_2 W_2]_{,11} - [w_1 w_2 W_2]_{,22}, \\ &[(W_1 + W_2)w_1]_{,1} + [(W_1 + W_2)w_2]_{,2} - \frac{c}{2} = 0. \end{aligned} \tag{3.78}$$

It is possible to show that the overdetermined differential system (3.78) is compatible only for particular choices of the strain energy function and only for special classes of materials. Knowles [72] gives necessary and sufficient condition in terms of the strain energy function for a homogeneous, isotropic, incompressible material to admit nontrivial states of anti-plane shear. For example, in the case of the following *rectilinear shear deformation*

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1), \tag{3.79}$$

the system (3.78) reduces to a single second-order differential equation for $w(X_1)$:

$$[(W_1 + W_2)w_1]_{,1} - \frac{c}{2} = 0, \tag{3.80}$$

where the subscript 1 stands for differentation with respect the argument X_1. In this case a formal solution of the balance equations is possible. This situation is similar to the situation discussed earlier for a telescopic shear deformation. Another example where positive results may occur is that of the *generalized neo-Hookean materials* (2.12). Here the overdetermined system (3.78) reduces to a single quasi-linear second-order partial differential equation

$$[(W_1)w_1]_{,1} + [(W_1)w_2]_{,2} - \frac{c}{2} = 0, \tag{3.81}$$

and then a formal solution of the balance equations is also possible.

From a mathematical point of view, the fact that a *pure* antiplane shear deformation cannot be sustained in an elastic material means that the overdetermined differential system (3.78), corresponding to the strain energy function we are using to model *real* materials, do not have common solutions. Therefore Mathematics says that the geometry and load condition of the problem does not allow a pure antiplane shear deformation. On the other hand, it may be possible to have a pure antiplane shear deformation coupled to *secondary* deformations (see [63]). For example, by coupling an *in-plane* deformation to the antiplane one, as

$$\begin{aligned} x_1 &= X_1 + u(X_1, X_2), \\ x_2 &= X_2 + v(X_1, X_2), \\ x_3 &= X_3 + w(X_1, X_2), \end{aligned} \tag{3.82}$$

where u, v are the *in-plane* smooth displacement functions. For every incompressible elastic material, the balance equations $\text{div}\boldsymbol{T} = \mathbf{0}$ now reduce to a determined system of partial differential equations. This does not mean that, for a generic material, it is not possible to deform the body as prescribed by our geometry and load condition, but it emphasizes that by semi-inverse methods it is not easy to understand when the equations lead to a deformation field that is more complex than an anti-plane shear. For generalized neo-Hookean materials, we have the following expressions for the Cauchy stress components

$$\begin{aligned} T_{11} &= -p + 2W_1[(1 + u_1)^2 + u_2^2], \\ T_{22} &= -p + 2W_1[v_1^2 + (1 + v_2)^2], \\ T_{33} &= -p + 2W_1[w_1^2 + w_2^2 + 1], \end{aligned} \tag{3.83}$$

$$T_{12} = 2W_1[(1+u_1)v_1 + u_2(1+v_2)],$$
$$T_{13} = 2W_1[(1+u_1)w_1 + u_2w_2],$$
$$T_{23} = 2W_1[v_1w_1 + (1+v_2)w_2].$$

It is clear that the stress components T_{13} and T_{23} involve a coupling of in-plane and out-of-plane deformations[7]. The boundary condition of traction may therefore couple the in-plane displacements with the out-of-plane displacement. It is only for special cases (for example of pure displacement boundary conditions) that in-plane and out-of-plane displacements may be decoupled.

3.2.3 Radial deformation

The following example shows how the semi-inverse method may be used to search for exact and analytical solutions which are not universal but relative universal (see Horgan [60]). Let us consider spherical polar coordinates for the radial deformation written as

$$r = r(R), \qquad \theta = \Theta, \qquad \phi = \Phi, \tag{3.84}$$

where (R, Θ, Φ) are the polar coordinates in the reference configuration and (r, θ, ϕ) are the polar coordinates in the current configuration, respectively, and $\mathrm{d}r/\mathrm{d}R > 0$. The polar components of the deformation gradient tensor associated with (3.84) are given by

$$\boldsymbol{F} = \mathrm{diag}(\mathrm{d}r/\mathrm{d}R, r/R, r/R) \tag{3.85}$$

and the principal stretches are thus $\lambda_1 = \mathrm{d}r/\mathrm{d}R$, $\lambda_2 = \lambda_3 = r/R$. Now, the equilibrium equations in the absence of body forces $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$ can be shown to reduce to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\hat{W}_1\right) - 2R\hat{W}_2 = 0, \tag{3.86}$$

which is a second-order nonlinear ordinary differential equation for $r(R)$. Six classes of compressible materials have received much attention in the literature; they are all examples of relative universal solutions for the solutions $r(R)$.

$$\text{Class I. } W = f(i_1) + b_1(i_2 - 3) + c_1(i_3 - 1), \quad f''(i_1) \neq 0, \tag{3.87}$$

where f is an arbitrary function of i_1, b_1 and c_1 are arbitrary constants and i_1, i_2, i_3 are the principal invariants of $\boldsymbol{V}$. This class represents the harmonic materials introduced by John [67]. In this case, on using the hypothesis $f''(i_1) \neq 0$, one finds that

$$r(R) = AR + \frac{B}{R^2}, \tag{3.88}$$

where A and B are constants of integration. Abeyaratne and Horgan [1] and Ogden [95] employed the deformation (3.88) to obtain closed-form solutions for pressurized

[7] It is possible to show that the pressure depends on the out-of-plane deformation and therefore that the normal stresses in (3.83) contain all the deformation fields.

hollow spheres composed of harmonic materials. Aboudi and Arnold [2] applied (3.88) to micromechanical modeling of multiphase composites.

$$\text{Class II. } W = a_2(i_1 - 3) + g(i_2) + c_2(i_3 - 1), \quad g''(i_2) \neq 0, \tag{3.89}$$

where g is an arbitrary function of i_2 and a_2 and c_2 are arbitrary constants. Here, one finds that

$$r^2(R) = AR^2 + \frac{B}{R}, \tag{3.90}$$

where A and B are constants of integration. Murphy [88] used the controllable deformation (3.90) to treat the problems of inflation and eversion of hollow spheres of class II materials. Aboudi and Arnold [2] utilized (3.90) in their recent study of micromechanics of multiphase composites.

$$\text{Class III. } W = a_3(i_1 - 3) + b_3(i_2 - 3) + h(i_3), \quad h''(i_3) \neq 0, \tag{3.91}$$

where h is an arbitrary function of i_3 and a_3 and b_3 are arbitrary constants. This class of materials are called *generalized Varga materials* [58]. Here, one finds that

$$r^3(R) = AR^3 + B, \tag{3.92}$$

where A and B are constants of integration. Horgan [58] used the controllable deformation (3.92) to illustrate the phenomenon of *cavitation* for compressible materials in a particularly tractable setting. Aboudi and Arnold [2] utilized (3.92) in their micromechanics analysis of composites undergoing finite deformation. Murphy [87] introduced the next three material classes.

$$\text{Class IV. } W = a_4 i_1 i_2 + b_4 i_1 + c_4 i_2 + d_4 i_3 + e_4, \quad a_4 \neq 0, \tag{3.93}$$

where a_4, b_4, c_4, d_4, e_4 are arbitrary constants. Here, one finds that

$$r^3(R) = \frac{(A + BR^3)^2}{R^3}, \tag{3.94}$$

where A and B are constants of integration.

$$\text{Class V. } W = a_5 i_2 i_3 + b_5 i_1 + c_5 i_2 + d_5 i_3 + e_5, \quad a_5 \neq 0, \tag{3.95}$$

where a_5, b_5, c_5, d_5, e_5 are arbitrary constants. Here, one finds that

$$r^5(R) = \frac{(A + BR^3)^2}{R}, \tag{3.96}$$

where A and B are constants of integration.

$$\text{Class VI. } W = a_6 i_1 i_3 + b_6 i_1 + c_6 i_2 + d_6 i_3 + e_6, \quad a_6 \neq 0, \tag{3.97}$$

where a_6, b_6, c_6, d_6, e_6 are arbitrary constants. Here, one finds that

$$r^2(R) = AR^2 + \frac{B}{R}, \tag{3.98}$$

where A and B are constants of integration. This deformation field is identical to that given for Class II (see (3.89)).

This type of investigation was proposed by Currie and Hayes [25] where the search for exact solutions starts from a different point of view. They search for special solutions by choosing a deformation whose geometry is completely known a priori; in doing so they are solving Ericksen's problems *in miniature*: they are searching all the corresponding relative universal solutions[8].

Other typical applications of semi-inverse investigations are concerned for example in finding, for a given deformation (fixed a priori), the general form of the strain energy for which such deformation is a controllable solution. This is a sort of *inverse problem*: find the elastic materials (i.e. the functional form of the strain energy function) for which a given deformation field is controllable (i.e. for which the deformation is a solution to the equilibrium equations in the absence of body force).

Both problems are very difficult to solve and generally only partial results are available. The influential papers by Knowles [75] and Currie and Hayes [25] have stimulated the developement of a large amount of research on closed-form solutions in nonlinear elasticity. Beatty, Boulanger, Carroll, Chadwick, Hill, Horgan, Murphy, Ogden, Polignone, Rajagopal, Saccomandi, Wineman, and many others have determined a long list of exact solutions for special classes of constitutive equations. We refer to the recent books edited by Fu and Ogden [43] and by Hayes and Saccomandi [55] for an overview of this activity.

Although several authors have used such *inverse procedures*, and many solutions have been derived by using such methods, it is not easy to find a definition describing the true power of these methods. Further, no general mathematical theory can be applied, at least at first, sight, because they are a sort of *heuristics methods*. Only Lie group theory can provide a general, algorithmic, and efficient method for obtaining exaxt solutions of partial differential equations by a reduction method. It shares many similarities with the semi-inverse method. For this reason many authors have tried to find a relationship between Lie's classical method of reduction and the semi-inverse method, but the standard Lie method of symmetry reduction is not always applicable; it has to be generalized to recover all the solutions obtainable via ad hoc reduction methods. Olver and Rosenau [96] introduced the concept of *weak symmetry*, based on the analytic properties of the overdetermined system, and made it clear that a group theory nature is indeed possible for every solution of a given partial differential equation[9]. But it is still not known how to obtain the relevant groups.

[8] The expression "in miniature" is taken from a paper by Knowles [75] where the author tries to find non-homogeneous universal solutions in the family of anti-plane shear deformations.

[9] In [114] Saccomandi, considering the Navier-Stokes equations, shows how it is necessary to resort to the idea of weak symmetries to recover all the solutions found by the semi-inverse method.

Notes

A list of some suitable of inverse methods, useful to solve boundary value problems in elasticity, are given in this chapter. We have also underlined the important contributions of these methods to continuum mechanics in understanding the non-linear behaviour of materials (or fluids in the case of fluid dynamics), overcoming the difficulty in solving boundary value problem by direct methods. By inverse procedure, in addition to those homogeneous, five families of universal solutions are been found (they are listed in Section 3.1.2) where we have not widely discussed families 2-5 to save space but we refer to Section 57 of [127] for more details. The first investigation about universal solutions dates back to 1954, when Ericksen [33] was able to find several families of universal inhomogeneous deformations. However the proof of Ericksen was not complete in two points:

1. when two principal stretches of the deformation are equal and at least one of the principal invariants is not constant;

2. when all the principal invariants of the deformation are constants.

The first point has been completely resolved by Marris and Shiau [80] who showed that if two principal stretches are equal then the universal deformations are homogeneous or enclosed in Family 2. As regards the second point the final answer is still lacking but further developments on this problem are contained in the work on universal solutions for the elastic dielectric by Singh and Pipkin [119]. As a by-product of this research a new family of deformation with constant invariants has been discovered (Family 5).

Although the search for solutions of boundary value problems by use of inverse methods has been important and fundamental, on the other hand we are nonetheless of the opinion that some solutions have been the source of possible confusion in the field, and that some investigations are even incorrect in their use of the semi-inverse method (see [28]). In the next chapter we develop our point of view further, by analysing in detail some inhomogeneous solutions for compressible materials subjected to isochoric deformations.

Chapter 4

Isochoric deformations of compressible materials

Rubbers and elastomers are highly deformable solids, which have the remarkable property of preserving their volume through any deformation. This permanent isochoricity, incorporated mathematically into the equations of continuum mechanics through the concept of internal constraint of incompressibility (see (1.39) or (1.40)), has led to the discovery of several exact solutions in isotropic finite elasticity, most notably to the controllable or universal solutions of Rivlin and co-workers (e.g. Rivlin [106]).

Subsequently, Ericksen [33] examined the problem of finding all such solutions. He found that there are no controllable finite deformations in isotropic compressible elasticity, except for homogeneous deformations [34]. The impact of that result on the theory of nonlinear elasticity was quite important, and for many years there has been "the false impression that the only deformations possible in an elastic body are the universal deformations" (see [25]). However, around the same time as the publication of Ericksen's result, there was considerable activity in trying to find solutions for nonlinear elastic materials using the semi-inverse method. A summary of these earlier results may be found in the monograph by Green and Adkins [49].

Even though for homogeneous isotropic incompressible nonlinearly elastic solids, the simplified kinematics arising from the constraint of no volume change has facilitated the analytic solution of a wide variety of boundary-value problems, the situation is quite different for compressible materials. Firstly, the absence of the isochoric constraint leads to more complicated kinematics. Secondly, since the only controllable deformations are the homogeneous deformations, the discussion of inhomogeneous deformations has to be confined to a particular strain energy function or class of strain energy functions. Nevertheless, some progress has been made in recent years in the development of analytic forms for the deformation and in the solution of boundary value problems. One strategy to find some exact solutions for compressible elastic materials consists in taking inspiration from isochoric solutions for incompressible materials, and to seek similar solutions in compressible elastic materials. However, it is obvious that the isochoric deformations of an incompressible elastic body have differents loads than the isochoric deformations of a compressible elastic body, because they will in general produce changes in volume when applied in compressible materials. A review of isochoric problems can be found in [60].

4.1 Pure torsion

In the incompressible isotropic theory of nonlinear elasticity, the problem of finite torsion was first considered by Rivlin [106, 107, 108], while relevant experimental issues were discussed in [105, 112]. Rivlin showed that finite torsion is a universal deformation (see Family 3 of universal solution in (3.53) when $A = C = F = 1,\ B = E = 0$). By virtue of the constraint of zero volume change, the deformation is that of *pure torsion* so that there is no extension in the radial direction and the cross-section of the cylinder remains circular. We know, by Ericksen result [34], that finite torsion is not sustainable, however, in all compressible isotropic elastic materials. In fact the deformation here is more complicated and in general, there will be some radial extension, see Polignone and Horgan [97] and Kirkinis and Ogden [70]. Those two articles present the torsional problem for the strain energy written either in terms of the principal invariants I_1, I_2, I_3, or in terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ of the left Cauchy stress tensor $\boldsymbol{B}$, or in terms of the principal invariants i_1, i_2, i_3 of $\boldsymbol{V}$. Polignone and Horgan [97] obtain a necessary condition for a pure torsion to be possible without imposing the zero traction on the lateral surface. Kirkinis and Ogden [70] find new necessary and sufficient conditions on the strain energy function for pure torsion with zero traction on the lateral surface of the cylinder.

The torsion problem in the compressible case is discussed in other important works as well. For example it is discussed from both theoretical and experimental viewpoints in [48] or in [49], where a formula is derived for the couple required to maintain the deformation in respect of an arbitrary (isotropic) strain energy function. Slight compressibility effects are investigated in [38], using the general theory of small deformations superimposed on a large deformation for the Blatz-Ko material model and for the Levinson-Burgess material in [78]. Currie and Hayes [25] determined constitutive relations for which pure torsion is sustainable and proposed a general class of materials, which includes the Hadamard material. The Blatz-Ko material for foam polyurethane elastomers has been studied recently in respect of pure torsion by various authors (see, for example, [8, 9, 20]). Loss of ellipticity for this material model during a pure torsional deformation was examined by Horgan and Polignone [99].

4.1.1 Formulation of the torsion problem

Let us consider the torsional deformation of an elastic solid circular cylinder of radius A due to applied twisting moments at its ends,

$$r = r(R), \qquad \theta = \Theta + \tau Z, \qquad z = Z, \tag{4.1}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, $\mathrm{d}r/\mathrm{d}R > 0$, and the constant $\tau > 0$ is the *twist* per unit undeformed length. Let us consider the strain energy function in terms of the first three principal invariants of $\boldsymbol{B}$, $W = \bar{W}(I_1, I_2, I_3)$. The deformation gradient tensor $\boldsymbol{F}$ for (4.1) is given by

$$\begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & \tau r \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.2}$$

and the physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} r'^2 & 0 & 0 \\ 0 & r^2/R^2 + \tau^2 r^2 & \tau r \\ 0 & \tau r & 1 \end{bmatrix}, \quad \begin{bmatrix} r'^{-2} & 0 & 0 \\ 0 & R^2/r^2 & -\tau R^2/r \\ 0 & -\tau R^2/r & 1 + \tau^2 R^2 \end{bmatrix}, \tag{4.3}$$

respectively. The first three principal strain invariants are

$$\begin{aligned} I_1 &= 1 + r'^2 + \frac{r^2}{R^2} + \tau^2 r^2, \\ I_2 &= \frac{r^2}{R^2} + r'^2 + \frac{r'^2 r^2}{R^2} + \tau^2 r'^2 r^2, \\ I_3 &= \frac{r'^2 r^2}{R^2}. \end{aligned} \tag{4.4}$$

Substituting (4.3) into (1.36), we obtain the physical components of the Cauchy stress

$$\begin{aligned} T_{rr} &= \beta_0 + \beta_1 r'^2 + \beta_{-1} r'^{-2}, \\ T_{\theta\theta} &= \beta_0 + \beta_1 \left(\frac{r^2}{R^2} + \tau^2 r^2 \right) + \beta_{-1} \frac{R^2}{r^2}, \\ T_{zz} &= \beta_0 + \beta_1 + \beta_{-1}(1 + \tau^2 R^2), \\ T_{\theta z} &= \beta_1(\tau r) - \frac{\tau R^2}{r} \beta_{-1}, \\ T_{r\theta} &= 0, \quad T_{rz} = 0. \end{aligned} \tag{4.5}$$

In the present case, the equilibrium equations, in absence of body forces, $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$, reduce to the single equation

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0. \tag{4.6}$$

Since $r = r(R)$, it is possible to consider the stress as a function of the reference co-ordinate R, i.e., $\boldsymbol{T} = \boldsymbol{T}(R)$ instead of $\boldsymbol{T} = \boldsymbol{T}(r)$. In this case the chain rule gives

$$\frac{\partial T_{rr}}{\partial R} + \frac{r'}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0. \tag{4.7}$$

By $(4.5)_{1,2}$, (1.38) and (4.4) we obtain a single ordinary nonlinear equation for $r(R)$,

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{Rr'}{r}(\bar{W}_1 + \bar{W}_2) + \frac{rr'}{R}(\bar{W}_3 + \bar{W}_2) + \tau^2 Rr'r\bar{W}_2 \right] + \left(\frac{Rr'^2}{r^2} - \frac{1}{R} \right)(\bar{W}_1 + \bar{W}_2) - \tau^2 R\bar{W}_1 = 0, \tag{4.8}$$

where the $\bar{W}_i$ $(i = 1, 2, 3)$ are evaluated at (4.4). For a solid circular cylinder of initial radius A subjected to end torques only, the boundary conditions of traction-free lateral surface are satisfied when

$$T_{rr}(A) = 0, \tag{4.9}$$

since $T_{r\theta} = T_{rz} \equiv 0$ by $(4.5)_5$. In addition, to ensure that $\boldsymbol{F}$ is bounded, we impose the following regularity condition

$$r(R) = \mathcal{O}(R) \text{ as } R \to 0. \tag{4.10}$$

Thus the two-point boundary value problem consists in solving (4.8) for $r(R)$ on $0 < R < A$ subject to the conditions (4.9) and (4.10).

The same problem has been written by Kirkinis and Ogden [70] in terms of the principal stretches in Eulerian principal axes[1] but in a more general form than here. These authors consider the case of torsional deformation superimposed on a uniform extension,

$$r = r(R), \qquad \theta = \Theta + \lambda_z \tau Z, \qquad z = \lambda_z Z, \tag{4.11}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, $\mathrm{d}r/\mathrm{d}R > 0$, the constant $\tau > 0$ is the twist per unit undeformed length, and λ_z is the uniform axial stretch. Here we consider the strain energy function in terms of the principal stretches, $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$. The deformation gradient tensor $\boldsymbol{F}$ for (4.11) has components,

$$\begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & \lambda_z \tau r \\ 0 & 0 & \lambda_z \end{bmatrix}, \tag{4.12}$$

and the physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} r'^2 & 0 & 0 \\ 0 & r^2/R^2 + \lambda_z^2 \tau^2 r^2 & \lambda_z^2 \tau r \\ 0 & \lambda_z^2 \tau r & \lambda_z^2 \end{bmatrix}, \tag{4.13}$$

$$\begin{bmatrix} r'^{-2} & 0 & 0 \\ 0 & R^2/r^2 & -\tau R^2/r \\ 0 & -\tau R^2/r & 1/\lambda_z^2 + \tau^2 R^2 \end{bmatrix}, \tag{4.14}$$

respectively. Let $\boldsymbol{\mu}^i$, $i = 1, 2, 3$, be the unit Eulerian principal axes associated with this deformation. We see that $\boldsymbol{e}_r$ is the Eulerian principal axis associated with the principal stretch $\boldsymbol{\mu}^1$ and hence

$$\lambda_1 = r'. \tag{4.15}$$

We may express the remaining two principal directions in terms of the cylinder polar axes $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$. Thus, we write

$$\boldsymbol{\mu}^2 = \cos\phi \boldsymbol{e}_\theta + \sin\phi \boldsymbol{e}_z, \quad \boldsymbol{\mu}^3 = -\sin\phi \boldsymbol{e}_\theta + \cos\phi \boldsymbol{e}_z, \tag{4.16}$$

[1] Ogden [95], Section 5.2.5, writes the same problem using the Lagrangian principal axes.

where ϕ defines the orientation of the axes $\boldsymbol{\mu}^2,\ \boldsymbol{\mu}^3$ relative to $\boldsymbol{e}_\theta,\ \boldsymbol{e}_z$. By defining the following rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \tag{4.17}$$

considering the left stretch tensor $\boldsymbol{V}$ in $(1.10)_2$, and by comparing $\boldsymbol{V}^2$ and $\boldsymbol{B}$ as in $(1.11)_1$, we obtain the connections

$$\begin{aligned} &\lambda_2^2\cos^2\phi + \lambda_3^2\sin^2\phi = \frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2, \\ &\lambda_2^2\sin^2\phi + \lambda_3^2\cos^2\phi = \lambda_z^2, \\ &(\lambda_2^2 - \lambda_3^2)\sin\phi\cos\phi = \lambda_z^2\tau r, \end{aligned} \tag{4.18}$$

from which we deduce that

$$\begin{aligned} &\lambda_2^2 + \lambda_3^2 = \frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2 + \lambda_z^2, \\ &(\lambda_2^2 - \lambda_z^2)(\lambda_z^2 - \lambda_3^2) = \lambda_z^4\tau^2 r^2, \\ &\lambda_2\lambda_3 = \frac{\lambda_z r}{R}. \end{aligned} \tag{4.19}$$

Further, we obtain the explicit expression for ϕ as

$$\cos 2\phi = \frac{\lambda_2^2 + \lambda_3^2 - 2\lambda_z^2}{\lambda_2^2 - \lambda_3^2}. \tag{4.20}$$

Since the Cauchy stress tensor $\boldsymbol{T}$ is coaxial in the isotropic case with the left Cauchy-Green strain tensor $\boldsymbol{B}$, we may express it in terms of its principal stresses $T_1,\ T_2,\ T_3$ through

$$\begin{aligned} T_{rr} &= T_1, & T_{\theta z} &= (T_2 - T_3)\cos\phi\sin\phi, \\ T_{\theta\theta} &= T_2\cos^2\phi + T_3\sin^2\phi, & T_{zz} &= T_2\sin^2\phi + T_3\cos^2\phi. \end{aligned} \tag{4.21}$$

By (4.20) and (4.21) we obtain the following connection

$$\lambda_z^2\tau r(T_{\theta\theta} - T_{zz}) = \left(\frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2 - \lambda_z^2\right) T_{\theta z}. \tag{4.22}$$

The principal stresses are given by (1.42) and we use (4.21) to obtain from equation (4.7)

$$\frac{\mathrm{d}}{\mathrm{d}R}(R\hat{W}_1) = \frac{\lambda_z}{\lambda_2\lambda_3}\frac{\lambda_2(\lambda_2^2 - \lambda_z^2)\hat{W}_2 - \lambda_3(\lambda_3^2 - \lambda_z^2)\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \tag{4.23}$$

where $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$, $(i, = 1, 2, 3)$ are evaluated at the values given by (4.15) and (4.19). In this case the boundary condition to be satisfied is

$$T_{rr}(A) = T_1(A) = 0. \tag{4.24}$$

4.1.2 Pure torsion: necessary and sufficient condition

On setting $r = R$ in equation (4.8), Polignone and Horgan [97] obtain a necessary condition on $\bar{W}$ for pure torsion to be possible. From (4.4), for pure torsional deformation ($r = R$), we obtain

$$I_1 = I_2 = 3 + \tau^2 R^2, \qquad I_3 = 1, \tag{4.25}$$

and so the deformation is *isochoric*, and by (4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\bar{W}_1 + \bar{W}_3 + (2 + \tau^2 R^2)\bar{W}_2\right] - \tau^2 R \bar{W}_1 = 0. \tag{4.26}$$

On employing the chain rule, (4.26) may be written as

$$2(3+\tau^2R^2)\bar{W}_{21} + 2(2+\tau^2R^2)\bar{W}_{22} + 2(\bar{W}_{31} + \bar{W}_{32} + \bar{W}_{11}) + 2\bar{W}_2 - \bar{W}_1 = 0, \tag{4.27}$$

where $\bar{W}_{ij} = \partial^2\bar{W}/(\partial I_i \partial I_j)$ $(i, j = 1, 2, 3)$ are evaluated at the values (4.25). The condition (4.27) is therefore a necessary condition on $\bar{W}$ for pure torsion to be possible (an equivalent condition was obtained by Currie and Hayes [25]).

Kirkinis and Ogden [70], on setting $r = \lambda_z^{-1/2} R$ in order to have isochoric deformation for the torsion superimposed on uniform extension, obtain from (4.15) and $(4.19)_{(1,3)}$

$$\lambda_1 = \lambda_z^{-1/2}, \quad \lambda_2\lambda_3 = \lambda_z^{1/2}, \quad \lambda_2^2 + \lambda_3^2 = \lambda_z^2 + \lambda_z^{-1} + \lambda_z \tau^2 R^2. \tag{4.28}$$

In terms of the stretches, the equations (4.21) are given by

$$\begin{aligned}
T_{rr} &= \lambda_z^{-1/2}\hat{W}_1, \quad T_{\theta z} = \sqrt{(\lambda_2^2 - \lambda_z^2)(\lambda_z^2 - \lambda_3^2)}\,\frac{\lambda_2\hat{W}_2 - \lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \\
T_{\theta\theta} &= \frac{(\lambda_2^2 - \lambda_z^2)\lambda_2\hat{W}_2 - (\lambda_3^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \\
T_{zz} &= \frac{(\lambda_z^2 - \lambda_3^2)\lambda_2\hat{W}_2 + (\lambda_2^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}.
\end{aligned} \tag{4.29}$$

On use of (4.28), the equilibrium equation (4.23) specializes to

$$\begin{aligned}
(\lambda_2^2 + \lambda_3^2 - \lambda_z^2 - \lambda_z^{-1})\frac{\lambda_2\hat{W}_{12} - \lambda_3\hat{W}_{13}}{\lambda_2^2 - \lambda_3^2} + \hat{W}_1 &= \\
\lambda_z^{1/2}\frac{(\lambda_2^2 - \lambda_z^2)\lambda_2\hat{W}_2 - (\lambda_3^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}&,
\end{aligned} \tag{4.30}$$

in which the derivatives of $\hat{W}$ are evaluated for (4.28). Equation (4.30) provides a necessary condition for the strain energy to admit the deformation considered and generalizes (4.27) for $\lambda_z = 1$ to the case $\lambda_z \neq 1$. When $\lambda_z = 1$, equation (4.28) reduces to

$$\gamma(\lambda\hat{W}_{12} - \lambda^{-1}\hat{W}_{13}) + (\lambda + \lambda^{-1})\hat{W}_1 = \lambda^2\hat{W}_2 + \lambda^{-2}\hat{W}_3, \tag{4.31}$$

where γ is defined by

$$\gamma = \lambda - \lambda^{-1} = \tau R, \tag{4.32}$$

setting $\lambda_2 = \lambda,\ \lambda_3 = \lambda^{-1}$. Equations (4.27) and (4.31) are clearly equivalent, but they do not guarantee that the zero-traction boundary condition on the lateral surface of the cylinder is satisfied and therefore, in general, appropriate radial tractions need to be supplied in order to maintain the deformation. By $(4.29)_1$, and since λ_z is constant, T_{rr} depends on the deformation only through the combination τR. On setting $T_{rr}(R) = T(\tau R)$, on the lateral surface, we then have $T_{rr}(A) = T(\tau A)$. Thus, the lateral traction vanish if

$$T_{rr}(A) = T(\tau A) = 0, \tag{4.33}$$

for all $\tau \geq 0$. From that condition follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_{rr}(A) = T'(\tau A)A = 0, \tag{4.34}$$

for all $\tau > 0$, and therefore, for any fixed $\tau > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}R} T_{rr}(R) = T'(\tau R)\tau = 0, \tag{4.35}$$

for all $0 < R < A$. Thus, $T_{rr}(R)$ is constant, and since it vanishes for $R = A$, $T_{rr} \equiv 0$ and from (4.6), it follows that $T_{\theta\theta} \equiv 0$ also. From $(4.29)_{(1,2)}$ we deduce that

$$\hat{W}_1 \equiv 0, \quad \lambda_2(\lambda_2^2 - \lambda_z^2)\hat{W}_2 - \lambda_3(\lambda_3^2 - \lambda_z^2)\hat{W}_3 = 0, \tag{4.36}$$

where the derivatives of $\hat{W}$ are evaluated for the stretches given by (4.28). The conditions (4.36) are necessary and sufficient for the strain energy function to admit the combined isochoric torsion and uniform extension with zero tractions on the lateral surface of the cylinder. To derive $(4.36)_2$, we used the inequality $\lambda_2 \neq \lambda_3$ (otherwise by (4.28) the trivial situation $\tau = 0,\ \lambda_z = 1,\ \lambda_2 = \lambda_3 = 1$ is verified). When $\lambda_z = 1$, the conditions (4.36) reduce to

$$\hat{W}_1 \equiv 0, \qquad \lambda^2\hat{W}_2 + \lambda^{-2}\hat{W}_3 = 0, \tag{4.37}$$

evaluated for $\lambda_2 = \lambda,\ \lambda_3 = \lambda^{-1},\ \lambda - \lambda^{-1} = \tau R$. Conditions (4.31) are obviously implied by (4.37).

If the strain energy W is written in terms of the principal invariants i_1, i_2, i_3 (see (1.32)), then the Cauchy stress components for pure torsional deformation become

$$\begin{aligned}
T_{rr} &= \tilde{W}_1 + (i-1)\tilde{W}_2 + \tilde{W}_3, \quad T_{\theta z} = \frac{\gamma}{i-1}\left(\tilde{W}_1 + \tilde{W}_2\right), \\
T_{\theta\theta} &= \frac{1}{i-1}\left(\tilde{W}_1(\gamma^2+2) + (i-2)(i+1)\tilde{W}_2 + (i-1)\tilde{W}_3\right), \\
T_{zz} &= \frac{1}{i-1}\left(2\tilde{W}_1 + (i+1)\tilde{W}_2 + (i-1)\tilde{W}_3\right),
\end{aligned} \tag{4.38}$$

where $\tilde{W}_j$ $(j = 1, 2, 3)$ are the derivatives of $\tilde{W}$ with respect to i_1, i_2, i_3, respectively, and evaluated for

$$i = i_1 = i_2 = \lambda + \lambda^{-1} + 1, \quad \gamma = \lambda - \lambda^{-1}, \quad i_3 = 1. \tag{4.39}$$

Here, the equilibrium equation (4.27) and (4.31) become

$$(\tilde{W}_{11} + i\tilde{W}_{12} + (i-1)\tilde{W}_{22} + \tilde{W}_{31} + \tilde{W}_{32})(i+1) + i(\tilde{W}_2 - \tilde{W}_1) = 0. \tag{4.40}$$

When $T_{rr} = T_{\theta\theta} = 0$ we obtain, after some rearrangement, the necessary and sufficient conditions as

$$i\tilde{W}_1 + \tilde{W}_2 = 0, \qquad (i^2 - i - 1)\tilde{W}_1 - \tilde{W}_3 = 0, \tag{4.41}$$

where, here, $\tilde{W}$ depends on i_1, i_2, i_3 and the derivatives are evaluated for (4.39).

In the case of pure torsion, the resultant axial force N on any cross-section of the cylinder and the resultant moment M are related by

$$N = -\tau M, \tag{4.42}$$

independently of which strain energy function is used. Thus, (4.42) establishes another example of a *universal relation*[2]. To show the relation (4.42), we consider the definitions

$$N = 2\pi \int_0^A T_{zz} R \, \mathrm{d}R, \qquad M = 2\pi \int_0^A T_{\theta z} R^2 \, \mathrm{d}R, \tag{4.43}$$

where the integrals (which are independent of Z) are taken over any cross-section of the cylinder. The relationship (4.22) at $\lambda_z = 1,\ T_{\theta\theta} = 0,\ r = R$ reduces to

$$T_{zz} = -\tau R T_{\theta z}, \tag{4.44}$$

and from (4.43), the relation (4.42), therefore, holds.

4.1.3 Some examples

It has been shown by Beatty [8] and by Carroll and Horgan [21] that pure torsion is possible for the following Blatz-Ko material (2.40)

$$\bar{W}(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right). \tag{4.45}$$

In fact, here the stress response equation takes the simple form

$$\boldsymbol{T} = \mu\left(\boldsymbol{I} - I_3^{-1/2}\boldsymbol{B}^{-1}\right), \tag{4.46}$$

and the equation (4.8) reduces to

$$3Rr^3r'' - r^3r' + R^3r'^4 = 0, \tag{4.47}$$

where the prime refers to the ordinary derivative with respect to R. The equation (4.47) is a second-order nonlinear ordinary differential equation where the parameter τ does not appear. The components of the stress T_{rr} and $T_{\theta\theta}$ do not contain

[2]It may be compared to the universal relation $(3.29)_2$, because pure torsion is an example of locally simple shear of magnitude $\gamma = \tau R$ in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_z)$ plane.

τ and it is clear that $r = R$ is a solution of (4.47) that verify (4.10). In this case, from (4.46), it is easy to see that

$$\begin{aligned} T_{rr} &= 0, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= 0, & T_{rz} &= 0, \\ T_{zz} &= -\mu\tau^2 R^2, & T_{\theta z} &= \mu\tau R, \end{aligned} \tag{4.48}$$

and so the boundary free traction condition (4.9) is satisfied. For the general Blatz-Ko material (2.38), the necessary condition (4.26) holds if and only if $f = 0$. In this case, the traction free boundary condition is also satisfied.

Consider the Hadamard material (2.21): to ensure the normalization conditions (1.34) and (1.64), the arbitrary function H in (2.21) must satisfy

$$H(1) = 0, \qquad H'(1) + c_1 + 2c_2 = 0. \tag{4.49}$$

One can see that the necessary condition (4.27) is satisfied if and only if

$$2c_2 = c_1. \tag{4.50}$$

The stress components for the Hadamard material are given, by (1.36), as

$$\begin{aligned} T_{rr} &= 2c_2\tau^2 R^2, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= 2(c_1 + c_2)\tau^2 R^2, & T_{\theta z} &= 2(c_1 + c_2)\tau R, \\ T_{zz} &= 0, & T_{rz} &= 0. \end{aligned} \tag{4.51}$$

Thus, Hadamard materials cannot sustain pure torsion ($2c_2 = c_1$) with traction free lateral surface, except in the degenerate case where $c_2 = 0$.

The authors in [70] and in [97] try to find a more general form of strain energy to sustain pure torsion[3] with the difference that Polignone and Horgan [97] do not impose free boundary condition. They try to obtain materials where pure torsion may be possible. For example they start by requiring that

$$\bar{W}_{21} + \bar{W}_{22} = 0, \tag{4.52}$$

where the derivatives are evaluated in (4.25). Condition (4.52) is a good device to eliminate the explicit dependence of the parameter τ in the equation (4.27). In fact the explicit term $\tau^2 R^2$ vanishes identically when (4.52) is assumed. Since functions of the form $P(I_1 - I_2, I_3)$ clearly satisfy (4.52), Polignone and Horgan consider the following general form of strain energy function,

$$\begin{aligned} W = \frac{\mu}{2}\big[&P(I_1 - I_2, I_3) + Q(I_1, I_3) + R(I_2)S(I_3) \\ &+ H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)\big], \end{aligned} \tag{4.53}$$

where $\mu > 0$ is the infinitesimal shear modulus and P, Q, R, S, H_i ($i = 1, 2, 3$) are sufficiently smooth functions. The strain energy function (4.53) satisfies (4.52) if and only if $R(I_2)$ is a linear function. Thus, we assume that

$$R(I_2) = k_1 I_2 + k_2, \tag{4.54}$$

[3] This is an important task from the mathematical point of view, but afterward one must establish whether the material described by the model obtained describes reality or if it remains only an idealization.

where k_1 and k_2 are arbitrary constants. On redefining $S(I_3)$ and $H(I_3)$ to include these constants, we rewrite (4.53) as

$$W = \frac{\mu}{2}\big[P(I_1 - I_2, I_3) + Q(I_1, I_3) + I_2 S(I_3) \\ + H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)\big], \quad (4.55)$$

in order to satisfy (4.52). The normalization conditions (1.34) and (1.64) require that

$$P(0,1) + Q(3,1) + 3S(1) + H_3(1) = 0 \quad (4.56)$$

and

$$-P_1(0,1) + P_2(0,1) + \frac{\partial Q}{\partial I_1}(3,1) + \frac{\partial Q}{\partial I_3}(3,1) \\ + 3S'(1) + 2S(1) + H_1(1) + 2H_2(1) + H_3'(1) = 0, \quad (4.57)$$

where the subscripts 1 and 2 on P indicate the derivatives with respect to the first and second arguments, respectively. From (4.27) and (4.52), we ask that

$$2(\bar{W}_{21} + \bar{W}_{31} + \bar{W}_{32} + \bar{W}_{11}) + 2\bar{W}_2 - \bar{W}_1 = 0. \quad (4.58)$$

On substitution from (4.55) into (4.58), one finds that

$$-3P_1(0,1) + \left(2\frac{\partial^2 Q}{\partial I_1^2} + 2\frac{\partial^2 Q}{\partial I_1 \partial I_3} - \frac{\partial Q}{\partial I_1}\right)_{I_1 = 3+\tau^2 R^2,\ I_3 = 1} \\ + 2\left(S'(1) + S(1)\right) + 2H_1'(1) - H_1(1) + 2H_2'(1) + 2H_2(1) = 0. \quad (4.59)$$

Rather than describe the most general class of functions P, Q, S, H_i ($i = 1, 2, 3$) for which (4.58) holds, Polignone and Horgan [97] indicate some possibilities. One possibility is to search for Q such that

$$2\frac{\partial^2 Q}{\partial I_1^2} + 2\frac{\partial^2 Q}{\partial I_1 \partial I_3} - \frac{\partial Q}{\partial I_1} = 0, \quad (4.60)$$

holds. One solution of (4.60) is given in the following form

$$Q(I_1, I_3) = \alpha e^{\beta(I_3 - 1)} e^{(1/2 - \beta)(I_1 - 3)}, \quad (4.61)$$

to within an arbitrary additive function of I_3, that we may include in $H_3(I_3)$. The parameters α and $\beta(\neq 1/2)$ are constants. To find a possible form for $S(I_3)$, one might seek S such that

$$S'(1) + S(1) = 0, \quad (4.62)$$

so that (4.59) is further simplified. Thus, it is possible to set

$$S = k e^{-(I_3 - 1)}, \quad (4.63)$$

where k is a constant. By the previous choices for Q and S, the condition (4.59) reduces to

$$-3P_1(0,1) + 2H_1'(1) - H_1(1) + 2H_2'(1) + 2H_2(1) = 0, \quad (4.64)$$

and the normalization condition (4.56) and (4.57) read as

$$
\begin{aligned}
& P(0,1) + \alpha + 3k + H_3(1) = 0, \\
& - P_1(0,1) + P_2(0,1) + \frac{\alpha}{2} - k + H_1(1) + 2H_2(1) + H_3'(1) = 0.
\end{aligned}
\tag{4.65}
$$

Thus, a material described by a strain energy function $\bar{W}$ of the form (4.55), with Q and S chosen as in (4.61) and (4.63), respectively, can sustain pure torsional deformations provided (4.64) holds. This is one possible way to generate a benchmark of possible strain energy functions. On every setting for the unknown function P, Q, S, H one could check afterwards if the strain energy function found satisfies the boundary condition as well.

An other investigation concerns the energy functions of the form

$$
\tilde{W}(i_1, i_2, i_3) = f(i_1)h_1(i_3) + g(i_2)h_2(i_3) + h_3(i_3), \tag{4.66}
$$

which is one generalization of the material described by classes I, II and III in (3.87), (3.89) and (3.91). Polignone and Horgan [97] show that the first two classes cannot sustain pure torsion (in general). Instead the third class verifies the condition (4.40) if and only if $a_3 = b_3$ but a uniformly distributed tensile loading would be required on the lateral surface of the cylinder.

4.2 Pure axial shear

The pure axial shear (also called telescopic shear) problem is a particular form of axisymmetric anti-plane shear. It has been introduced in Section (3.2.2) for the case of incompressible isotropic nonlinear elastic cylinder. Here, we are investigating when this isochoric deformation can be sustained for compressible homogeneous isotropic materials. In general, for an arbitrary compressible material, the cylinder will undergo both a radial $r(R)$ deformation, and an axial deformation $w(R)$. For an arbitrary incompressible, isotropic and homogeneous, hyperelastic material, Rivlin [107] has shown that the telescopic shear problem leads to a nonlinear ordinary differential equation for the radial displacement $w(R)$, whose solution may be obtained only upon specification of the strain energy function. Necessary and sufficient conditions on the form of the strain energy function for the compressible and incompressible cases, for which nontrivial states of anti-plane shear may be admissible, have been derived by Knowles [72, 74], but the mathematical structure used to derive the conditions for compressible material excludes the axisymmetric case. Here, our main references are Jiang and Beatty [64] and Polignone and Horgan [98], but some examples for telescopic shear in the compressible case are also described by Agarwal [3] and by Mioducowski and Haddow [82]. In [98], necessary conditions on the strain energy function W for pure axial shear to be possible are established by seeking solutions of the governing equations for which $r = R$. Two conditions on W are obtained in the form of a second-order and first-order nonlinear ordinary differential equation for axial displacement $w(R)$, whose solutions must be compatible. In [64], a single necessary and sufficient condition is obtained in order that the material may support pure axial shear, instead.

4.2.1 Formulation of the axial shear problem

Let us consider the axisymmetric finite axial shear deformation of an isotropic compressible nonlinearly elastic hollow circular cylinder with inner surface $R = A$ and outer surface $R = B$,

$$r = r(R), \qquad \theta = \Theta, \qquad z = Z + w(R), \tag{4.67}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, and $\mathrm{d}r/\mathrm{d}R > 0$. We set the inner surface $R = A$ to be bonded to a rigid cylinder so that

$$r(A) = A, \qquad w(A) = 0. \tag{4.68}$$

The deformation (4.67) may be achieved either by prescribing $r(R)$ and $w(R)$ on the outer surface $R = B$, or by applying a uniformly distributed axial shear traction to the outer surface of the cylinder and assuming that the radial traction is zero there,

$$T_{rr}(B) = 0, \qquad T_{rz}(B) = T_0, \tag{4.69}$$

where T_0 is a given constant. Let us assume that the cylinder is sufficiently long so that end effects are negligible and that the strain energy function is given in terms of the first three principal invariants of $\boldsymbol{B}$:

$$W = \bar{W}(I_1, I_2, I_3). \tag{4.70}$$

Corresponding to the deformation field (4.67), we have

$$\boldsymbol{F} = \begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & 0 \\ w' & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} r'^2 & 0 & r'w' \\ 0 & r^2/R^2 & 0 \\ r'w' & 0 & 1 + w'^2 \end{bmatrix}, \tag{4.71}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} (w'^2 + 1)/r'^2 & 0 & -w'/r' \\ 0 & R^2/r^2 & 0 \\ -w'/r' & 0 & 1 \end{bmatrix}, \tag{4.72}$$

where $r' = \mathrm{d}r/\mathrm{d}R$ and $w' \equiv \mathrm{d}w/\mathrm{d}R$. The first three principal invariants are given by

$$\begin{aligned} I_1 &= 1 + \frac{r^2}{R^2} + r'^2 + w'^2, \\ I_2 &= r'^2 + \frac{r^2}{R^2}\left(1 + r'^2 + w'^2\right), \\ I_3 &= r'^2 \frac{r^2}{R^2}. \end{aligned} \tag{4.73}$$

Substitution from $(4.71)_2$ and (4.72) into (1.36) yields the physical components of the Cauchy stress $\boldsymbol{T}$ as

$$
\begin{aligned}
T_{rr} &= \beta_0 + \beta_1 r'^2 + \beta_{-1}\frac{w'^2+1}{r'^2},\\
T_{\theta\theta} &= \beta_0 + \beta_1\frac{r^2}{R^2} + \beta_{-1}\frac{R^2}{r^2},\\
T_{zz} &= \beta_0 + \beta_1(w'^2+1) + \beta_{-1},\\
T_{rz} &= \beta_1 r'w' - \beta_{-1}\frac{w'}{r'},\\
T_{r\theta} &= 0, \quad T_{\theta z} = 0.
\end{aligned}
\tag{4.74}
$$

Because $r = r(R)$, it is convenient to consider that $\boldsymbol{T} = \boldsymbol{T}(R)$. The equilibrium equations in the absence of body force, $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$, for this deformation, reduce to the following two equations:

$$
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0, \qquad \frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} = 0. \tag{4.75}
$$

We observe that equation $(4.75)_2$ can also be written in the form

$$
\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rz}) = 0, \tag{4.76}
$$

so that on integrating and using $(4.74)_4$, $(4.73)_3$ and (1.38), we arrive at the following first-order nonlinear ordinary differential equation

$$
w'\left(RW_1 + \frac{r^2}{R}W_2\right) = K, \tag{4.77}
$$

where K is a constant and W_1, W_2 are the derivatives of W with respect to I_1 and I_2, respectively, evaluated at the values (4.73). The constant K appearing in (4.77) can now be expressed in terms of T_0. In fact, by $(4.74)_4$, (4.77) and (4.69), we find that

$$
K = \frac{r(B)T_0}{2}. \tag{4.78}
$$

Equation $(4.75)_1$, after using a chain rule to differentiate with respect to R, may be written as the following second-order nonlinear ordinary differential equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R}&\left(\frac{Rr'}{r}W_1 + \left(\frac{Rr'}{r} + \frac{rr'}{R}\right)W_2 + \frac{rr'}{R}W_3\right)\\
&+ W_1\left(\frac{Rr'^2}{r^2} - \frac{1}{R}\right) + W_2\left(\frac{Rr'^2}{r^2} - \frac{1}{R} - \frac{w'^2}{R}\right) = 0,
\end{aligned}
\tag{4.79}
$$

where again the derivatives W_i $(i = 1,2,3)$ are evaluated at the values (4.73). Equations (4.77) and (4.79) are a coupled pair of nonlinear ordinary differential equations for the unknowns functions $r(R)$ and $w(R)$.

4.2.2 Pure axial shear: necessary and sufficient conditions

Polignone and Horgan [98] obtain necessary conditions on the strain energy function for pure axial shear to be possible by setting $r = R$ in the equations (4.77) and (4.79). When $r = R$, first we know by (4.73) that

$$I_1 = I_2 = 3 + w'^2, \qquad I_3 = 1, \tag{4.80}$$

and so the deformation is isochoric. From (4.79), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}(W_1 + 2W_2 + W_3) - \frac{w'^2}{R} W_2 = 0. \tag{4.81}$$

Employing the chain rule, and setting $w' \neq 0$ to have a nontrivial solution, this last equation can be written as

$$2(W_{11} + 2W_{22} + 3W_{12} + W_{13} + W_{23})w'' - W_2 \frac{w'}{R} = 0. \tag{4.82}$$

From (4.77), we obtain the necessary condition

$$(W_1 + W_2)w' = \frac{BT_0}{2R}. \tag{4.83}$$

In (4.82) and in (4.83) the derivatives are evaluated in (4.80). By differentiating both sides of (4.83) with respect to R and using the chain rule, we obtain

$$(W_1 + W_2)w'' = -\frac{BT_0}{2R^2} - 2(W_{11} + 2W_{12} + W_{22})w'^2 w'', \tag{4.84}$$

where the derivatives are evaluated at values (4.80). When $r = R$, the corresponding stress components in (4.74) become

$$\begin{aligned}
T_{rr} &= \beta_0 + \beta_1 + \beta_{-1}(w'^2 + 1), \\
T_{\theta\theta} &= \beta_0 + \beta_1 + \beta_{-1}, \\
T_{zz} &= \beta_0 + \beta_1(w'^2 + 1) + \beta_{-1}, \\
T_{rz} &= (\beta_1 - \beta_{-1})w', \\
T_{r\theta} &= 0, \quad T_{\theta z} = 0,
\end{aligned} \tag{4.85}$$

where the β_i ($i = -1, 0, 1$) are evaluated at values (4.80). Setting $r = R$, the boundary condition $(4.68)_1$ is satisfied and so from $(4.68)_2$ and $(4.69)_1$, the remaining boundary conditions are

$$w(A) = 0, \quad (W_1 + 2W_2 + W_3)\,|_{I_1 = I_2 = 3 + w'^2,\ I_3 = 1,\ R = B} = 0, \tag{4.86}$$

respectively.

In [64], the problem is introduced in the reference configuration. From (1.21), the physical components of the first Piola-Kirchhoff stress tensor are given by

$$\begin{aligned}
&(T_R)_{RR} = (T_R)_{ZZ} = 2(W_1 + 2W_2 + W_3), \quad (T_R)_{RZ} = -2w'(W_2 + W_3), \\
&(T_R)_{\Theta\Theta} = 2(W_1 + (2 + w'^2)W_2 + W_3), \quad (T_R)_{ZR} = 2w'(W_2 + W_2), \\
&(T_R)_{R\Theta} = (T_R)_{\Theta R} = (T_R)_{Z\Theta} = (T_R)_{\Theta Z} = 0,
\end{aligned} \tag{4.87}$$

where $W_i \equiv \partial W/\partial I_i,\ (i = 1, 2, 3)$ are evaluated at (4.80). The equilibrium equations, in the absence of body forces, reduce to the following radial and axial equilibrium equations

$$\begin{aligned} &R\frac{\mathrm{d}}{\mathrm{d}R}\left(W_1 + 2W_2 + W_3\right) = w'^2 W_2, \\ &\frac{\mathrm{d}}{\mathrm{d}R}\left(R(T_R)_{ZR}\right) = 0. \end{aligned} \tag{4.88}$$

Similarly to the definition in (3.24), after setting $w' = k$, we define the shear stress response function as

$$\tau(k) \equiv (T_R)_{ZR} = k\mu(k^2) \tag{4.89}$$

and the shear response function as

$$\mu(k^2) \equiv 2(W_1 + W_2). \tag{4.90}$$

We may therefore rewrite (4.88) as

$$\begin{aligned} &R\frac{\mathrm{d}}{\mathrm{d}R}\left(W_1 + 2W_2 + W_3\right) = k^2 W_2, \\ &\frac{\mathrm{d}}{\mathrm{d}R}\left(R\tau(k)\right) = 0. \end{aligned} \tag{4.91}$$

As in (3.26), by the empirical inequality (1.46), we know that

$$\mu(k^2) > 0, \quad \forall k. \tag{4.92}$$

We observe that the shear strain $k(R)$ vanishes identically if either the shear strain itself or its derivative $\mathrm{d}k/\mathrm{d}R$ vanishes at a single location in $[A, B]$. In fact by $(4.91)_2$ and (4.89), we obtain that

$$Rk\mu(k^2) = h, \tag{4.93}$$

where h is an integration constant. If $[A, B]$ contains the origin, the statement is trivial, because (4.92) holds. Thus, if there exist a point $0 \neq R_0 \in [A, B]$ such that $k(R_0) = 0$, by (4.92), it is necessary to have $h = 0$ and therefore $k \equiv 0$. If there exist a point $0 \neq R_1 \in [A, B]$ such that $\mathrm{d}k(R_1)/\mathrm{d}R = 0$, by differentiation of (4.93) with respect to R, we have

$$R\frac{\mathrm{d}k(R)}{\mathrm{d}R}\frac{\mathrm{d}}{\mathrm{d}k}[\tau(k(R))] = -\tau(k(R)), \tag{4.94}$$

which with the aid of (4.93), may be written as

$$\frac{\mathrm{d}k(R)}{\mathrm{d}R}\frac{\mathrm{d}}{\mathrm{d}k}[\tau(k(R))] = -\frac{h}{R^2}, \tag{4.95}$$

and since $\mathrm{d}k(R_1)/\mathrm{d}R = 0$ we therefore obtain the constant $h = 0$, deducing as in the previous case that $k \equiv 0$.

The necessary and sufficient condition for a compressible, isotropic and homogeneous, hyperelastic material to be capable of sustaining nontrivial, pure axial shear deformation whose strain energy function W satisfies (4.92) is the following condition

$$(W_1 + W_2)\left[W_{11} + I_1 W_{12} + (I_1 - 1)W_{22} + W_{13} + W_{23} + \tfrac{1}{2}W_2\right] = \\ (I_1 - 3)\left[W_1(W_{12} + W_{22}) - W_2(W_{11} + W_{21})\right], \quad (4.96)$$

for $I_1 = I_2 \geq 3,\ I_3 = 1$. By using (4.80), and recalling that the strain energy function W depends on the shear strain k only through the invariants I_1 and I_2, it follows that (4.96) admits the following representation

$$(W_1 + W_2)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -kW_2\frac{\mathrm{d}}{\mathrm{d}k}\left[k(W_1 + W_2)\right]. \quad (4.97)$$

Recalling the definition of the shear stress response function in (4.89) and since we may suppose that k and its derivative with respect to R never vanishes (otherwise by previous consderations the only solution $w(R)$ would be a constant), we may rewrite (4.97) as

$$\tau(k)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -k^2 W_2\frac{\mathrm{d}\tau(k)}{\mathrm{d}k}. \quad (4.98)$$

To prove sufficiency, we need to show that every solution of the equation $(4.91)_2$ also satisfies the radial equilibrium equation $(4.91)_1$ when the condition (4.92) and (4.98) are identically satisfied. Since the equilibrium equation $(4.91)_2$ may be written in the form (4.94), after substitution from (4.94) into (4.98), we obtain the equilibrium equation $(4.91)_1$ and sufficiency is hence shown.

To prove the necessary condition, we consider a solution $\bar{w}$ of both equations $(4.91)_1$ and $(4.91)_2$. Since the strain energy function depends on k trough the invariants I_1 and I_2, we may rewrite Equation $(4.91)_1$ as

$$R\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3)\frac{\mathrm{d}k}{\mathrm{d}R} = k^2 W_2. \quad (4.99)$$

Because equation $(4.91)_2$ is equivalent to (4.94), we use (4.94) in (4.99) and we obtain

$$\tau(k)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -k^2 W_2\frac{\mathrm{d}\tau(k)}{\mathrm{d}k}. \quad (4.100)$$

Thus (4.98) is obtained and the necessary condition is therefore proved. In order to attain this result, a division by μ was necessary, but we recall that this is always possible because (4.92) holds.

4.2.3 Some examples

Let us consider the Hadamard material (2.21). It follows from (4.90) that the shear response function for the Hadamard material (2.21), is

$$\mu(k^2) = 2(c_1 + c_2) > 0, \quad (4.101)$$

a constant, and that the shear stress response function (4.89) is

$$\tau(k) = 2k(c_1 + c_2), \tag{4.102}$$

a linear dependence of k. Here, in accordance with (4.96), it follows immediately that non-trivial, controllable, axial pure shear deformations are possible in every Hadamard material (2.21) for which

$$\frac{1}{2}(W_1 + W_2)W_2 = (c_1 + c_2)c_2 = 0. \tag{4.103}$$

Since $(c_1 + c_2) > 0$,

$$c_2 = 0 \tag{4.104}$$

is a necessary and sufficient condition for the pure axial shear to be controllable in an Hadamard material. When $c_2 = 0$ and $c_1 > 0$, from $(4.91)_2$ or (4.83) the out-of-plane displacement $w(R)$ is given by

$$w(R) = \frac{BT_0}{2c_1} \ln\left(\frac{R}{A}\right), \tag{4.105}$$

and from $(4.85)_{3,4}$, the nonzero stress components are

$$T_{zz} = \frac{B^2 T_0^2}{2c_1 R^2}, \quad T_{rz} = \frac{BT_0}{R}. \tag{4.106}$$

It is readily seen that (4.96) fails for a Blatz-Ko material (2.40), which is capable of sustaining pure torsion (see Section 4.1.3).

In searching for a more general class of material for which the pure axial shear is possible, Polignone and Horgan [98] require that the strain energy W satisfy the following condition

$$W_{11} + 2W_{12} + W_{22} = 0, \tag{4.107}$$

where the derivatives are evaluated in (4.80). By (4.84), and since we are assuming that $W_1 + W_2 > 0$, we obtain

$$w'' = -\frac{BT_0}{2R^2(W_1 + W_2)}. \tag{4.108}$$

On employing (4.107), (4.108), (4.83) in (4.82), we find that

$$2(W_{22} + W_{12} + W_{13} + W_{23}) + W_2 = 0. \tag{4.109}$$

Thus, Polinone and Horgan [98] start by considering the following form of the strain energy

$$W = \frac{\mu}{2}\left[P(I_1 - I_2, I_3)(I_1 - 3) + Q(I_1 - I_2, I_3)(I_2 - 3) + R(I_1 - I_2, I_3)\right] \tag{4.110}$$

where $\mu > 0$ is the infinitesimal shear modulus, and P, Q, R are sufficiently smooth functions. This form of strain energy function certainly verifies the conditions (4.107). The normalization conditions (1.34) and $(1.64)_1$ are satisfied by (4.110) if

$$R(0, 1) = 0, \tag{4.111}$$

and

$$P(0,1) + 2Q(0,1) - R_1(0,1) + R_2(0,1) = 0, \tag{4.112}$$

where the subscripts $1, 2$ indicate the derivatives with respect to the first and second arguments, respectively. The task now is to find conditions on the functions P, Q, R such that condition (4.109) is satisfied. Hence, on substitution from (4.110) in (4.109), one finds that

$$\begin{aligned} 2\left[P_2(0,1) + Q_2(0,1) - (P_1(0,1) + Q_1(0,1))\right] \\ + Q(0,1) - R_1(0,1) - \left[P_1(0,1) + Q_1(0,1)\right] w'^2 = 0. \end{aligned} \tag{4.113}$$

Since we are searching for w' as a varying function (otherwise (4.82) and (4.83) are not compatible), the condition (4.113) implies that

$$P_1(0,1) + Q_1(0,1) = 0 \tag{4.114}$$

and

$$2P_2(0,1) + 2Q_2(0,1) + Q(0,1) - R_1(0,1) = 0. \tag{4.115}$$

In order that $\mu > 0$, i.e. $W_1 + W_2 > 0$, it is necessary that

$$P(0,1) + Q(0,1) > 0. \tag{4.116}$$

Thus a cylindrical tube composed of a material described by a strain energy function W of the form (4.110), with P, Q chosen so that (4.114) is satisfied, can sustain pure axial shear provided (4.115) and (4.116) hold. From (4.83) and the boundary condition $(4.68)_2$, we obtain the solution

$$w(R) = \frac{BT_0}{\mu\left(P(0,1) + Q(0,1)\right)} \ln\left(\frac{R}{A}\right) \tag{4.117}$$

for any W of the form (4.110). To satisfy the boundary condition $(4.69)_1$, by (4.86), (4.112) and (4.114), one finds that

$$P_2(0,1) + Q_2(0,1) = 0. \tag{4.118}$$

Combining (4.118) and (4.115), we obtain

$$Q(0,1) - R_1(0,1) = 0. \tag{4.119}$$

Thus, in summary, provided that (4.114), (4.116), (4.118), and (4.119) hold, any compressible material described by (4.110) allows pure axial shear of the tube arising from a uniform shear traction applied to its outer surface, with the radial traction vanishing there. By application also of (4.112), from (4.85), we find that the only nonzero stresses are then

$$\begin{aligned} T_{zz} &= \frac{B^2 T_0^2}{\mu\left(P(0,1) + Q(0,1)\right) R^2}, \\ T_{rz} &= \frac{BT_0}{R}. \end{aligned} \tag{4.120}$$

In the same way as for the pure torsion deformation, here, it is possible to give some explicit example of (4.110) when P, Q, R are chosen. Polignone and Horgan [98] give some examples. One of these is the following strain energy,

$$W = \frac{\mu}{2}\gamma\Big[e^{\alpha(I_1-I_2)}e^{\beta(I_3-1)}(I_1-3) + e^{-\alpha(I_1-I_2)}e^{-\beta(I_3-1)}(I_2-3) \\ + e^{(I_1-I_2)}e^{-2(I_3-1)} - 1\Big],$$

with $\alpha \neq 0,\ \beta \neq 0,\ \gamma > 0$ arbitrary constants.

4.3 Some other meaningful isochoric deformations

A third isochoric deformation for compressible materials that has been investigated in a similar fashion is that of *azimuthal shear* (or circular shear) of a cylindrical tube,

$$r = R, \qquad \theta = \Theta + g(R), \qquad z = Z, \tag{4.121}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, and the inner surface of the tube is bonded to a rigid cylinder. The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction on the outer surface together with zero radial traction or by subjecting the outer surface to a prescribed angular displacement, with zero radial displacement. For compressible materials, we know by Ericksen's result [34] that azimuthal shear is not a universal solution and that in general, it is accompained by a radial deformation. These axisymmetric fields are governed by a coupled pair of nonlinear ordinary differential equations, one of which is second-order and the other first-order. Azimuthal shear, therefore, cannot be sustained by all compressible materials, unless certain auxiliary conditions on the strain energy function are satisfied. That problem has been examined by Beatty and Jiang [10], Haughton [52], Jiang and Ogden [66] and Polignone and Horgan [100].

The *generalized azimuthal shear* is an isochoric deformation of the form

$$r = R, \qquad \theta = \Theta + g(R, Z), \qquad z = Z, \tag{4.122}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively. This deformation (or its $Z-$independent specialization) may also appear under the names of *circular* or *rotational* shear. For compressible materials, that problem has been investigated by Kirkinis and Tsai [71].

The isochoric deformation consisting of the composition of the shearing deformation (4.67) (with $r(R) = R$) and (4.121) is called *helical shear* and it is described by

$$r = R, \qquad \theta = \Theta + g(R), \qquad z = Z + w(R), \tag{4.123}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively. This last problem has been examined by Beatty and Jiang [11].

4.4 Nearly isochoric deformations for compressible materials

In the previous sections we have seen an example of how a strategy of applying the semi-inverse method, while dealing with complex models, generalizes forms of solutions already known within the framework of a simpler theory. Motivated by the results obtained in the incompressible case, we have tried to understand what happens in the compressible case. By doing so, many important exact solutions for special classes of compressible elastic materials have been obtained. In [28], we emphasized that great care has to be exercised in using semi-inverse method in continuum mechanics to delineate classes of constitutive equations that admit a particular class of deformations and motions. Sometimes, the admissibility of a given deformation field is considered to delineate special classes of constitutive laws. We pointed out that the classes of constitutive equations thus identified from the standpoint that it may admit a type of deformation may lead to models that exhibit *physically unacceptable mechanical behavior.*

To illustrate the dangers inherent to merely turning the mathematical crank to determine classes of constitutive equations where a certain class of deformations are possible, we now consider the torsion of a cylindrical shaft (§4.4.1), the axisymmetric anti-plane shear of a cylindrical tube (§4.4.2), and then the propagation of transverse waves (§4.4.3) in a compressible nonlinear elastic material. We show that great care has to be exercised in appealing to the semi-inverse method. The first and third examples are extracted from our recent work [28]. In the first and second examples, we consider some static deformations with the help of which we can lay bare the confusion that has been created in seeking semi-inverse solutions. By considering torsional deformation and axial axisymmetric shear of a cylindrical shaft and tube, respectively, we discuss step by step the criticism concerning the mistakes that have been made as well as the possible errors that can be committed. Then in the third example, we consider the propagation of transverse bulk waves (primary motion), which, according to general nonlinear elasticity theory, must always be coupled to a longitudinal wave (secondary motion). Instead of considering what happens within the context of the linearized theory, a second-order theory and then the general nonlinear setting, we consider a top-to-bottom approach. We derive the general equations and, assuming that the amplitude of the displacements is of order ϵ, we show that at the first order we recover the results of the linearized theory and that at a higher order of approximation, we may have some insight into the coupling between the various modes of deformation. Here, the interesting point is the occurence of the phenomena of resonance between the primary and secondary fields.

Let us recall that it has been possible to determine the most general class of compressible materials for which pure torsion is a controllable deformation in the case of a circular solid cylinder. This means that for the constitutive equations that allow the deformation in question, the balance equations are satisfied for the pure torsion deformation. The next step is to ensure that the lateral surface of the circular cylinder is traction-free. Now, because simple torsion is an isochoric deformation, we have to ensure that the lateral boundary has to be traction-free while the volume remains constant. There is no reason to expect that this situation is automatically complied with in a compressible material. It is more natural to expect that when the lateral boundary of the cylinder is traction-free, the volume change has to be non-zero. In some sense, the behavior of a class of compressible materials such that pure torsion is controllable is *extraordinary.* We now investigate quantitatively the meaning of this sort of unusual possibility.

To make this claim quantitative, let us observe that any *idealized material* characterized by special mathematical properties cannot be clearly identified in the *real world.* That is, all mathematical models have to be viewed as approximations and one has to evaluate how well such models represent reality. We have to make some determination of what we will find acceptable in terms of an approximate answer. Such a determination cannot be totally subjective and one has to have some sort of agreement amongst those developing and using such models. Whether the criticism concerning the inapplicability of certain models is appropriate or otherwise needs to be judged by the modeller.

For example, let us suppose that we wish to consider the mathematical assumption that $W = W(I_2, I_3)$ only with regard to a specific body. This is exactly the constitutive assumption made by Blatz and Ko [8] in their celebrated model for foamed polyurethane elastomeric foams. It is imperative, when we make such an assumption, to check whether the experimental data backs the validity of the *mathematical* relationship

$$\partial W/\partial I_1 = 0. \tag{4.124}$$

Because, the first derivatives of the strain energy function are the mechanical quantities directly related to the stress, the relation (4.124) is indeed the correct way to check the constitutive assumption $W = W(I_2, I_3)$, for example in a biaxial experiment. It is clear that in the *real world*, our measurement in itself introduces an *uncertainty* with regard to the measured quantity, and that the accuracy of measurement is such that any measurement of the mechanical quantity $\partial W/\partial I_1$ to check the (4.124) will deliver a real number ϵ different from zero. It is not merely the prerogative of the modeller to say when ϵ is sufficiently small enough to be considered zero but, and as always, any theoretical assumption is an approximation and making such an approximation is an *art*. Roughly speaking, in a nonlinear theory, just because a certain quantity is small it does not follow that everything else connected with this quantity is or remains small. For this reason, we must be very careful in considering constitutive assumptions generated by purely mathematical arguments such as the ones arising from the semi-inverse method[4].

On the other hand, it is clear that approximations must be consistent and for the specific problem under consideration the following problem arises. If a given problem depends on various parameters α_i, $i = 1, ..., n$ and depends on a *small* parameter ϵ such that for $\epsilon = 0$ the secondary deformation may be ignored, then the small ϵ-approximation is consistent if for $\epsilon << 1$ the secondary field is neglible for any admissible value of the parameters α_i.

[4]We point out that this procedure is exactly the reverse of the constitutive assumption that comes out from a rigorous mathematical definition of some physical intuition. Notable examples of this last situation are the concept of frame indifference and material symmetry. In this case we start by the evidence provided by our observations in the real world and we then try to translate this into mathematics; in the former case we force mathematics to fit into the real world.

4.4.1 Nearly pure torsion of compressible cylinder

Let us consider a compressible cylinder of radius A subjected to the torsional deformation (4.1). We refer to the pure torsion of the cylindrical shaft as the "primary deformation", while by "secondary deformation" we mean the radial displacement $r(R)$. This means

$$\max_{R\in[0,A]} \left| \frac{r(R)}{R} - 1 \right| \approx O(\epsilon), \tag{4.125}$$

or $\sqrt{I_3} \approx 1$ for all $R \in [0, A]$ and for any other parameters (α_i for previous reference).

Now let us consider the classical Blatz-Ko material (2.40), with the strain energy function

$$W = \frac{\mu}{2} \left[\left(\frac{I_2}{I_3} - 3 \right) + 2(\sqrt{I_3} - 1) \right], \tag{4.126}$$

where μ is a constant, the initial shear modulus. This model is of the form $W = W(I_2, I_3)$ and it is well known (see Section 4.1.3) that for the class of materials described by the strain energy function given by (4.126), the isochoric simple torsion deformation is controllable.

Let us consider a more general strain energy function than (4.126), i.e.

$$W = k(I_1 - 3) + \frac{\mu}{2} \left[\left(\frac{I_2}{I_3} - 3 \right) + 2(1 - 2k/\mu)(\sqrt{I_3} - 1) \right], \tag{4.127}$$

where k and μ are constants. The strain energy function (4.127) differs from (4.126) by a term linear in I_1 and a *null-Lagrangian* term $\sqrt{I_3}$ (see Haughton [53]) such that the usual restrictions imposed by the normalization conditions are satisfied. Clearly as $k \to 0$ we recover (4.126) from (4.127).

The derivatives of the strain energy function (4.127) with respect to the invariants are

$$W_1 = k, \qquad W_2 = \frac{\mu}{2I_3}, \qquad W_3 = \frac{\mu}{2} \left(\frac{1 - 2k/\mu}{\sqrt{I_3}} - \frac{I_2}{I_3^2} \right). \tag{4.128}$$

Now it is possible to evaluate via a suitable experiment the magnitude of the parameter k and to decide if the assumption $W_1 = 0$ is reasonable on the basis of fitting the experimental data. If $k = 0$, then the model (4.127) reduces to (4.126). Our point is that this model is so special that it is not possible to ensure that the predictions of the mechanical response are not in contradiction with the assumption $k = 0$.

To make this point more quantitative, the next step is to introduce the dimensionless independent variable $\zeta = R/A \in [0, 1]$, the dimensionless dependent variable

$$F(\zeta) = r/A \tag{4.129}$$

and the quantities

$$\hat{\tau} = A\tau, \quad \hat{k} = k/\mu. \tag{4.130}$$

The introduction of (4.128), evaluated for the specific deformation under consideration, in (4.8), leads to the equation

$$\hat{k}\left(\frac{\zeta F''}{F}+\frac{F'}{F}-\hat{\tau}^2\zeta-\frac{1}{\zeta}\right)+\frac{3}{2}\frac{\zeta F''}{FF'^4}+\frac{\zeta^3}{2F^4}-\frac{1}{2FF'^3}=0. \tag{4.131}$$

(Here $F'=\mathrm{d}F/\mathrm{d}\zeta$). Moreover, from $(4.5)_1$, the dimensionless radial stress component associated with the deformation, for the model (4.127) is given by

$$\hat{T}_{\zeta\zeta}(\zeta)=1-2\hat{k}+2\hat{k}\frac{F'^2}{\sqrt{I_3}}-\frac{F'^{-2}}{\sqrt{I_3}}. \tag{4.132}$$

Therefore, for a solid circular cylinder initially of radius A subjected to end torques only, the boundary value problem of interest here is given by equation (4.131), subject to the conditions $\hat{T}_{\zeta\zeta}(1)=0$ (i.e. $T_{rr}(A)=0$) and $F(\zeta)\to 0$ as $\zeta\to 0$. We point out that the isochoric solution $F(\zeta)=\zeta$ is controllable for the model (4.127) if and only if $k=0$ and in this case, $\hat{T}_{\zeta\zeta}(1)=0$.

It seems unlikely that one can obtain an explicit exact solution for equation (4.131), and even a numerical solution for the boundary value problem under investigation is not easy to obtain because the boundary condition on $\zeta=1$ is nonlinear and of mixed type. For this reason, we consider an approximate $\mathcal{O}(\hat{k})$ solution. A straighforward computation gives

$$F(\zeta)\approx\zeta+\hat{k}\frac{\hat{\tau}^2\zeta}{24}\left(2\zeta^2-5\right), \tag{4.133}$$

and the $\mathcal{O}(\hat{k})$ volume approximation is

$$J\approx 1+\mathcal{V}(\hat{\tau}^2,\zeta)\hat{k}, \tag{4.134}$$

where

$$\mathcal{V}(\hat{\tau}^2,\zeta)=\frac{(4\zeta^2-5)\hat{\tau}^2}{12}$$

is the local variation of volume at order $\hat{k}$. The maximum of this variation is

$$\left|\mathcal{V}(\hat{\tau}^2,0)\right|=\frac{5}{12}\hat{\tau}^2. \tag{4.135}$$

Because equations (4.133) and (4.135) depend not only on $\hat{k}$ but also on τ^2, and because the two parameters are independent, it is clear that the approximation $\hat{k}=0$ may be not consistent.

Now, imagine that you are able to evaluate via an experiment the parameter $\hat{k}$ and that you discover that this parameter is small. It is clear that the experimentally determined number may be never small enough to justify the model corresponding to $\hat{k}=0$ and only the modeller can choose to set $\hat{k}=0$, or do otherwise. Our computation shows that such an assumption might be dangerous under certain circumstances. Indeed, while the limiting model for $\hat{k}\to 0$ predicts that during torsion the variation of volume is null, this is not always the case even for very small $\hat{k}$. To show this we generated pictures in 4.1, where two different coaxial cylinders are considered to describe the situation evoked. The external cylinder is represented in the picture by only its external surface through a circumference line of radius $R = 1$. It is the cylinder where no deformation occurs in the reference configuration. The dark circle stays in place for the cylinder in the current configuration, after a torsional deformation (4.1) is imposed. Now it can be appreciated, depending on the amount of torsion $\hat{\tau}$ imposed, how the radius reduces with the law (4.133) and consequent change of volume occurs. In Figures 4.1 a-b) the approximation value $\hat{k} = 0.05$ is considered, and the amounts of torsion are $\hat{\tau} = 2$ and $\hat{\tau} = 2.5$ respectively. When a small value for $\hat{k}$ is slightly increased to $\hat{k} = 0.1$, the reduction of the radius for the deformed cylinder is more appreciated. See Figures 4.1 c-d) where the parameters of the torsion are $\hat{\tau} = 2$ and $\hat{\tau} = 2.5$, respectively. Clearly, the use of the model (4.126) is fraught with danger because it is too special.

This situation is peculiar to all the constitutive models that are identified by enforcing special mechanical behaviors via purely mathematical properties, such as the controllability of isochoric deformations within the context of a theory to describe the response of compressible bodies.

4.4.2 Nearly pure axial shear of compressible tube

Let us consider a compressible tube of internal and external radii A and B, respectively, subjected to an axial axisymmetric shear deformation (4.67). Here, in order to search for pure axial shear deformation, we refer the out-displacement $w(R)$ as "primary deformation" while we refer to the radial displacement $r(R)$ as "secondary field". Similarly to the previous section, this means that

$$\max_{R\in[A,B]} \left| \frac{r(R)}{R} - 1 \right| \approx \mathcal{O}(\epsilon),$$

or $\sqrt{I_3} \approx 1$ for all $R \in [A, B]$ and for any other parameters. Now let us consider the classical Hadamard material (2.21), with strain energy function that we rewrite here as

$$W = c_1(I_1 - 3) + k(I_2 - 3) + H(I_3), \tag{4.136}$$

where $c_1 > 0$ and $k \geq 0$ are material constants. Clearly if $k = 0$, the model (4.136) satisfies the necessary and sufficient condition (4.104) and the material will be therefore capable of sustaining pure axial shear, for every function $H(I_3)$ satisfying the normalization conditions on the strain energy (1.34) and (1.64).

The derivatives of the strain energy function (4.136) with respect to the invariants are

$$W_1 = c_1, \quad W_2 = k, \quad W_3 = H'(I_3). \tag{4.137}$$

Now it is possible to evaluate, via a suitable experiment, the magnitude of the parameter k with respect to the parameter μ and to decide if the assumption $W_2 = 0$ is reasonable on the basis of fitting the experimental data. Let us consider the model (2.23) proposed by Levinson and Burgess [79] as special case of strain energy function (4.136),

$$\begin{aligned} W = {} & \left(\frac{\mu}{2} - k\right)(I_1 - 3) + k(I_2 - 3) \\ & + \tfrac{1}{2}(-2k + \lambda + \mu)(I_3 - 1) - (\lambda + 2\mu)(\sqrt{I_3} - 1), \end{aligned} \tag{4.138}$$

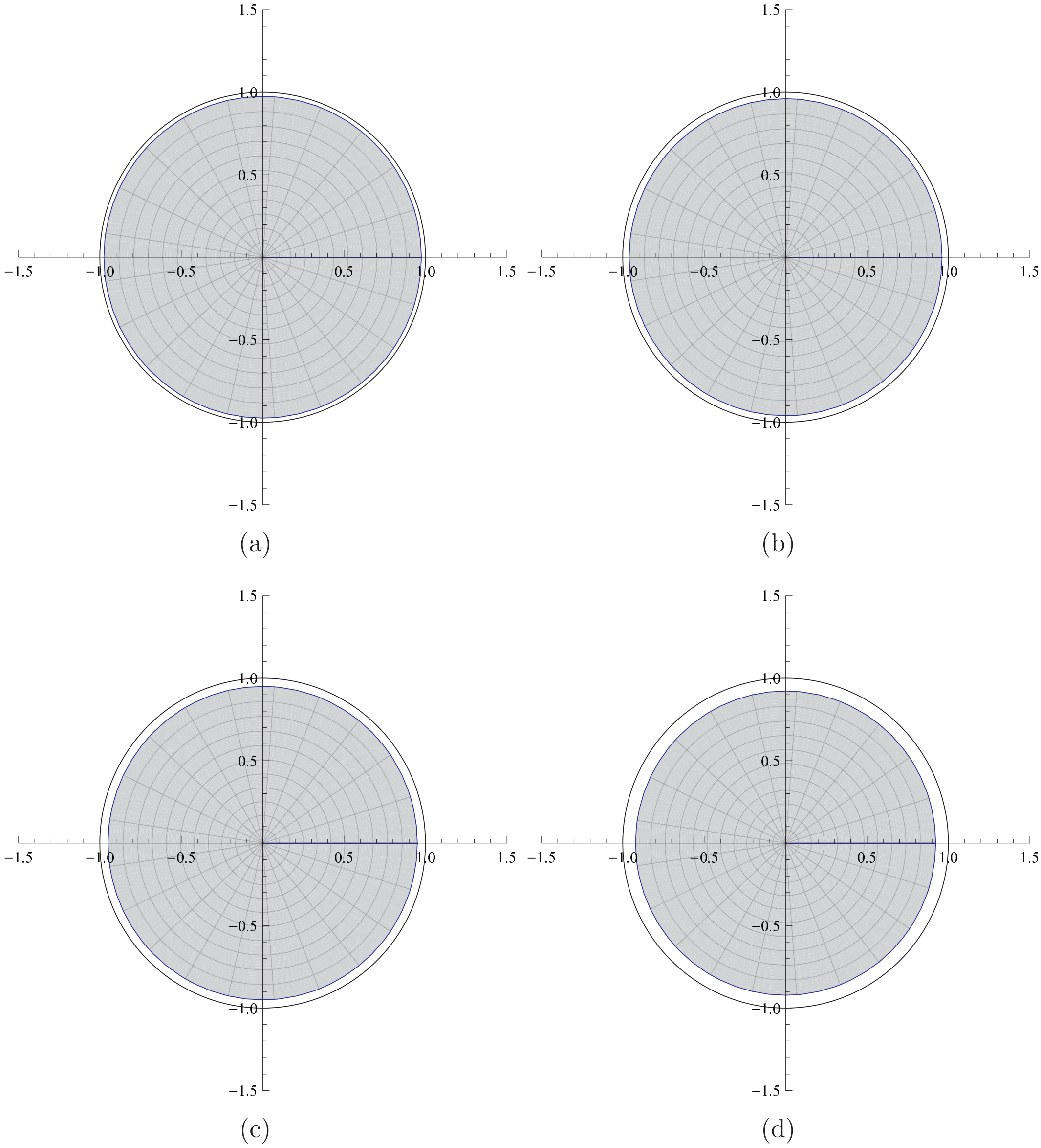


Figure 4.1: View of the transverse sections of two perturbed Blatz-Ko cylinder (4.127): the first one is a cylinder in the reference configuration when no deformation is applied (in the figure represents only its lateral surface through the circumference line of radius $R = 1$), and the second one is a cylinder in the current configuration when an amount of torsion is applied (in the figure, it is represented by a meshed cylinder of radius (4.133)) for a) $\hat{k} = 0.05,\ \hat{\tau} = 2.0$, b) $\hat{k} = 0.05,\ \hat{\tau} = 2.5$, c) $\hat{k} = 0.1,\ \hat{\tau} = 2.0$, d) $\hat{k} = 0.1,\ \hat{\tau} = 2.5$.

where λ and μ are the Lamé constants of linear elasticity. At $k = 0$ the model (4.138) reduces to

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{1}{2}(\lambda + \mu)(I_3 - 1) - (\lambda + 2\mu)(\sqrt{I_3} - 1) \tag{4.139}$$

which satisfies the necessary and sufficient condition to sustain pure axial shear. In this last case ($k = 0$) it is easy to obtain the expression for the displacement

$w(R)$ and the stress field (see Section 4.2.3). The next step is to introduce the dimensionless independent variable $\zeta = R/B$, the dimensionless dependent variables

$$F(\zeta) = r/B, \quad \hat{w} = w/B, \tag{4.140}$$

and independent variables

$$\begin{aligned} \eta = A/B, \quad \hat{k} = k/\mu, \quad \hat{\lambda} = \lambda/\mu, \\ \hat{K} = K/(B\mu), \quad \hat{\boldsymbol{T}} = \boldsymbol{T}/\mu, \quad \hat{T}_0 = T_0/\mu, \end{aligned} \tag{4.141}$$

so that $\eta \leq \zeta \leq 1$. The derivatives of the strain energy (4.138) with respect to the invariants are

$$W_1 = \frac{\mu}{2} - k, \quad W_2 = k, \quad W_3 = \tfrac{1}{2}\left(-2k + \lambda + \mu\right) - \frac{\lambda + 2\mu}{2\sqrt{I_3}}. \tag{4.142}$$

The introduction of the dimensionless variables and of (4.142) evaluated for the specific deformation under consideration in (4.79), leads to the equation

$$\begin{aligned} &- 2\hat{k}\zeta\hat{w}'^2 F + \zeta F(-1 + (\hat{\lambda} + 1)F'^2) \\ &\qquad - (\hat{\lambda} + 1)F^2(F' - \zeta F'') + \zeta^2(F' + \zeta F'') = 0, \end{aligned} \tag{4.143}$$

and in (4.77) leads to the equation

$$2k\hat{w}'[F^2 - \zeta^2] + \zeta(\zeta\hat{w}' - 2\hat{K}) = 0. \tag{4.144}$$

For $\hat{k} = 0$, we know that a solution for pure axial shear is $F(\zeta) = \zeta$. Here, we consider an approximation $\mathcal{O}(\hat{k})$ solution, in the spirit of the previous section. Let us assume that

$$F(\zeta) = \zeta + \hat{k}g(\zeta), \tag{4.145}$$

where g is an unknown dimensionless function of ζ. The problem is to solve both the equilibrium equations (4.143) and (4.144) for the unknowns F and $\hat{w}$ such that the following boundary conditions, equivalent to (4.68) and $(4.69)_1$,

$$g(\eta) = 0, \qquad \hat{w}(\eta) = 0, \qquad \hat{T}_{rr}(1) = 0, \tag{4.146}$$

are satisfied. Using (4.145) and $(4.146)_1$ we obtain (at first order) the boundary conditions that g must satisfy:

$$g(\eta) = 0, \quad (\lambda + 2)g'(1) + \lambda g(1) = 0. \tag{4.147}$$

The approximation equilibrium equations (4.143) and (4.144) (at first order) become

$$\zeta\left[(\hat{\lambda} + 2)(g' + \zeta g'') - 2\hat{w}'^2\right] - (\hat{\lambda} + 2)g = 0, \tag{4.148}$$

and

$$\tfrac{1}{2}\zeta\hat{w}' = \hat{K}, \tag{4.149}$$

respectively. The solution of (4.149) satisfying also the boundary condition $\hat{w}(\eta) = 0$ is given by

$$\hat{w}(\eta) = 2\hat{K}\ln\left(\frac{\zeta}{\eta}\right). \tag{4.150}$$

Using (4.150) in (4.148), the equilibrium equation in the unknown g is given by

$$(\hat{\lambda}+2)\zeta[\zeta(\zeta g''+g')-g]-8\hat{K}^2=0 \tag{4.151}$$

from which we obtain

$$g(\zeta)=\frac{1+\zeta^2}{2\zeta}d_1+\frac{\zeta^2-1}{2\zeta}d_2-2\hat{K}^2\frac{1+2\ln\zeta}{\left(\hat{\lambda}+2\right)\zeta}, \tag{4.152}$$

where d_1 and d_2 are integration constants obtained by the boundary conditions (4.147). From (4.78), making use of the solution (4.145), we obtain the value $\hat{K}$ as

$$\hat{K}=\frac{(\hat{\lambda}+1+\eta^2)\hat{T}_0}{(\hat{\lambda}+1+\eta^2)+\left[(\hat{\lambda}+1+\eta^2)[(\hat{\lambda}+1+\eta^2)+2\hat{k}\hat{T}_0^2(\eta^2-1-2\ln\eta)]\right]^{1/2}} \tag{4.153}$$

The $\mathcal{O}(\hat{k})$ volume change approximation is

$$J\approx 1+\hat{k}\hat{T}_0^2\frac{2\zeta^2\log\eta+(\hat{\lambda}+2)\zeta^2-1-\eta^2-\hat{\lambda}}{(\hat{\lambda}+2)(\hat{\lambda}+1+\eta^2)\zeta^2}. \tag{4.154}$$

It is interesting to study the behaviour of J when $\zeta\to\eta$, because it attains the maximum of this variation there,

$$J(\eta)\approx 1+\hat{k}\hat{T}_0^2\frac{2\eta^2\log\eta+(\hat{\lambda}+1)\eta^2-1-\hat{\lambda}}{(\hat{\lambda}+2)(\hat{\lambda}+1+\eta^2)\eta^2}. \tag{4.155}$$

Because $\eta<1$ is arbitrary, if we consider an approximation of $J(\eta)$ for small $\eta=\delta$, we obtain that

$$J(\delta)\approx 1-\hat{k}\frac{\hat{T}_0^2}{(\hat{\lambda}+2)\delta^2}. \tag{4.156}$$

Since (4.156) depends not only on $\hat{k}$ but also on the square of the traction $\hat{T}_0^2$, and because the two parameters are independent, it is clear that here as in the previous example, the approximation $\hat{k}=0$ may not be consistent. For example, if we are able to evaluate via an experiment the parameter $\hat{k}$ and we discover that this parameter is small, say $\hat{k}=0.01$, then we may in our upcoming numerical simulations take $A=B/10$, so that $\eta=0.1$, and assume $\hat{\lambda}=1$ of the same magnitude of μ. After these assumptions, the formula (4.156) becomes

$$J(\delta)\approx 1-\frac{1}{3}\hat{T}_0^2, \tag{4.157}$$

and cleary we can imagine that the isochoric assumption $J=1$ might be very dangerous when the magnitude of the traction $|\hat{T}_0|$ moves away from zero (see Figure (4.2)), because the dependence is quadratic[5].

[5]The formula (4.157) is a good approximation when small $\hat{k}$, δ and $\hat{T}_0$ are considered, to avoid zero or negative volume variation.

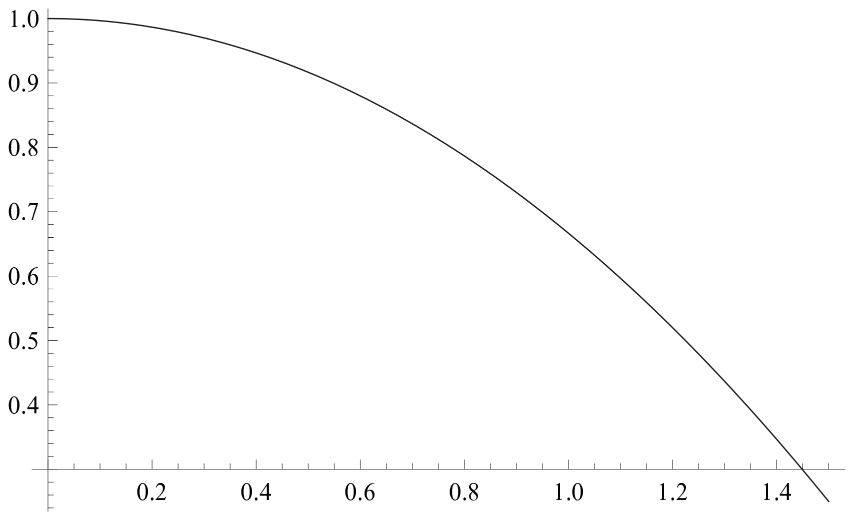


Figure 4.2: Plot of $J(\delta)$ when the assumption $\hat{k} = 0.01$, $\delta = 0.1$, $\hat{\lambda} = 1$ (see formula (4.157)) against $\hat{T}_0$ running from zero ($J = 1$) to 1.5 ($J \approx 0.25$).

4.4.3 Another example: transverse and longitudinal waves

Another important example emphasizing that if we ignore the full scope of the deformation, we may be misguided and we may miss real and interesting phenomena, is given by the propagation of longitudinal and transverse waves.

Introducing the Cartesian coordinates (X_1, X_2, X_3) in the undeformed configuration and the Cartesian coordinates (x_1, x_2, x_3) in the current configuration, we consider the motion given by

$$x_1 = u(X_1, t), \quad x_2 = X_2 + v(X_1, t), \quad x_3 = X_3, \tag{4.158}$$

where the longitudinal wave u and the transverse wave v must be determined from the balance equation. The principal invariants: I_1, I_2 and I_3, are given by

$$I_1 = 2 + u_{X_1}^2 + v_{X_1}^2, \quad I_2 = 1 + 2u_{X_1}^2 + v_{X_1}^2, \quad I_3 = u_{X_1}^2. \tag{4.159}$$

The equations of motion (1.24) in the absence of body forces, reduce to the two scalar equations

$$\begin{aligned} \rho_r \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial X_1} \left[2 \left(W_1 + 2W_2 + W_3 \right) u_{X_1} \right], \\ \rho_r \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial X_1} \left[2 \left(W_1 + W_2 \right) v_{X_1} \right]. \end{aligned} \tag{4.160}$$

Here the strain energy W is a function of $u_{X_1}^2$ and $v_{X_1}^2$.

We remark that in the linearized limit, (4.160) reduces to the classical *uncoupled* systems of linear wave equations (Atkin and Fox [4]).

If we consider the case $u(X_1, t) \equiv X_1$, equations (4.160) reduce, in the general case, to an overdetermined system of two differential equations in the single unknown v. Therefore it seems, at least at first sight, that it is not possible to ensure the existence of a transverse wave in the nonlinear theory for any material within the constitutive class (1.36). It is possible that for *special* classes of materials, this overdetermined system may have a solution. For example this is the case for Hadamard materials (2.21). In the case of Hadamard materials, because $u \equiv X_1$ and $I_3 = 1$, we find that (4.160) reduces

$$\rho_r \frac{\partial^2 v}{\partial t^2} = \mu \frac{\partial^2 v}{\partial X_1^2}. \tag{4.161}$$

In this case, the system is compatible and the transverse wave solution may be computed by solving a linear differential equation, as in the linearized theory of elasticity.

Now let us consider for the Hadamard material the case where the longitudinal wave $u(X_1, t)$ is of order ϵ, where $|\epsilon| << 1$. Then we consider the model (2.23),

$$H(I_3) = (\lambda + \mu)(I_3 - 1) - (\lambda + 2\mu)\left(\sqrt{I_3} - 1\right), \tag{4.162}$$

proposed by Levinson and Burgess [79]. Now equations (4.160) become

$$\rho_r \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial X_1^2}, \qquad \rho_r \frac{\partial^2 v}{\partial t^2} = \mu \frac{\partial^2 v}{\partial X_1^2}. \tag{4.163}$$

In this case we find that the equations are the same as in the linearized theory: they are uncoupled.

We take a further step and we consider a small coupling, i.e. we modify the constitutive equation (2.21) to be

$$\begin{aligned} W = {} & c_1(I_1 - 3) + c_2(I_2 - 3) + (\lambda + \mu)(I_3 - 1) \\ & - (\lambda + 2\mu)\left(\sqrt{I_3} - 1\right) + kI_3(I_1 - I_3 - 2), \end{aligned} \tag{4.164}$$

where k is the coupling parameter and

$$c_1 = \frac{1}{2}(\lambda + 2\mu - 4k), \qquad c_2 = \frac{1}{2}(2k - \lambda - \mu). \tag{4.165}$$

In this case we compute

$$\rho_r \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial X_1^2} + 2k \frac{\partial}{\partial X_1}\left(v_{X_1}^2 u_{X_1}\right), \tag{4.166}$$

and

$$\rho_r \frac{\partial^2 v}{\partial t^2} = (\mu - 2k)\frac{\partial^2 v}{\partial X_1^2} + 2k \frac{\partial}{\partial X_1}\left(u_{X_1}^2 v_{X_1}\right). \tag{4.167}$$

Clearly the term $\partial(u_{X_1}^2 v_{X_1})/\partial X_1$ in the right hand side of (4.167) may be (at least at first sight) ignored because the amplitude u is small. This means that we may consider the system of equations (4.166) and (4.167) as being decoupled. This is indeed a way to justify the Hadamard material (2.21), which is a model predicting an exact decoupling. As we have already remarked, any experimental determination of the coupling k may lead to k being *small* but never zero.

To make the idea rigorous, we must (at least) require that, given a set of boundary conditions (for example $u = v = 0$ at $X_1 = 0$ and L), the initial condition is such that $u(X_1, 0) \approx \mathcal{O}(\epsilon^n)$ with suitable $n \geq 1$ and that we have a suitable a priori bound on the solution such that for any time we ensure $u(X_1, t) \approx \mathcal{O}(\epsilon)$. Then, if this a priori bound exists, the initial conditions satisfy the requirements and when k is small it is possible to consider the transverse waves as being decoupled from the longitudinal waves.

The point is that it is clear from the structure of the equations that this bound cannot exist for all the admissible range of parameters. Let $k \approx \mathcal{O}(\epsilon)$. When the longitudinal motion is small, a better approximation than the linear one is to neglect the term $2k\partial_{X_1}\left(u_{X_1}^2 v_{X_1}\right)$ in (4.167) (which is $\mathcal{O}(\epsilon^3)$), but to maintain the coupling term in (4.166) (which is $\mathcal{O}(\epsilon^2)$). In this case (4.167) is a classical linear wave equation; introducing $c_T^2 = (\mu - 2k)/\rho_r$ this equation admits solutions of the usual form

$$v(X_1, t) = \sum_{n=1}^{\infty} \left[A_n \cos(k_n^T t) + B_n \sin(k_n^T t)\right] \sin(n\pi X_1/L),$$

where

$$k_n^T = n\pi c_T/L \tag{4.168}$$

is the transverse wave number of the $nth-$mode and A_n, B_n are integration constants such that the initial condition $u(X_1, 0) \approx \mathcal{O}(\epsilon^n)$ is verified. If we introduce this solution for $v(X_1, t)$ into (4.166) we obtain for $u(X_1, t)$ a linear but non-autonomous equation for which is possible to search for solutions in the form

$$u(X_1, t) = \sum_{n=1}^{\infty} \eta_n(t) \sin(n\pi X_1/L),$$

where $\eta_n(t)$ are unspecified functions of t. Using standard methods of nonlinear oscillations (Nayfeh and Mook [89]) we obtain a reduction of the equations to an infinite system of coupled ordinary differential equations in the unknowns η_n. These equations are non-autonomous and they display autoparametric resonance phenomena for some values of the various parameters. Therefore, an a priori bound is impossible. This means that it does not matter how small the longitudinal motions are, because after a certain time their amplitude cannot be neglected and a full coupling between transverse and longitudinal motions must be considered. Therefore, the Hadamard model is much too special to be considered as a reasonable idealization of real elastic bodies.

Phenomena of this kind are quite common in classical mechanics. For example in the framework of the elementary and classical theory for holonomic systems, it is well known that unstable normal modes may not contribute to the approximate linear theory. This happens for modes that are "latent" at the initial time. Nevertheless, the higher orders neglected in the Lagrangian can awaken these latent unstable modes, and bring the system away from equilibrium[6].

[6] A simple and clear example of a mechanical system displaying wake-up of latent modes is reported in page 133 of Biscari et al. [16].

Notes

Isochoric deformations play on important role in solid mechanics and here we have appealed to them to illustrate our thesis in the context of nonlinear elasticity. To simplify the exposition, we have only considered the theory of unconstrained nonlinear isotropic elasticity. But our remarks are completely general and apply (with some modifications) in general to the use of semi-inverse methods in continuum mechanics. For example Jiang and Beatty [65] find also necessary and sufficient conditions on the strain energy function for homogeneous and compressible, *anisotropic* hyperelastic materials to sustain controllable, axisymmetric helical shear deformations. Thus we think that one needs to exercise a great deal of prudence in ensuring that the results obtained by using the semi-inverse method make sense.

Chapter 5

Secondary deformations in nonlinear elasticity

In Section 3.2.2, we have shown that in the incompressible case, the general antiplane shear (3.64) can not be always sustained unless the axisymetric case is considered. This means that when the geometry of the material deformed moves away from axial symmetry, we can describe the deformation only for special particular materials. For example, consider the case of an elastic material filling the annular region between two coaxial cylinders, with the following boundary-value problem: hold fixed the outer cylinder and pull the inner cylinder by applying a tension in the axial direction. A solution to this problem, valid for every incompressible isotropic elastic solid, is obtained by assuming a priori that the deformation field is a pure axial shear. However if consider the corresponding problem for non-coaxial cylinders, thereby losing the axial symmetry, then it is clear that we cannot expect the material to deform as prescribed by a pure axial shear deformation. Knowles' result [72] tells us that now the boundary-value problem can be solved with a general anti-plane deformation (not axially symmetric) only for a subclass of incompressible isotropic elastic materials.

Moreover, in Sections 4.1 and 4.2, we have underlined how it is not always possible for a compressible material to sustain pure torsion and pure axial shear, respectively, unless some particular forms of strain energy functions are considered, because in general they might be accompanied by the radial deformation. In Section 4.4, some explanations about the dangers in forcing the compressible material to have some special behaviours were given.

Of course, these restrictions do not mean that, for a generic material, it is not possible to deform the solid as prescribed by our boundary conditions, but rather that, in general, these lead to a deformation field that is more complex for example than a pure torsion or than an anti-plane shear. Hence, we also expect secondary deformations: a clear difficult task to understand in solving boundary-value problem by appealing only to a semi-inverse procedure.

The theory of non-Newtonian fluid dynamics has generated a substantial literature about secondary flows, see for example Fosdick and Serrin [40]. In 1956, Ericksen [35] conjectured that purely rectilinear flows would be possible only in pipes of circular cross sections or cross sections made of straight lines and circles, secondary flows being necessarily present in pipes of arbitrary cross sections. In

1973, Fosdick and Serrin [40] proved a more precise version of the Ericksen's conjecture: requiring certain technical assumptions on regularity concerning the material properties, they showed that unless the material functions characterizing the fluid satisfied certain special relationships, the cross section ought to be a circle or the annular region between two concentric circles.

In solid mechanics, Fosdick and Kao [39] were the first to explore the counterpart to Ericksen's conjecture in fluids within the context of nonlinear elasticity. Denoting by $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ an orthonormal basis in rectangular Cartesian coordinates, they consider a cylindrical domain, whose generators are parallel to the axis $\boldsymbol{i}_3$, with bounded and connected cross section A having boundary

$$\partial A = \bigcup_{i=0}^{n} \partial A_i \tag{5.1}$$

consisting of $n+1$ sufficiently smooth non intersecting closed curves, where ∂A_0 is the external boundary of A which encloses all other inner boundaries ∂A_i $(i = 1, \ldots, n)$. Fosdick and Kao [39] assume the displacement $\boldsymbol{u}$ to be decomposed into an axial component $w = w(X_1, X_2)$ and a cross sectional component $\boldsymbol{v} = \boldsymbol{v}(X_1, X_2)$ and consider the following boundary condition

$$w = \begin{cases} 0 \text{ on } \partial A_0 \\ w_i \text{ on } \partial A_i \quad (i = 1, \ldots, n) \end{cases} \tag{5.2}$$

and $\boldsymbol{v} = \boldsymbol{0}$ on ∂A. First, they show that in general, rectilinear shear ($\boldsymbol{v} = \boldsymbol{0}$) of cylinders is not always possible, unless the cross-section is a circle or the annular region between two concentric circles. Then, they analyse the problem which includes not only an axial shear deformation but also the possibility of cross-sectional distortion. They use the *specific driving force* (applied shear) a, as small parameter and consider the following perturbation problem for uniformly infinitesimal boundary data,

$$\begin{gathered} a = \varepsilon \bar{a}, \quad w_i = \varepsilon \bar{w}_i \ (i = 1, 2, \ldots, n) \\ w = \sum_{i=1}^{n} w^i \varepsilon^i, \quad \boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{v}^i \varepsilon^i, \end{gathered} \tag{5.3}$$

where $\varepsilon \ll 1$ is a real non-negative number, $\bar{a}$ and $\bar{w}_i$ $(i = 1, 2, \ldots, n)$ are constant numbers independent of ε which carry, respectively, the dimensions of force per unit volume and displacement, and w^i, $\boldsymbol{v}^i$ $(i = 1, 2, \ldots, n)$ are functions depending of the same arguments of w and $\boldsymbol{v}$, respectively. It follows immediately from the assumed form for the displacement field that when $\varepsilon = 0$, there is no displacement and this has to be expected as there is no driving force. Using (5.3), the balance equations and the constraint of incompressibility, Fosdick and Kao [39] find $\boldsymbol{v}^i = \boldsymbol{0}$ for $i = 1, 2, 3$ and show that $\boldsymbol{v}^4$ is necessary different from zero. Thus secondary deformations appear at only fourth order.

Another possibility of perturbation approach in order to investigate for secondary deformations is given by departure from circular symmetry. Mollica and Rajagopal [83] showed that in this last case the secondary deformations appear at first order when the driving force is a fixed value placed without restrictions, the perturbation parameter being the departure from circularity. The deformation that they consider takes place between two infinite cylinders eccentrically placed and it can be driven by an axial pressure gradient or by the axial motion of one of the boundaries. They use the eccentricity ε which is the distance between the centers of cylinders as the perturbation parameter. In a Cartesian coordinates system (X, Y, Z), the equations for two cylinders, whose radii are R_1 and R_2, with $R_1 < R_2$, are

$$\begin{aligned} &X^2 + Y^2 = R_2^2, \\ &(X - |\overrightarrow{O_1O_2}|)^2 + Y^2 = R_1^2, \end{aligned} \tag{5.4}$$

where $|\overrightarrow{O_1O_2}|$ is the eccentricity and they let

$$\varepsilon = \frac{|\overrightarrow{O_1O_2}|}{R_1}, \tag{5.5}$$

be a dimensionless small parameter. Let (R, Θ, Z) and (r, θ, z) be cylindrical coordinates in reference and current configuration, respectively. Then they consider a deformation of the form

$$\begin{aligned} &r = R + \varepsilon v(R, \Theta) + o(\varepsilon), \\ &\theta = \Theta + \varepsilon w(R, \Theta) + o(\varepsilon), \\ &z = Z + f(R) + \varepsilon g(R, \Theta) + o(\varepsilon). \end{aligned} \tag{5.6}$$

At order zero ($\varepsilon = 0$), (5.6) is not the undeformed state (here it is therefore different from previous case (5.3)), but it is an axially symmetric deformation. For the problem under investigation, they assume that the outer cylinder is fixed while the inner cylinder translates in the axial direction by a fixed amount f_w. Thus, denoting by $\mathcal{C}_1$ and $\mathcal{C}_2$ the inner and outer cylinders, respectively, from (5.6), they set

$$\begin{aligned} &\varepsilon v(R, \Theta)|_{\mathcal{C}_1} = \varepsilon w(R, \Theta)|_{\mathcal{C}_1} = 0, \\ &\varepsilon v(R, \Theta)|_{\mathcal{C}_2} = \varepsilon w(R, \Theta)|_{\mathcal{C}_2} = 0, \\ &f(R) + \varepsilon g(R, \Theta)|_{\mathcal{C}_1} = f_w, \\ &f(R) + \varepsilon g(R, \Theta)|_{\mathcal{C}_2} = 0. \end{aligned} \tag{5.7}$$

After that, they suppose that the incompressible material is described by the following strain energy function

$$W(I_1, I_2) = \frac{\delta_1}{2b} \left\{ \left[1 + \frac{b}{n}(I_1 - 3) \right]^n - 1 \right\} - \frac{\delta_2}{2} \left\{ I_2 - 3 \right\}, \tag{5.8}$$

where δ_1, δ_2, b, n are material parameters. When $n = 1$, the model (5.8) reduces to the classical Mooney-Rivlin model (2.3), while if $\delta_2 = 0$, it reduces to the power-law model (2.16). As well, they set

$$\delta_1 > 0, \quad \delta_2 < 0, \quad 1 > n > \frac{1}{2}, \quad n > 1, \quad b > 0, \tag{5.9}$$

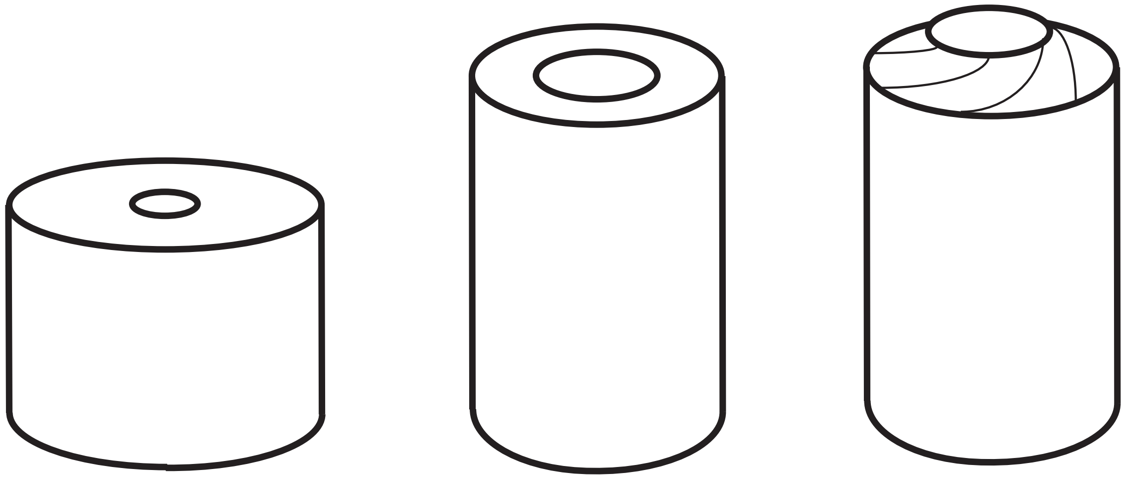


Figure 5.1: Shrink fit of an elastic tube, followed by the combination of simple torsion and helical shear. (The figure does not respect scales among the various deformations).

such that from the analysis of Fosdick and Kao [39], the material cannot exhibit a purely axial displacement when subjected to axial shear, and secondary displacements are therefore necessary. Using (5.6), the boundary conditions (5.7), the balance equations and the constraint of incompressibility, Mollica and Rajagopal [83] establish that secondary deformations at first order in ε are possible when the driving force is not small and the annular region deviates slightly from axial symmetry.

In the next section we consider a complex deformation field in isotropic incompressible elasticity, to point out by an explicit example (extracted from our work [27]) the situations just evoked, and to elaborate on their possible impact on solid mechanics. The deformation field takes advantage of the radial symmetry; therefore we find it convenient to visualize it by considering an elastic cylinder.

5.1 An analytic example of secondary deformations

For a better understanding of the "real" situation we evoke, let us imagine that a corkscrew has been driven through a cork (the cylinder) in a bottle. The inside of the bottleneck is the outer rigid cylinder and the idealization of the gallery carved out by the corkscrew constitutes the inner coaxial rigid cylinder. Our first deformation is purely radial, originated from the introduction of the cork into the bottleneck and then completed when the corkscrew penetrates the cork (a so-called *shrink fit problem*, which is a source of elastic residual stresses here). We call A, B the respective inner and outer radii of the cork in the reference configuration and $r_1 > A$, $r_2 < B$ their current counterparts. Then we follow with a simple torsion combined to a helical shear, in order to model pulling the cork out of the bottleneck in the presence of a contact force. Figure 5.1 sketches this deformation.

Of course, we are aware of the shortcomings of our modelling with respect to the description of a "real" cork-pulling problem, because no cork is an infinitely long cylinder, nor is a corkscrew perfectly straight. In addition, traditional corks made from bark are anisotropic (honeycomb-shaped mesoscopic structure) and possess the remarkable (and little-known) property of having an infinitesimal Poisson ratio equal to zero, see the review article by Gibson *et al.* [46]. However we note that *polymer corks* have appeared on the world wine market; they are made of elastomers, for which incompressible, isotropic elasticity seems like a reasonable framework (indeed the documentation of these synthetic wine stoppers indicates that they lengthen during the sealing process)[1].

5.1.1 Equilibrium equations

Consider a long hollow cylindrical tube composed of an isotropic incompressible nonlinearly elastic material. At rest, the tube is in the region

$$A \le R \le B, \qquad 0 \le \Theta \le 2\pi, \qquad -\infty \le Z \le \infty, \tag{5.10}$$

where (R, Θ, Z) are the cylindrical coordinates associated with the undeformed configuration, and A and B are the inner and outer radii of the tube, respectively.

Consider the general deformation obtained by the combination of radial dilatation, helical shear and torsion as

$$r = r(R), \qquad \theta = \Theta + g(R) + \tau Z, \qquad z = \lambda Z + w(R), \tag{5.11}$$

where (r, θ, z) are the cylindrical coordinates in the deformed configuration; τ is the amount of torsion; and λ is the stretch ratio in the Z-direction. Here, g and w are yet unknown functions of R only. (The classical case of torsion deformation (4.1) corresponds to $w = g = 0,\ \lambda = 1$.) Hidden inside (5.11) is the *shrink fit deformation*

$$r = r(R), \qquad \theta = \Theta, \qquad z = \lambda Z, \tag{5.12}$$

which is (5.11) without any torsion or helical shear ($\tau = g = w \equiv 0$). The physical components of the deformation gradient $\boldsymbol{F}$ and of its inverse $\boldsymbol{F}^{-1}$ are then

$$\begin{bmatrix} r' & 0 & 0 \\ rg' & r/R & r\tau \\ w' & 0 & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r\lambda/R & 0 & 0 \\ rw'\tau - rg'\lambda & r'\lambda & -rr'\tau \\ -rw'/R & 0 & rr'/R \end{bmatrix}, \tag{5.13}$$

respectively. Note that we used the incompressibility constraint in order to compute $\boldsymbol{F}^{-1}$; it states that $\det \boldsymbol{F} = 1$, so that

$$r' = \frac{R}{\lambda r}. \tag{5.14}$$

In our first deformation, the cylindrical tube is pressed into a cylindrical cavity with inner radius $r_1 > A$ and outer radius $r_2 < B$. It follows by integration of the equation (5.14) that

$$r(R) = \sqrt{\frac{R^2}{\lambda} + \alpha}, \tag{5.15}$$

[1]We hope that this study provides a first step toward a nonlinear alternative to the linear elasticity testing protocols presented in the international standard ISO 9727. We also note that low-cost *shock absorbers* often consist of a moving metal cylinder, glued to the inner face of an elastomeric tube, whose outer face is glued to a fixed metal cylinder [56].

where now

$$\alpha = \frac{B^2 r_1^2 - A^2 r_2^2}{B^2 - A^2}, \qquad \lambda = \frac{B^2 - A^2}{r_2^2 - r_1^2}. \tag{5.16}$$

We compute the physical components of the left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ from (5.13) and find its first three principal invariants as

$$\begin{aligned} I_1 &= (r')^2 + (rg')^2 + (r/R)^2 + (r\tau)^2 + \lambda^2 + (w')^2, \qquad (5.17)\\ I_2 &= (r\lambda/R)^2 + (rw'\tau - rg'\lambda)^2 + (rw'/R)^2 + (R/r)^2 + (1/\lambda)^2 + (R\tau/\lambda)^2, \end{aligned}$$

and of course, $I_3 = 1$. For a general incompressible hyperelastic solid, the Cauchy stress tensor $\boldsymbol{T}$ is given by (1.40). Having computed $\boldsymbol{B}^{-1} = (\boldsymbol{F}^T)^{-1}\boldsymbol{F}^{-1}$ from (5.13), we find that the components of $\boldsymbol{T}$ are

$$\begin{aligned} T_{rr} &= -p + 2W_1(r')^2 - 2W_2\left[(r\lambda/R)^2 + (rw'\tau - rg'\lambda)^2 + (rw'/R)^2\right],\\ T_{\theta\theta} &= -p + 2W_1\left[(rg')^2 + (r/R)^2 + (r\tau)^2\right] - 2W_2(R/r)^2,\\ T_{zz} &= -p + 2W_1[\lambda^2 + (w')^2] - 2W_2\left[(1/\lambda)^2 + (R\tau/\lambda)^2\right], \qquad (5.18)\\ T_{r\theta} &= 2W_1(rr'g') - 2W_2(w'\tau - g'\lambda)R,\\ T_{rz} &= 2W_1(r'w') - 2W_2\left[rRg'\tau - rRw'\tau^2/\lambda - rw'/(\lambda R)\right],\\ T_{\theta z} &= 2W_1(rw'g' + r\lambda\tau) + 2W_2(r'R\tau). \end{aligned}$$

Finally the equilibrium equations, in the absence of body forces, are: div $\boldsymbol{T} = \boldsymbol{0}$; for fields depending only on the radial coordinate as shown here, they reduce to

$$\begin{aligned} \frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{T_{rr} - T_{\theta\theta}}{r} &= 0,\\ \frac{\mathrm{d}T_{r\theta}}{\mathrm{d}r} + \frac{2}{r}T_{r\theta} &= 0, \qquad (5.19)\\ \frac{\mathrm{d}T_{rz}}{\mathrm{d}r} + \frac{1}{r}T_{rz} &= 0. \end{aligned}$$

5.1.2 Boundary conditions

Now consider the inner face of the tube: we assume that it is subject to a vertical pull,

$$T_{rz}(A) = T_0^A, \qquad T_{r\theta}(A) = 0, \tag{5.20}$$

say. Then by integrating the second and third equations of equilibrium $(5.19)_{2,3}$, we find that

$$T_{rz}(r) = \frac{r_1}{r}T_0^A, \qquad T_{r\theta}(r) = 0. \tag{5.21}$$

The outer face of the tube (in contact with the glass in the cork/bottle problem) remains fixed, so that

$$w(B) = 0, \qquad g(B) = 0, \qquad T_{rr}(B) = T_0, \tag{5.22}$$

say. In addition to the axial traction applied on its inner face, the tube is subject to a resultant axial force N (say) and a resultant moment M (say),

$$N = \int_0^{2\pi}\int_{r_1}^{r_2} T_{zz} r \mathrm{d}r\mathrm{d}\theta, \qquad M = \int_0^{2\pi}\int_{r_1}^{r_2} T_{\theta z} r^2 \mathrm{d}r\mathrm{d}\theta. \tag{5.23}$$

Note that the traction T_0 of (5.22) is not arbitrary but is instead determined by the *shrink fit pre-deformation* (5.12), by requiring that $N = 0$ when $T_0^A = \tau = g = w \equiv 0$ (this process is detailed in the Section 5.1.3 for the neo-Hookean material). Therefore, T_0 is connected with the stress field experienced by the cork when it is introduced in the bottleneck. In the rest of this explanation we aim at presenting results in dimensionless form. To this end, we normalize the strain energy function W and the Cauchy stress tensor $\boldsymbol{T}$ with respect to μ, the infinitesimal shear modulus; hence we introduce $\overline{W}$ and $\overline{\boldsymbol{T}}$ defined by

$$\overline{W} = \frac{W}{\mu}, \qquad \overline{\boldsymbol{T}} = \frac{\boldsymbol{T}}{\mu}. \tag{5.24}$$

Similarly we introduce the following non-dimensional variables,

$$\eta = \frac{A}{B}, \quad \overline{R} = \frac{R}{B}, \quad \overline{r}_i = \frac{r_i}{B}, \quad \overline{w} = \frac{w}{B}, \quad \overline{\alpha} = \frac{\alpha}{B^2}, \quad \overline{\tau} = B\tau, \tag{5.25}$$

so that $\eta \leq \overline{R} \leq 1$. Also, we find from (5.16) that

$$\overline{\alpha} = \frac{\overline{r}_1^2 - \eta^2 \overline{r}_2^2}{1 - \eta^2}, \qquad \lambda = \frac{1 - \eta^2}{\overline{r}_2^2 - \overline{r}_1^2}. \tag{5.26}$$

Turning to our cork or shock absorber problems, we imagine that the inner metal cylinder is introduced into a pre-existing cylindrical cavity (this precaution ensures a one-to-one correspondence of the material points between the reference and the current configurations). In our upcoming numerical simulations, we take $A = B/10$ so that $\eta = 0.1$; we consider that the outer radius is shrunk by 10%, $r_2 = 0.9B$, and that the inner radius is doubled, $r_1 = 2A$; finally, we apply a traction, the magnitude of which is half the infinitesimal shear modulus: $|T_0^A| = \mu/2$. This gives

$$\overline{\alpha} \simeq 3.22 \times 10^{-2}, \qquad \lambda \simeq 1.286, \qquad \overline{T}_0^A = -0.5. \tag{5.27}$$

At this point it is possible to state clearly our main observation. A first glance at the boundary conditions, in particular at the requirements that g be zero on the outer face of the tube, gives the expectation that $g \equiv 0$ everywhere is a solution to our boundary-value problem, at least for some simple forms of the constitutive equations. In what follows, we find that, for the neo-Hookean solids, $g \equiv 0$ is indeed a solution, whether there is a torsion τ or not. However if the solid is not neo-Hookean, then it is necessary that $g \neq 0$ when $\tau \neq 0$, and the picture becomes more complex. For this reason, we classify as "purely academic" the question:

> *Which is the most general strain-energy density for which it is possible to solve the above boundary value problem with* $g \equiv 0$?

Indeed, there is no "real world" material, the behaviour of which is ever going to be described *exactly* by that strain-energy density (supposing it exists). Instead a more pertinent issue to raise for "real word applications" is whether we are able to evaluate the importance of latent (secondary) stress fields, because they are bound to be woken up (triggered) by the deformation.

5.1.3 neo-Hookean materials

First, we consider the special strain energy function which generates the class of neo-Hookean materials (2.1). Note that here and hereafter, we use the non-dimensional quantities introduced previously, from which we drop the overbar for convenience. Hence, the components of the (non-dimensional) stress field in a neo-Hookean material reduce to

$$\begin{aligned} T_{rr} &= -p + (r')^2, & T_{\theta\theta} &= -p + (rg')^2 + (r/R)^2 + (r\tau)^2, \\ T_{zz} &= -p + \lambda^2 + (w')^2, & T_{r\theta} &= rr'g', \\ T_{rz} &= r'w', & T_{\theta z} &= rg'w' + r\lambda\tau. \end{aligned} \tag{5.28}$$

Substituting into (5.21) we find that

$$w' = \lambda r_1 T_0^A / R, \qquad g' = 0, \tag{5.29}$$

and by integration, using (5.22), that

$$w = \lambda r_1 T_0^A \ln R, \qquad g = 0. \tag{5.30}$$

In Figure 5.2a, we present a rectangle in the tube at rest, which is delimited by $0.1 \leq R \leq 1.0$ and $0.0 \leq Z \leq 1.0$. Then it is subject to the deformation corresponding to the numerical values of (5.27). To generate Figure 5.2b, we computed the resulting shape for a neo-Hookean tube, using (5.11), (5.15), and (5.30).

Now that we know the full deformation field, (see (5.11) and (5.30)), we can compute $T_{rr} - T_{\theta\theta}$ from (5.28) and deduce T_{rr} by integration of $(5.19)_1$, with initial condition $(5.22)_3$. Then the other field quantities follow from the rest of (5.28). Finally, we find in turn that

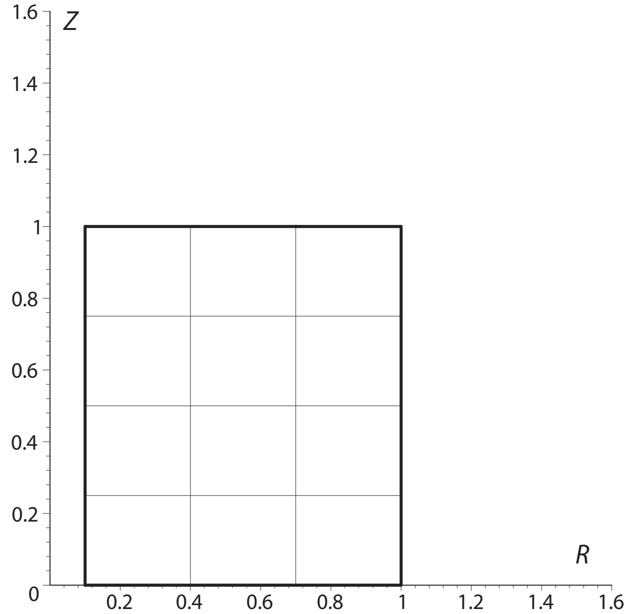
$$\begin{aligned} T_{rr} &= \frac{1}{2\lambda}\left\{\ln\frac{\lambda r_2^2 R^2}{R^2+\alpha\lambda} + (R^2-1)\left[\frac{\alpha}{r_2^2\,(R^2+\alpha\lambda)} + \tau^2\right]\right\} + T_0, \\ T_{\theta\theta} &= T_{rr} + \left(\frac{R^2}{\lambda}+\alpha\right)\left(\frac{1}{R^2}+\tau^2\right) - \frac{R^2}{\lambda(R^2+\alpha\lambda)}, \\ T_{zz} &= T_{rr} + \lambda^2\left(1+\frac{r_1^2(T_0^A)^2}{R^2}\right) - \frac{R^2}{\lambda(R^2+\alpha\lambda)} \end{aligned} \tag{5.31}$$

(where we used the identity $1+\alpha\lambda = \lambda r_2^2$, see (5.15) with $R=1$), and that

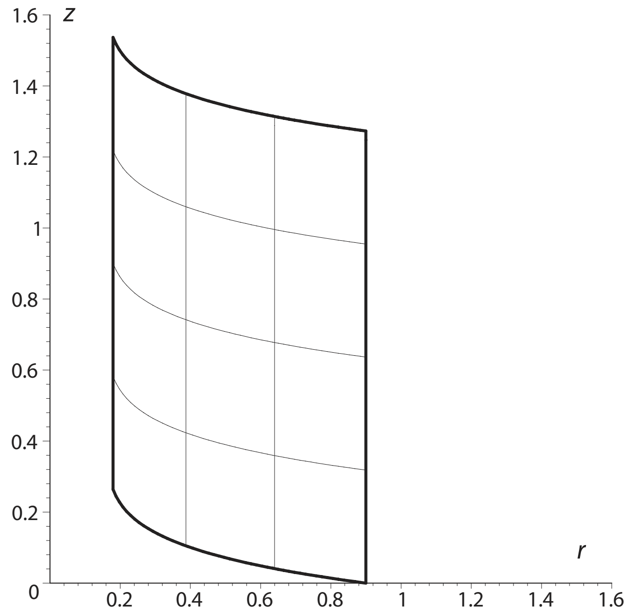
$$T_{r\theta} = 0, \qquad T_{rz} = \frac{r_1}{\sqrt{\dfrac{R^2}{\lambda}+\alpha}} T_0^A, \qquad T_{\theta z} = \lambda\tau\sqrt{\frac{R^2}{\lambda}+\alpha}. \tag{5.32}$$

The constant T_0 is fixed by the shrink fit pre-deformation (5.12), imposing that $N = 0$ when $\tau = g = w = T_0^A \equiv 0$, or

$$(T_0+\lambda^2)(1-\eta^2) + \frac{1}{\lambda}\int_\eta^1 \left[\ln\frac{\lambda r_2^2 R^2}{R^2+\alpha\lambda} + \frac{\alpha(R^2-1)}{r_2^2(R^2+\alpha\lambda)} - \frac{2R^2}{R^2+\alpha\lambda}\right] R\,\mathrm{d}R = 0. \tag{5.33}$$



(a)



(b)

Figure 5.2: (a, b) Pulling on the inside face of a neo-Hookean tube. Here the vertical axis is the symmetry axis of the tube.

Using (5.33) and (5.23), (5.31), (5.32), we find the following expressions for the resultant moment,

$$M = \pi(r_2^4 - r_1^4)\lambda\tau/2, \tag{5.34}$$

and for the axial force,

$$N = 2\pi\lambda r_1^2 |\ln\eta| \left(T_0^A\right)^2 - \frac{\pi}{4}(r_2^2 - r_1^2)^2\tau^2. \tag{5.35}$$

We now have a clear picture of the response of a neo-Hookean solid to the deformation (5.11), with the boundary conditions of Section 5.1.2. First, we saw that here the contribution $g(R)$ is not required for the azimuthal displacement, whether there is a torsion τ or not. Also, if a moment $M \neq 0$ is applied, then the tube suffers an amount of torsion $\tau \neq 0$ proportional to M. On the other hand, if the tube is pulled by the application of an axial force only ($N \neq 0$) and no moment ($M = 0$), then $\tau = 0$ and no azimuthal shear occurs at all.

5.1.4 Generalized neo-Hookean materials

As a first broadening of the neo-Hookean strain-energy density (2.1), we consider generalized neo-Hookean materials (2.12). To gain access to the Cauchy stress components in this context, it suffices to take $W_2 = 0$ and $W_1 = W'$, where the prime stands for the derivatives of W with respect to the first invariant, in equations (5.18). In particular,

$$T_{r\theta} = 2rr'g'W', \tag{5.36}$$

and the integrated equation of equilibrium $(5.21)_2$ gives $g' = 0$. By integrating, with $(5.22)_2$ as an initial value, we find that

$$g \equiv 0. \tag{5.37}$$

Hence, just as in the neo-Hookean case, azimuthal shear can be avoided altogether, whether there is a torsion τ or not. We are left with an equation for the axial shear, namely $(5.21)_1$, which can be written as

$$2W'(I_1)w'(R) = \frac{\lambda r_1}{R}T_0^A. \tag{5.38}$$

Obviously the same steps as those taken for neo-Hookean solids may be followed here for any given strain energy density (2.12), but now by resorting to a numerical treatment. Horgan and Saccomandi [62] show, through some specific examples of hardening generalized neo-Hookean solids, how rapidly involved the analysis becomes, even when there is only helical shear and no shrink fit. Instead, we simply point out some striking differences between our present situation and the neo-Hookean case. We remark that I_1 is of the form $(5.17)_1$ at $g \equiv 0$, i.e.

$$I_1 = \lambda^2 + \frac{R^2}{\lambda(R^2 + \alpha\lambda)} + \left(\frac{R^2}{\lambda} + \alpha\right)\left(\frac{1}{R^2} + \tau^2\right) + [w'(R)]^2. \tag{5.39}$$

It follows that (5.38) is a nonlinear differential equation for w', in contrast to the neo-Hookean case. Another contrast is that the axial shear w is now intimately coupled to the torsion parameter τ, and that this dependence is a *second-order effect* (τ appears above as τ^2).

A similar problem where the azimuthal shear has not been ignored, but the axial shear has been considered null, i.e. $w \equiv 0$ has been recently considered by Wineman [130].

5.1.5 Mooney–Rivlin materials

In this section, we specialize the general equations of section 5.1.1 to the Mooney–Rivlin form of the strain energy function (2.3), which in its non-dimensional form reads

$$W = \frac{I_1 - 3 + m(I_2 - 3)}{2(1+m)}, \tag{5.40}$$

so that

$$2W_1 = \frac{1}{1+m}, \qquad 2W_2 = \frac{m}{1+m}, \tag{5.41}$$

where $m > 0$ is a material parameter, distinguishing the Mooney–Rivlin material from the neo-Hookean material (2.1), and also allowing a dependence on the second principal strain invariant I_2, in contrast to the generalized neo-Hookean solids of the previous section. Then the integrated equations of equilibrium (5.21) read

$$\begin{aligned}
&\left(R + m\tau^2 r^2 R + mr^2/R\right) w' - (m\tau\lambda r^2 R)g' = (1+m)\lambda r_1 T_0^A, \\
&(m\tau\lambda)w' - (1+m\lambda^2)g' = 0.
\end{aligned} \tag{5.42}$$

First we ask ourselves if it is possible to avoid torsion during the pulling of the inner face. Taking $\tau = 0$ above gives

$$(R + mr^2/R)w' = (1+m)\lambda r_1 T_0^A, \qquad g' = 0. \tag{5.43}$$

It follows that here it is indeed possible to solve our boundary value problem. We find

$$w = \lambda r_1 T_0^A \, \frac{\lambda(1+m)}{2(\lambda+m)} \ln\left[\frac{m\alpha\lambda + (\lambda+m)R^2}{m\alpha\lambda + (\lambda+m)}\right], \qquad g = 0. \tag{5.44}$$

However if $\tau \neq 0$, then it is necessary that $g \neq 0$, otherwise $(5.42)_2$ gives $w' = 0$ while $(5.42)_1$ gives $w' \neq 0$, a contradiction. This constitutes the first departure from the neo-Hookean and generalized neo-Hookean behaviours: *torsion ($\tau \neq 0$) is necessarily accompanied by azimuthal shear ($g \neq 0$).* In the case $\tau \neq 0$, we introduce the function $\Lambda = \Lambda(R)$ defined as

$$\Lambda(R) = (R + mr^2/R)(1+m\lambda^2) + m\tau^2 r^2 R, \tag{5.45}$$

(recall that $r = r(R)$ is given explicitly in (5.15)). We then solve the system (5.42) for w' and g' as

$$w' = (1+m)(1+m\lambda^2)\lambda\frac{T_0^A}{\Lambda(R)}r_1, \qquad g' = m(1+m)\lambda^2\frac{T_0^A}{\Lambda(R)}\tau r_1, \tag{5.46}$$

making clear the link between g and τ. Thus for the Mooney–Rivlin material, the azimuthal shear g is a *latent* mode of deformation; it is *woken up* by any amount of torsion τ. Recall that, at first sight, the azimuthal shear component of the deformation (5.11) seemed inessential to satisfy the boundary conditions, especially in view of the boundary condition $g(1) = 0$. However, a non-zero W_2 term in the constitutive equation clearly couples the effects of a torsion and an azimuthal shear, as displayed explicitly by the presence of τ in the expression for g' above. It is perfectly possible to integrate equations (5.46) in the general case, but to save space we do not reproduce the resulting long expressions. With them, we generated the deformation field picture of Figure 5.3(a,b) and Figure 5.4(a,b). There we took the numerical values of (5.27) for α, λ, T_0^A; we took a Mooney–Rivlin solid with $m = 5.0$; we imposed a torsion of amount $\tau = 0.5$; and we looked at the deformation field in the plane $Z = 1$ (reference configuration) and $z = \lambda$ (current configuration).

Although the secondary fields appear to be slight in the picture, they are nonetheless truly present and cannot be neglected. To show this, we consider a perturbation method to obtain simpler solutions and to understand the effect of the coupling, by taking m small. Then integrating (5.46), we find at first order that

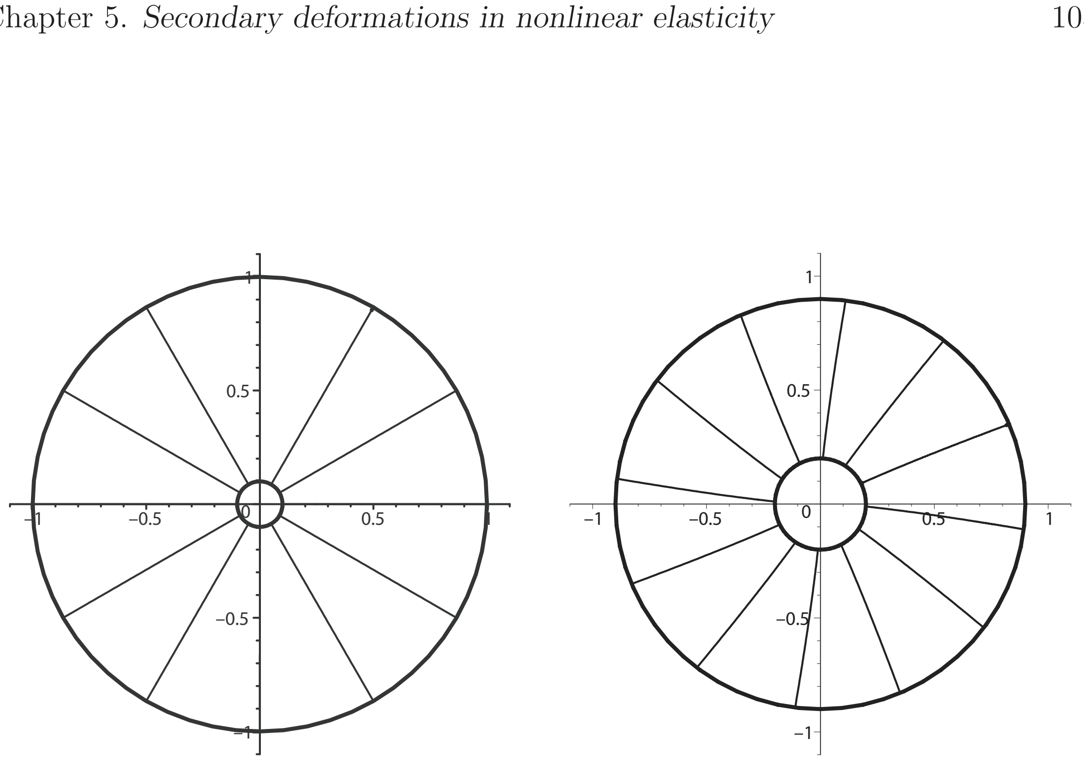
$$\begin{aligned}
\frac{w}{r_1 T_0^A} &\simeq (1+m)\lambda \ln R - \frac{1}{2} m \left[\tau^2 R^2 + 2(1+\tau^2\alpha\lambda)\ln R - \alpha\lambda/R^2 - \tau^2 + \alpha\lambda\right], \\
\frac{g}{r_1 T_0^A} &\simeq \lambda^2 \tau m \ln R.
\end{aligned} \tag{5.47}$$

Hence, the secondary field g exists even for a nearly neo-Hookean solid (if m is small, then g is of order m.) Interestingly, we also note that the azimuthal shear g in (5.47) varies in a homogeneous and linear manner with respect to the torsion parameter τ and in a quadratic manner with respect to the axial stretch λ, showing that the presence of this secondary deformation field cannot be neglected when the effects of both the prestress and the torsion are taken into account. To complete the picture, we use the first-order approximations

$$2W_1 \simeq 1 - m, \qquad 2W_2 \simeq m, \tag{5.48}$$

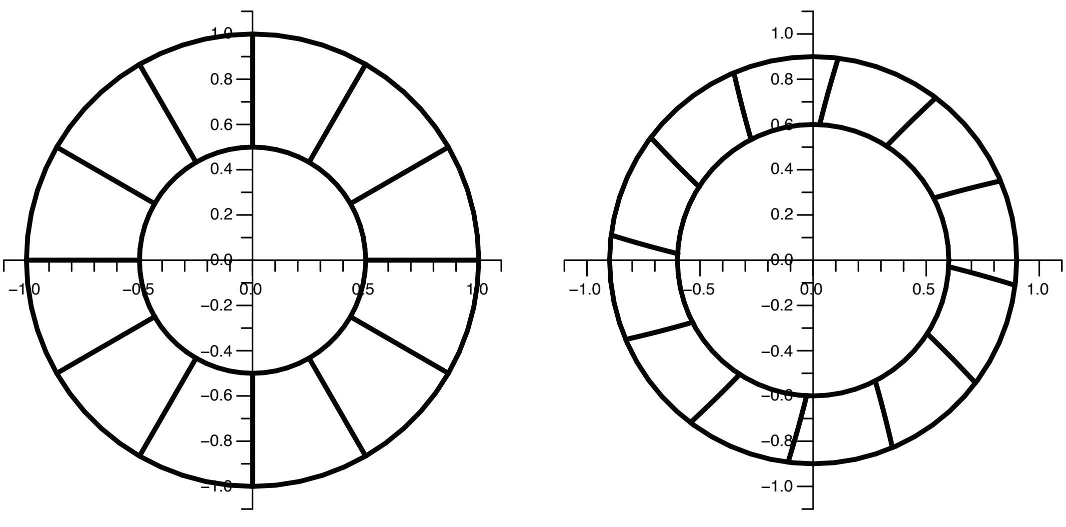
to obtain the stress field as

$$\begin{aligned}
T_{rr} &\simeq -p + (1-m)(r')^2 - m\left\{(r\lambda/R)^2 + \left[(r\tau)^2 + (r/R)^2\right]\left(\lambda r_1 T_0^A\right)^2/R^2\right\}, \\
T_{\theta\theta} &\simeq -p + (1-m)\left[(r/R)^2 + (r\tau)^2\right] - m(R/r)^2, \\
T_{zz} &\simeq -p + \left(\lambda T_0^A r_1\right)^2\left[\left(1+2m\lambda^2\right)\frac{1}{R^2} - \frac{2}{R}\left(\frac{\tau^2 r^2}{R} + \frac{r^2}{R^3} + \frac{\lambda^2}{R} - \frac{1}{2R}\right)m\right] \\
&\qquad + (1-m)\lambda^2 - m\left[(1/\lambda)^2 + (R\tau/\lambda)^2\right], \\
T_{r\theta} &\simeq rr'g' - m\lambda r_1 T_0^A \tau, \\
T_{rz} &\simeq (1-m)(r'w') + m\lambda r_1 T_0^A\left[rR\tau^2/\lambda + r/(\lambda R)\right]/R, \\
T_{\theta z} &\simeq (1-m)r\lambda\tau + \lambda r r_1 T_0^A g'/R + m(r'R\tau).
\end{aligned} \tag{5.49}$$



(a) (b)

Figure 5.3: (a, b) Pulling on the inside face of a Mooney–Rivlin tube, with a clockwise torsion. We have setted $m = 5.0,\ A = B/10,\ r_1 = 2A,\ r_2 = 0.9B,\ \tau = 0.5,\ |T_0^A| = \mu/2$.



(a) (b)

Figure 5.4: (a, b) Pulling on the inside face of a Mooney–Rivlin tube, with a clockwise torsion. We have setted $m = 5.0,\ A = B/2,\ r_1 = 0.6B,\ r_2 = 0.9B,\ \tau = 0.5,\ |T_0^A| = \mu/2$.

5.2 Final remarks

In non-Newtonian fluid mechanics and in turbulence theory, the existence of shear-induced normal stresses on planes transverse to the direction of shear is at the root of some important phenomena occurring in the flow of fluid down pipes of non-circular cross section (see [40]). In other words, pure parallel flows in tubes without axial symmetry are possible only when we consider the classical theory of Navier-Stokes equations or the linear theory of turbulence or tubes of circular cross section.

In nonlinear elasticity theory, similar phenomena are reported. Hence Fosdick and Kao [39] and Mollica and Rajagopal [83] show that, for general isotropic incompressible materials, an anti-plane shear deformation of a cylinder with non axial-symmetric cross section causes a secondary in-plane deformation field, because of normal stress differences. In compressible nonlinear elasticity pure torsion is possible only in a special class of materials, but we know that torsion plus a radial displacement is possible in all compressible isotropic elastic materials.

Now a further example is given in the literature from our recent work [27], where axial symmetry holds and the boundary conditions suggest that an axial shear deformation field is sufficient to solve the boundary value problem, but nevertheless, the normal stress difference wakes up a latent azimuthal shear deformation.

In conclusion, from these notes, it comes out that it is not really as crucial to determine the class of materials for which a given deformation field is possible, as it may be crucial to classify all the latent deformations associated with a given deformation field in such a way that this field is controllable for the entire class of materials. Indeed, no "real" material, even when we accept that its mechanical behaviour is purely elastic, is ever going to be described exactly by a special choice of strain-energy. Looking for special classes of materials for which special deformations fields are admissible may mislead us in our understanding of the nonlinear mechanical behaviour of materials.

5.3 A nice conjecture in solid mechanics

In the example discussed for secondary deformation, we have used a strong analogy with a cork-pulling problem, by modelling a cork as an incompressible rubber-like material. When we try to apply the previous results to the extraction of a cork from the neck of a bottle, the following remarks seem to be relevant. From the elementary theory of Coulomb friction, it is known that the pulled cork starts to move when, in modulus, the friction force exerted on the neck surface is equal to the normal force times the coefficient of static friction. In our case this means that

$$\sqrt{|T_{rz}(1)|^2 + |T_{r\theta}(1)|^2} = f_S\,|T_{rr}(1)| = f_S\,|T_0|\,, \tag{5.50}$$

where f_S is the coefficient of static friction. Using (5.21), we find that the elements of the left handside of equality (5.50) are

$$T_{rz}(1) = (r_1/r_2)T_0^A, \qquad T_{r\theta}(1) = 0. \tag{5.51}$$

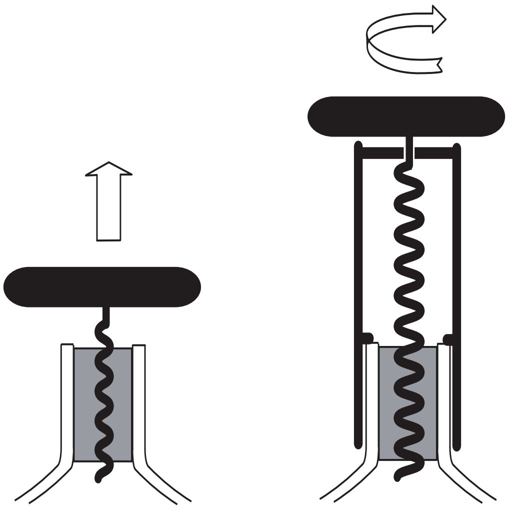


Figure 5.5: There are two main types of corkscrews: one that relies on pulling only (left) and one that adds a twist to the cork-pulling action (right). The analysis developed, indicates that the second type is more efficient.

Now, our main concern is to understand if it is better to apply a moment $M \neq 0$ when uncorking a bottle, than to pull only. First we suppose the cork is described by a neo-Hookean model (2.1). Then, to address this question, we note that the left-hand side of inequality (5.50) increases when $|T_0^A|$ increases; on the other hand, combining (5.34) and (5.35), we have

$$(T_0^A)^2 = \frac{\left[N + \frac{1}{\pi\lambda^2(r_1^2+r_2^2)^2}M^2\right]}{(2\pi\lambda r_1^2|\ln\eta|)}. \tag{5.52}$$

It is now clear, that for a fixed value of T_0^A, in the case $M \neq 0$, it is necessary to apply an axial force, the intensity of which is less than the one in the case $M = 0$. Moreover, Equation (5.52) shows that $\left(T_0^A\right)^2$ grows linearly with N but quadratically with M^2. With respect to efficient cork-pulling, the conclusion is that adding a twisting moment to a given pure axial force is more advantageous than solely increasing the vertical pull. Moreover, we observe that a moment is applied by using a lever and this is always more convenient from an energetic point of view. Recall that we made several simplifying assumptions to reach these results: not only infinite axial length, incompressibility, and isotropy, but also the choice of a truly special strain energy function.

In the end, we evoke a classic wine party dilemma:

> *Which kind of corkscrew system requires the least effort to uncork a bottle?*

Figure 5.5 sketches the two working principles commonly found in commercial corkscrews. The most common type (on the left) relies on pulling only (directly or

[2]Using the stress field (5.49) it is straightforward, but long and cumbersome, to derive the analogue for a Mooney–Rivlin solid with a small m of relation (5.52) (which was established for neo-Hookean solids). However, nothing truly new is gained from these complex formulae with respect to the simple neo-Hookean case, and we do not pursue this aspect any further.

through levers) and the other type (on the right) relies on a combination of pulling and twisting. Notwithstanding the shortcomings of this model with respect to an actual uncorking, we are confident that we have provided a scientific argument to those wine amateurs who favour the second type of corkscrews over the first.

Notes

In this chapter we have emphasized another important aspect in the use of the semi-inverse method: the emergence of secondary flows in fluid dynamics and of latent deformations (secondary fields) in solid mechanics. Navier-Stokes fluids or isotropic incompressible hyperelastic materials are clearly constructions of the mind. No real-world fluid is exactly a Navier-Stokes fluid and no real-world elastomer can be precisely characterized from a given elastic strain energy (in fact, the experimental data associated with the extension of a rubber band can be approximated by several, widely different, strain energy functions). It is fundamental to keep this observation in mind in order to understand that the results obtained by a semi-inverse method can be misleading at times. For example, we know that a Navier-Stokes fluid can move by parallels flows in a cylindrical tube of arbitrary section. To derive this result, we use the semi-inverse method by considering that the velocity possesses a non-zero component only along the generatrix of the cylinder and that this field is a function of the section variables only; then the Navier-Stokes equations are reduced to a linear parabolic equation which we solve by taking no-slip boundary conditions. This picture is specific to Navier-Stokes fluids. In fact, if the relation between the stress and the stretching is not linear, then a fluid can flow in a tube by parallel flows if and only if the tube possesses cylindrical symmetry (see Fosdick and Serrin [40]). If the tube is not cylindrically symmetric, then what is going on? Clearly any real fluid may flow in a tube, irrespective of whether it is a Navier-Stokes fluid or not. In reality we observe the birth of secondary flows, i.e. flows in the section of the cylinder. The true, meaningful problem is to understand when these secondary flows can be or cannot be neglected; it is not to determine for which special theory secondary flows disappear.

Here, an analogous phenomenon in non-linear elasticity is derived where the counterpart to secondary flows is the notion of latent deformations, i.e. deformations that are woken up from particular boundary conditions. Boundary conditions allow semi-inverse simple solutions for special materials, but for general materials they pose very difficult tasks. Many studies (see Chapter 4) sought to characterize the special strain energy functions for which particular classes of deformations turn out to be possible (or, using a standard terminology, turn out to be controllable). For example: which elastic compressible isotropic materials support simple isochoric torsion? In fact, it is of no utility to understand which materials possess this property, because these materials do not exist. It is far more important to understand which complex geometrical deformation accompanies the action of a moment twisting a cylinder. The range of results to be derived possesses meaningful applications, most importantly in biomechanics. In hemo-dynamics, it is often assumed that the arterial wall deforms according to simple geometric fields, but this hypothesis does not take into account several fundamental factors. A specific example is the effect of torsion on microvenous anastomotic patency and early thrombolytic phenomenon (see Selvaggi et al. [116]).

While there exists a remarkable literature on secondary flows in fluid dynamics, most notably by Rivlin, Ericksen, and Green (see for example the classic paper [109]), very little is known in solid mechanics about latent deformations. The main references in that area are: Fosdick and Kao [39], Mollica and Rajagopal [83], Horgan and Saccomandi [63].

Of course, our work [27] is another result to add to the previous ones, enriching therefore the literature of latent deformations in solid mechanics. The article [27] has been noticed in Science magazine [31/08/07, 317, no 5842, 1151, DOI: 10.1126/science.317.5842.1151a] where the paper has been qualified

> *A very nice application of the theory of nonlinear elasticity,*

and noticed also by the Daily Telegraph [22/08/07] and La Recherche [11/07].

Chapter 6

Euler buckling for compressible cylinders

One of the first, and most important, problems to be tackled by the theory of linear elasticity is that of the buckling of a column under an axial load. Using Bernoulli's beam equations, Euler found the critical load of compression N_{cr} leading to the buckling of a slender cylindrical column of radius B and length L. As recalled by Timoshenko and Gere [124], Euler looked at the case of an ideal column with (built in)-(free) end conditions. What is now called "Euler buckling", or the "*fundamental case* of buckling of a prismatic bar" [124] corresponds to the case of a bar with hinged-hinged end conditions. The corresponding critical load is given by

$$\frac{N_{\text{cr}}}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2, \tag{6.1}$$

where E is the Young modulus. The extension of this formula to the case of a thick column is a non-trivial, even sophisticated, problem of non-linear three-dimensional elasticity. In general, progress can be only made by using reductive (rod, shells, etc.) theories. However, there is another choice of boundary conditions for which the criterion (6.1) is valid: namely, the case where both ends are "guided" or "sliding" (the difference between the two cases lies in the shape of the deflected bar, which is according to the half-period of a sine in one case and of a cosine in the other case). In exact non-linear elasticity, there exists a remarkable three-dimensional analytical solution to this problem (due to Wilkes [129]) which describes a small-amplitude (incremental) deflection superimposed upon the large homogeneous deformation of a cylinder compressed between two lubricated platens. In this case, the Euler formula can be extended to the case of a column with finite dimensions, for arbitrary constitutive law.

The exact incremental solution allows for an explicit derivation of Euler's formula at the *onset of non-linearity*, which combines third-order elastic constants with a term in $(B/L)^4$. Goriely et al. [47], showed that for an incompressible cylinder,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96}\left(\frac{20}{3}E + 9\mathcal{A}\right)\left(\frac{B}{L}\right)^4, \tag{6.2}$$

where $\mathcal{A}$ is Landau's third-order elasticity constant. This formula clearly shows that *geometrical non-linearities* (term in $(B/L)^4$) are intrinsically coupled to *physical non-linearities* (term in $\mathcal{A}$) for this problem. (For the connection between Euler's theory of buckling and incremental incompressible nonlinear elasticity, see the early works of Wilkes [129], Biot [13], Fosdick and Shield [41], and the references collected in [47].)

Now, in third-order incompressible elasticity, there are two elastic constants: the shear modulus μ $(= E/3)$ and $\mathcal{A}$ (see (2.45)). In third-order compressible elasticity, there are five elastic constants: λ and μ, the (second-order) Lamé constants (or equivalently, E and ν, Young's modulus and Poisson's ratio), and $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the (third-order) Landau constants (see (2.43)). Euler's formula at order $(B/L)^2$, equation (6.1), involves only one elastic constant, E. It is thus natural to ask whether Poisson's ratio, ν, plays a role in the non-linear correction to Euler formula of $(B/L)^4$, the next-order term. The final answer is found in our recent work [26] as formula (6.32) below, which shows that the non-linear correction involves all five elastic constants.

6.1 Finite compression and buckling

In this section, we recall the equations governing the homogeneous compression of a cylinder in the theory of exact (finite) elasticity and we also use the form of some incremental solutions that is, of some small-amplitude static deformations which may be superimposed upon the large compression and which indicate the onset of instability for the cylinder.

The mathematical method used to determine the solutions for incremental solution is another example for the semi-inverse method. In fact our ansatz for the components of the mechanical displacement will be in looking for incremental static solutions that are periodic along the circumferential and axial directions, and have yet unknown radial variations (see formula (6.10)).

6.1.1 Large deformation

We take a cylinder made of a hyperelastic, compressible, isotropic solid with strain energy function $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ say, with radius B and length L in its undeformed configuration. We denote by (R, Θ, Z) the coordinates of a particle in the cylinder at rest, in a cylindrical system.

Then we consider that the cylinder is subject to the following deformation,

$$r = \lambda_1 R, \qquad \theta = \Theta, \qquad z = \lambda_3 Z, \tag{6.3}$$

where (r, θ, z) are the current coordinates of the particle initially at (R, Θ, Z), λ_1 is the radial stretch ratio and λ_3 is the axial stretch ratio. Explicitly, $\lambda_1 = b/B$ and $\lambda_3 = l/L$, where b and l are the radius and length of the deformed cylinder, respectively. The physical components of $\boldsymbol{F}$, the corresponding deformation gradient, are:

$$\boldsymbol{F} = \mathrm{Diag}\,(\lambda_1, \lambda_1, \lambda_3)\,, \tag{6.4}$$

showing that the deformation is equi-biaxial and homogeneous (and thus, universal).

The (constant) Cauchy stresses required to maintain the large homogeneous compression are given by formula (1.42),

$$T_i = J^{-1}\lambda_i W_i, \qquad i = 1, 2, 3 \quad \text{(no sum)}, \tag{6.5}$$

where $W_i \equiv \partial W/\partial\lambda_i$. In our case, $T_1 = T_2$ because the deformation is equi-biaxial, and $T_1 = T_2 = 0$ because the outer face of the cylinder is free of traction. Hence

$$T_1 = T_2 = \lambda_1^{-1}\lambda_3^{-1}W_1 = 0, \qquad T_3 = \lambda_1^{-2}W_3. \tag{6.6}$$

Note that we may use the first equality to express one principal stretch in terms of the other (provided, of course, that inverses can be performed).

6.1.2 Incremental equations

Now we recall the equations governing the equilibrium of incremental solutions, in the neighborhood of the finite compression. They read in general as (1.80)

$$\text{div } \boldsymbol{\Sigma}^T = \mathbf{0}, \tag{6.7}$$

where $\boldsymbol{\Sigma}$ is the incremental nominal stress tensor. It is related to $\boldsymbol{u}$, the incremental mechanical displacement, through the incremental constitutive law (1.81), that in component form can be written as

$$\Sigma_{ji} = \mathcal{A}_{0jilk} u_{k,l}, \tag{6.8}$$

where the comma denotes partial differentiation with respect to the current coordinates and $\boldsymbol{\mathcal{A}}_0$ is the fourth-order tensor of incremental elastic moduli. Its non-zero components, in a coordinate system aligned with the principal axes of strain, are given by (1.84). Note that here, some of these components are not independent one from another because $\lambda_1 = \lambda_2$ and $T_1 = T_2 = 0$. In particular, we find that

$$\begin{aligned}
&\mathcal{A}_{01212} = \mathcal{A}_{02121} = \mathcal{A}_{01221}, \qquad \mathcal{A}_{02323} = \mathcal{A}_{01313} = \mathcal{A}_{01331} = \mathcal{A}_{02332},\\
&\mathcal{A}_{02222} = \mathcal{A}_{01111}, \qquad \mathcal{A}_{02233} = \mathcal{A}_{01133}, \qquad \mathcal{A}_{03232} = \mathcal{A}_{03131},\\
&\mathcal{A}_{01122} = \mathcal{A}_{01111} - 2\mathcal{A}_{01212}.
\end{aligned} \tag{6.9}$$

6.1.3 Incremental solutions

We look for incremental static solutions that are periodic along the circumferential and axial directions, and have yet unknown radial variations. Thus our ansatz for the components of the mechanical displacement is the same as Wilkes's [129]:

$$\begin{aligned}
u_r &= U_r(r)\cos n\theta \cos kz,\\
u_\theta &= U_\theta(r)\sin n\theta \cos kz,\\
u_z &= U_z(r)\cos n\theta \sin kz,
\end{aligned} \tag{6.10}$$

where $n = 0, 1, 2, \ldots$ is the *circumferential mode number*; k is the *axial wavenumber*; the subscripts (r, θ, z) refer to components within the cylindrical coordinates (r, θ, z); and all upper-case functions are functions of r alone.

Dorfmann and Haughton [31] show that the following displacements $\boldsymbol{U}^{(1)}$, $\boldsymbol{U}^{(2)}$, and $\boldsymbol{U}^{(3)}$ are solutions to the incremental equations (6.7),

$$\boldsymbol{U}^{(1)}(r),\, \boldsymbol{U}^{(2)}(r) = \left[I_n'(qkr), -\frac{n}{qkr} I_n(qkr), -\frac{(\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131})}{q(\mathcal{A}_{01313} + \mathcal{A}_{01133})} I_n(qkr) \right]^T, \tag{6.11}$$

and

$$\boldsymbol{U}^{(3)}(r) = \left[\frac{1}{r} I_n(q_3kr), -\frac{q_3k}{n} I_n'(q_3kr), 0 \right]^T, \tag{6.12}$$

where $q = q_1, q_2$ in turn and I_n is the modified Bessel function of order n. Here q_1, q_2, and q_3 are the square roots of the roots q_1^2, q_2^2 of the following quadratic in q^2:

$$\begin{aligned} \mathcal{A}_{01313}\mathcal{A}_{01111}q^4 + [(\mathcal{A}_{01133} + \mathcal{A}_{01313})^2 - \mathcal{A}_{01313}\mathcal{A}_{03131} \\ - \mathcal{A}_{03333}\mathcal{A}_{01111}]q^2 + \mathcal{A}_{03333}\mathcal{A}_{03131} = 0, \end{aligned} \tag{6.13}$$

and of the root of the following linear equation in q^2

$$\mathcal{A}_{01212}q^2 - \mathcal{A}_{03131} = 0, \tag{6.14}$$

respectively.

From (6.8) we find that the incremental traction on planes normal to the axial direction has components of the same form as that of the displacements, namely

$$\begin{aligned} \Sigma_{rr} &= S_{rr}(r) \cos n\theta \cos kz, \\ \Sigma_{r\theta} &= S_{r\theta}(r) \sin n\theta \cos kz, \\ \Sigma_{rz} &= S_{rz}(r) \cos n\theta \sin kz, \end{aligned} \tag{6.15}$$

say, where again the functions $S_{rr}, S_{r\theta}, S_{rz}$ are functions of r alone. Then we find that the traction solutions corresponding to the solutions (6.11) and (6.12) are given by

$$\begin{aligned} r\boldsymbol{S}^{(1)}(r), \quad r\boldsymbol{S}^{(2)}(r) = \Bigg[& 2\mathcal{A}_{01212} I_n'(qkr) \\ & - \left(2\mathcal{A}_{01212}\frac{n^2}{qkr} + qkr\mathcal{A}_{01111} - \frac{kr\mathcal{A}_{01133}\,(\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131})}{q\,(\mathcal{A}_{01313} + \mathcal{A}_{01133})} \right) I_n(qkr), \\ 2n\mathcal{A}_{01212} & \left(\frac{I_n(qkr)}{qkr} - I_n'(qkr) \right), -kr\mathcal{A}_{01313} \left(\frac{\mathcal{A}_{01111}q^2 - \mathcal{A}_{03131}}{\mathcal{A}_{01313} + \mathcal{A}_{01133}} + 1 \right) I_n'(qkr) \Bigg]^T, \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} r\boldsymbol{S}^{(3)}(r) = \Bigg[& 2\mathcal{A}_{01212} \left(\frac{I_n(q_3kr)}{r} - q_3kI_n'(q_3kr) \right), \\ \mathcal{A}_{01212} & \left(\frac{2q_3k}{n} I_n'(q_3kr) - \left(\frac{2n}{r} + \frac{q_3^2k^2r}{n} \right) I_n(q_3kr) \right), -k\mathcal{A}_{01313}I_n(q_3kr) \Bigg]^T. \end{aligned} \tag{6.17}$$

The general solution to the incremental equations of equilibrium is thus of the form

$$r\boldsymbol{S}(r) = \left[r\boldsymbol{S}^{(1)}(r) \middle| r\boldsymbol{S}^{(2)}(r) \middle| r\boldsymbol{S}^{(3)}(r) \right] \boldsymbol{c}, \tag{6.18}$$

where $\boldsymbol{S} \equiv [S_{rr}, S_{r\theta}, S_{rz}]^T$, and $\boldsymbol{c}$ is a constant three-component vector. Note that we use the quantity $r\boldsymbol{S}$ for the traction (instead of $\boldsymbol{S}$), because it is the Hamiltonian conjugate to the displacement in cylindrical coordinates [117].

Now when the cylinder is compressed (by platens say), its end faces should stay in full contact with the platens so that the first incremental boundary condition is

$$u_z = 0, \qquad \text{on} \quad z = 0, l, \tag{6.19}$$

which leads to [31, 47]

$$k = m\pi/l, \tag{6.20}$$

for some integer m, the *axial mode number*. From (6.15), we now see that on the thrust faces, we have

$$\Sigma_{rz} = 0, \qquad \text{on} \quad z = 0, l, \tag{6.21}$$

which means that the end faces of the column are in sliding contact with the thrusting platens. In other words, in the limit of a slender column, we recover the Euler strut with *sliding-sliding*, or *guided-guided* end conditions. In Figure 6.1, we show the first two axi-symmetric and two asymmetric modes of incremental buckling.

The other boundary condition is that the cylindrical face is free of incremental traction: $\boldsymbol{S}(b) = \boldsymbol{0}$. This gives

$$\Delta \equiv \det \left[b\boldsymbol{S}^{(1)}(b) \middle| b\boldsymbol{S}^{(2)}(b) \middle| b\boldsymbol{S}^{(3)}(b) \right] = 0. \tag{6.22}$$

6.2 Euler buckling

6.2.1 Asymptotic expansions

We now specialize the analysis to the asymmetric buckling mode $n = 1$, $m = 1$, corresponding to the Euler buckling with guided-guided end conditions, in the limit where the axial compressive stretch λ_3 is close to 1 (the other modes are not reached for slender enough cylinders). To this end, we only need to consider the third-order elasticity expansion of the strain energy density (2.43). (Note that there are other, equivalent, expansions based on other invariants, such as the ones proposed by Murnaghan [86], Toupin and Bernstein [125], Bland [17], or Eringen and Suhubi [37], see Norris [91] for the connections.)

To measure how close λ_3 is to 1, we introduce ϵ, a small parameter proportional to the slenderness of the deformed cylinder,

$$\epsilon = kb = \pi b/l. \tag{6.23}$$

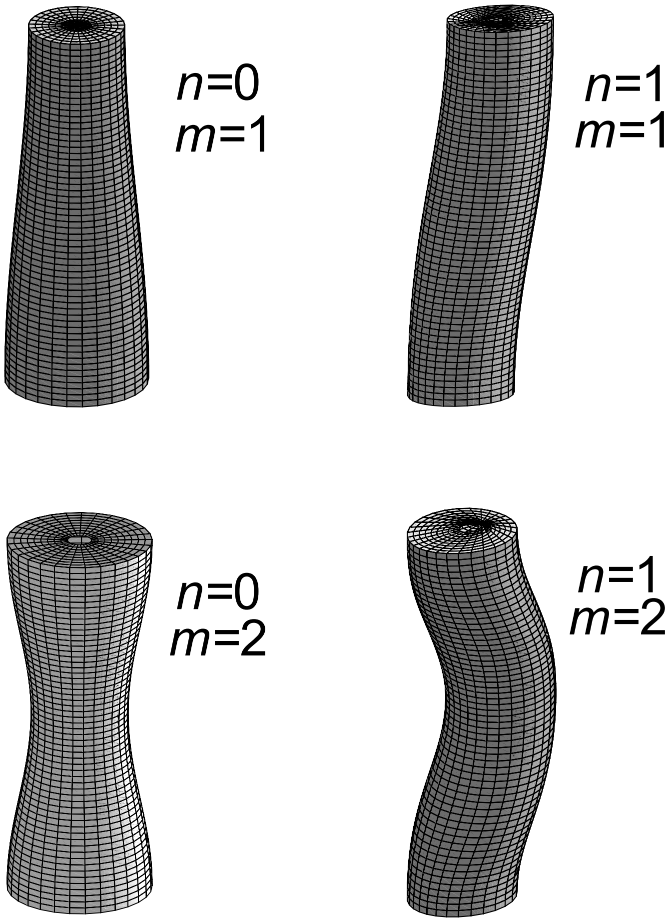


Figure 6.1: First two axi-symmetric and two asymmetric modes of buckling for a compressed strut with guided-guided end conditions: n is the circumferential mode number and m the axial mode number. For slender enough cylinders, the $n = 1$, $m = 1$ mode is the first mode of buckling.

Then we expand the radial stretch λ_1 and the critical buckling stretch λ_3 in terms of ϵ up to order M,

$$\lambda_1 = \lambda_1(\epsilon) = 1 + \sum_{p=1}^{M} \alpha_p \epsilon^p + \mathcal{O}(\epsilon^{M+1}), \qquad \lambda_3 = \lambda_3(\epsilon) = 1 + \sum_{p=1}^{M} \beta_p \epsilon^p + \mathcal{O}(\epsilon^{M+1}), \tag{6.24}$$

say, where the α's and β's are to be determined shortly. Similarly, we expand Δ in powers of ϵ,

$$\Delta = \sum_{p=1}^{M_d} d_p \epsilon^p + \mathcal{O}(\epsilon^{M_d+1}),$$

and solve each order $d_p = 0$ for the coefficients α_p and β_p, making use of the condition $T_1 = 0$. We find that α_p and β_p vanish identically for all odd values of p, and that λ_1 and λ_3, up to the fourth-order in ϵ, are given by

$$\lambda_1 = 1 + \alpha_2 \epsilon^2 + \alpha_4 \epsilon^4 + O(\epsilon^6), \qquad \lambda_3 = 1 + \beta_2 \epsilon^2 + \beta_4 \epsilon^4 + O(\epsilon^6), \tag{6.25}$$

with α_2 and α_4 given by

$$\alpha_2 = \frac{\nu}{4}, \tag{6.26}$$

and

$$\alpha_4 = -\frac{\nu(1+\nu)}{32} - \frac{(1+\nu)(1-2\nu)}{16E}\left[\nu^2\mathcal{A} + (1-2\nu+6\nu^2)\mathcal{B} + (1-2\nu)^2\mathcal{C}\right] - \nu\beta_4, \tag{6.27}$$

wherein

$$\beta_2 = -\frac{1}{4}, \tag{6.28}$$

and

$$\beta_4 = \frac{29+39\nu+8\nu^2}{96(1+\nu)} + \frac{1}{16E}\left[(1-2\nu^3)\mathcal{A} + 3(1-2\nu)(1+2\nu^2)\mathcal{B} + (1-2\nu)^3\mathcal{C}\right].$$

Note that we switched from Lamé constants to Poisson's ratio and Young's modulus for these expressions, using the connections (1.53).

6.2.2 Onset of nonlinear Euler buckling

The analytical results presented above are formulated in terms of the current geometrical parameter ϵ, defined in (6.23). In order to relate these results to the classical form of Euler buckling, we introduce the initial geometric slenderness B/L. Recalling that $\epsilon = \pi b/l$, $\lambda_3 = l/L$, and $b = \lambda_1 B$, we find that

$$\epsilon\lambda_3 = \pi\lambda_1(B/L). \tag{6.29}$$

We expand ϵ in powers of B/L, and solve (6.29) to obtain

$$\begin{aligned}\epsilon &= \pi(B/L) + (\alpha_2-\beta_2)\pi^3(B/L)^3 + \mathcal{O}\left((B/L)^4\right)\\ &= \pi(B/L) + (1+\nu)(\pi^3/4)(B/L)^3 + \mathcal{O}\left((B/L)^4\right).\end{aligned} \tag{6.30}$$

Second, we wish to relate the axial compression to the current axial load N. To do so, we integrate the axial stress over the faces of the cylinder,

$$N = -2\pi\int_0^b rT_3\mathrm{d}r = -\pi b^2 T_3 = -\pi\lambda_1^2 B^2 T_3, \tag{6.31}$$

because T_3 is constant, given by $(6.6)_2$.

Finally, in order to write the nonlinear buckling formula, we expand λ_1 and λ_3 in (6.31), using (6.25), and then expand ϵ in powers of the slenderness (B/L), using (6.30). It gives the desired expression for the first *non-linear correction to Euler formula*,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96}\delta_{\text{NL}}\left(\frac{B}{L}\right)^4, \tag{6.32}$$

where

$$\delta_{\text{NL}} = 2\frac{13+12\nu-2\nu^2}{(1+\nu)}E + \\ 12\left[(1-2\nu^3)\mathcal{A} + 3(1-2\nu)(1+2\nu^2)\mathcal{B} + (1-2\nu)^3\mathcal{C}\right]. \tag{6.33}$$

We now check this equation against its incompressible counterpart (6.2). Theoretical considerations and experimental measurements [22, 29, 94, 131], show that in the incompressible limit, E and $\mathcal{A}$ remain finite,

$$\nu \to 1/2, \quad (1-2\nu)\mathcal{B} \to -E/3, \quad (1-2\nu)^3\mathcal{C} \to 0. \tag{6.34}$$

It is then a simple exercise to verify that (6.32) is indeed consistent with (6.2) in those limits.

6.2.3 Examples

Table 6.1: Lamé constants and Landau third-order elastic moduli for five solids (10^9 N$\cdot$m^{-2})

material	λ	μ	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Polystyrene	1.71	0.95	−10	−8.3	−10.6
Steel Hecla 37	111	82.1	−358	−282	−177
Aluminium 2S	57	27.6	−228	−197	−102
Pyrex glass	13.5	27.5	420	−118	132
SiO_2 melted	15.9	31.3	−44	93	36

To evaluate the importance of the non-linear correction, we computed the critical axial stretch ratio of column buckling for two solids. In Table 6.1, we list the second- and third-order elastic constants of five compressible solids, as collected by Porubov [101] (in the Table we converted the "Murnaghan constants" given by Porubov to Landau constants $\mathcal{A}, \mathcal{B}, \mathcal{C}$). Figure 6.2 shows the variations of λ_3 with the squared slenderness $(B/L)^2$, for pyrex and silica (two last lines of Table 1).

Notes

The present analysis can be considered an extension of the article [47] from incompressible solids to compressible solids. It provides an asymptotic formula for the critical value of the load for the Euler buckling problem, with guided-guided (sliding-sliding) end conditions. This formula was checked both in the incompressible limit and on particular cases against the exact value of the buckling

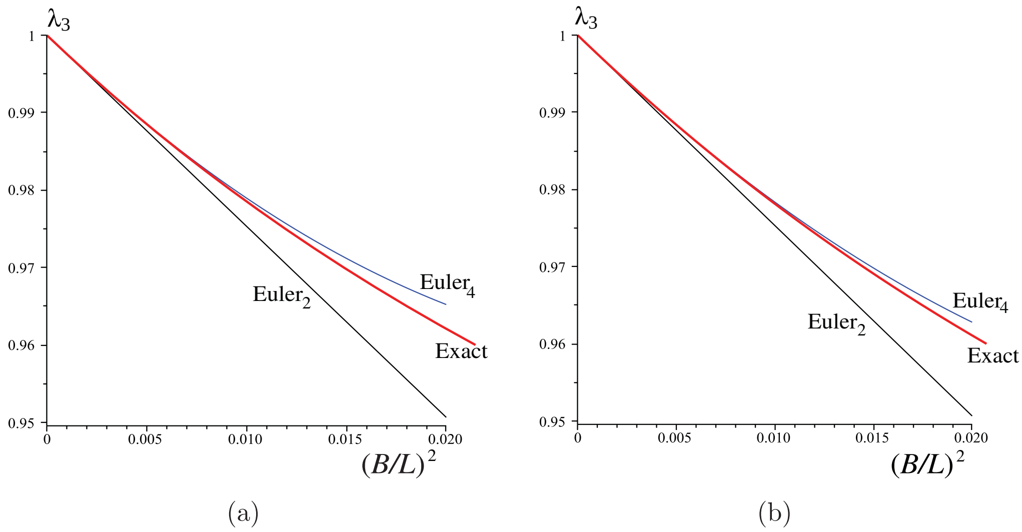


Figure 6.2: Comparison of the different Euler formulas obtained by expanding the exact solution to order 2 (classical Euler buckling formula, plot labeled "Euler$_2$") and to order 4 (plot labeled "Euler$_4$"), for pyrex (a) and for silica (b).

obtained from the exact solutions. Not surprisingly it reinforces the universal and generic nature of the Euler buckling formula, as the correction is small for most systems even when nonlinear elastic effects and nonlinear geometric effects are taken into account. It would be of great interest to see if these effects could be observed experimentally.

We note that Goriely et al. [47] write the equations for the linearized problem of instability using the *Stroh formalism* [122]. They adapt the work of Shuvalov [117] on waves in anisotropic cylinders to develop a Stroh-like formulation of the problem. From a computational perspective, the Stroh formalism is particularly well suited and well behaved (Biryukov [14]; Fu [42]) and if numerical integration was required it would provide an ideal representation of the governing equation.

Here we could have also presented the linearized problem of instability for the compressible cylinder within the Stroh formalism. We have omitted this formalism because it was not essential for our discussion.

Appendix A

Articles in the press relating our work [27]

Our article [27] has been noticed by the international press and web sites.

On August 21, 2007 the Proceedings of the Royal Society of London A published our article [27] online and the following day it was just noticed by *The Daily Telegraph* [August 22, 2007]. A few days later [August 31, 2007] *Science* magazine [Vol. 317, no 5842, 1151, DOI: 10.1126/science.317.5842.1151a] reported its results too. Our research has also been noticed in *La recherche* magazine [November, 2007, no. 413].

Some websites which liked our results are: *UK Wine Show* [August 25, 2007, http://www.thirtyfifty.co.uk], *The Math Gateway* of the Mathematical Association of America [October 9, 2007, http://mathdl.maa.org/mathDL] and the website *Cyberpresse.ca* [October 31, 2007, http://www.cyberpresse.ca].

Professor Sir Michael Berry, Editor of the Proceedings of the Royal Society A, later included our article in a list of "outliers", [Editorial Proceedings of The Royal Society A, January 8, 2010 466:1-2; published online before print November 11, 2009, doi:10.1098/rspa.2009.0535].

In the following pages, reproduce these press clippings.

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How to remove the cork from a wine bottle

Last Updated: 12:01am BST 22/08/2007

It is more efficient to remove a cork from a bottle by a combination of twist and pull, rather than pulling alone, according to a team of French and Italian researchers.

A team from Lecce University in Italy and the University of Pierre and Marie Curie in Paris has explored the background mathematics in a paper entitled: "The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity."

Second rate? Scientists prefer more complex instruments

In today's Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, Michel Destrade, Riccardo de Pascalis, and Prof Giuseppe Saccomandi say their arcane workings are able to "answer a classic wine party dilemma: which kind of corkscrew system requires the least effort to uncork a bottle?"

After 16 pages of mathematical calculations in which a cork is modelled as "an incompressible rubber-like material", they conclude that their work "backs wine amateurs who favour a system relying on a combination of pulling and twisting over a system relying on pulling."

They point out that the most common corkscrews rely on pulling only - either directly or through levers - but these are inferior to the rarer kinds that rely on a combination of pulling and twisting.

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The authors conclude that they "are confident that they have provided a scientific argument to those wine experts who favour the second type of corkscrew over the first."

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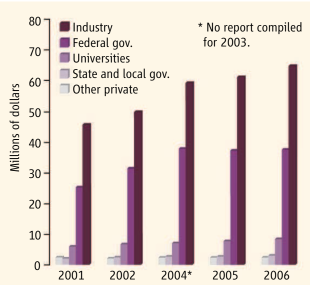
RANDOMSAMPLES

EDITED BY ROBERT COONTZ

Health Research Funding: No Relief in Sight

Some policy wonks have suggested that foundations and other private sources will compensate for the flat National Institutes of Health (NIH) budget (*Science*, 11 May, p. 817). That's wishful thinking, says Research!America, a nonprofit group in Alexandria, Virginia, that tracks U.S. health research funding. Its latest analysis (below) shows that nonindustry private funding represented 2% of the $116 billion spent on U.S. health research in 2006 and has been "completely flat" since 2001, says Research!America policy analyst Stacie Propst.



Spending by industry has risen slightly since NIH's budget stalled at about $29 billion after 2004, but Propst predicts a dip because industry research funding typically follows federal patterns with a lag of a few years. The proportion of each U.S. health care dollar that now goes to research is 5.5 cents and falling, Propst adds; meanwhile, countries such as the United Kingdom and Singapore, although still behind the United States, are expanding their investments. "The trends are not good," says Research!America President Mary Woolley.

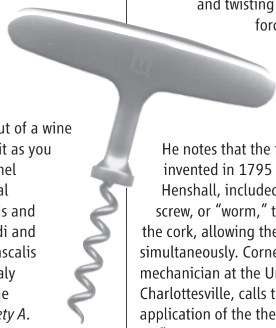
Filet of Zebrafish

NET WATCH

Long a favorite of developmental biologists, the zebrafish is now catching on with researchers studying cancer, drug addiction, and numerous other conditions. A new anatomical atlas for this scientific school is FishNet from the Victor Chang Cardiac Research Institute in Sydney, Australia.

The reference, which features 36,000 images captured using optical projection tomography, is the first to detail the fish's structure from embryo to adult. For each stage, visitors can call up lengthwise or cross-sectional slices, many of which include labels that pinpoint nascent organs and other features. Additional image sets highlight the developing nervous system and the skeleton. >> www.FishNet.org.au

Crisp, With a Hint of Calculus



It's official: A cork will come out of a wine bottle more easily if you twist it as you pull. That's what physicist Michel Destrade of the French national research agency, CNRS, in Paris and engineer Giuseppe Saccomandi and mathematician Riccardo De Pascalis of the University of Lecce in Italy reported last week online in the *Proceedings of the Royal Society A*.

The team analyzed the problem to underscore that solids can deform in counterintuitive ways. For example, they show that a cork can twist internally even if it is pulled straight up. Such "secondary deformations" should not be overlooked, Destrade says. As a sidelight, the team also showed that pulling and twisting extracts the cork with less force than pulling alone.

That result won't surprise enophiles, says Rajendra Kanodia, proprietor of the Web site Corkscrew.com. He notes that the first patented corkscrew, invented in 1795 by Englishman Samuel Henshall, included a disk just above the screw, or "worm," that butts up against the cork, allowing the user to twist and pull it simultaneously. Cornelius Horgan, an applied mechanician at the University of Virginia, Charlottesville, calls the analysis "a very nice application of the theory of nonlinear elasticity," which is currently undergoing a renaissance with its applications to biological materials.

No Mean Cat Feat

Researchers working in central China have photographed one of the world's most poorly studied mammals, the Chinese mountain cat. First described by scientists in 1892, the cat (*Felis bieti*) is known only from a few skins in museums and six live animals in Chinese zoos, says Jim Sanderson, a mammalogist and founder of the Small Cat Conservation Alliance. In May 2003, Sanderson and colleagues Yin Yufeng and Drubgyal (his single Tibetan name) set out to find it in the wild. The effort paid off this summer, when their camera traps on the Tibetan Plateau in northwestern Sichuan Province caught eight photos of the cats hunting at night. Sanderson hopes the images will encourage conservation of the cat.

CREDITS (TOP TO BOTTOM): RAJENDRA KANODIA/CORKSCREW.COM; RESEARCH!AMERICA; J. SANDERSON

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CURIOSA

Arbres à ours

Dans les forêts de Colombie-Britannique, des « arbres à ours » ont été repérés où les grizzlis aiment à se frotter, mordillant et griffant l'écorce. Ce sont toujours quelques mêmes troncs qui sont choisis, selon une expérimentation menée durant les nuits de printemps de 2005 et 2006 à l'aide de caméras cachées. Quatre arbres ont été ainsi la cible d'ours qui s'y sont frotté 52 fois. Les bêtes étaient principalement des mâles dominants, ce qui fait penser aux chercheurs que chaque animal passant près d'un de ces arbres a à cœur d'effacer toutes traces de possibles rivaux. Mais les auteurs de l'étude ne savent toujours pas pourquoi les ours choisissent tel ou tel arbre.

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Maison vendue après la virgule

Les publicitaires savent depuis des lustres qu'à 99,99 euros un bien apparaît plus attractif aux yeux du consommateur qu'à 100 euros. C'est la même chose pour le prix des maisons, selon une nouvelle étude de l'université américaine Cornell. En Floride du Sud et à Long Island (État de New York), une maison cotée à 484 700 dollars a été vendue, au final, 1 300 dollars de plus qu'une autre affichant un prix de départ de 485 000 dollars. Selon les chercheurs, lors du processus d'achat, le nombre précis sera perçu comme plus bas et plus « définitif » que le nombre rond, ce qui découragerait les tentatives de marchandage. Mais cette tendance a-t-elle résisté à la crise des « subprimes » ?

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Poussez pas le bouchon

Les amateurs de bonnes bouteilles l'ont maintes et maintes fois expérimenté : le bouchon se déloge plus facilement si l'on effectue une torsion avec le tire-bouchon en même temps que l'on tire. La chose vient d'être étudiée en laboratoire par le physicien français Michel Destrade et deux de ses collègues de l'université de Lecce en Italie. L'objectif était d'étudier comment certains solides se déforment de manière parfois contre-intuitive. Par exemple, un bouchon se vrille alors qu'il est tiré tout droit, dans ce que l'équipe appelle une « déformation secondaire ». Commentaire très sobre d'un spécialiste de mécanique appliquée de l'université de Virginie : ce travail est une *« application très intéressante de la théorie de l'élasticité non linéaire ».*

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Ballon taille 42

Plus une balle se rapproche d'une sphère parfaite, plus précise sera sa trajectoire. Un Sud-Africain, géologue et fou de foot, a décidé de s'attaquer à la question après que, durant la dernière Coupe du Monde, des goals se sont plaints de « trajectoires imprévisibles » des ballons Adidas « Teamgeist » à 14 faces utilisés pendant cette compétition. Jos Luris propose un ballon à 42 faces, combinaison la plus équilibrée selon lui : 12 faces d'un dodécaèdre pentagonal et 30 d'un triacontraèdre rhombique où les faces sont en forme de losanges. Le professeur émérite de l'université de Johannesburg a envoyé son *design* à Adidas, espérant ainsi le voir testé par la firme lors de la prochaine Coupe du monde, en 2010 en Afrique du Sud. Mais Adidas a botté en touche : elle défend son Teamgeist, *« un ballon parfaitement rond alliant précision et contrôle ».*

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Glaneuses-acheteuses

La psychologie évolutionniste ne cesse de nous éclairer sur la nature profonde de l'élément féminin. Voici qu'elle nous explique pourquoi, quand il s'agit de faire les courses, la femme fait mieux que l'homme pour repérer les différentes nourritures qui s'offrent à la vue dans un marché, de la même façon, supposent-ils, qu'elle glanait avec brio les baies comestibles dans son groupe de chasseurs-cueilleurs. L'étude de chercheurs de l'université de Californie portant sur 86 individus montre que les femmes se rappellent mieux de l'emplacement des aliments. Mais s'agissait-il de paquets de pâtes, ou bien de packs de bière ?

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Cigales en bout de ligne

L'année dernière au Japon, plus de 1 000 pannes ont affecté un réseau de communication pourtant ultramoderne avec ses lignes en fibres optiques. Responsables, les cigales *Cryptotympana facialis* qui utilisent les fils pour pondre leurs œufs. Connu au Japon sous le nom de Kumazemi, l'insecte de 7 centimètres de long prolifère en zone urbaine. Il semble confondre le câble en fibre optique avec les brindilles mortes sur lesquelles il se plaît. Parade imaginée par les ingénieurs de Nippon Telegraph and Telephon : gainer les fibres d'un revêtement en polyuréthane qui imite l'écorce vivante que, généralement, les cigales évitent.

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Wine News

Research shows pull and twist corkscrew is best by Sandra Clement Saturday, August 25, 2007

It is easier to remove a cork from a wine bottle by twisting and pulling rather than pulling alone, according to a team of Italian and French researchers.

The team from Lecce University and the Centre National de la Recherche Scientifique used the theory of nonlinear elasticity to answer a classic wine party dilemma: which kind of corkscrew system requires the least effort to uncork a bottle? Researchers Riccardo De Pascalis, Michel Destrade and Giuseppe Saccomandi provided a mathematical argument – running to some 16 pages – which backed 'those wine amateurs who favour a system relying on a combination of pulling and twisting over a system relying on pulling only' because the 'calculations indicated that it requires more force to pull only than to pull and twist'.

The team, whose paper appears in this month's *Proceedings of the Royal Society: Mathematical, Physical and Engineering Sciences*, points out that corkscrews based on the pull-only option are more common but the ease of opening with the second option is reflected in some of the commercial names, such as 'magic corkscrew' and 'easy corkscrew'. However, the twist-and-pull kind aren't as rare as hen's teeth – you'll find one at Ikea.

Interviews and News for Anjou-Gamay

Research shows pull and twist corkscrew is best

It is easier to remove a cork from a wine bottle by twisting and pulling rather than pulling alone, according to a team of Italian and French researchers.

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Giving a Cork a Mathematical Twist

October 9, 2007

Well, now we know: It takes less effort to remove a cork from a bottle by pulling *and* twisting than by just pulling it. The evidence appears in a 15-page mathematical paper published recently in the *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*.

Modeling a cork as an "incompressible rubber-like material," Michel Destrade of the Université Pierre et Marie Curie in Paris, along with mathematician Riccardo de Pascalis and engineer Giuseppe Saccomandi (both from the University of Lecce, in Italy), showed that a cork being pulled from a bottle can twist internally. Adding your own twist to this "secondary deformation" makes the task that much easier.

Their paper, titled "The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity," shows that solids can deform in counterintuitive ways. As reported in *Science*, applied mathematician Cornelius Horgan of the University of Virginia called the researchers' analysis a "very nice application of the theory of nonlinear elasticity."

We'll drink to that!

Source: *Science*, Aug. 31, 2007; Telegraph, Aug. 22, 2007.

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Le mercredi 31 octobre 2007

Des chercheurs se penchent sur le tire-bouchon

Agrandir

Agence Science-Presse

Une équipe de physiciens franco-italienne a étudié le plus sérieusement du monde la manière idéale de déboucher une bouteille de vin.

Sa conclusion: le bouchon sortira plus facilement de la bouteille si vous le tordez en même temps que vous tirez. Michel Destrade et Giuseppe Saccomandi profitaient de cette «énigme» pour réfléchir sur la façon dont un solide (ici, le bouchon) peut se déformer de manière «contre-intuitive»: par exemple, le bouchon de liège se déforme par l'intérieur, même s'il est «tiré» par une force extérieure.

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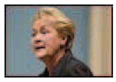

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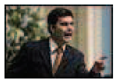

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PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2010) **466**, 1–2
doi:10.1098/rspa.2009.0535
Published online 11 November 2009

Editorial

In my last editorial, I explained that in order to cope with the large number of papers that were being submitted to *Proc. R. Soc. A*, every paper we received would be pre-assessed by a member of the Editorial Board, and referees for papers that passed this hurdle were asked to apply stringent quality standards. These procedures, initiated by my predecessor Trevor Stuart, have now proved successful in several ways. Referees no longer waste time reporting on papers that have no chance of being accepted—sometimes because they are poor, but more often because although they contain good science their content is deemed suitable for a more specialist journal. Publication has been greatly accelerated: the current receipt to acceptance time is 90 days. Moreover, the rejection rate has been brought down from an unrealistic height of more than 80 per cent to a more reasonable 72 per cent (getting the right rate is delicate: with 100 per cent, nobody would submit papers; with 0 per cent, we would be publishing trash).

These improvements would not have occurred without the efforts of Joanna Harries and Louise Gardner in the journal office, the Society's publications production staff and, of course, indispensable advice from the Editorial Board.

Proc. R. Soc. A aims to publish papers across the whole of the physical sciences. I am interpreting this very widely. Recently, we have published serious scientific studies of a painting by Monet (interpreting the position of the sun to determine where and when it was painted; Baker & Thornes 2006), Viking navigation (to determine whether they could have used the polarization of skylight; Hedegus *et al.* 2007), erasing toner on office paper (to enable it to be re-used; Counsell & Allwood 2009), stability of the Millennium bridge (MacDonald 2009), dynamics of golf swings (Sharp 2009), efficiency of gaits (Srinivasan & Ruina 2007), stresses (in the cork) during the opening of wine bottles (De Pascalis *et al.* 2007), etc. These outliers supplement our core papers, reporting substantial, occasionally seminal, advances in quantum physics, engineering, information science, materials science, pure and applied mathematics, and chemistry (for which we are at last starting to attract papers in numbers commensurate with the scientific importance of that subject).

A curse of researchers, publishers and editors is the fashionable emphasis on bibliometric indicators. Chief among these evils is the impact factor. Ours is increasing but still rather low (currently 1.7). But the impact factor is a measure only of short-term success (citations over the preceding 2 years); for *Proc. R. Soc. A*, a better indicator is the citation half-life. Ours is off-scale: greater than ten years, reflecting our aim of publishing slow-burning, long-lasting, papers.

In this anniversary year, celebrating 350 years since the foundation of the Royal Society, we plan to publish a series of invited articles, contributed by world-leading authorities across the range of subjects that we cover. The first

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of these, 'Nanostructured Bainite', by H.K.D.H. Bhadeshia (Bhadeshia 2010), appears in this issue. These articles will reinforce our position as one of the best, as well as the oldest, journals of physical science.

Michael Berry

References

Baker, J. & Thornes, J. E. 2006 Solar position within Monet's Houses of Parliament. *Proc. R. Soc. A* **462**, 3775–3788. (doi:10.1098/rspa.2006.1754)

Bhadeshia, H. K. H. D. 2010 Nanostructured Bainite. *Proc. R. Soc. A* **466**, 2113. (doi:10.1098/rspa.2009.0407)

Counsell, T. A. M. & Allwood, J. M. 2009 Using solvents to remove a toner print so that office paper might be reused. *Proc. R. Soc. A* **465**, 3839–3858. (doi:10.1098/rspa.2009.0144)

De Pascalis, R., Destrade, M. & Saccomandi, G. 2007 The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity. *Proc. R. Soc. A* **463**, 2945–2959. (doi:10.1098/rspa.2007.0010)

Hegedüs, R., Åkesson, S., Wehner, R. & Horváth, G. 2007 Could Vikings have navigated under foggy and cloudy conditions by skylight polarization? On the atmospheric optical prerequisites of polarimetric Viking navigation under foggy and cloudy skies. *Proc. R. Soc. A* **463**, 1081–1095. (doi:10.1098/rspa.2007.1811)

Macdonald, J. H. G. 2009 Lateral excitation of bridges by balancing pedestrians. *Proc. R. Soc. A* **465**, 1055–1073. (doi:10.1098/rspa.2009.0367)

Sharp, R. S. 2009 On the mechanics of the golf swing. *Proc. R. Soc. A* **465**, 551–570. (doi:10.1098/rspa.2008.0304)

Srinivasan, M. & Ruina, A. 2007 Idealized walking and running gaits minimize work. *Proc. R. Soc. A* **463**, 2429–2446. (doi:10.1098/rspa.2007.0006)

Appendix B

To generate Pictures 2.1-2.2, we used for the stretch λ and the Biot stress $f = t/\lambda$ (with t the principal stress) the following data

λ	1	1.02	1.12	1.24	1.39	1.58	1.9	2.18	2.42
f	0	0.26	1.37	2.3	3.23	4.16	5.1	6	6.9
λ	3.02	3.57	4.03	4.76	5.36	5.75	6.15	6.4	
f	8.8	10.7	12.5	16.2	19.9	23.6	27.4	31	
λ	6.6	6.85	7.05	7.15	7.25	7.4	7.5	7.6	
f	34.8	38.5	42.1	45.8	49.6	53.3	57	64.4	

To generate Pictures 2.3-2.4, we used for the stretch λ and for the Biot stress divided for λ, f/λ, the following data

λ	1	1.027	1.065	1.115	1.14	1.20	1.31	1.42	1.68
f/λ	0	0.92	1.50	2.17	2.30	2.77	3.38	3.65	3.93
λ	1.94	2.49	3.03	3.43	3.75	4.07	4.26	4.45	
f/λ	4.01	3.93	4.17	4.28	4.64	4.94	5.27	5.54	

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Bibliography

[1] Abeyaratne, R., Horgan, C. O., *The pressurized hollow sphere problem in finite elastostatics for a class of compressible materials*, International Journal of Solids and Structures, 20 (8): 715–723 (1984).

[2] Aboudi, J., Arnold, S. M., *Micromechanical modeling of the finite deformation of thermoelastic multiphase composites*, Mathematics and Mechanics of Solids, 5 (1): 75–99 (2000).

[3] Agarwal, V. K., *On finite anti-plane shear for compressible elastic circular tube*, Journal of Elasticity, 9 (3): 311–319 (1977).

[4] Atkin, R. J., Fox, N., *An Introduction to the Theory of Elasticity*, Longman Scientific and Technical, Harlow, Essex (1980).

[5] Barré de Saint-Venant, A. J., *Mémoire sur la torsion des prismes*, Journal de Mathématique, 2 (1) (1856).

[6] Barré de Saint-Venant, A. J. C., *Mémoire sur la flexion des prismes*, volume 14, Mm. Savants Étrangers (1855).

[7] Batra, R. C., *Deformation produced by a simple tensile load in an isotropic elastic body*, Journal of Elasticity, 6 (1): 109–111 (1976).

[8] Beatty, M. F., *Topics in finite elasticity: Hyperelasticity of rubber, elastomers, and biological tissues-With examples*, Applied Mechanics Reviews, 40 (12): 1699–1734 (1987).

[9] Beatty, M. F., *Introduction to nonlinear elasticity*, Nonlinear effects in fluids and solids, volume 45 of *Math. Concepts Methods Sci. Engrg.*, 13–112, Plenum, New York (1996).

[10] Beatty, M. F., Jiang, Q., *On compressible materials capable of sustaining axisymmetric shear deformations. II. Rotational shear of isotropic hyperelastic materials*, Quarterly Journal of Mechanics and Applied Mathematics, 50 (2): 211–237 (1997).

[11] Beatty, M. F., Jiang, Q., *On compressible materials capable of sustaining axisymmetric shear deformations. Part 3: Helical shear of isotropic hyperelastic materials*, Quarterly of Applied Mathematics, 577 (4): 681–697 (1999).

[12] Beatty, M. F., Stalnaker, D. O., *The Poisson function of finite elasticity*, Journal of Applied Mechanics, 53 (108): 807–813 (1986).

[13] Biot, M. A., *Surface instability of rubber in compression*, Applied Scientific Research, 12: 168–182 (1963).

[14] Biryukov, S. V., *Impedance method in the theory of elastic surface waves*, Akademiya Nauk SSSR. Akusticheskiĭ Zhurnal, 31 (5): 583–590 (1985).

[15] Biscari, P., Omati, C., *Stability of generalized Knowles solids*, IMA Journal of Applied Mathematics, to appear (2010).

[16] Biscari, P., Poggi, C., Virga, E. G., *Mechanics Notebook*, Liguori Editore, Napoli (1999).

[17] Bland, D. R., *Nonlinear Dynamic Elasticity*, Blaisdell Publishing Co. [Ginn and Co.], Waltham, Mass.-Toronto, Ont.-London (1969).

[18] Blatz, P. J., Ko, W. L., *Application of finite elastic theory to the deformation of rubbery materials*, Transactions of the Society of Rheology, 6 (1): 223–252 (1962).

[19] Carlson, D. E., *On the role of inverse methods in nonlinear elasticity*, Ima, P. S. (Editor), Abstracts from the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations, volume 98, Beatty M., Brezis H., Ericksen J., Kinderlehrer D., (Eds) (1984).

[20] Carroll, M. M., *Finite strain solutions in compressible isotropic elasticity*, Journal of Elasticity, 20 (1): 65–92 (1988).

[21] Carroll, M. M., Horgan, C. O., *Finite strain solutions for a compressible elastic solid*, Quarterly of Applied Mathematics, 48 (4): 767–780 (1990).

[22] Catheline, S., Gennisson, J. L., Fink, M., *Measurement of elastic nonlinearity of soft solid with transient elastography*, Journal of the Acoustical Society of America, 114 (6 Pt 1): 3087–3091 (2003).

[23] Clebsch, A., *Théorie de l'Élasticité des Corps Solides*, Dunod, Paris (1883), traduite par M. M. Barré de Saint-Venant et Flamant, avec des Notes tendues de M. Barré de Saint-Venant, Reprinted by Johnson Reprint Corporation, New York, (1966).

[24] Currie, P., Hayes, M., *Longitudinal and Transverse Waves in Finite Elastic Strain. Hadamard and Green Materials*, 5 (2): 140–161 (1969).

[25] Currie, P. K., Hayes, M. A., *On non-universal finite elastic deformations in Finite Elasticity*, Finite Elasticity, Proc. of IUTAM Symposium on Finite Elasticity, 143–150, Springer, Martinus Nijhoff, The Hague (1981).

[26] De Pascalis, R., Destrade, M., Goriely, A., *Nonlinear correction to the Euler buckling formula for compressed cylinders with guided-guided end conditions*, Journal of Elasticity, to appear (2010).

[27] De Pascalis, R., Destrade, M., Saccomandi, G., *The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity*, Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences, 463 (2087): 2945–2959 (2007).

[28] De Pascalis, R., Rajagopal, K. R., Saccomandi, G., *Remarks on the use and misuse of the semi-inverse method in the nonlinear theory of elasticity*, Quarterly Journal of Mechanics and Applied Mathematics, 62 (4): 451–464 (2009).

[29] Destrade, M., Ogden, R., *On the third- and fourth-order constants of incompressible isotropic elasticity*, submitted.

[30] Destrade, M., Saccomandi, G. (Editors), *Waves in Nonlinear Pre-Stressed Materials*, CISM International Centre for Mechanical Sciences, Number 424, Springer-Verlag Wien, Vienna (2007).

[31] Dorfmann, A., Haughton, D. M., *Stability and bifurcation of compressed elastic cylindrical tubes*, International Journal of Engineering Science, 44 (18-19): 1353–1365 (2006).

[32] Dorfmann, A., Muhr, A., *Constitutive Models for Rubber*, Balkema, Rotterdam (1999).

[33] Ericksen, J. L., *Deformations possible in every isotropic, incompressible, perfectly elastic body*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 5: 466–489 (1954).

[34] Ericksen, J. L., *Deformations possible in every compressible, isotropic, perfectly elastic material*, Journal of Mathematics and Physics, 34: 126–128 (1955).

[35] Ericksen, J. L., *Overdetermination of the speed in rectilinear motion of non-Newtonian fluids*, Quarterly of Applied Mathematics, 14: 318–321 (1956).

[36] Ericksen, J. L., *Special Topics in Elastostatics*, volume 17 of *Advances in Applied Mechanics*, 189 – 244, Elsevier (1977).

[37] Eringen, A. C., Şuhubi, E. S., *Elastodynamics. Vol. I. Finite motions*, Academic Press, New York-London (1974).

[38] Faulkner, M. G., Haddow, J. B., *Nearly isochoric finite torsion of a compressible isotropic elastic circular cylinder*, Acta Mechanica, 13 (3-4): 245–253 (2005).

[39] Fosdick, R. L., Kao, B. G., *Transverse deformations associated with rectilinear shear in elastic solids*, Journal of Elasticity, 8 (2): 117–142 (1978).

[40] Fosdick, R. L., Serrin, J., *Rectilinear steady flow of simple fluids*, Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences, 332: 311–333 (1973).

[41] Fosdick, R. L., Shield, R. T., *Small bending of a circular bar superposed on finite extension or compression*, Archive for Rational Mechanics and Analysis, 12: 223–248 (1963).

[42] Fu, Y. B., *An explicit expression for the surface-impedance matrix of a generally anisotropic incompressible elastic material in a state of plane strain*, International Journal of Non-Linear Mechanics, 40 (2-3): 229 – 239 (2005), Special Issue in Honour of C. O. Horgan.

[43] Fu, Y. B., Ogden, R. W. (Editors), *Nonlinear Elasticity: Theory and Applications*, volume 283, London Mathematical Society Lecture Notes, Cambridge University Press (2001).

[44] Fung, Y. C., *Elasticity of soft tissues in simple elongation*, American Journal of Physiology, 213: 1532–1544 (1967).

[45] Gent, A. N., *A new constitutive relation for rubber*, Rubber Chemistry and Technology, 69: 59–61 (1996).

[46] Gibson, L. J., Easterling, K. E., Ashby, M. F., *The structure and mechanics of cork*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 377 (1769): 99–117 (1981).

[47] Goriely, A., Vandiver, R., Destrade, M., *Nonlinear Euler buckling*, Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences, 464 (2099): 3003–3019 (2008).

[48] Green, A. E., *Finite elastic deformation of compressible isotropic bodies*, Proceedings of the Royal Society. London. Series A. Mathematical and Physical Sciences, 227: 271–278 (1955).

[49] Green, A. E., Adkins, J. E., *Large Elastic Deformations*, Second edition, revised by A. E. Green, Clarendon Press, Oxford (1970).

[50] Gurtin, M. E., *An Introduction to Continuum Mechanics*, volume 158 of *Mathematics in Science and Engineering*, Academic Press, New York (1981).

[51] Hadamard, J., *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*, Paris: A. Hermann. XIII u. 375 S. 8 (1903).

[52] Haughton, D. M., *Circular shearing of compressible elastic cylinders*, Quarterly Journal of Mechanics and Applied Mathematics, 46 (3): 471–486 (1993).

[53] Haughton, D. M., *Using null strain energy functions in compressible finite elasticity to generate exact solutions*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 59 (4): 730–749 (2008).

[54] Hayes, M., Horgan, C. O., *On the Dirichlet problem for incompressible elastic materials*, Journal of Elasticity, 4 (1): 17–25 (2001).

[55] Hayes, M. A., Saccomandi, G. (Editors), *Topics in Finite Elasticity*, CISM International Centre for Mechanical Sciences, Number 424, Springer-Verlag Wien, Vienna (2001).

[56] Hill, J. M., *The effect of precompression on the load-deflection relations of long rubber bush mountings*, Journal of Applied Polymer Science, 19 (3): 747–755 (1975).

[57] Holzapfel, G. A., *Nonlinear Solid Mechanics. A Continuum Approach for Engineering*, John Wiley & Sons Ltd., Chichester (2000).

[58] Horgan, C. O., *Void nucleation and growth for compressible non-linearly elastic materials: An example*, International Journal of Solids and Structures, 29 (3): 279–291 (1992).

[59] Horgan, C. O., *Anti-plane shear deformations in linear and nonlinear solid mechanics*, SIAM Review, 37 (1): 53–81 (1995).

[60] Horgan, C. O., *Nonlinear Elasticity: Theory and Applications*, chapter Equilibrium solutions for compressible nonlinear elasticity, 135–159, number 283 in London Mathematical Society Lecture Note Series, Cambridge University Press (2001).

[61] Horgan, C. O., Saccomandi, G., *Anti-plane shear deformations for non-Gaussian isotropic, incompressible hyperelastic materials*, Proceedings. Series A. Mathematical, Physical and Engineering Sciences, 457 (2012): 1999–2017 (2001).

[62] Horgan, C. O., Saccomandi, G., *Helical shear for hardening generalized neo-Hookean elastic materials*, Mathematics and Mechanics of Solids, 8 (5): 539–559 (2003), dedicated to Raymond W. Ogden on the occasion of his 60th birthday.

[63] Horgan, C. O., Saccomandi, G., *Superposition of generalized plane strain on anti-plane shear deformations in isotropic incompressible hyperelastic materials*, Journal of Elasticity, 73 (1-3): 221–235 (2003).

[64] Jiang, Q., Beatty, M. F., *On compressible materials capable of sustaining axisymmetric shear deformations. Part 1: Anti-plane shear of isotropic hyperelastic materials*, Journal of Elasticity, 39 (1): 75–95 (1995).

[65] Jiang, Q., Beatty, M. F., *On Compressible Materials Capable of Sustaining Axisymmetric Shear Deformations. Part 4: Helical Shear of Anisotropic Hyperelastic Materials*, Journal of Elasticity, 62 (1): 47–83 (2001).

[66] Jiang, X., Ogden, R., *On azimuthal shear of a circular cylindrical tube of compressible elastic material*, Quarterly Journal of Mechanics and Applied Mathematics, 51 (1): 143–158 (1998).

[67] John, F., *Plane strain problems for a perfectly elastic material of harmonic type*, Communications on Pure and Applied Mathematics, 13: 239–296 (1960).

[68] John, F., *Plane elastic waves of finite amplitude. Hadamard materials and harmonic materials*, Communications on Pure and Applied Mathematics, 19: 309–341 (1966).

[69] Jones, D. F., Treloar, L. R. G., *The properties of rubber in pure homogeneous strain*, Journal of Physics D: Applied Physics, 8 (11): 1285–1304 (1975).

[70] Kirkinis, E., Ogden, R. W., *On extension and torsion of a compressible elastic circular cylinder*, Mathematics and Mechanics of Solids, 7 (4): 373–392 (2002).

[71] Kirkinis, E., Tsai, H., *Generalized azimuthal shear deformations in compressible isotropic elastic materials*, SIAM Journal on Applied Mathematics, 65 (3): 1080–1099 (2005).

[72] Knowles, J. K., *On finite anti-plane shear for incompressible elastic materials*, Australian Mathematical Society Journal. Series B. Applied Mathematics, 19 (4): 400–415 (1975/76).

[73] Knowles, J. K., *The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids*, International Journal of Fracture, 13 (5): 611–639 (1977).

[74] Knowles, J. K., *A note on anti-plane shear for compressible materials in finite elastostatics*, Australian Mathematical Society Journal. Series B. Applied Mathematics, 20 (1): 1–7 (1977/78).

[75] Knowles, J. K., *Universal states of finite anti-plane shear: Ericksen's problem in miniature*, American Mathematical Monthly, 86 (2): 109–113 (1979).

[76] Landau, L. D., Lifshitz, E. M., *Theory of Elasticity*, Third revised edition, Pergamon, New York (1986).

[77] Leipholz, H. H. E., *Theory of Elasticity*, Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics of Elastic Stability, 1, Noordhoff International Publishing, Leyden (1974).

[78] Levinson, M., *Finite torsion of slightly compressible rubberlike circular cylinders*, International Journal of Non-Linear Mechanics, 7 (4): 445 – 463 (1972).

[79] Levinson, M., Burgess, I., *A comparison of some simple constitutive relations for slightly compressible rubber-like materials*, International Journal of Mechanical Sciences, 13 (6): 563 – 572 (1971).

[80] Marris, A. W., Shiau, J. F., *Universal deformations in isotropic incompressible hyperelastic materials when the deformation tensor has equal proper values*, Archive for Rational Mechanics and Analysis, 36 (2): 135–160 (1970).

[81] Martin, S. E., Carlson, D. E., *The behavior of elastic heat conductors with second-order response functions*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 28 (2): 311–329 (1977).

[82] Mioduchowski, A., Haddow, J. B., *Finite telescopic shear of a compressible hyperelastic tube*, International Journal of Non-Linear Mechanics, 9 (3): 209 – 220 (1974).

[83] Mollica, F., Rajagopal, K. R., *Secondary deformations due to axial shear of the annular region between two eccentrically placed cylinders*, Journal of Elasticity, 48 (2): 103–123 (1997).

[84] Mooney, M., *A theory of large elastic deformation*, Journal of Applied Physics, 11 (9): 582–592 (1940).

[85] Murnaghan, F. D., *Finite Deformations of an Elastic Solid*, American Journal of Mathematics, 59 (2): 235–260 (1937).

[86] Murnaghan, F. D., *Finite deformation of an elastic solid*, John Wiley & Sons Inc., New York (1951).

[87] Murphy, J. G., *Some new closed-form solutions describing spherical inflation in compressible finite elasticity*, IMA Journal of Applied Mathematics, 48 (3): 305–316 (1992).

[88] Murphy, J. G., *Inflation and eversion of spherical shells of a special compressible material*, Journal of Elasticity, 30 (3): 251–276 (1993).

[89] Nayfeh, A., Mook, D. T., *Nonlinear Oscillations*, John Wiley & Sons, New York (1979).

[90] Neményi, P., *Recent Developments in Inverse and Semi-Inverse Methods in the Mechanics of Continua*, volume 2 of *Advances in Applied Mechanics*, 123 – 151, Elsevier (1951).

[91] Norris, A. N., *Finite amplitude waves in solids*, Hamilton, M. F., Blackstock, D. T. (Editors), Nonlinear Acoustics, 263–277, Academic Press, San Diego (1999).

[92] Obata, Y., Kawabata, S., Kawai, H., *Mechanical properties of natural rubber vulcanizates in finite deformation*, Journal of Polymer Science Part A-2: Polymer Physics, 8 (6): 903–919 (1970).

[93] Ogden, R. W., *Large deformation isotropic elasticity - On the correlation of theory and experiment for incompressible rubberlike solids*, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 326 (1567): 565–584 (1972).

[94] Ogden, R. W., *On isotropic tensors and elastic moduli*, Mathematical Proceedings of the Cambridge Philosophical Society, 75: 427–436 (1974).

[95] Ogden, R. W., *Non-Linear Elastic Deformations*, Dover, New York (1997), Reprint of Ellis Harwood Ltd, Chichester, (1984).

[96] Olver, P. J., Rosenau, P., *The construction of special solutions to partial differential equations*, Physics Letters. A, 114 (3): 107–112 (1986).

[97] Polignone, D. A., Horgan, C. O., *Pure torsion of compressible non-linearly elastic circular cylinders*, Quarterly of Applied Mathematics, 49 (3): 591–607 (1991).

[98] Polignone, D. A., Horgan, C. O., *Axisymmetric finite anti-plane shear of compressible nonlinearly elastic circular tubes*, Quarterly of Applied Mathematics, 50 (2): 323–341 (1992).

[99] Polignone, D. A., Horgan, C. O., *A note on the pure torsion of a circular cylinder for a compressible nonlinearly elastic material with nonconvex strain-energy*, Journal of Elasticity, 37 (2): 167–178 (1994).

[100] Polignone, D. A., Horgan, C. O., *Pure azimuthal shear of compressible nonlinearly elastic circular tubes*, Quarterly of Applied Mathematics, 52 (1): 113–131 (1994).

[101] Porubov, A. V., *Amplification of Nonlinear Strain Waves in Solids*, volume 9 of *Series on Stability Vibration and Control of Systems, Series A*, World Scientific, Singapore (2003).

[102] Poynting, J. H., *On pressure perpendicular to the shear planes in finite pure shears, and on the lengthening of loaded wires when twisted*, Royal Society of London Proceedings Series A, 82: 546–559 (1909).

[103] Rajagopal, K., Srinivasa, A., *On the response of non-dissipative solids*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 463 (2078): 357–367 (2007).

[104] Rajagopal, K. R., *The elasticity of elasticity*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 58 (2): 309 – 317 (2007).

[105] Rivlin, R. S., *Torsion of a rubber cylinder*, Journal of Applied Physics, 18 (5): 444–449 (1947).

[106] Rivlin, R. S., *Large elastic deformations of isotropic materials. IV. Further developments of the general theory*, Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences, 241: 379–397 (1948).

[107] Rivlin, R. S., *Large elastic deformations of isotropic materials. VI. Further results in the theory of torsion, shear and flexure*, Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences, 242: 173–195 (1949).

[108] Rivlin, R. S., *A note on the torsion of an incompressible highly-elastic cylinder*, Mathematical Proceedings of the Cambridge Philosophical Society, 45: 485–487 (1949).

[109] Rivlin, R. S., *The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids*, Quarterly of Applied Mathematics, 15: 212–215 (1957).

[110] Rivlin, R. S., *Some topics in finite elasticity*, Proceedings of the First Naval Symposyum on Structural Mechanics, Pergamon Press, New York (1960).

[111] Rivlin, R. S., *Collected Papers of R.S. Rivlin*, Springer-Verlag, New York (1997).

[112] Rivlin, R. S., Saunders, D. W., *Large elastic deformations of isotropic materials. VII. Experiments on the deformation of rubber*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 243 (865): 251–288 (1951).

[113] Saccomandi, G., *Universal results in finite elasticity*, Nonlinear elasticity: theory and applications, volume 283 of *London Math. Soc. Lecture Note Ser.*, 97–134, Cambridge Univ. Press, Cambridge (2001).

[114] Saccomandi, G., *A personal overview on the reduction methods for partial differential equations*, Note di Matematica, 23 (2): 217–248 (2004/05).

[115] Saccomandi, G., Ogden, R. W. (Editors), *Mechanics and Thermomechanics of Rubberlike Solids*, volume 8 of *CISM International Centre for Mechanical Sciences*, Springer-Verlag Wien, New York (2004).

[116] Selvaggi, G., Anicic, S., Formaggia, L., *Mathematical explanation of the buckling of the vessels after twisting of the microanastomosis*, Microsurgery, 26 (7): 524–528 (2006).

[117] Shuvalov, A. L., *A sextic formalism for three-dimensional elastodynamics of cylindrically anisotropic radially inhomogeneous materials*, The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences, 459 (2035): 1611–1639 (2003).

[118] Signorini, A., *Trasformazioni termoelastiche finite*, Annali di Matematica Pura ed Applicata, 39: 147–201 (1955).

[119] Singh, M., Pipkin, A. C., *Note on Ericksen's problem*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 16 (5): 706–709 (1965).

[120] Singh, S., *Small finite deformations of elastic dielectrics*, Quarterly of Applied Mathematics, 25: 275–284 (1967).

[121] Spencer, A. J. M., *Continuum Mechanics*, Dover Publications Inc., Mineola, NY (2004), reprint of the 1980 edition [Longman, London; MR0597343].

[122] Stroh, A. N., *Steady state problems in anisotropic elasticity*, Journal of Mathematical Physics, 41: 77–103 (1962).

[123] Thomson (Lord Kelvin), W., Tait, P. G., *Treatise on Natural Philosophy, Part I*, Cambridge (1867), [The second edition, which appeared in 1879, has been reprinted by Dover Publications, N. Y., under the redundant as well as gratuitous title *Principles of Mechanics and Dynamics*].

[124] Timoshenko, S. P., Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill, New York (1961).

[125] Toupin, R. A., Bernstein, B., *Sound waves in deformed perfectly elastic materials. Acoustoelastic effect*, Journal Acoustical Society of America, 33: 216–225 (1961).

[126] Treloar, L. R. G., *Stress-strain data for vulcanised rubber under various types of deformation*, Transactions of the Faraday Society, 40: 59 – 70 (1944).

[127] Truesdell, C., Noll, W., *The Nonlinear Field Theories of Mechanics*, Springer-Verlag, Berlin, 2nd edition (1992).

[128] Valanis, K. C., Landel, R. F., *The strain-energy function of a hyperelastic material in terms of the extension ratios*, Journal of Applied Physics, 38 (7): 2997–3002 (1967).

[129] Wilkes, E. W., *On the stability of a circular tube under end thrust*, Quarterly Journal of Mechanics and Applied Mathematics, 8: 88–100 (1955).

[130] Wineman, A., *Some results for generalized neo-Hookean elastic materials*, International Journal of Non-Linear Mechanics, 40 (2-3): 271 – 279 (2005), special Issue in Honour of C.O. Horgan.

[131] Wochner, M. S., Hamilton, M. F., Ilinskii, Y. A., Zabolotskaya, E. A., *Cubic nonlinearity in shear wave beams with different polarizations*, Journal of the Acoustical Society of America, 123 (5): 2488–2495 (2008).