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GENERICALLY SECANT AND SECANT FAMILIES

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Abstract. *In the paper we state the notions of generalically secant and secant families of varieties in Stoka's integral geometry theory, showing some results on their union and developing a few examples.*

Let ω be a set of varieties belonging to a family $\mathcal{F}_q$, and consisting of p-dimensional varieties V_p, defined in an n-dimensional space X_n by analytic equations

$$F^{\lambda}(x_1, x_2, \ldots, x_n, A_1, A_2, \ldots, A_q), \quad \lambda = 1, 2, \ldots, n-p.$$

Here $x_1, x_2, \ldots, x_n$ are coordinates in X_n, while $A_1, A_2, \ldots, A_q$ are essential parameters, that is coordinates in the parameter space Y_q.

If G_r is a group of transformations acting on X_n, and depending on r parameters, then the invariant measures of ω with respect to G_r, whether these exist, are

$$\mu_{G_r}(\omega) = \int_{\underline{\omega}(\underline{A})} |\Phi(A_1, A_2, \ldots, A_q)| dA_1 \wedge dA_2 \wedge \ldots \wedge dA_q, \tag{1}$$

where $\underline{\omega} \subseteq Y_q$ is the set of parameters which determines ω; Φ is an invariant integral function, that is one of the possible solution of Deltheil system

$$\sum_{k=1}^{q} \frac{\partial(\xi_k^h(A_1, A_2, \ldots, A_q)\Phi)}{\partial A_k} = 0, \quad h = 1, 2, \ldots, r, \tag{2}$$

being $\xi_k^h(A_1, A_2, \ldots, A_q)$ coefficients of infinitesimal transformations of $\mathcal{F}_q$ with respect to the group H_r, isomorphic to G_r **[1]**, **[2]**, **[6]**, **[7]**, **[8]**, **[9]**.

Let $\mathcal{F}_q = \mathcal{F}_q(\underline{A})$ be measurable with density $d\psi = d\psi(\underline{A}) = \Phi(\underline{A}) \wedge^q d\underline{A}$, and H_s a measurable subgroup of the group H_r isomorphic to the maximal group of invariance G_r of $\mathcal{F}_q$, $(q \leq s \leq r)$. Then there exist q relative components of G_r, namely $dD_1, dD_2, \ldots, dD_q$, such that the density of $\mathcal{F}_q$ with respect to the group G_s, to which H_s is isomorphic, is given by

$$d\psi = [dD_1 dD_2 \ldots dD_q] = dD_1 \wedge dD_2 \wedge \ldots \wedge dD_q = \wedge^q d\underline{D} \tag{3}$$

This statement, pointed out in the previous form in **[3]**, is a consequence of results of Santaló and Stoka **[6]**, **[8]**. It shows that, up to relations

$$\alpha_k(A_1, A_2, \ldots, A_q, D_1, D_2, \ldots, D_q) = 0 \quad k = 1, 2, \ldots, q, \tag{4}$$

it is always possible to obtain a change of parameters

$$A_k = A_k(D_1, D_2, \ldots, D_q) \quad k = 1, 2, \ldots, q, \tag{5}$$

such that

$$\Phi(A_1, A_2, \ldots, A_q) det \left[\frac{\partial \underline{A}}{\partial \underline{D}}\right] = 1.$$

Consequently, Φ being an invariant integral function, we have

$$\mu_{G_r}(\omega) = \int_{\underline{\omega}(\underline{A})} |\Phi(A_1, A_2, \ldots, A_q)| dA_1 \wedge dA_2 \wedge \ldots \wedge dA_q =$$

$$= \int_{\underline{\omega}(\underline{D})} |\Phi(A_1, A_2, \ldots, A_q)| det \left[\frac{\partial \underline{A}}{\partial \underline{D}}\right] dD_1 \wedge dD_2 \wedge \ldots \wedge dD_q = \int_{\underline{\omega}(\underline{D})} dD_1 \wedge dD_2 \wedge \ldots \wedge dD_q \tag{6}$$

then, the density of $\mathcal{F}_q$ may be written as the exterior product of the new parameters. These are called *normal parameters* **[10]**.

Remark 1. Since the relative components of a group are determined up to linear transformations with constant coefficients, these keep normal parameters, which consequently are not unique. So, though it is always possible to represent a family with normal parameters, not all normalizations are allowed, in sense that replacements (5) may restrict the family $\mathcal{F}_q(\underline{A})$ on a weak-subfamily $\overline{\mathcal{F}}_q(\underline{D})$ **[3]**,**[5]**. Moreover, the maximal group of invariance $\overline{G}$ of a weak-subfamily is not necessarily the same G_r as that of the family **[5]**. Consequently the normalized density might be meaningless for $\overline{\mathcal{F}}_q(\underline{D})$ with respect to G_s, whether G_s is not contained in $\overline{G}$. In this case the normalized density works for $\overline{\mathcal{F}}_q(\underline{D})$ with respect to $G_s \cap \overline{G}$.

Definition 1. *Two families of varieties $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$, depending respectively on q and t essential parameters, $\{\underline{A}\} = \{A_1, A_2, \ldots, A_q\}$ and $\{\underline{B}\} = \{B_1, B_2, \ldots, B_t\}$, are said to be generically secant families if there exist s independent relations*

$$\alpha_k(\underline{A}, \underline{B}) = 0 \quad k = 1, 2, \ldots, s \quad 0 \leq s \leq min(q, t),$$

among $\{\underline{A}\}$ and $\{\underline{B}\}$.

Remark 2. $\{\underline{A}\}$ and $\{\underline{B}\}$ being essential parameters, the number s cannot be greater than $min(q, t)$. If $s = 0$ then the families are independent **[4]**.

A special case is obtained when s parameters appear both in $\{\underline{A}\}$ and in $\{\underline{B}\}$. If this occurs we say simply that $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$ are *secant families*.

Definition 2. *The number s of independent relations (or common parameters) between the parameter sets of two generically secant families (or secant families) $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$, is called the intersection order of $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$.*

When we do not need to treat separately the cases of generically secant and secant families, of intersection order s, we say simply that the families have intersection order s.

Remark 3. The way to get families of intersection order s is to start from any pair of independent families of varieties, and to introduce, if possible, s relations (in particular s equalities) among their essential parameters. Here we emphasize that intersection is related to parameter spaces, that is, for given lists of parameters $\underline{A}$ and $\underline{B}$, the related varieties of $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$ may have in X_n empty intersection.

If we take the union family of two generically secant families of varieties, $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$, of intersection order s, then the parameter space of $\mathcal{F}_q(\underline{A}) + \mathcal{F}_t(\underline{B})$ can be described, in the cartesian product $Y_q \times Y_t$, by a $(q+t-s)$-dimensional variety V_{q+t-s}. If $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$ are secant families, then V_{q+t-s} is a linear subspace of $Y_q \times Y_t$.

When $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$, $q \leq t$, have intersection order s, to each variety $W \in \mathcal{F}_t(\underline{B})$ are associated ∞^{q-s} varieties belonging to $\mathcal{F}_q(\underline{A})$. The union family $\mathcal{F}_{q+t-s} = \mathcal{F}_q(\underline{A}) + \mathcal{F}_t(\underline{B})$, may be regarded as the family of ∞^t sets of ∞^{q-s} pairs of this kind.

Definition 3. *Each set of varieties of the union family $\mathcal{F}_{q+t-s} = \mathcal{F}_q(\underline{A}) + \mathcal{F}_t(\underline{B})$ of two families of intersection order s, $q \leq t$, obtained for any fixed $W \in \mathcal{F}_t$, is said to be an s-secant figure.*

The s-secant figures give, in X_n, a description of the $(q+t-s)$-dimensional variety which determines, in $Y_q \times Y_t$ the parameter space of the union family.

Theorem 1. *Let $\mathcal{F}_q(A_1, A_2, \ldots, A_{q-s}, B_1, B_2, \ldots, B_s)$ and $\mathcal{F}_t(B_1, B_2, \ldots, B_s, C_1, C_2, \ldots, C_{t-s})$ be two measurable secant families of varieties, having intersection order s, whose respective densities are*

$$d\Phi_q(A_1, A_2, \ldots, A_{q-s}, B_1, B_2, \ldots, B_s) =$$

$$\alpha_{q-s}(A_1, A_2, \ldots, A_{q-s})\beta_s(B_1, B_2, \ldots, B_s)dA_1 \wedge \ldots \wedge dB_s,$$

$$d\Psi_t(B_1, B_2, \ldots, B_s, C_1, C_2, \ldots, C_{t-s}) =$$

$$\beta_s(B_1, B_2, \ldots, B_s)\gamma_{t-s}(C_1, C_2, \ldots, C_{t-s})dB_1 \wedge \ldots \wedge dC_{t-s}.$$

Then the family $\mathcal{F}_{q+t-s}(\underline{A}, \underline{B}, \underline{C}) = \mathcal{F}_q(\underline{A}, \underline{B}) + \mathcal{F}_t(\underline{B}, \underline{C})$ takes the density

$$d\mu_{q+t-s}(\underline{A}, \underline{B}, \underline{C}) = \alpha_{q-s}(\underline{A})\beta_s(\underline{B})\gamma_{t-s}(\underline{C}) \wedge^{q-s} d\underline{A} \wedge^s d\underline{B} \wedge^{t-s} d\underline{C}, \tag{7}$$

for every group of invariance G_r with respect to which $\mathcal{F}_q(\underline{A}, \underline{B})$ and $\mathcal{F}_t(\underline{B}, \underline{C})$ are both measurable.

Proof. Let G_r be a group of invariance of $\mathcal{F}_{q+t-s}$ with respect to which $\mathcal{F}_q$ and $\mathcal{F}_t$ are measurable. The Deltheil system related to $\mathcal{F}_q$ through G_r is

$$\sum_{k=1}^{q-s} \frac{\partial(\xi_k^h(\underline{A}, \underline{B})\Phi(\underline{A}, \underline{B}))}{\partial A_k} + \sum_{j=1}^{s} \frac{\partial(\xi_{q-s+j}^h(\underline{A}, \underline{B})\Phi(\underline{A}, \underline{B}))}{\partial B_j} = 0, \quad h = 1, 2, \ldots, r. \tag{8}$$

The only solution is $\Phi(\underline{A},\underline{B}) = \alpha_{q-s}(\underline{A})\beta_s(\underline{B})$. The Deltheil system related to $\mathcal{F}_t$ through G_r is

$$\sum_{j=1}^{s} \frac{\partial(\eta_j^h(\underline{B},\underline{C})\Psi(\underline{B},\underline{C}))}{\partial B_j} + \sum_{i=1}^{t-s} \frac{\partial(\eta_{s+i}^h(\underline{B},\underline{C})\Psi(\underline{B},\underline{C}))}{\partial C_j} = 0, \quad h = 1,2,\dots,r. \tag{9}$$

The only solution is $\Psi(\underline{B},\underline{C}) = \beta_s(\underline{B})\gamma_{t-s}(\underline{C})$. The Deltheil system related to $\mathcal{F}_q + \mathcal{F}_t$ through G_r is

$$\sum_{k=1}^{q-s} \frac{\partial\left[\xi_k^h(\underline{A},\underline{B})\delta(\underline{A},\underline{B},\underline{C})\right]}{\partial A_k} + \sum_{j=1}^{s} \frac{\partial\left[(\xi_{q-s+j}^h(\underline{A},\underline{B}) + \eta_j^h(\underline{B},\underline{C}))\delta(\underline{A},\underline{B},\underline{C})\right]}{\partial B_j} +$$

$$+ \sum_{i=1}^{t-s} \frac{\partial\left[\eta_{s+i}^h(\underline{B},\underline{C})\delta(\underline{A},\underline{B},\underline{C})\right]}{\partial C_j} = 0 \quad h = 1,2,\dots,r. \tag{10}$$

If we set $\delta(\underline{A},\underline{B},\underline{C}) = \alpha_{q-s}(\underline{A})\beta_s(\underline{B})\gamma_{t-s}(\underline{C})$, the left hand side of (10) becomes

$$\gamma_{t-s}(\underline{C}) \sum_{k=1}^{q-s} \frac{\partial\left[\xi_k^h(\underline{A},\underline{B})\alpha_{q-s}(\underline{A})\beta_s(\underline{B})\right]}{\partial A_k} + \gamma_{t-s}(\underline{C}) \sum_{j=1}^{s} \frac{\partial\left[\xi_{q-s+j}^h(\underline{A},\underline{B})\alpha_{q-s}(\underline{A})\beta_s(\underline{B})\right]}{\partial B_j} +$$

$$+\alpha_{q-s}(\underline{A}) \sum_{j=1}^{s} \frac{\partial\left[\eta_j^h(\underline{B},\underline{C})\beta_s(\underline{B})\gamma_{t-s}(\underline{C})\right]}{\partial B_j} + \alpha_{q-s}(\underline{A}) \sum_{i=1}^{t-s} \frac{\partial\left[\eta_{s+i}^h(\underline{B},\underline{C})\beta_s(\underline{B})\gamma_{t-s}(\underline{C})\right]}{\partial C_i}, \tag{11}$$

$h = 1,2,\dots,r$. Consequently (11) is equal to zero, and then

$$d\mu_{q+t-s}(\underline{A},\underline{B},\underline{C}) = \delta(\underline{A},\underline{B},\underline{C}) \wedge^{q-s} d\underline{A} \wedge^{s} d\underline{B} \wedge^{t-s} d\underline{C} =$$

$$= \alpha_{q-s}(\underline{A})\beta_s(\underline{B})\gamma_{t-s}(\underline{C}) \wedge^{q-s} d\underline{A} \wedge^{s} d\underline{B} \wedge^{t-s} d\underline{C},$$

is a density for $\mathcal{F}_{q+t-s}$ with respect to G_r. □

Remark 4. Theorem 1 shows that a set $\omega \subseteq \mathcal{F}_{q+t-s}$ can be measured with respect to G_r by

$$\mu_{G_r}(\omega) = \int_{\underline{\omega}} \alpha_{t-s}(\underline{A})\beta_s(\underline{B})\gamma_{q-s}(\underline{C}) \wedge^{t-s} d\underline{A} \wedge^{s} d\underline{B} \wedge^{q-s} d\underline{C}, \tag{12}$$

(other measures may exist), where $\underline{\omega}$ is the set of points corresponding to ω in the parameter space $Y_q \times Y_t$. If H_r is transitive on this space, then $\mathcal{F}_{q+t-s}$ is measurable with respect to G_r with this measure.

Remark 5. Theorem 1 holds even if the secant families are not measurable, on the condition that these assume the described densities with respect to suitable groups of invariance of $\mathcal{F}_{q+t-s}$.

Remark 6. If $s = 0$ then the families $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_q(\underline{B})$ are taken independently, and Theorem 1 gives the same result as shown in **[4]**.

Remark 7. Theorem 1 holds even if the common parameters $\{B_1, B_2, \ldots, B_s\}$ are not all essential for both families, but only for one of them, because they are all essential in $\mathcal{F}_{q+t-s}$ even in this case. However if this occurs, the intersection order is not equal to s, but to the number of parameters, among $\{B_1, B_2, \ldots, B_s\}$, which are essential in both families.

We can consider of the family of intersections between two families $\mathcal{F}_q(\underline{A}, \underline{B})$ and $\mathcal{F}_t(\underline{B}, \underline{C})$ having intersection order s. Its elements are obtained, for every list $\underline{A}, \underline{B}, \underline{C}$, by intersecting the correspondent varieties of $\mathcal{F}_t(\underline{A}, \underline{B})$ and $\mathcal{F}_q(\underline{B}, \underline{C})$. The groups of invariance of this family are the same as those of $\mathcal{F}_q(\underline{A}, \underline{B}) + \mathcal{F}_t(\underline{B}, \underline{C})$, since a group of transformations keeps belongings relations. Then we have the same Deltheil systems, and consequently densities obtained by Theorem 1 can also be related to the family of intersections between $\mathcal{F}_q(\underline{A}, \underline{B})$ and $\mathcal{F}_t(\underline{B}, \underline{C})$

$$\mathcal{F}_{t+q-s} \begin{cases} \mathcal{F}_t(\underline{A}, \underline{B}) \\ \mathcal{F}_q(\underline{B}, \underline{C}). \end{cases}$$

Here the brace means intersection, while $\mathcal{F}_q(\underline{A}, \underline{B})$+$\mathcal{F}_t(\underline{B}, \underline{C})$ is the family of systems of $\mathcal{F}_q(\underline{A}, \underline{B})$ and $\mathcal{F}_t(\underline{B}, \underline{C})$. Both notations are often used for the family of systems. However by the previous arguments, as regards measurability problems we can refer to each one of them.

Remark 8. As emphasized in Remark 3, the family of intersections between two families, having intersection order s, might be empty. In this case we can formally associate densities to an empty family of varieties. Indeed the empty family is invariant with respect to every group of transformations, and so it can take every density.

Following the early classification given by Stoka **[9]**, a family of varieties is measurable or not measurable whether it assumes a single density, or more, with respect to its transitive groups of invariance. This classification can be refined, as shown in **[5]**, pointing out different kinds of measurability or not measurability.

Definition 4. *We say that a family of varieties is of class* **A** **[5]**, *if the first Stoka's condition holds.*

Theorem 2. *Let $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$, $q \leq t$, be two families of class* **A**, *whose respective densities are*

$$d\Phi_q(\underline{A}) = \alpha_q(\underline{A}) \wedge^q d\underline{A},$$

$$d\Psi_t(\underline{B}) = \beta_t(\underline{B}) \wedge^t d\underline{B}.$$

If the same conditions (possibly no conditions) hold for s pairs of normal parameters $\{\underline{D}\}$ and $\{\underline{E}\}$, obtained by $\{\underline{A}\}$ and $\{\underline{B}\}$ respectively, $(0 \leq s \leq q)$, then $\mathcal{F}_q(\underline{D})$ and $\mathcal{F}_t(\underline{B})$ can be made generically secant of order s, such that their union family $\mathcal{F}_{q+t-s}(\underline{D}, \underline{B})$ takes the invariant integral function $\beta_t(\underline{B})$ with respect to its maximal group of invariance. Analogously, $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{E})$ can be made generically secant of order s, such that their union family $\mathcal{F}_{q+t-s}(\underline{A}, \underline{E})$ takes the invariant integral function $\alpha_q(\underline{A})$ with respect to its maximal group of invariance.

Proof. We can always normalize parameters $\underline{A}$ by q relations

$$\gamma_k(\underline{A}, \underline{D}) = 0, \quad k = 1, 2, \ldots, q, \tag{13}$$

in such a way that $\mathcal{F}_q(\underline{D})$ is the whole $\mathcal{F}_q(\underline{A})$, and its density is given by

$$\wedge^q d\underline{D} = dD_1 \wedge dD_2 \wedge \ldots \wedge dD_{q-s} \wedge \ldots \wedge dD_q.$$

The same holds for $\mathcal{F}_t(\underline{B})$, so we normalize also parameters $\underline{B}$ by t relations

$$\delta_h(\underline{B}, \underline{E}) = 0, \quad h = 1, 2, \ldots, t \tag{14}$$

this gives the family in the form $\mathcal{F}_t(\underline{E})$, and the density is

$$\wedge^t d\underline{E} = dE_1 \wedge dE_2 \wedge \ldots \wedge dE_s \wedge \ldots \wedge dE_t.$$

Up to rename the parameters, we can suppose that the s pairs of normal parameters depending on the same conditions are D_i, E_j, $(q - s + 1 \leq i \leq q, \quad 1 \leq j \leq s)$. So we can put

$$D_{q-s+1} = E_1, D_{q-s+2} = E_2, \ldots, D_q = E_s, \tag{15}$$

which makes $\mathcal{F}_q(\underline{D})$ and $\mathcal{F}_t(\underline{E})$ secant families of intersection order s. By the previous theorem, the family $\mathcal{F}_{q+t-s}(\underline{D}, \underline{E}) = \mathcal{F}_q(\underline{D}) + \mathcal{F}_t(\underline{E})$, takes the density

$$dD_1 \wedge dD_2 \wedge \ldots \wedge dD_{q-s} \wedge^t d\underline{E}.$$

Relations (14) and identifications (15) turn out to be s independent relations

$$\rho_m(\underline{D}, \underline{B}) = 0, \quad m = 1, 2, \ldots, s, \tag{16}$$

among parameters $\underline{D}$ and $\underline{B}$ consequently $\mathcal{F}_q(\underline{D})$ and $\mathcal{F}_t(\underline{B})$ are generically secant families of intersection order s, and $\mathcal{F}_{q+t-s}(\underline{D}, \underline{B}) = \mathcal{F}_q(\underline{D}) + \mathcal{F}_t(\underline{B})$ takes the density

$$d\chi_{q+t-s}(\underline{B}, D_1, D_2, \ldots, D_{q-s}) = dD_1 \wedge dD_2 \wedge \ldots \wedge dD_{q-s} \wedge^t d\underline{E} =$$

$$= \beta_t(\underline{B}) \wedge^t d\underline{B} \wedge dD_1 \wedge dD_2 \wedge \ldots \wedge dD_{q-s},$$

with respect to its maximal group of invariance. In the same way we see that $\mathcal{F}_{q+t-s}(\underline{A}, \underline{E})$ takes the density $\alpha_q(\underline{A}) \wedge^q d\underline{A} \wedge dE_{s+1} \wedge dE_{s+2} \wedge \ldots \wedge dE_t$, with respect to its maximal group of invariance. □

Remark 9. If $s = 0$, Theorem 2 gives the same result as shown in **[4]**.

Remark 10. The assumption that $\mathcal{F}_q(\underline{A})$ and $\mathcal{F}_t(\underline{B})$ are of class **A** can be weakened, and densities can be taken only with respect to certain groups of invariance of these families. Of course, in that case, the density $d\chi_{q+t-s}(\underline{B}, D_1, D_2, \ldots, D_{q-s})$ must be related to the intersection of the chosen groups.

Remark 11. Theorem 2 does not ensure that $\mathcal{F}_{q+t-s}$ is of class **A**. It points out that $d\chi_{q+t-s}$ is one among the possible densities assumed by the family with respect to its maximal group of invariance G_r; however other densities may exist if the group H_r, isomorphic to G_r, is not transitive. This surely occurs if $r < q + t - s$.

Examples.

1. Let $\mathcal{F}_3(u, v, w)$ be the family of planes in the Euclidean space E_3

$$\mathcal{F}_3 : ux + vy + wz + 1 = 0. \tag{17}$$

It is measurable with respect to the group of congruences of E_3, G_6, with density **[1]**, **[11]**

$$d\Phi_3 = \frac{du \wedge dv \wedge dw}{(u^2 + v^2 + w^2)^2}. \tag{18}$$

With the change of parameters

$$u = \frac{sin\theta cos\theta}{-p}, v = \frac{sin\theta sin\phi}{-p}, w = \frac{cos\theta}{-p},$$

the equation (17) becomes

$$\mathcal{F}_3(\theta, \phi, p) : xsin\theta cos\phi + ysin\theta sin\phi + zcos\theta - p = 0, \tag{19}$$

and (18) turns in

$$d\overline{\Phi_3} = |sin\theta| d\phi \wedge d\theta \wedge dp. \tag{20}$$

Let $\mathcal{F}_4(a, b, c, R)$ be the family of spheres in the projective space P_3

$$\mathcal{F}_4 : (x - a)^2 + (y - b)^2 + (z - c)^2 - R^2 = 0. \tag{21}$$

This family is of class **A** **[8]**, with density

$$d\Psi_4 = R^{-4} da \wedge db \wedge dc \wedge dR. \tag{22}$$

The maximal group of invariance of $\mathcal{F}_4$ is the group of similarities, which contains the group of congruences, and $\mathcal{F}_4(a, b, c, R)$ takes the same density (22) on this group.

Now we consider in E_3 the families (19) and (21), depending, respectively, on parameters θ, ϕ, p and a, b, c, R. The relation

$$\alpha(a, b, c, p, \theta, \phi) : p - a(cos\phi sin\theta) - b(sin\phi sin\theta) - c(cos\theta) = 0,$$

allows us to write $\mathcal{F}_3(\theta, \phi, p)$ in the form

$$\mathcal{F}_5(\theta, \phi, a, b, c) : (x - a)cos\phi sin\theta + (y - b)sin\phi sin\theta + (z - c)cos\theta = 0, \tag{23}$$

where only 1 among a,b,c is essential. By Theorem 1 and Remark 7, $\mathcal{F}_5(\theta,\phi,a,b,c)$ and $\mathcal{F}_4(a,b,c,R)$ become secant families of intersection order 1, and the family

$$\mathcal{F}_6 = \mathcal{F}_5(\theta,\phi,a,b,c) + \mathcal{F}_4(a,b,c,R),$$

takes the density

$$d\mu_6(R,a,b,c,\theta,\phi) = \frac{|sin\theta|}{R^{-4}} dR \wedge da \wedge db \wedge dc \wedge d\theta \wedge d\phi,$$

with respect to the group of congruences.

The same density works, with respect to the group of congruences, on the family of intersections between $\mathcal{F}_5(\theta,\phi,a,b,c)$ and $\mathcal{F}_4(a,b,c,R)$

$$\mathcal{F}_6 \begin{cases} (x-a)^2 + (y-b)^2 + (z-c)^2 - R^2 = 0 \\ (x-a)cos\phi sin\theta + (y-b)sin\phi sin\theta + (z-c)cos\theta = 0, \end{cases}$$

that is the family of circumferences of the space.

2. Let $\mathcal{F}_2(A_1,A_2) : A_1x + A_2y + 1 = 0$ be the family of straight lines of the Euclidean plane E_2. It is of class **A [7]**, **[12]**, with density

$$\frac{1}{(A_1^2 + A_2^2)^{\frac{3}{2}}} dA_1 \wedge dA_2. \tag{24}$$

The change of parameters

$$A_1 = -\frac{cosD_1}{D_2}, A_2 = -\frac{sinD_1}{D_2},$$

gives the family in the form

$$\mathcal{F}_2(D_1,D_2) : cosD_1x + sinD_1y - D_2 = 0, \quad (D_2 \geq 0).$$

The density becomes

$$dD_1 \wedge dD_2.$$

so D_1 and D_2 are normal parameters for the family of straight lines.

Now let $\mathcal{F}_3(B_1,B_2,B_3) : x^2+y^2-2B_1x-2B_2y+B_3 = 0, (B_1^2+B_2^2-B_3 \geq 0)$ be the family of circumferences of the plane E_2, which is of class **A [8]**, with density

$$\frac{1}{(B_1^2 + B_2^2 - B_3)^2} dB_1 \wedge dB_2 \wedge dB_3. \tag{25}$$

We can normalize the parameters $\underline{B}$ with the following relations **[8]**

$$E_1 - B_1 = 0, E_2 - B_2 = 0, E_3(B_1^2 + B_2^2 - B_3) - 1 = 0.$$

With respect to the normal parameters, the family $\mathcal{F}_3$ has the form

$$\mathcal{F}_3(E_1, E_2, E_3) : (x - E_1)^2 + (y - E_2)^2 = \frac{1}{E_3},$$

and its density is

$$dE_1 \wedge dE_2 \wedge dE_3.$$

We wish to point out the s-secant figures, $s = 0, 1, 2$, which are determined by the union family $\mathcal{F}_{5-s} = \mathcal{F}_2(D_1, D_2) + \mathcal{F}_3(B_1, B_2, B_3)$ of straight lines and circumferences. When these have intersection order s, by Theorem 2 $\mathcal{F}_{5-s}$ takes the invariant integral function $\frac{1}{(B_1^2+B_2^2-B_3)^2}$ with respect to the group of Euclidean motions.

I. $s = 0$. In this case $\mathcal{F}_5 = \mathcal{F}_2(D_1, D_2) + \mathcal{F}_3(B_1, B_2, B_3)$ is the independent union of straight lines and circumferences, so the 0-secant figure is obtained by taking, for every circumference, the whole family of straight lines, (fig.1).

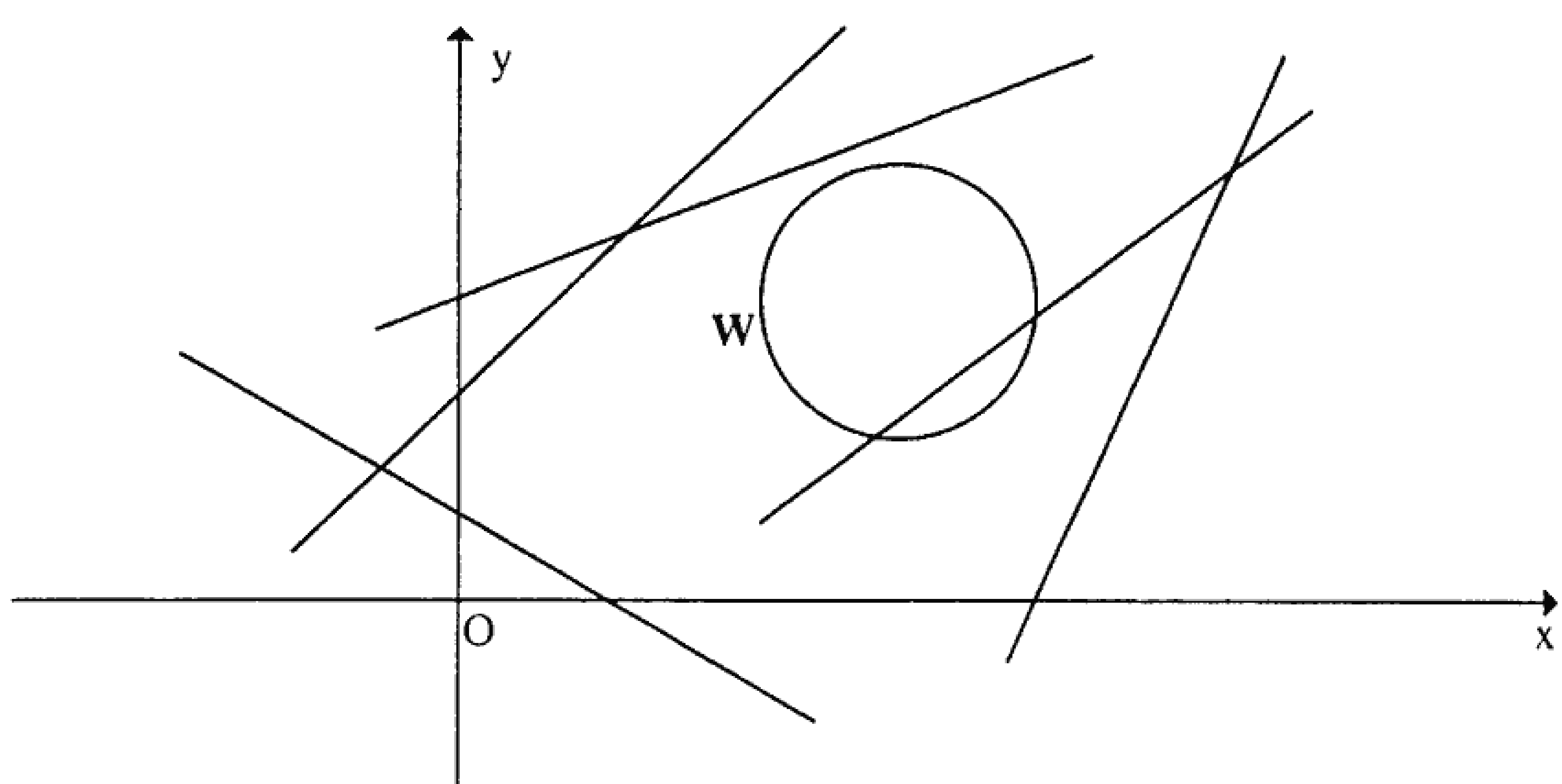

fig.1

II. $s = 1$. Now the families of straight lines and circumferences are generically secant of intersection order 1. This is possible in various ways by taking one between parameters D_1, D_2 equal to one among parameters E_1, E_2, E_3, but not all identifications are allowed. We have the following possibilities:

II a). $D_1 = E_1$. This gives $D_1 = B_1$, so the 1-secant figure consists of a circumference of centre $C(B_1, B_2)$ and the ∞^1 straight lines intersecting the y-axis under an angle $\sigma = x_C$, (fig.2).

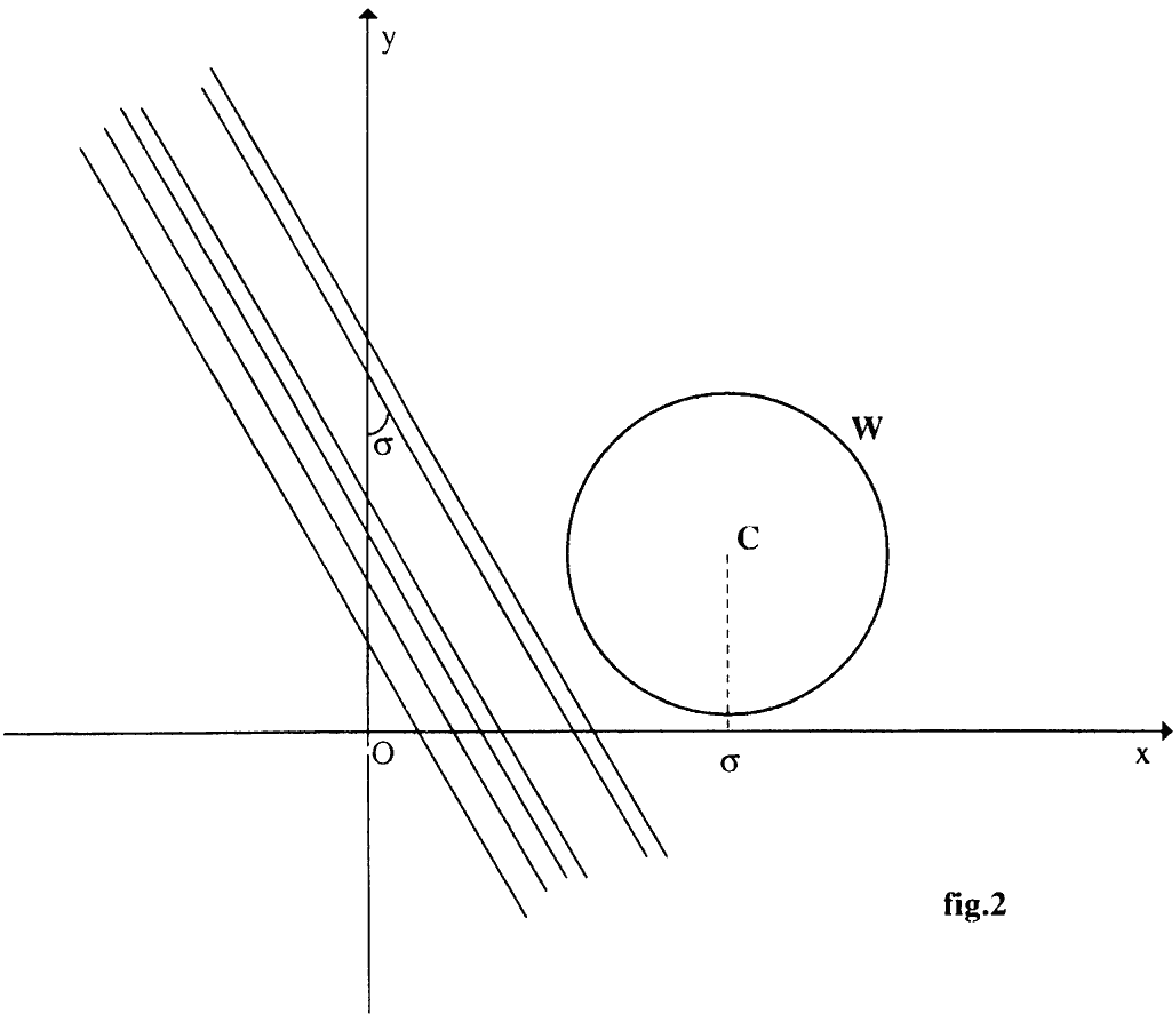

fig.2

II b). $D_1 = E_2$. In this case we have $D_1 = B_2$, and the 1-secant figure is determined by joining to a circumference of centre $C(B_1, B_2)$ the ∞^1 straight lines intersecting the y-axis under an angle $\sigma = y_C$ (fig.3).

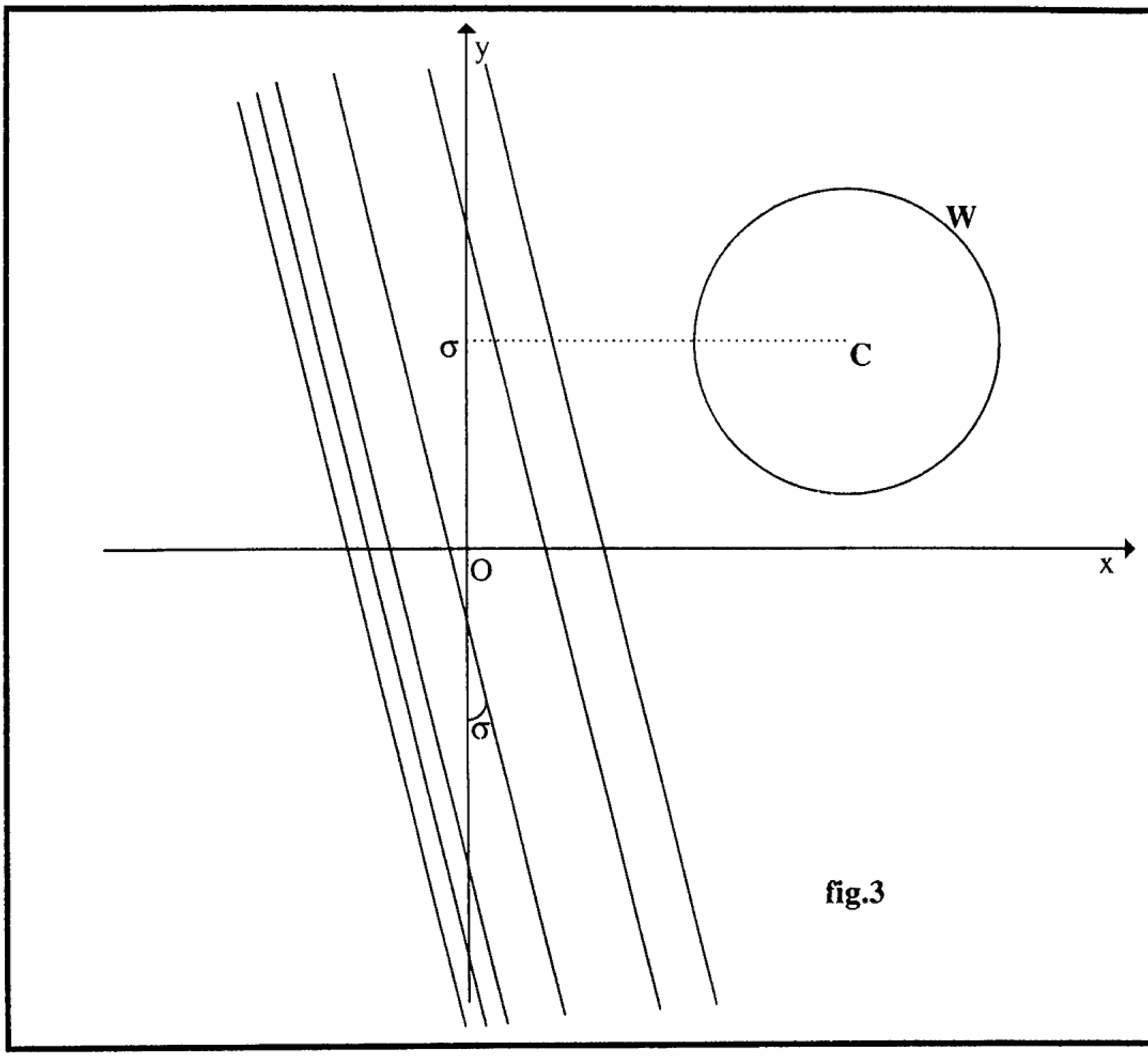

fig.3

II c). $D_1 = E_3$. This case is not allowed, being $E_3 \geq 0$, while D_1 can take all real values. The same for cases $D_2 = E_1$ and $D_2 = E_2$, since $D_2 \geq 0$ but E_1 and E_2 can be negative.

II d). $D_2 = E_3$. The 1-secant figure consists of a circumference of radius R and the ∞^1 straight lines whose distance from the origin is $\frac{1}{R}$ (fig.4).

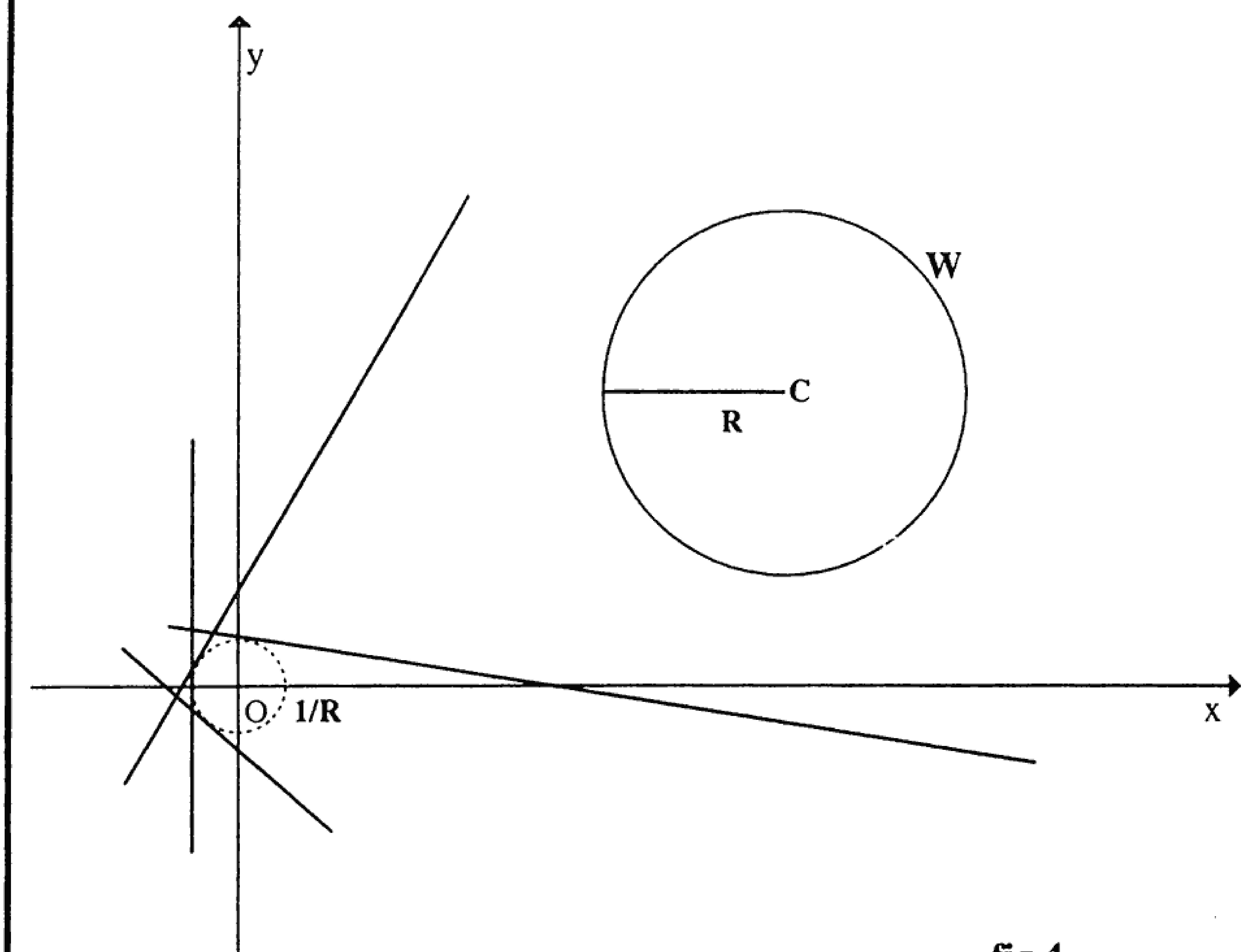

fig.4

III. $s = 2$. This case occurs in two ways:

III a). $D_1 = E_1$ and $D_2 = E_3$;

III b). $D_1 = E_2$ and $D_2 = E_3$.

In **III a)** the 2-secant figure is the pair determined by a circumference with centre $C(B_1, B_2)$ and radius R, and the straight line which intersects the y-axis under an angle $\sigma = x_c$ and whose distance from the origin is $\frac{1}{R}$ (fig.5).

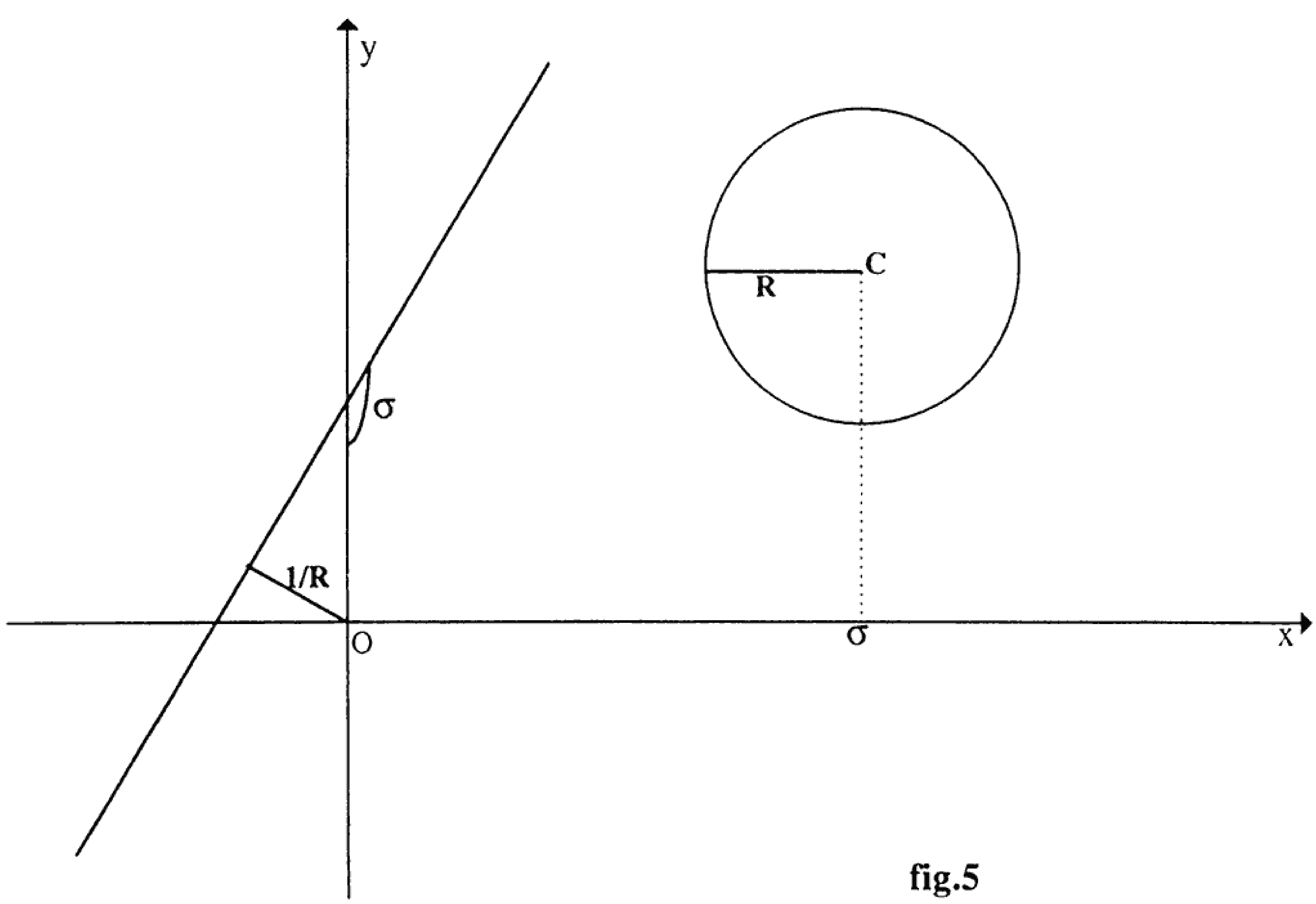

fig.5

In **III b)** the 2-secant figure is the pair determined by a circumference with centre $C(B_1, B_2)$ and radius R, and the straight line which intersects the y-axis under an angle $\sigma = y_c$ and whose distance from the origin is $\frac{1}{R}$ (fig.6).

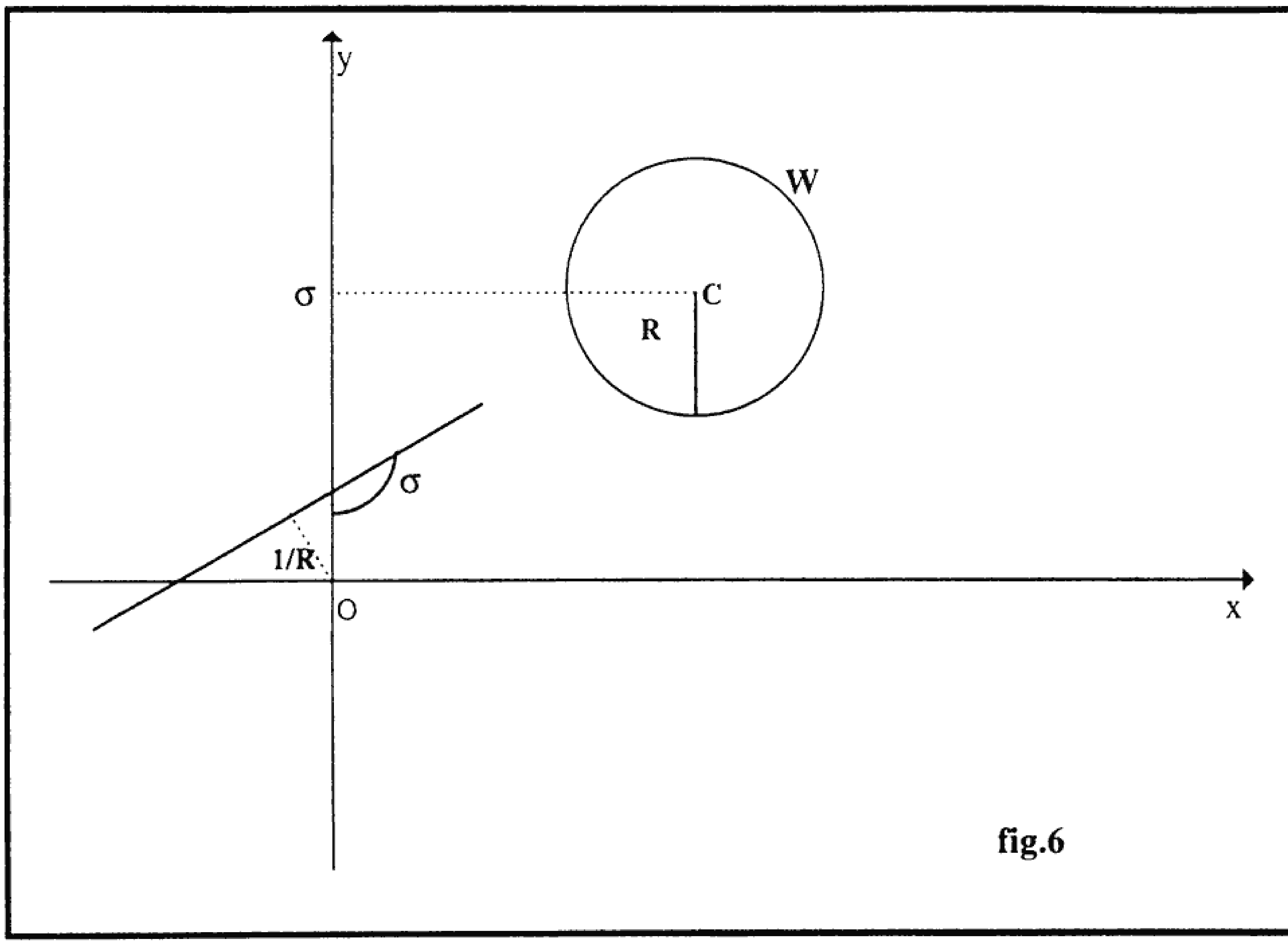

fig.6

Remark 12. When the identifications among normal parameters is forbidden, we can relate the arguments to the possible weak subfamilies on which these are allowed. Then Theorem 2 works again with respect to the maximal group of invariance of the union family of the weak subfamilies, which may be different from the maximal group of invariance of the union family of the starting families. For instance it becomes trivial in all the cases pointed out in the previous **II c**).

3. By Remark 1, it is always possible to normalize the parameters of a measurable family of varieties, but not all normalization are allowed. An example is given by taking, in the Euclidean plane E_2, the family of straight lines $\mathcal{F}_2 : A_1 x + A_2 y + 1 = 0$, **[5]**.

Indeed, if we do the change of parameters

$$tgD_1 = \frac{A_2}{A_1}, \quad D_2 = k + \frac{1}{\sqrt{A_1^2 + A_2^2}},$$

where $k > 0$ is a given constant, we obtain again the normalization of the density of $\mathcal{F}_2$, but now the family is restricted on the weak subfamily $\overline{\mathcal{F}}_2$, consisting of all straight lines whose distance from the origin is greater or equal k. Since the maximal group of invariance of $\overline{\mathcal{F}}_2$ does not contain any translation, the normalized density cannot be related to $\overline{\mathcal{F}}_2$ with respect to the group of Euclidean motions.

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