Non-existence of non-trivial generic warped product in Kaehler manifolds

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Abstract. Warped product manifolds provide excellent setting to model space time near black holes or bodies with large gravitational force [7]. Recently there have been studies to explore the existence of warped products in certain settings [5], [9]. To continue the sequel, non existence of generic warped product submanifolds in a Kaehler manifold is established extending the results of B. Y. Chen [5] and B. Sahin [9].

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1 Introduction

The study of geometry of warped product manifolds was introduced by R. L. Bishop and B. O’Neill [2]. These manifolds emerge in a natural manner. For instance the best relativistic model of the Schwarzschild space time that describes the outer space around a massive star or a black hole is a warped product manifold. Bishop and O’Neill obtained various fundamentally important results for warped product manifolds. Recently the study got impetus with B. Y. Chen’s work on warped product CR-submanifolds of a Kaehler manifold (cf. [5], [6], [7]). He studied CR-Submanifolds of a Kaehler manifold which are warped products of a holomorphic and totally real submanifolds $N_T$ and $N^\perp$ respectively. He proved that a warped product submanifold $N_\perp \times_f N_T$ in a Kaehler manifolds is simply a CR-product whereas the warped product $N_T \times_f N_\perp$ is nontrivial. B. Sahin [9] extended the study by proving that the semi-slant warped product submanifolds $N_T \times_f N_\theta$ and $N_\theta \times_f N_T$ in Kaehler manifolds are trivial in the
sense that they are simply Riemannian product of $N_T$ and $N_0$ where $N_0$ denotes a proper slant submanifold of the underlying Kaehler manifold. Our aim in the present note is to extend the study by considering the warped products $N_T \times_f N^0$ and $N^0 \times_f N_T$ of a Kaehler manifold $\overline{M}$ where $N^0$ is an arbitrary submanifold of $\overline{M}$.

2 Preliminaries

Let $\overline{M}$ be a Kaehler manifold with an almost complex structure $J$ and Levi-Civita connection $\overline{\nabla}$. If $M$ is a submanifold of $\overline{M}$, then the Gauss and Weingarten formulae are given respectively by

\[
\overline{\nabla}_U V = \nabla_U V + h(U, V), \tag{1}
\]
\[
\overline{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi, \tag{2}
\]
for any vector fields $U, V$ tangent to $M$ and $\xi$ normal to $M$, where $\nabla$ and $\nabla^\perp$ denote the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ respectively. $h$ is the second fundamental form and $A$ is the shape operator of the immersion of $M$ into $\overline{M}$. The two are related by

\[
g(A_\xi U, V) = g(h(U, V), \xi), \tag{3}
\]
where $g$ denotes the metric on $\overline{M}$ as well as the one induced on $M$.

For any vector field $U$ tangent to $M$, we put

\[
JU = PU + FU, \tag{4}
\]
where $PU$ and $FU$ are respectively the tangential and normal components of $JU$.

The covariant differentiation of the tensor $P$ is defined by

\[
(\overline{\nabla}_U P)V = \nabla_U PV - P \nabla_U V. \tag{5}
\]

As $\overline{M}$ is Kaehler, by using (1), (2), (4) and (5), we obtain

\[
(\overline{\nabla}_U P)V = A_{FV} U + th(U, V). \tag{6}
\]

1 Definition. Let $M$ be a submanifold of a Kaehler manifold $\overline{M}$ and for $x \in M$, $D_x = T_x(M) \cap JT_x(M)$ be the maximal complex subspace of the tangent space $T_x(M)$. If $D : x \rightarrow D_x$ defines a $C^\infty$-distribution on $M$, known as holomorphic distribution, then $M$ is called a \textit{generic submanifold} (cf. [4]).
For a generic submanifold $M$ of a Kaehler manifold, the tangent bundle $TM$, can be decomposed as

$$TM = D \oplus D^0,$$

where $D^0$ denotes the orthogonal complementary distribution of $D$ and is known as \textit{purely real distribution}. A generic submanifold is a holomorphic submanifold if $D^0 = \{0\}$ and is called a \textit{purely real submanifold} if $D = \{0\}$. Further, if $D^0$ is totally real, the generic submanifold is a CR-submanifold. Thus, a generic submanifold provides a generalization of holomorphic, totally real, purely real and a CR-submanifold. It can also be observed that a purely real distribution $D^0$ on a submanifold $M$, is a slant distribution if the angle $\theta(Z) \in [0, \pi/2]$ between $D^0_x$ and $JZ$ is constant for each $Z \in D^0_x$ and $x \in M$. A purely distribution $D^0$ on $M$ is called a \textit{proper purely real distribution} if $\theta(Z) \neq \pi/2$ for any $Z \in D^0$.

On a generic submanifold of a Kaehler manifold, a) $PD = D$, b) $PD^0 \subset D^0$, c) $FD = \{0\}$.

\textbf{2 Theorem.} [4] Let $M$ be a generic submanifold of a Kaehler manifold $\mathcal{M}$. Then the holomorphic distribution $D$ on $M$ is integrable if and only if

$$g(h(JX,Y),FZ) = g(h(X,JY),FZ),$$

for each $X,Y \in D$ and $Z \in D^0$.

\textbf{3 Definition.} Let $B$ and $F$ be two Riemannian manifolds with Riemannian metric $g_B$ and $g_F$ respectively and $f > 0$ a smooth function on $B$. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \rightarrow B$ and $\eta : B \times F \rightarrow F$. The \textit{warped product} $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric such that

$$||U||^2 = ||d\pi U||^2 + f^2(\pi(x)||d\eta U||^2,$$

for any tangent vector $U$ on $B \times F$. In other words, the Riemannian metric $g$ on a warped product manifold $B \times_f F$ is given by

$$g = g_B + f^2 g_F.$$

The function $f$ is called the \textit{warping function} of the warped product. For a warped product $N \times_f N^0$, we may consider $D$ and $D^0$ the distributions determined by the vectors tangent to the leaves and fibres respectively. That is, $D$ is obtained from tangent vectors of $N$ via the horizontal lift and $D^0$ is obtained by tangent vectors of $N^0$ via the vertical lift.

A warped product $N \times_f N^0$ is said to be trivial if its warping function $f$ is constant. A trivial generic warped product $N_T \times_f N^0$ is nothing but a
generic product $N_T \times N_f^0$, where $N_f^0$ is the manifold with metric $f^2g_{N_0}$ which is homothetic to the original metric $g_{N_0}$ on $N_0$.

R. L. Bishop and B. O’Neill [2] obtained the following basic results for warped product manifolds.

4 Theorem. [2] Let $M = B \times_f F$ be a warped product manifold. Then for any $X, Y \in D$ and $V, W \in D^0$,

(i) $\nabla_X Y \in D$,
(ii) $\nabla_X V = \nabla_V X = (X \ln f)V$,
(iii) $\nabla_V W = -\frac{g(V, W)}{f} \nabla f$.

$\nabla f$ is the gradient of $f$ and is defined as $g(\nabla f, U) = Uf$.

3 Warped product generic submanifolds in a Kaehler manifold

In this section, we study generic submanifolds of a Kaehler manifold $M$ which are warped products of the form $N^0 \times_f N_T$ and $N_T \times_f N^0$ where $N_T$ is a holomorphic submanifold and $N^0$ is a proper purely real submanifold of $M$.

5 Theorem. A Kaehler manifold does not admit non-trivial warped products with one of the factors a holomorphic submanifold.

Proof. Let $N_T$ be a holomorphic submanifold of a Kaehler manifold $M$ and $N^0$ an arbitrary submanifold.

Consider the warped product $M = N^0 \times_f N_T$ in a Kaehler manifold $M$. Then by Theorem 4

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X,$$

for each $X \in TN_T$ and $Z \in TN^0$. Thus

$$g(X, \nabla_J X Z) = 0.$$

Making use of (1), (2), (4) and the fact that $M$ is Kaehler, we deduce from the above equation that

$$0 = g(JX, \nabla_J X JZ) = g(JX, \nabla_J X PZ) - g(h(JX, JX), FZ),$$

which on applying formula (7) yields that

$$g(h(JX, JX), FZ) = (PZ \ln f)\|X\|^2. \quad (8)$$
Now, by (5), (6) and (7), we obtain
\[(PZ \ln f)X - (Z \ln f)PX = A_{FZ}X + th(X, Z).\]

On taking inner product with \(Y \in TN_T\), the above equation gives
\[(Z \ln f)g(X, Y) - (PZ \ln f)g(PX, Y) = g(h(JX, Y), FZ).\] (9)

Interchanging \(X\) and \(Y\) in the above equation and adding the resulting equation in (9) while taking account of Theorem 2, we obtain
\[(Z \ln f)g(X, Y) = g(h(JX, Y), FZ).\] (10)

In particular, we have
\[g(h(JX, JX), FZ) = 0.\] (11)

Now, by (8) and (11), it follows that
\[PZ \ln f = 0,\]
for each \(Z \in TN^0\). This shows that \(f\) is constant and thus \(M\) is a Riemannian product of \(N^0\) and \(N_T\).

Let now \(M\) be the warped product submanifold \(N_T \times_f N^0\) of \(\overline{M}\). Then for any \(X \in TN_T\) and \(Z \in TN^0\), by Theorem 4
\[\nabla_X Z = \nabla_Z X = (X \ln f)Z,\] (12)
and therefore
\[(\nabla_X P)Z = 0,\]
\[(\nabla_Z P)X = (PX \ln f)Z - (X \ln f)PZ.\]

The above equations, in view of (6) yield
\[A_{FZ}X + th(X, Z) = 0,\] (13)
and,
\[(PX \ln f)Z - (X \ln f)PZ = th(X, Z).\] (14)

Thus, we have
\[A_{FZ}X = (X \ln f)PZ - (PX \ln f)Z.\] (15)
Taking inner product with \(PZ\) in (15) and making use of (13), we obtain
\[g(h(X, PZ), FZ) = g(h(X, Z), FPZ) = (X \ln f)\|PZ\|^2.\] (16)
On the other hand as $\overline{M}$ is Kaehler and formula (13) holds we have

\[ g(\nabla PZ JZ, JX) = 0. \]

Which on using (1), (2) and (4) yields

\[ g(\nabla PZ PZ, JX) = g(A FZ PZ, JX). \]

The above equation on taking account of (5) gives

\[ g(h(X, PZ), FZ) = -(X \ln f) ||PZ||^2. \]  (17)

By (16), (17) and the assumption that $N^0$ is a proper purely real submanifold, we get

\[ X \ln f = 0. \]

This proves that the product $N_T \times f N^0$ is trivial.

QED

References