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About Fefferman-Einstein metrics

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Abstract. This is an expository article related to a talk, which I gave at the *International Conference in honour of Prof. Kowalski* on *Recent Advances in Differential Geometry* in Lecce/Italy, June 2007. The article describes the construction of Fefferman-Einstein metrics, which will be presented in an explicit form. Our construction is motivated by tractor calculus and the notion of conformal holonomy. We will sketch the background of this motivation as well.

Keywords: conformal and CR-geometry, Fefferman construction, tractor calculus, conformal holonomy

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1 Introduction

CR-geometry of hypersurface type is closely related to conformal geometry via the Fefferman construction. This construction was originally invented by C. Fefferman in [10] for boundaries of pseudoconvex domains in $\mathbb{C}^{m+1}$ in order to study their geometric properties and invariant theory. He showed that a trivial circle bundle over a pseudoconvex boundary admits a certain Lorentzian metric whose conformal class is invariant under biholomorphisms of the domain. The construction was later extended to abstract CR-structures by an intrinsic approach due to J.M. Lee, which assigns to any pseudo-Hermitian structure a so-called Fefferman metric on the canonical circle bundle. It was pointed out that a Fefferman metric is never Einstein (cf. [16]).

We aim to describe in this expository article that under certain conditions on the underlying pseudoconvex CR-space a Fefferman metric on the canonical circle bundle is not Einstein, but, in fact, the corresponding Fefferman conformal class does contain (at least locally) Einstein metrics. To be concrete, we will explain here that the Fefferman metric to any transverse symmetric, pseudo-Einstein structure is (locally) conformally related to an Einstein metric. Thereby, we will show that the transverse symmetric, pseudo-Einstein structures can be derived from Kähler-Einstein spaces. The construction itself is rather simple and we are able to explicitly compute the Ricci-tensor in order to show that our metrics are Einstein. However, the idea for the Fefferman-Einstein construction is originally motivated by conformal tractor calculus and the notion of conformal holonomy.

We will proceed as follows. In Section 2 we recall basic notions of CR-geometry and pseudo-Hermitian geometry. The intrinsic Fefferman construction is reviewed in the subsequent section. In Section 4 we explain our construction of Fefferman-Einstein metrics. In particular, Theorem 4 will provide explicit local forms for these metrics. At this point we already have achieved our main result. However, in the final two sections we aim to explain the motivation and idea for the construction. For this purpose, we introduce in Section 5 conformal Cartan geometry and tractor calculus. In particular, we will define the notion of conformal holonomy, which is an invariant for any conformal space. The conformal holonomy is suitable to detect conformally Einstein metrics (cf. Theorem 5). The conformal holonomy is also suitable to characterise Fefferman spaces (cf. Theorem 6). Finally, in Section 6 we explain the correspondence of parallel standard tractors in conformal and CR-geometry. The parallel standard tractors of CR-geometry correspond to the transverse symmetric, pseudo-Einstein structures (cf. Theorem 7), which is the key to our explicit construction.

Since this is an expository article we will omit any proofs. The main result (Theorem 4) originates from [21] and is proved there in detail. Our display of tractor calculus, which is basic for the motivation, is very much influenced by [4] (cf. also [25, 2]). The notion of conformal holonomy and applications are discussed e.g. in [1, 18, 20, 19].

2 CR-geometry and pseudo-Hermitian structures

We recall in brief some basic notions concerning CR-geometry and pseudo-Hermitian structures. We will pay special attention to the condition of transverse symmetry. For more substantial informations about these topics we refer e.g. to [24, 26, 16, 17, 9, 6, 13].

Let M^n denote a smooth manifold of odd dimension $n = 2m + 1$. An integrable CR-structure of hypersurface type on M is given by a pair (H, J) consisting of a contact distribution H in TM of codimension 1 and a complex structure J on H, i.e., $J^2 = -id|_H$, which is subject to the integrability conditions $[JX, Y] + [X, JY] \in \Gamma(H)$ and

$$J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0$$

for all $X, Y \in \Gamma(H)$. The Lie bracket of vector fields induces the (tensorial) Levi bracket

$$\begin{array}{rccl} L: & H \times H & \to & Q := TM/_H \ , \\ & (X, Y) & \mapsto & \pi_Q[X, Y] \ , \end{array}$$

where $\pi_Q : TM \to TM/H$ denotes the natural projection onto the quotient. Since H is a contact distribution, the Levi bracket L is non-degenerate. We usually assume that the quotient $Q = TM/_H$ is a trivial real line bundle over M. In this case one can see that L is the imaginary part of a Hermitian form on (H, J). We always assume here that this Hermitian form is positive definite and in this case we call (M, H, J) a strictly pseudoconvex CR space (of hypersurface type).

A nowhere vanishing real 1-form $\theta \in \Omega^1(M)$ on a strictly pseudoconvex CR manifold (M, H, J) is called a pseudo-Hermitian structure if $\theta|_H \equiv 0$. The 1-form θ is necessarily a contact form. If such a θ is given on (M, H, J) then its exterior differential $d\theta$ gives rise to the Levi form $L_\theta(\cdot,\cdot) := d\theta(\cdot, J\cdot)$, which is by assumption a definite scalar product on H. The existence of a pseudo-Hermitian structure θ is guaranteed by the triviality of the quotient Q. In practice, we choose only such θ, for which $L_\theta(\cdot,\cdot)$ is positive defined and in this case we call the data (M, H, J, θ) a (strictly pseudoconvex) pseudo-Hermitian space.

To any pseudo-Hermitian structure θ on a strictly pseudoconvex CR-space (M, H, J) belongs a Reeb field T_θ, which is uniquely determined by the conditions

$$\iota_T\theta \equiv 1 \qquad \text{and} \qquad \iota_T d\theta \equiv 0 \ .$$

For convenience, we set $J(T) := 0$. Moreover, we obtain a Riemannian metric g_θ on M, which is defined as $g_\theta := L_\theta + \theta \circ \theta$, and a covariant derivative

$$\nabla^W : \mathfrak{X}(M) \longrightarrow \Gamma(T^*M \otimes TM) \ ,$$

which is called the Tanaka-Webster connection, and which is uniquely determined by $\nabla^W g_\theta = 0$, i.e., ∇^W is metric for g_θ, and by the condition

$$Tor^W(X,Y) = L_\theta(JX,Y)\cdot T \qquad \text{for all } X,Y \in \Gamma(H) \quad \text{and}$$
$$Tor^W(T,X) = -\tfrac{1}{2}([T,X] + J[T,JX]) \qquad \text{for all } X \in \Gamma(H) \ ,$$

where $Tor^W(X,Y) := \nabla^W_X Y - \nabla^W_Y X - [X,Y]$ is the usual torsion tensor. In addition, for this connection the properties $\nabla^W\theta = 0$ and $\nabla^W \circ J = J \circ \nabla^W$ hold.

The curvature tensor R^W of a Tanaka-Webster connection ∇^W is given for $X, Y, Z, V \in TM$ by

$$R^W(X,Y,Z,V) := g_\theta([\nabla^W_X, \nabla^W_Y]Z - \nabla^W_{[X,Y]}Z, V) \ .$$

This tensor has the symmetry properties

$$R^W(X,Y,Z,V) = -R^W(Y,X,Z,V) = -R^W(X,Y,V,Z),$$
$$R^W(X,Y,JZ,V) = -R^W(X,Y,Z,JV) \ .$$

There is also a notion of Ricci curvature for pseudo-Hermitian structures. It is called the Webster-Ricci curvature tensor and can be defined by

$$Ric^W(X,Y) := i\sum_{\alpha=1}^{m} R^W(e_\alpha, Je_\alpha, X, Y)$$

with respect to a local orthonormal frame

$$\{e_1, Je_1, \cdots, e_m, Je_m\}$$

of L_θ on H. (This contraction is independent from the choice of frame.) The Webster-Ricci curvature Ric^W is by definition skew-symmetric with values in the purely imaginary numbers $i\mathbb{R}$. Moreover, we have the Webster scalar curvature:

$$scal^W := i\sum_{\alpha=1}^{m} Ric^W(e_\alpha, Je_\alpha)\ .$$

The function $scal^W$ is real on (M,H,J,θ).

Finally, in this section, let us assume that the Reeb field T_θ to a pseudo-Hermitian structure θ on (M,H,J) is an (infinitesimal) transverse symmetry, i.e., the torsion part $Tor^W(T,X)$ vanishes for all $X \in H$. In this case we call θ a TSPH structure on (M,H,J). The condition that T_θ is a transverse symmetry implies $\mathcal{L}_T g_\theta = 0$, which means that T_θ is a Killing vector field for the metric g_θ. Under the assumption of transverse symmetry the connection ∇^W and the Levi-Civita connection ∇^{g_θ} to the metric g_θ on M compare by

$$B^\theta \ := \ \nabla^W - \nabla^{g_\theta} \ = \ \frac{1}{2}\left(d\theta\cdot T - (\theta\otimes J + J\otimes\theta)\right)\ .$$

Using this comparison tensor one can explicitly compute the relation of the Webster curvature tensor R^W and the Riemannian curvature tensor R^{g_θ}. This comparison shows the following symmetry properties for R^W under the assumption of transverse symmetry.

1 Lemma. *(cf. [21]) Let θ be a TSPH structure on (M,H,J). Then the Webster curvature tensor R^W satisfies*

$$R^W(X,Y,Z,V) + R^W(Y,Z,X,V) + R^W(Z,X,Y,V) = 0$$

for all $X,Y,Z,V \in TM$. In particular, it is

$$\begin{aligned} R^W(X,Y,Z,V) &= R^W(Z,V,X,Y) \qquad \text{and} \\ R^W(X,JY,JZ,V) &= R^W(JX,Y,Z,JV)\ . \end{aligned}$$

Moreover, we find the following formula for the relation of the Ricci curvatures. It is

$$Ric^{g_\theta}(X,Y) = iRic^W(X,JY) - \frac{1}{2}g_\theta(X,Y) \tag{1}$$

for any $X,Y \in H$.

3 The intrinsic Fefferman construction

We review here the intrinsic Fefferman construction for abstract (pseudo-convex) CR-structures established by J.M. Lee in [16] (cf. also [3]).

So let (M^n, H, J) be a strictly pseudoconvex CR space of dimension $n = 2m+1$ with pseudo-Hermitian form θ. We denote by T_{10} the i-eigenspace of the $\mathbb{C}$-linear extension of J to $H \otimes \mathbb{C}$ in the complexified tangent bundle $TM^{\mathbb{C}}$. The canonical line bundle of the CR space (M, H, J) is given by

$$\Lambda^{m+1,0}M \ := \ \{ \ \rho \in \Lambda^{m+1}T^*M \otimes \mathbb{C} : \ \iota_X \rho = 0 \ \text{ for all } X \in T_{01} = \overline{T_{10}} \ \} \ .$$

The positive real numbers $\mathbb{R}_+$ act by multiplication on $K^* := \Lambda^{m+1,0}M \setminus \{0\}$, which denotes the canonical line bundle with deleted zero section. We set $F_c := K^*/_{\mathbb{R}_+}$, which is the total space of the canonical S^1-principal bundle over (M, H, J) with projection $\pi_c : F_c \to M$, whose fibre action is induced by multiplication with complex numbers of the unit circle S^1 in $\mathbb{C}$.

The Tanaka-Webster connection ∇^W to θ on M gives rise via extension to a covariant derivative on $\Lambda^{m+1,0}M$. This covariant derivative is seen to be induced by a uniquely defined connection 1-form $A^W : TF_c \to i\mathbb{R}$ on the canonical S^1-principal bundle. By adding a normalisation term to A^W we obtain the connection form

$$A_\theta := A^W - \frac{i}{2(m+1)} scal^W \theta \ ,$$

where θ and $scal^W$ denote the pull-backs via π_c to F_c of the corresponding objects on M. We call the latter 1-form A_θ the Weyl connection to θ on F_c. The Fefferman metric to θ on the canonical bundle F_c is then defined by

$$f_\theta \ := \ L_\theta - i\frac{4}{m+2}\theta \circ A_\theta \ .$$

Obviously, this is a non-degenerate, symmetric 2-tensor on F_c of signature $(1, n)$, i.e., f_θ is a Lorentzian metric on F_c over the strictly pseudoconvex space (M^n, H, J, θ). The important point about this construction is that the conformal class $[f_\theta]$ does not depend on the choice of the pseudo-Hermitian structure θ, i.e., the conformal space $(F_c, [f_\theta])$ is a CR-invariant of (M, H, J). In fact, rescaling the pseudo-Hermitian form by $\tilde{\theta} = e^{2\phi}\theta$ with some real function ϕ on M produces the conformally changed Fefferman metric $f_{\tilde{\theta}} = e^{2\phi} f_\theta$.

We remark that there exists the following characterisation result for Fefferman metrics on integrable CR-spaces due to G. Sparling and C.R. Graham (cf. [23, 14]).

2 Theorem. *(Sparling's characterisation) Let (F^{n+1}, f) be a Lorentzian space of dimension $n+1 \geq 4$. Suppose that f admits a Killing vector j (i.e., $\mathcal{L}_j f = 0$) such that*

(1) $f(j,j) = 0$, i.e., j is lightlike,

(2) $\iota_j W^f = 0$ and $\iota_j C^f = 0$, where W^f and C^f denote resp. the Weyl and Cotton tensor of f,

(3) $Ric^f(j,j) > 0$ on F.

Then f is locally isometric to the Fefferman metric of some integrable, pseudoconvex CR-space (M,H,J) of hypersurface type and dimension n (equipped with pseudo-Hermitian structure θ).

On the other hand, any Fefferman metric of an integrable, pseudoconvex CR-space (M,H,J) of hypersurface type admits a Killing vector field j satisfying (1) to (3).

4 Fefferman-Einstein metrics

Computing the Ricci-tensor Ric^{f_θ} of any Fefferman metric of a pseudo-Hermitian space (M,H,J,θ) shows that f_θ is never an Einstein metric (cf. [16]). However, we want to show here that the Fefferman metric f_θ of a so-called TSPE structure θ is conformally related to an Einstein metric (cf. [21]).

Due to Lee's definition in [17], a pseudo-Hermitian structure θ on a CR-space (M,H,J) is called a pseudo-Einstein structure if

$$Ric^W = -i\frac{scal^W}{m} \cdot d\theta \ .$$

Further, we say now that θ is a TSPE structure if it is pseudo-Einstein and, simultaneously, the corresponding Reeb vector T_θ defines a transverse symmetry, i.e., $Tor^W(T_\theta, X) = 0$ for all $X \in H$. We will derive in the following a description and construction principle for TSPE structures. This consideration will be of local nature. First, let (M,H,J,θ) be a TSPE space. Since T_θ is a transverse symmetry, we can locally on a suitable neighbourhood U (of any point of M) take the quotient by the integral curves of T_θ. This gives rise to a smooth quotient manifold N of dimension $2m$ with projection $\pi_U : U \to N$. Since the metric g_θ on M and the complex structure J on H are invariant under the local flow of the vector field T_θ on U, the projection π_U induces naturally a metric h on N and a complex structure, which we denote again by $J : TN \to TN$.

In fact, taking into account that $\pi_U : (U, g_\theta) \to (N,h)$ is a Riemannian submersion and (1), shows that the Ricci tensor Ric^h of h on N is given by

$$Ric^h(X,Y) = iRic^W(X^*, JY^*) \tag{2}$$

for all $X, Y \in TN$, where X^* and Y^* denote the corresponding horizontal lifts for the submersion π_U. Obviously, this relation shows that the metric h is Einstein if and only if θ is a TSPE structure. The relation for the scalar curvatures is $scal^h = 2 \cdot scal^W$. Moreover, using the comparison tensor B^θ and the submersion property of π_U, it is straightforward to see that J on the quotient N is ∇^h-parallel. We conclude that any TSPE structure θ on a CR-space (M, H, J) determines naturally a Kähler-Einstein structure (h, J) of Riemannian signature on a local quotient space N.

On the other hand, let us consider a Kähler-Einstein space (N^{2m}, h, J) of dimension $2m$ and Riemannian signature. The Kähler-Einstein space N is equipped with the anti-canonical S^1-bundle

$$\mathcal{S}_{ac}(N) := P(N) \times_{det_{\mathbb{C}}} S^1$$

with projection π_{ac}, where $P(N)$ denotes the $\mathrm{U}(n)$-reduction of the orthonormal frame bundle of (N, h, J). The Levi-Civita connection of the Kähler metric h induces a connection form $\rho_{ac} : T\mathcal{S}_{ac}(N) \to i\mathbb{R}$ on the anti-canonical S^1-bundle. Moreover, the metric h induces the Kähler form $\omega \in \Omega^2(N)$ and the Ricci form $\varsigma \in \Omega^2(N)$. Since h is a Kähler-Einstein metric both 2-forms ω, ς are closed. Hence we can find locally some 1-forms α and β such that $d\alpha = \omega$ and $d\beta = \varsigma$. With the lift of these 1-forms to $\mathcal{S}_{ac}(N)$ we define

$$\theta \ := \ i\rho_{ac} - \pi_{ac}^*\alpha + \pi_{ac}^*\beta \ .$$

This 1-form θ is not merely a local connection, but, in fact, it is by definition a (local) contact form on $\mathcal{S}_{ac}(N)$. In particular, we can lift the complex structure J on N to the horizontal distribution H_θ, which is given by the local connection θ. As result we obtain a (local) CR-structure (H, J) on $\mathcal{S}_{ac}(N)$ with corresponding pseudo-Hermitian form θ. This CR-structure (H, J) is by construction integrable, since the Nijenhuis tensor of J vanishes on N, and therefore, also on $\mathcal{S}_{ac}(N)$.

We can make some further observations about our local construction of (H, J) on $\mathcal{S}_{ac}(N)$. First, we note that the fundamental vector field T of the S^1-action along $\mathcal{S}_{ac}(N)$ in the vertical direction (normed by $\theta(T) = 1$) is a transverse symmetry of the constructed (H, J). This implies that we have a reconstruction procedure for a Kähler metric on N as described at the beginning of this section. In fact, by construction of (H, J, θ) it is clear that the reconstructed Kähler space is isomorphic to our initial space (N, h, J). This shows by use of (2) that (H, J, θ) is (locally) a pseudoconvex TSPE structure on $\mathcal{S}_{ac}(N)$. Furthermore, we note that our local construction of (H, J, θ) on $\mathcal{S}_{ac}(N)$ admits a gauge freedom in form of the choice of the local 1-forms α and β. However, it is easy to see that the (local) isomorphism class of the constructed pseudo-Hermitian structure (H, J, θ) on $\mathcal{S}_{ac}(N)$ does not depend on the chosen gauges α, β. Altogether, this leads us to the conclusion that we have found a natural (local) 1-to-1-correspondence between TSPE spaces on one side and Kähler-Einstein spaces on the other side. We manifest this statement in the following theorem. Thereby, notice that for $scal^h \neq 0$ the 1-form $i\rho_{ac} - \pi_{ac}^*\alpha + \pi_{ac}^*\beta$ is gauge equivalent to $i\frac{2m}{scal^h}\rho_{ac}$. We will find the latter gauge more convenient for the following considerations.

3 Theorem. *Let (N, h, J) be a Riemannian Kähler-Einstein space of dimension $2m$ with scalar curvature $scal^h$.*

(1) If $scal^h \neq 0$ then the anti-canonical S^1-principal bundle

$$\mathcal{S}_{ac}(N) = P(N) \times_{det_{\mathbb{C}}} S^1$$

with connection 1-form

$$\theta := i\frac{2m}{scal^h}\rho_{ac} \ ,$$

where ρ_{ac} is the Levi-Civita connection to h, and with induced horizontal CR-structure (H, J) is a pseudoconvex TSPE space with

$$scal^W = \frac{1}{2}scal^h \neq 0 \ .$$

(2) If $scal^h = 0$ and the Kähler form is given by $\omega = d\alpha$ for some (local) 1-form α on N then $(\mathcal{S}_{ac}(N), H, J)$ with (local) TSPE structure $\theta = i\rho_{ac} - \pi_{ac}^*\alpha$ is Webster-Ricci flat.

Locally, any TSPE space (M, H, J, θ) is isomorphic to one of these two models depending on the Webster scalar curvature $scal^W$.

In the next step we simply construct the Fefferman space (F_c, f_θ) over a TSPE space (M, H, J, θ), which is (resp., can be locally) presented in the form (1) or (2) of Theorem 3. In this construction the total space F_c is a torus bundle, the product of the canonical $\mathcal{S}_c$ and the anti-canonical circle bundle $\mathcal{S}_{ac}$ over an underlying Kähler-Einstein space (N, h, J). In particular, we have the 1-forms ρ_{ac} and ρ_c on F_c, which arise as the lifts of the h-induced Levi-Civita connections on $\mathcal{S}_{ac}$, resp., on $\mathcal{S}_c$. Using Lemma 1 for the condition of transverse symmetry, the curvature form $\Omega^W = dA^W$ on F_c of the Tanaka-Webster connection is seen to be equal to $-\pi_c^*Ric^W$. This relation and the fact that θ is pseudo-Einstein is responsible for the following simple form of the Ricci curvature of the Fefferman metric, which we denote here by $f_h := f_\theta$ in order to indicate its natural relation with the underlying Kähler-Einstein metric h:

$$\begin{aligned} Ric^{f_h} &= \tfrac{-2m}{(m+2)^2}\rho_c^2 && \text{if } \ scal^h = 0 \ , \\ Ric^{f_h} &= \tfrac{scal^h}{2(m+1)} f_h - \tfrac{2m}{(m+2)^2}(\rho_c + \rho_{ac})^2 && \text{if } \ scal^h \neq 0 \ . \end{aligned}$$

Having derived these formulae for the Ricci curvature one can almost guess the conformal factor, which rescales the metric f_h to on Einstein metric in the Fefferman conformal class. In fact, let us choose a local coordinate function t on the torus bundle F_c such that

$$\begin{aligned} dt &= i(m+2)\rho_c && \text{when } \ scal^h = 0, \quad \text{resp.,} \\ dt &= i(m+2)\cdot(\rho_c + \rho_{ac}) && \text{when } \ scal^h \neq 0 \ . \end{aligned}$$

Using the standard formula for the conformal transformation rule of the Ricci curvature we immediately see that the conformally changed metric

$$\tilde{f}_h := cos^{-2}(t) f_h$$

is an Einstein metric (on its domain of definition). Rewriting f_h in a convenient way, we obtain the following explicit local forms for Fefferman-Einstein metrics with Lorentzian signature.

4 Theorem. *Let (N, h, J) be a Riemannian Kähler-Einstein space of dimension $2m$ with scalar curvature $scal^h$.*

(1) If $scal^h = 0$ and the Kähler form is $\omega = d\alpha$ for some 1-form α on N, then the metric

$$\tilde{f}_h = cos^{-2}(t)\cdot\big(\ \pi^* h + 4dt \circ (\pi^*\alpha + ds)\ \big)$$

on $N \times \{\ (s,t)\ :\ -\frac{\pi}{2} < t < \frac{\pi}{2}\ \} \subset N \times \mathbb{R}^2$ (with natural projection π onto N) is Ricci-flat and conformally related to a Fefferman metric.

(2) If $scal^h \neq 0$ then the metric

$$\tilde{f}_h = cos^{-2}(t)\cdot\left(\ \pi^* h - \frac{4m(m+1)}{scal^h}\cdot\big(\ dt^2 + \frac{\rho_{ac}^2}{(m+1)^2}\big)\right)$$

on $\mathcal{S}_{ac}(N)\times(-\frac{\pi}{2},\frac{\pi}{2})$, where $(\mathcal{S}_{ac}(N), \pi, N)$ is the anti-canonical S^1-bundle over N with Levi-Civita connection $\rho_{ac} : T\mathcal{S}_{ac}(N) \to i\mathbb{R}$, is Einstein with $scal^{\tilde{f}_h} = \frac{2m+1}{2m}\cdot scal^h$ and conformally related to a Fefferman metric.

On the other hand, if a Fefferman metric f_h of Lorentzian signature over an integrable CR-space is locally conformally Einstein, then any Einstein metric $\tilde{f} \in [f_h]$ can be brought locally into the form (1) or (2).

We have to add some remarks about this theorem. First, we note that the Fefferman metric $cos^2(t)\tilde{f}_h$ *minus* the parallel line element dt^2 in the Ricci non-flat situation (2) is locally isometric to the Sasaki-Einstein metric constructed on the anti-canonical line bundle over the Kähler-Einstein space (N, h, J). Also note that the conformal factor $cos^{-2}(t)$ is not defined on the entire torus bundle F_c. In fact, there does not exist a global Fefferman-Einstein metric on F_c. However, locally in the neighbourhood of any point of F_c a conformal Einstein scale does exist (depending on where the coordinate function t is set to be zero). Finally, we remark that so far we gave no proof or reason why the reversed statement, that any Fefferman-Einstein metric can be brought into the form (1) or (2) of Theorem 4, shall be true. The justification for this statement will become clear in the final section.

5 Conformal tractor calculus

We introduce here conformal tractor calculus with canonical connection via Cartan geometry. In particular, we define the notion of conformal tractor holonomy. Our explanations are not more than a dispatch. We refer to [8, 25, 15, 2, 11, 7] for the necessary background.

The orthogonal group $G := \mathrm{O}(2, n)$ acts in a standard way on $\mathbb{R}^{2,n}$, which denotes the $(n+2)$-dimensional pseudo-Euclidean space of signature $(2, n)$. Then we denote by $\mathbb{P}L$ the projectivation of the lightcone in $\mathbb{R}^{2,n}$, which is a homogeneous space G/P with fixed point group P of a real line in $\mathbb{P}L$. In fact, the space $\mathbb{P}L$ admits a G-invariant conformal structure and can be used as the flat model of Lorentzian conformal geometry. We call $\mathbb{P}L$ the Möbius space of signature $(1, n-1)$ and G the Möbius group (which acts effectively only after factoring through its centre $\mathbb{Z}_2$). The Lie algebra $\mathfrak{g} = \mathfrak{so}(2, n)$ of the Möbius group G is equipped with a $|1|$-grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ induced by the parabolic subgroup $P \subset G$. The Lie algebra $\mathfrak{p}$ is given by $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. We also set $\mathfrak{p}_+ := \mathfrak{g}_1$.

Now let (F^n, c) be an arbitrary n-dimensional space with conformal structure c given by the conformal class $[g]$ of a Lorentzian metric g on F. It is a matter of fact that the conformal structure c gives rise in a natural and unique way to a principal P-bundle $\mathcal{P}(F)$ over F with projection π and a canonical Cartan connection, i.e., a P-equivariant absolute parallelism on $\mathcal{P}(F)$,

$$\omega_{nor} : T\mathcal{P}(F) \;\to\; \mathfrak{g}\ .$$

We denote by $\Omega := d\omega_{nor} + \frac{1}{2}[\omega_{nor}, \omega_{nor}]$ the curvature of the canonical Cartan connection. The curvature Ω gives rise to a curvature function

$$\kappa : \mathcal{P}(F) \ \rightarrow \ Hom(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \ ,$$

and the Cartan connection ω_{nor} is uniquely determined by the normalisation condition $\partial^* \kappa = 0$, where ∂^* denotes the Kostant-codifferential computing the Lie algebra cohomology of $\mathfrak{g}_{-1} \cong \mathfrak{g}/\mathfrak{p}$ with values in $\mathfrak{g}$. We call $(\mathcal{P}(F), \omega_{nor})$ the canonical conformal Cartan geometry on (F, c).

The spaces $\mathbb{R}^{2,n}$ and $\mathfrak{g}$ are in a natural way G-modules (and hence P-modules), which give rise to associated vector bundles of $\mathcal{P}(F)$. In fact, we define

$$\mathcal{T} \ := \ \mathcal{P} \times_P \mathbb{R}^{2,n} \qquad \text{and} \qquad \mathcal{A} := \mathcal{P} \times_P \mathfrak{g}$$

and call $\mathcal{T}$ the conformal standard tractor bundle, whereas $\mathcal{A}$ is called the adjoint tractor bundle of (F, c). Since the tractor bundles $\mathcal{T}$ and $\mathcal{A}$ are induced by G-modules, the canonical Cartan connection ω_{nor} gives naturally rise to covariant derivatives ∇^{nor} acting on sections of $\mathcal{T}$ and $\mathcal{A}$. We call ∇^{nor} on $\mathcal{T}$ the tractor connection. This connection has a uniquely defined holonomy group, which we denote by $Hol(\mathcal{T})$. Its Lie algebra is denoted by $\mathfrak{hol}(\mathcal{T})$. The holonomy group $Hol(\mathcal{T})$ and its algebra are by construction conformal invariants of the underlying space (F, c). We also want to note here that the tractor bundle $(\mathcal{T}, \nabla^{nor})$ for a conformal space (F, c) can be defined directly without the approach of Cartan geometry. In fact, tractor calculus and Cartan geometry are formulations of conformal geometry, which are on an equal footing (cf. [25, 2]).

The tractor bundles $\mathcal{T}$ and $\mathcal{A}$ and the tractor holonomy have some interesting features, which we want to explain next. First, let us consider $\mathcal{T}$ with the tractor connection ∇^{nor}. The bundle $\mathcal{T}$ admits an invariant scalar product $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ and an invariant filtration $\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$, where $\mathcal{T}^1$ is a lightlike real line bundle and $\mathcal{T}^0$ the corresponding orthogonal subspace. With respect to a compatible metric $\tilde{g} \in c$ this filtration of the standard tractor bundle $\mathcal{T}$ splits into $\mathbb{R} \oplus TF \oplus \mathbb{R}$ and the tractor connection ∇^{nor} acts on a standard tractor $t = (f, \psi, h) \in \Gamma(\mathcal{T})$ with respect to this splitting and $X \in TF$ by

$$\nabla^{nor}_X t \ = \ \begin{pmatrix} Xh \\ \nabla^{\tilde{g}}_X \psi \\ Xf \end{pmatrix} + \begin{pmatrix} 0 & \mathsf{P}^{\tilde{g}}(X, \cdot) & 0 \\ X & 0 & -\mathsf{P}^{\tilde{g}}(X, \cdot)^{\sharp} \\ 0 & -\tilde{g}(X, \cdot) & 0 \end{pmatrix} \begin{pmatrix} h \\ \psi \\ f \end{pmatrix} \ .$$

Thereby, $\nabla^{\tilde{g}}$ denotes the Levi-Civita connection of $\tilde{g}$, $\mathsf{P}^{\tilde{g}} := \frac{1}{n-2}(\frac{scal^{\tilde{g}}}{2(n-1)} - Ric^{\tilde{g}})$ is the Schouten tensor and $\mathsf{P}^{\tilde{g}}(X, \cdot)^{\sharp}$ denotes the dual vector to $\mathsf{P}^{\tilde{g}}(X, \cdot)$. Note that the content of the curvature Ω of the tractor connection ∇^{nor} is given with respect to $\tilde{g} \in c$ by the Weyl curvature tensor $W^{\tilde{g}}$ and the Cotton tensor $C^{\tilde{g}}$.

Let us consider now the tractor equation $\nabla^{nor}t = 0$ for a standard tractor $t \in \Gamma(\mathcal{T})$. With respect to the metric $g \in c$ and the corresponding splitting $t = (f, \psi, h)$ this equation is equivalent to the single second order PDE

$$\textit{trace-free part of } \ (Hess^g(f) - f \cdot \mathsf{P}^g) \ = \ 0 \ .$$

It is well known that a function $f \in C^\infty(F)$ solves this equation if and only if $\tilde{g} := f^{-2}g$ is an Einstein metric in the conformal class c. Note that if a solution f admits a zero on F then $\tilde{g}$ is not globally defined on F. In general, we call a conformal class c almost-Einstein if it admits a compatible Einstein metric up to singular points on F (cf. [12]). The first conclusion here in the framework of tractor calculus is that a conformal class c is almost-Einstein if and only if there exists a ∇^{nor}-parallel standard tractor $t \in \Gamma(\mathcal{T})$. The interesting point in relation with the conformal holonomy is the observation that a ∇^{nor}-parallel standard tractor t exists on F if and only if the action of $Hol(\mathcal{T})$ fixes some vector γ in the standard module $\mathbb{R}^{2,n}$.

5 Theorem. *Let (F, c) be a conformal space with standard tractor bundle $\mathcal{T}$ and tractor holonomy $Hol(\mathcal{T})$. Then the conformal class c is almost-Einstein if and only if $Hol(\mathcal{T})$ fixes a vector in $\mathbb{R}^{2,n}$.*

Let us also consider the adjoint tractor bundle $\mathcal{A}$ on (F, c). The adjoint tractor bundle $\mathcal{A}$ is filtered by $\mathcal{A} \supset \mathcal{A}^0 \supset \mathcal{A}^1$, where $\mathcal{A}^0$ is the associated bundle to the P-module $\mathfrak{p}$ and $\mathcal{A}^1$ is associated to $\mathfrak{p}_+$. The quotient $\mathcal{A}/\mathcal{A}^0$ is canonically identified with the tangent space TF on F and we denote the corresponding projection by $\Pi : \mathcal{A} \to TF$. It is a matter of fact that, if $A \in \Gamma(\mathcal{A})$ is an adjoint tractor, which solves the tractor equation

$$\nabla^{nor} A \ = \ \Omega(\ \cdot\ , \Pi(A))\ , \tag{3}$$

then $\Pi(A)$ is a conformal Killing vector field on (F, c), i.e., an infinitesimal conformal isometry. On the other hand, it is also true that the 2-jet of a conformal Killing vector field V on (F, c) gives uniquely rise to an adjoint tractor A_V solving (3) such that $\Pi(A_V) = V$. This establishes a description of conformal Killing vector fields by use of tractor calculus. Finally, note that the standard action of $\mathfrak{p}$ on $\mathbb{R}^{2,n}$ is P-equivariant, which implies the existence of an invariant action of $\mathcal{A}$ on the standard tractors in $\mathcal{T}$. We denote this action by $\bullet : \mathcal{A} \otimes \mathcal{T} \to \mathcal{T}$.

6 The motivation from conformal holonomy

Having tractor calculus at hand we are able to explain now the motivation for our Fefferman-Einstein construction as outlined in Section 4. In particular, we can sketch a proof for the reversed statement in Theorem 4. The arguments about CR-tractor calculus and its relation to conformal tractor calculus that we use in this section work along the lines of [4].

Let us start with a conformal holonomy consideration for our motivation. As we have seen at the end of the previous section, an adjoint tractor A on a conformal space (F^{n+1}, c) of Lorentzian signature with dimension $n+1 = 2m+2$, which solves $\nabla^{nor} A = \Omega(\cdot, \Pi(A))$, corresponds to the conformal Killing vector field $\Pi(A)$. Now let us consider a solution $\mathcal{J}$ in the adjoint tractors on (F, c) of the equation

$$\nabla^{nor}\mathcal{J} = 0 \qquad \text{with} \qquad \mathcal{J} \bullet \mathcal{J} = -id|_{\mathcal{T}} \;.$$

Such a solution $\mathcal{J}$ is nothing else, but a ∇^{nor}-parallel orthogonal complex structure on $\mathcal{T}$ and the projection $j := \Pi(\mathcal{J})$ is a conformal Killing vector field on (F, c). In fact, one can easily see that j is lightlike and has no zeros on F, i.e., we can choose (at least locally) a compatible metric $\tilde{g} \in c$, for which j is a lightlike Killing vector field. Since $\Omega(j, \cdot) = 0$ implies $\iota_j W^{\tilde{g}} = 0$ and $\iota_j C^{\tilde{g}} = 0$, we see that Sparling's characterisation applies to $\tilde{g}$. We conclude that $\tilde{g} \in c$ is (locally) on F the Fefferman metric of an underlying pseudoconvex space (M, H, J) equipped with some pseudo-Hermitian structure θ.

Note that the existence of a solution $\mathcal{J}$ of the above tractor equation is equivalent to the reduction of the conformal holonomy $Hol(\mathcal{T})$ to some subgroup of the unitary group $\mathrm{U}(1, m+1)$. In combination with Theorem 5 we understand now that if a conformal space (F^{n+1}, c) has a conformal holonomy $Hol(\mathcal{T})$, which is reduced to $\mathrm{U}(1, m+1)$ and, in addition, fixes a vector in $\mathbb{R}^{2,2m+2}$, then the conformal class has to be Fefferman-Einstein (up to singularities). Since there is no reason to believe that such a conformal holonomy reduction does not exist for a conformal space (apart from the trivial holonomy in the conformally flat case), the idea is born to generate spaces with such holonomies.

To proceed with this idea, we need a version of tractor calculus for CR-geometries and a relation to conformal tractors via the Fefferman construction. First, let us consider the group $\tilde{G} := \mathrm{SU}(1, m+1)$, which acts in the standard way on $\mathbb{C}^{m+2}$ equipped with the indefinite Hermitian product $(\cdot, \cdot)$ of signature $(1, m+1)$. The parabolic subgroup $\tilde{P} \subset \tilde{G}$ is defined to be the stabiliser of a complex null line in $\mathbb{C}^{m+2}$ under the standard action of $\tilde{G}$. Then the pair $(\tilde{G}, \tilde{P})$ describes a Kleinian model of pseudoconvex CR-geometry. Note that we have a natural inclusion $\tilde{\iota} : \tilde{G} \to G$ via the identification $\mathbb{C}^{m+2} \cong \mathbb{R}^{2m+4}$, where $G = \mathrm{O}(2, 2m+2)$ is the Möbius group.

Now let us consider an integrable, pseudoconvex CR-space (M^n, H, J) of hypersurface type and dimension $n = 2m+1$ with a choice of $(m+2)$nd root $\mathcal{E}$ for the complexified line bundle $\Lambda^m_{\mathbb{C}} H \otimes Q$, i.e., $\mathcal{E}^{\otimes^{m+2}} = \Lambda^m_{\mathbb{C}} H \otimes Q$. In this situation, there belongs a uniquely defined parabolic Cartan geometry $(\tilde{\mathcal{P}}, \omega_{CR})$ of type $(\tilde{G}, \tilde{P})$ to the CR-structure (H, J) on M, where the canonical Cartan

connection
$$\omega_{CR}:\ T\tilde{\mathcal{P}}\ \rightarrow\ \mathfrak{su}(1,m+1)$$
is again (as in the conformal case) uniquely determined by a normalisation condition on the corresponding curvature Ω_{CR}.

The intersection $\tilde{G}\cap P$, where P is the parabolic subgroup of conformal geometry in G, is contained in the parabolic $\tilde{P}$, and $\tilde{P}/(\tilde{G}\cap P)$ is a real projective line. We set $\tilde{F}_c := \tilde{\mathcal{P}}/(\tilde{G}\cap P)$. Then it is clear that $\tilde{F}_c$ is a circle bundle with projection $\tilde{\pi}_c$ over (M,H,J) and $\tilde{\mathcal{P}}$ is a principal $(\tilde{G}\cap P)$-bundle over $\tilde{F}_c$. We can extend this principal bundle $\tilde{\mathcal{P}}$ by the parabolic P in order to obtain a principal P-bundle $\mathcal{P}(\tilde{F}_c)$ over $\tilde{F}_c$. The canonical Cartan connection ω_{CR} on $\tilde{\mathcal{P}}$ extends as well by right translation to a Cartan connection $\omega : T\mathcal{P}(\tilde{F}_c) \rightarrow \mathfrak{g}$. Thus we obtain a pair $(\mathcal{P}(\tilde{F}_c),\omega)$, which is a naturally constructed Cartan geometry of conformal type (G,P) on the base $\tilde{F}_c$. In fact, it is straightforward to see that the Cartan geometry $(\mathcal{P}(\tilde{F}_c),\omega)$ generates in a natural way a conformal structure $[f]$ on the space $\tilde{F}_c$. One can show that this conformal structure $[f]$ on $\tilde{F}_c$ is locally conformally equivalent to the Fefferman conformal class $[f_\theta]$ defined on the canonical S^1-bundle F_c over (M,H,J). However, note that $(\tilde{F}_c,[f])$ and $(F_c,[f_\theta])$ (as introduced in Section 3) are in general not globally equivalent, since the bundle $\tilde{F}_c$ corresponds to a $(m+2)$nd root of the canonical line bundle $\Lambda^{m+1,0}M$. Nevertheless, we call here $(\tilde{F}_c,[f])$ the Fefferman space of (M,H,J) (with respect to a $(m+2)$nd root).

An extremely important feature of the above construction of the Cartan geometry $(\mathcal{P}(\tilde{F}_c),\omega)$ over $\tilde{F}_c$ is the fact that in case of an underlying integrable CR-space (M,H,J) the connection ω is identical to the canonical connection ω_{nor} of the conformal structure $[f]$ on $\tilde{F}_c$. This is the key observation here, which has important consequences for us. First, let us define the standard CR-tractor bundle over an arbitrary pseudoconvex CR-space (M,H,J) by
$$\mathcal{T}_{CR}\ :=\ \tilde{\mathcal{P}}\times_{\tilde{P}}\mathbb{C}^{m+2},$$
where $\mathbb{C}^{m+2}$ is the standard $\tilde{G}$-module. The canonical connection ω_{CR} induces a covariant derivative ∇^{CR} on $\mathcal{T}_{CR}$, which has a holonomy group $Hol(\mathcal{T}_{CR})$ sitting in $\mathrm{SU}(1,m+1)$. Now, since the right translation of ω_{CR} on $\mathcal{P}(\tilde{F}_c)$ is the canonical connection ω_{nor} of conformal geometry and $\Omega_{nor}(\chi,\cdot)=0$ for any vertical vector χ in the Fefferman fibration $\tilde{\pi}_c:\tilde{F}_c\rightarrow M$, it follows that (at least) the holonomy algebras $\mathfrak{hol}(\mathcal{T})$ and $\mathfrak{hol}(\mathcal{T}_{CR})$ of the tractor connections of conformal and CR-geometry are identical. Certainly, in case that $\tilde{\pi}_c:\tilde{F}_c\rightarrow M$ admits a global section the holonomy groups $Hol(\mathcal{T}_{CR})$ and $Hol(\mathcal{T})$ are identical as well.

The identification of the two holonomy algebras (resp., groups) has the following first consequence. Namely, it is obvious now that any Fefferman space, no matter if $(\tilde{F}_c, [f])$ or $(F_c, [f_\theta])$, has a conformal holonomy algebra $\mathfrak{hol}(\mathcal{T})$, which is reduced at least to $\mathfrak{su}(1, m+1)$ in $\mathfrak{so}(2, 2m+2)$. Together with Sparling's characterisation as used at the beginning of this section, we obtain a holonomy characterisation for Fefferman spaces in the general situation.

6 Theorem. *Let (F^{n+1}, c) be a simply connected conformal Lorentzian space of dimension $n+1 = 2m+2 \geq 4$ and let $Hol(\mathcal{T})$ be its conformal tractor holonomy. Then*

$$Hol(\mathcal{T}) \ \subset \ \mathrm{SU}(1, m+1)$$

if and only if c on F is locally the Fefferman conformal class over a pseudoconvex, integrable CR-space of hypersurface type. There exists no conformal space (F^{n+1}, c), whose tractor holonomy is identical to $\mathrm{U}(1, m+1)$.

Note that this theorem can also be proved by using tractor calculus only, without the detour of Sparling's characterisation and the reconstruction of CR-geometry (cf. [5, 22]).

It remains to discuss the Einstein condition in terms of CR-tractors. First, we observe that if a Fefferman space $(\tilde{F}_c, [f])$ over (M, H, J, θ) has a conformal holonomy group $Hol(\mathcal{T})$, which fixes a vector in the standard module, i.e., $(\tilde{F}_c, [f])$ is almost-Einstein, then the CR-holonomy $Hol(\mathcal{T}_{CR})$ has to fix a vector in $\mathbb{C}^{m+2}$. This fixed vector in $\mathbb{C}^{m+2}$ corresponds to a ∇^{CR}-parallel standard CR-tractor on (M, H, J). In fact, there is a natural 1-to-1-correspondence (at least locally) between ∇^{nor}-parallel standard conformal tractors on the Fefferman space $(\tilde{F}_c, [f])$ and ∇^{CR}-parallel standard CR-tractors on (M, H, J). The final step of our reasoning is now to clarify the meaning of a ∇^{CR}-parallel standard CR-tractor $\gamma \in \Gamma(\mathcal{T}_{CR})$ for the the underlying pseudoconvex CR-space (M, H, J). We will argue that γ corresponds (up to singularities) to a TSPE structure θ on (M, H, J).

For this purpose, we observe that the standard CR-tractor bundle $\mathcal{T}_{CR}$ admits an invariant filtration

$$\mathcal{T}_{CR} \ \supset \mathcal{T}^0_{CR} \ \supset \mathcal{T}^1_{CR} \ ,$$

where $\mathcal{T}^1_{CR}$ is a complex null line (associated to the $\tilde{P}$-stable line in $\mathbb{C}^{m+2}$) and $\mathcal{T}^0_{CR}$ is the corresponding orthogonal subspace in $\mathcal{T}_{CR}$. The quotient $\mathcal{T}_{CR}/\mathcal{T}^0_{CR}$ with projection $\tilde{\Pi}$ is isomorphic to the complex conjugated line bundle $\overline{\mathcal{E}}$ of the root $\mathcal{E}$, and $Q = TM/H$ naturally includes into $\overline{\mathcal{E}} \otimes \mathcal{E}$. We denote by $\overline{\sigma} := \tilde{\Pi}(\gamma)$ the corresponding section in $\overline{\mathcal{E}}$ to a standard CR-tractor $\gamma \in \Gamma(\mathcal{T}_{CR})$. Then the product $\overline{\sigma}\sigma$ is a section in Q, which determines in a unique way a pseudo-Hermitian form θ_γ on (M, H, J) (minus the singular zero set $\{\sigma = 0\}$). One can show that θ_γ is a TSPE structure if and only if γ is ∇^{CR}-parallel. On the other hand, the choice of a pseudo-Hermitian form θ on (M, H, J) gives rise via the

Reeb vector to a uniquely defined section $\tilde{T}_\theta$ in Q. Then there exist a unique $\sigma_\theta \in \Gamma(\mathcal{E})$ such that $\overline{\sigma}_\theta \sigma_\theta = \tilde{T}_\theta$ and the $(m+2)$nd power of the dual of σ_θ is a closed $(m+1,0)$-form on (M,H,J). Now, since $\mathcal{T}_{CR}$ is the 2-jet prolongation of its quotient $\overline{\mathcal{E}} = \mathcal{T}_{CR}/\mathcal{T}^0_{CR}$, there exists a natural CR-tractor D-operator $D : \Gamma(\overline{\mathcal{E}}) \to \Gamma(\mathcal{T}_{CR})$, which is a CR-invariant differential operator of 2nd order. It is a matter of fact that the application of D to $\overline{\sigma}_\theta$ is a ∇^{CR}-parallel standard tractor if and only if θ is a TSPE structure.

7 Theorem. *There exists a natural and unique* 1*-to-*1*-correspondence between TSPE structures θ and ∇^{CR}-parallel standard CR-tractors $\gamma \in \Gamma(\mathcal{T}_{CR})$ on any pseudoconvex integrable CR-space (M,H,J) (up to singular points).*

This completes our motivation for the construction of Fefferman-Einstein metrics as described in Theorem 4. In particular, it gives (a sketch) of a proof for the reversed statement therein.

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