Note di Matematica Vol. X - n. 2, 287-302 (1990)

ON THE CONDITIONS FOR THE PARALLELIZABILITY OF A COMPACT COMPLEX MANIFOLD

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Abstract. *In this note certain analytic and geometric conditions for the trivialization of a holomorphic vector bundle (on a compact complex space) are given. Applied to the holomorphic tangent bundle of a compact almost homogeneous complex manifold, these results yield parallelizability criteria for such manifolds. Especially, it is proved that a compaact, homogeneous, hermitian manifold with semi-negative scalar curvature is Ricci-flat and parallelizable. Similar results for manifolds admitting sufficiently many global holomorphic* 1*-forms are also obtained.*

1. INTRODUCTION

An m-dimensional (complex) torus is a quotient space of $\mathbb{C}^m$ by a discrete subgroup of maximal rank. As such it is a compact, homogeneous, Kähler manifold whose holomorphic tangent bundle is analytically trivial. A complex manifold with the latter property is called *parallelizable.* According to Wang [37], a (connected) compact complex manifold is parallelizable if and only if it is the homogeneous space of a complex Lie group by a discrete subgroup, and complex tori are the only compact, parallelizable manifolds which are Kählerian. On the other hand, Auslander [3] showed that there exists a compact, 2-dimensional Kähler manifold whose curvature tensor vanishes but its first Betti number is 2. Therefore the «parallelizability» of a manifold is not a consequence of zero curvature. This gives rise to the related question whether the «homogeneity» would be a consequence of zero curvature. That this is not the case is shown by the following (Lemma 3.1): A compact, hermitian manifold with semi-negative Ricci curvature is almost homogeneous if and only if it is a torus or parallelizable but not weakly Kählerian (i.e., not a $\mathscr{C}$-space in the sense of Fujiki [12]). Further, it is shown that an almost homogeneous manifold with semi-negative first Chern class is either parallelizable or not quasi-weakly Kählerian and has pseudo-trivial tangent bundle (Prop. 3.2). In the presence of homogeneity, however, a compact, hermitian manifold with semi-negative scalar curvature is necessarily Ricci-flat and parallelizable (Prop. 4.1).

A compact Riemann surface of positive genus is characterized by the fact that it admits a non-zero abelian differential of the first kind. In higher dimensions, it is natural to consider the analogous situation where the holomorphic cotangent bundle of a compact complex manifold is spanned at almost all points by global holomorphic 1-forms. Such a manifold is called, for convenience, *almost ample.* Example are given by the proper modifications (or semi-analytic coverings) of a compact parallelizable manifold. It turns out that an almost ample manifold is parallelizable if either it is quasi-weakly Kählerian with semi-positive first Chern class or its canonical bundle admits semi-negative Ricci curvature. As an alternative characterization of the complex tori, it is shown that among all almost ample Kähler manifolds, the tori are precisely those which admit an Albanese image of zero total scalar curvature with respect to some pseudo-Kähler metric (Prop. 4.8-(2)).

In preparation for §§ 3-4, some trivialization criteria for a holomorphic vector bundle are proved in § 2. As these might be of some use in other contexts (Cf. [27, p. 53] [15, Satz 1.1]), the base space of the bundle is allowed to have singularities (and this case is needed in Lemma 2.4). The proof given here depends on a Gauss-Bonnet-Chern type formula of Stoll [31, (7.3)]. This formula relates the current induced by the Ricci form of a hermitian bundle E to the divisor associated to a holomorphic section of the determinant bundle of E (see (2.6)).

The author is indebted to the referee for his valuable suggestions concerning the presentation of § 2. In particular, the idea of considering the family of induced bundles E_ρ (ρ being a polynomial representation of a general linear group) and the related isomorphism (2.7) was due to him.

2. TRIVIALIZATION OF HOLOMORPHIC VECTOR BUNDLES

Let X be a (reduced) complex space of dimension $m > 0$, and $E \to X$ a holomorphic vector bundle of rank r. A hermitian fiber metric h in E defines, in terms of a local frame $\{s_1, \ , s_r\}$ of E, a positive definite hermitian matrix $H = (h_{\alpha\beta})$, where

$$h_{\alpha\beta} = h(s_\alpha, s_\beta) = \overline{h}_{\beta\alpha}.$$

Restricted to X_{reg}, the manifold of simple points of X, the bundle E admits a unique hermitian connection whose connection and curvature matrices are (locally) given by

$$\omega = (\partial H) H^{-1}$$

$$\Omega = d\omega - \omega \wedge \omega = \overline{\partial}\omega.$$

It can be shown that the *total Chern form* ([9][29])

$$c(E; h) = \det(1 + (i/2\pi)\Omega)$$

does not depend on the choice of local frame fields and may be written

$$c(E; h) = \sum_{q=0}^{r} c_q(E; h),$$

where $c_q(E;h)$ is a closed, C^∞-form of type (q,q) on X. The de Rham class of the Chern form $c_q(E;h)$, denote $c_q(E)$, is called the q-th *Chern class of E* ([30]).

The Chern polynomial

$$C[E;h] = 1 + c_1(E;h)t + \ldots + c_r(E;h)t^r$$

admits a formal factorization ([17, p. 64]):

$$C[E;h] = \prod_{q=1}^{r}(1 + \lambda_q(E)t). \tag{2.1}$$

It follows from this that, if (E_j, h_j), $j = 1,2$, are hermitian vector bundles of rank r_j on X, then

$$c_1(E_1 \otimes E_2; h_1 \otimes h_2) = r_2 c_1(E_1; h_1) + r_1 c_1(E_2; h_2). \tag{2.2}$$

Also, for exterior powers of E, (2.1) implies that

$$c_1(\wedge^q E; \wedge^q h) = \binom{r-1}{q-1} c_1(E;h), \qquad (1 \le q \le r), \tag{2.3}$$

where $\wedge^q h$ denotes the induced hermitian metric on $\wedge^q E$.

Associated to a hermitian vector bundle $(E;h)$, there is the *Ricci form*

$$\text{Ric}\,(h) := -(1/4\pi)\text{dd}^c(\log\det\,(H)).$$

(where $\text{d}^c = i(\overline{\partial} - \partial)$). It is easy to see that Ric (h) is precisely the first Chern form of the determinant bundle of E relative to the induced hermtian metric. Thus by (2.3),

$$\text{Ric}(h) = c_1(E;h).$$

Let ϕ be a C^2-form of type $(m-1, m-1)$ on X. If $g : X \to \mathbb{C}$ is of class C^2, then

$$\phi \wedge \text{dd}^c g - g\,\text{dd}^c\phi = \text{d}(\phi \wedge \text{d}^c g - g\,\text{d}^c\phi). \tag{2.4}$$

In the following assume the space X is compact. If ϕ is dd^c-closed, the Stokes theorem and (2.4) imply that the integral

$$\text{Ric}(E;\phi) := \int_X \text{Ric}(h) \wedge \phi \tag{2.5}$$

is defined independently of the choice of hermitian metric h.

If $L \to X$ is a holomorphic line bundle and $\sigma \in H^0(X, L)$ a holomorphic section of L with $\sigma \not\equiv 0$ on any branch of X, then the vanishing multiplicity of σ induces a non-negative divisor, $D(\sigma)$, and accordingly, a current, $[D(\sigma)]$, on X (see [31, p. 52]). By Stoll [31, (7.3)], if $\sigma \in H^0(X, \det(E))$ and $\sigma \not\equiv 0$ on any branch of X, then a *Gauss-Bonnet-Chern* type formula

$$(2.6) \qquad [D(\sigma)](\phi) = \mathrm{Ric}(E; \phi) + (1/4\pi)\, dd^c[\log \|\sigma\|^2](\phi)$$

holds for all C^2-forms ϕ on X of type $(m-1, m-1)$. Here the left hand side of the equation is defined to be zero if σ is nowhere vanishing.

If $U \subseteq H^0(X, E)$, let $D(U)$ be the set of all $z \in X$ such that the evaluation map $\eta_z : U \to E_z$, $\eta_z(\sigma) = \sigma(z)$, is not surjective. The bundle E is said to be *semi-ample* (resp., *weakly ample*), if there exists a (finite dimensional) subspace $U \subseteq H^0(X, E)$ for which the degeneracy set $D(U)$ is nowhere dense (resp., empy) ([29]).

Let $\mathcal{P}_{r,N}$ be the set of all (non-constant) polynomial representations ρ of the general linear group $GL(r; \mathbb{C})$ with values in $GL(N; \mathbb{C})$, and set $\mathcal{P}_r = \cup_{N=1}^{\infty} \mathcal{P}_{r,N}$. If $\rho \in \mathcal{P}_r$ and $\{g_{ij}\}$ is a transition system of E relative to an open covering $\{U_j\}$ of X, set $g_{ij}^{[\rho]} = \rho(g_{ij})$. The system $\{g_{ij}^{[\rho]}\}$ defines a 1-cocyle and hence a vector bundle E_ρ on X. The family of the induced bundles E_ρ, $\rho \in \mathcal{P}_r$, includes the tensor, symmetric, and exterior powers of E as well as those vector bundles obtained by taking their compositions. In consequence of [22, p. 300], there exist, for each $\rho \in \mathcal{P}_r$, a positive integer $k = k(\rho)$ and an isomorphism

$$(2.7) \qquad \det(E_\rho) \to (\det(E))^k.$$

In the following, assume that X is irreducible. Let $h^0(E) = \dim H^0(X, E)$. For each $\rho \in \mathcal{P}_{r,1}$, define the ρ-genus *of* E by

$$g_\rho(E) = h^0((E^*)_\rho).$$

If E is non-singular, denote by K_X, the canonical bundle, and TX, the holomorphic tangent bundle, of X.

Let $L^n = \otimes^n L$ be the n-th tensor power of a line bundle $L \to X$. Notice that if $h^0(L^n) > 0$ and $h^0((L^*)^p) > 0$ for some positive integers n, p, then $L \to X$ is trivial.

Lemma 2.1. *If $E \to X$ is a semi-ample vector bundle admitting a non-zero ρ-genus, then $E \to X$ is trivial.*

Proof. By assumption, there exists a polynomial representation $\rho \in \mathcal{P}_{r,1}$ for which $h^0((E^*)_\rho) > 0$. Thus by (2.7), $h^0((\det(E^*))^k) > 0$ for some $k = k(\rho)$. On the other hand, by the semi-ampleness of E, one has $h^0((\det(E))^k) > 0$. It follows that the vector bundle $E \to X$ is trivial. Q.E.D.

A continuous $(1,1)$-form Ψ on X is said to be *semi-positive* (denoted ≥ 0) if in terms of local coordinates $z_1, \dots, z_m$ (at simple points of X),

$$\Psi = (i/2\pi) \sum_{\alpha,\beta} \gamma_{\alpha,\beta}\, \mathrm{d} z_\alpha \wedge \mathrm{d} \bar{z}_\beta,$$

where the matrix $(\gamma_{\alpha,\beta})$ is positive semi-definite everywhere. If, in addition, at some point of $X_{\rm reg}$, $(\gamma_{\alpha,\beta})$ is positive definite, then Ψ is said to be *quasi-positive* ([38, p. 403]). The first Chern class $c_1(E)$ is said to be *semi-positive* (resp., *quasi-positive* if $c_1(E)$ can be represented by a closed, semi-positive (resp., quasi-positive) $(1,1)$-form. A hermitian vector bundle (E,h) is said to have *semi-positive* (resp., *quasi-positive*) *Ricci curvature* if the Ricci form $\mathrm{Ric}(h)$ is semi-positive (resp., quasi-positive). The notions of *«semi-negative»* (resp., *«quasi-negative»* form (or Ricci curvature) are similarly defined.

A hermitian metric h on an open subset X_0 of $X_{\rm reg}$ is called a *pseudo-hermitian metric on* X if i) $D_h := X \setminus X_0$ is thin analytic in X, and ii) the associated fundamental form of h extends to a $(1,1)$-form ω_h of class C^2 on X. Further, a pseudo-hermitian metric h on X is called a *pseudo-Kähler metric*, (resp., *Kähler metric*), if the form ω_h is d-closed (resp., locally induced from a C^∞, d-closed, positive $(1,1)$-form defined on an imbedding space of X).

A C^2-form ϕ of type $(m-1, m-1)$ on X is called a *test form* if i) ϕ is positive, and ii) $\mathrm{dd}^c\phi = 0$. If, in place of i), ϕ is positive off a thin analytic subset, then ϕ is called an *almost-positive test form.* If X admits a Kähler form ω, then ω^{m-1} is obviously a test form. On an m-dimensional, compact homogeneous manifold, every invariant positive form of type $(m-1, m-1)$ is a test form. For an arbitrary compact complex manifold the existence of C^∞-test forms was established by Dektyarev [11].

Lemma 2.2. *(1) If for some almost-positive test form ϕ on X and some $\rho \in \mathscr{P}_\tau$*, Ric *$(E_\rho; \phi) < 0$ (resp., $= 0$), then $h^0(E_\tau) = 0$ (resp., ≤ 1), $\forall \tau \in \mathscr{P}_{\tau,1}$. (2) If $E \to X$ is semi-ample and for some $\rho \in \mathscr{P}_\tau$, the bundle E_ρ admits a hermitian metric of semi-negative Ricci curvature, then $E \to X$ is trivial.*

Proof. Assume for some $\rho \in \mathscr{P}_\tau$, the bundle E_ρ admits a hermitian metric h of semi-negative Ricci curvature. Let $\pi : \hat{X} \to X$ be a desingularization of X. Take a test form Ψ on $\hat{X}$. For the induced bundle $\widetilde{E_\rho} = \pi^* E_\rho$,

$$\mathrm{Ric}(\widetilde{E_\rho}; \Psi) = \int_{\hat{X}} \pi^* \mathrm{Ric}(h) \wedge \Psi \leq 0.$$

On the other hand, by (2.3), (2.6) and (2.7),

$$\begin{aligned}\operatorname{Ric}(\widetilde{E_\rho};\Psi) &= \operatorname{Ric}(\det(\widetilde{E_\rho});\Psi)\\ &= \operatorname{Ric}(\det((\widetilde{E})_\rho);\Psi)\\ &= \kappa(\rho)\operatorname{Ric}(\widetilde{E};\Psi).\end{aligned} \tag{2.8}$$

Consequently, if E is semi-ample, then $\operatorname{Ric}(\det(\widetilde{E});\Psi) = 0$, which implies that E is trivial.

Simillary, if $\operatorname{Ric}(E_\rho;\phi) < 0$ (resp., $= 0$) for some $\rho \in \mathscr{P}_\tau$ and some almost-positive test form ϕ, then $\operatorname{Ric}(E_\tau;\phi) < 0$ (resp., $= 0$), $\forall\tau \in \mathscr{P}_{\tau,1}$. By (2.6), the first case implies that $h^0(E_\tau) = 0$. Suppose now the bundle E_τ, $\tau \in \mathscr{P}_{\tau,1}$, admits non-trivial holomorphic sections ζ_1 and ζ_2. Pick $z \in X_{\mathrm{reg}}$ near which the form ϕ is positive. If $\operatorname{Ric}(E_\tau;\phi) = 0$, then $\zeta_j(z) \neq 0$ for $j = 1,2$. If $\zeta := \zeta_1(z)\zeta_2 - \zeta_2(z)\zeta_1$ is not identically zero, then by (2.6), $\operatorname{Ric}(E_\tau;\phi) > 0$, a contradiction. From this the assertion (1) follows. Q.E.D.

Remark. The above assertion (2) may also be proved using Kobayashi and Wu [20].

Corollary 2.3. *Let $L \to Y$ be a holomorphic line bundle on a normal, irreducible, compact complex space Y of dimension m. If L admits a hermitian metric of quasi-positive Ricci curvature, then there exists an integer N such that $H^m(Y,\mathscr{O}(L^n)) = 0$, for all $n \geq N$.*

Proof. Let $(\hat{Y},\pi)$ be a desingularization of Y, and ϕ a test form on $\hat{Y}$. Let $\widetilde{L} = \pi^*L$. Since $\operatorname{Ric}(\widetilde{L};\phi) > 0$, there exists a positive integer $k = k_\phi$ such that

$$n\operatorname{Ric}(\widetilde{L};\phi) + \operatorname{Ric}(T\hat{Y};\phi) > 0, \qquad \forall n \geq k.$$

In view of (2.2) and (2.3), the above inequality implies that

$$\operatorname{Ric}((\widetilde{L}^n)^* \otimes K_{\hat{Y}};\phi) < 0, \qquad \forall n \geq k,$$

Hence

$$H^0(\hat{Y},\mathscr{O}((\widetilde{L}^n)^* \otimes K_{\hat{Y}})) = 0$$

by Lemma 2.2-(1). It follows from Serre's duality theorem that $H^m(\hat{Y},\mathscr{O}(\widetilde{L^n})) = 0$. Consequently, by the normality of Y, $H^m(Y,\mathscr{O}(L^n)) = 0$, $\forall n \geq k$. Q.E.D.

The transcendence degree (over $\mathbb{C}$) of the field of meromorphic functions on X, $tr(X)$, is called the *algebraic dimension* of X. It is known that $tr(X) \leq \dim X$; moreover, if X

is projective algebraic, then $tr(X) = \dim X$. The space X is called a *Moushezon space* if $tr(X) = \dim X$. A Moishezon space needs not be Kählerian. According to Moishezon [25], there exist for each Moishezon space X a smooth projective algebraic variety X^* and a modification $\pi : X^* \to X$ such that π is obtained by a finite sequence of monoidal transformations with non-singular centers. This leads to the notion of a *weakly Kählerian space*, i.e., a $\mathscr{C}$-space in the sense of Fujiki [12] (see also [13] [35]). According to [12, 1.1], a weakly Kählerian space is necessarily a holomorphic image of a connected compact Kähler manifold. More generally, an irreducible, compact complex space X is called *quasi-weakly Kählerian*, if it is a holomorphic image of a compact, irreducible pseudo-Kähler space (Y, h) with codim $D_h \geq 2$.

Let $V = H^0(X, E)$, $X_0 = X \setminus D(V)$ and $E_0 = E|X_0$. If $E \to X$ is semi-ample, there is an exact sequence of holomorphic vector bundles on X_0:

$$0 \to N \to X_0 \times V \to E_0 \to 0.$$

If $h^0(E) = n+1 > r$, the map $\phi_V : X_0 \to G_p(V)$ by $\phi_V(x) = \mathbf{P}(N_x)$, $x \in X_0$, (here $p = n-r$ and $G_p(V)$ being the Grassmannian of $(p+1)$-planes in V), is holomorphic and has a meromorphic extension to X ([24, 4.1][29, 2.3]). The bundle E_0 is the pull-back under (the classifying map) ϕ_V of a universal quotient bundle $Q_p(V) \to G_p(V)$. Let Y be the closed graph of ϕ_V, and $\pi_V : Y \to G_p(V)$ the projection. Set $W_E := (\pi_V)^* Q_p(V)$.

For convenience, a holomorphic vector bundle W of rank r over a compact complex space Y is called *pseudo-trivial* if there exists a thin analytic subset S of Y such that with $Y' = Y \setminus S$, either W is trivial over Y' or there exist a holomorphic map π of a compact complex space $\widetilde{Y}$ onto Y and a weakly ample bundle $\widetilde{W} \to \widetilde{Y}$ such that i) $\pi : \pi^{-1}(Y') \to Y'$ is biholomorphic, ii) $\widetilde{W}|\pi^{-1}(Y')$ is isomorphic to $\pi^*(W|Y')$, and iii) all the Chern numbers $C^\alpha(\widetilde{W}) = 0$ (where $\alpha = (\alpha_1, \ldots, \alpha_r)$, the α_j's being non-negative integers with $\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = m$) (see [24, p. 81]).

A criterion for the trivialization of a weakly ample bundle on a projective space was given in [27, p. 53]. More generally one has the assertion (2)-(b) of the following:

Lemma 2.4. *(1) If X is pseudo-Kählerian and if $c_1(E)$ is quasi-negative, then $h^0(E_\tau) = 0$, $\forall \tau \in \mathscr{P}_{r,1}$. (2) Assume $E \to X$ is semi-ample. Then (a) either $E \to X$ is pseudo-trivial or X is a Moishezon space and none of the bundles $E_\rho \to X$ and $W_\rho \to Y$, $\rho \in \mathscr{P}_r$, where $W = W_E$, admits a semi-negative first Chern class; (b) if X is quasi-weakly Kählerian and if the bundle E admits a semi-negative first Chern class, then $E \to X$ is trivial.*

Proof. Let $\rho \in \mathscr{P}_r$, and κ be a hermitian metric on E_ρ. If $c_1(E_\rho) \leq 0$, then there exist C^∞-forms ξ and η on X such that ξ is closed, semi-negative, and $\mathrm{Ric}(\kappa) = \xi + d\eta$.

Assume at first that X is the holomorphic image (under π) of an n-dimensional, compact, irreducible pseudo-Kähler space (Y, h) with codim $D_h \geq 2$. Denoting by $\widetilde{E}_\rho$ the lifted hermitian bundle on Y, one has

$$\begin{aligned}\mathrm{Ric}(\widetilde{E}_\rho; \omega_h^{n-1}) &= \int_Y (\pi^*\xi + \mathrm{d}(\pi^*\eta)) \wedge \omega_h^{n-1} \\ &= \int_Y \pi^*\xi \wedge \omega_h^{n-1} \leq 0.\end{aligned}$$

Thus if E is semi-ample, (2.6) and (2.8) imply that $\widetilde{E}$, hence also E, is trivial. Similarly, if $c_1(E)$ is quasi-negative and X admits a pseudo-Kähler metric h, then one has $\mathrm{Ric}(E; \omega_h^{m-1}) < 0$. Hence by Lemma 2.2-(1), $h^0(E_\tau) = 0$, $\forall \tau \in \mathcal{P}_{r,1}$. This proves the assertion (1).

Assume now E is semi-ample but not pseudo-trivial. Then $h^0(E) > r$, and the induced bundle $W = W_E \to Y$ is weakly ample with at least one non-zero Chern number $C^\alpha(W)$. Thus for some branch Y_μ of Y, the restriction $\pi : Y_\mu \to X$ is surjective and $C^\alpha(W_\mu) \neq 0$, where $W_\mu = W|Y_\mu$. It follows from [24, 5.6] and [36, 3.9] that Y_μ, hence also X, is a Moishezon space. By the preceding, no bundles $E_\rho \to X$ and $W_\rho \to Y$, $\rho \in \mathcal{P}_r$, admit a semi-negative first Chern class. This completes the proof of the assertion (2). Q.E.D.

3. ALMOST HOMOGENEITY AND PARALLELIZABILITY

In the following, let M denote a connected, compact complex manifold of dimension $m > 0$. M is said to be *parallelizable* if it admits m global holomorphic vector fields linearly independent at every point. Such a manifold can be regarded, according to Wang [37, Thm. 1], as the compact coset space of a complex Lie group with a discrete isotropy subgroup. They are, in general, non-Kählerian. In fact, Wang [37, p. 776] has shown that a compact, parallelizable manifold is Kählerian if and only if it is a torus. An example of a non-Kählerian parallelizable manifold is given by the *Iwasawa manifold* G/Γ, where G is the Lie group of all complex matrices

$$g = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\Gamma \subseteq G$ is the discrete subgroup of all elements g whose entries are Gaussian integers. Note that the manifold G/Γ has a non-abelian fundamental group Γ, hence it is not a torus. Indeed, a non-Kählerian, compact parallelizable manifold is necessarily non-weakly Kählerian (Lemma 3.1).

The manifold M is said to be *almost homogeneous* if the automorphism group of biholomorphic transformations of M acts transtively on M exclusive (possibility) of a nowhere dense subset S; if S is empty, then M is *homogeneous.* By means of complex 1-parameter group of holomorphic transformations one can show that M is almost homogeneous (resp., homogeneous) if and only if its holomorphic tangent bundle is semi-ample (resp., weakly ample) (see [19, pp. 22-24] [28, p. 246]).

Every compact complex manifold admits a holomorphic map into a (possibility trivial) torus. In fact, there exist a complex torus, $\mathrm{Alb}(M)$, the *Albanese torus* of M, and a holomorphic map $\alpha : M \to \mathrm{Alb}(M)$, such that α is universal among all holomorphic maps of M into complex tori ([4]). The construction of the Albanese torus shows that the dimension of $\mathrm{Alb}(M)$ is at most equal to that of $H^0(M, \mathrm{d}\mathcal{O}_M)$, the space of closed holomorphic 1-forms on M. If M is weakly Kählerian, then $\dim \mathrm{Alb}(M) = \dim H^0(M, \mathrm{d}\mathcal{O}_M)$ ([36, 9. 22]).

Lemma 3.1. *Assume M is almost homogeneous and one of the following holds:*

(a) TM admits a non-zero ρ-genus,

(b) For some $\rho \in \mathcal{P}_m$, $(TM)_\rho$ admits a hermitian metric of semi-negative Ricci curvature.

Then M is either a torus or parallelizable but not weakly Kählerian.

Proof. Obviously, under the hypothesis (a), resp. (b), M is parallelizable by Lemma 2.1, resp. 2.2. Hence, if M is weakly Kählerian, then M is biholomorphic to a product $\mathrm{Alb}(M) \times Y$, where Y is a projective algebraic, rational manifold ([12, p. 255]). Consequently M is Kählerian. Thus the theorem of Wang ([ibid]) concludes the proof. Q.E.D.

If M is almost homogeneous and admits a Kähler metric, then the Albanese map α : $M \to \mathrm{Alb}(M)$ is a holomorphic fiber bundle whose typical fiber F is a connected, almost homogeneous Kähler manifold with vanishing first Betti number (see [26, 2.11 & 2.13]). Thus $c_1(F)$ is not quasi-negative. It turns out that, if $c_1(F)$ is semi-negative or a torsion (integral) class, then F is zero dimensional:

Proposition 3.2. *Assume M is almost homogeneous with semi-negative first Chern class. Then either M is parallelizable or M is not quasi-weakly Kählerian and has pseudo-trivial tangent bundle.*

Proof. This is an immediate consequence of Lemmas 2.4-(2). Q.E.D.

Remark. The above Proposition sharpens a result of Aeppli [1, Remark 3] (where, by assumption, M is homogeneous under the action of a compact transformation group and $c_1(M) = 0$).

Corollary 3.3. *Assume M is homogeneous. (1) If M has algebraic dimension zero, then M is a torus or parallelizable but non-weakly Kählerian. (2) If M is a symmetric spaace and has either algebraic dimension zero or semi-negative first Chern class, then M is a torus.*

Proof. By the structure theorem of Grauert and Remmert [14, Satz 3], M is a holomorphic fiber bundle over a projective algebraic variety Y with connected, parallelizable fibers; moreover, the function field of Y is isomorphic to that of M. Therefore, if M has algebraic dimension zero, then Y is a single point. Thus the assertion (1) follows from Lemma 3.1.

According to Borel [7, 2.4], a homogeneous, symmetric space M is biholomorphic to a product, $T \times B$, where T is a torus and B is a projective algebraic, rational manifold. Thus M is Kählerian. If, in addition, $c_1(M)$ is semi-negative, then M is a torus by Proposition 3.2 and Lemma 3.1. If M has algebraic dimension zero, then the same is true by (1). Q.E.D.

4. SCALAR CURVATURE AND PARALLELIZABILITY

Let M be a complex manifold of dimension $m > 0$, and g a hermitian metric on TM with fundamental form ω. An analogue of the Gaussian curvature in higher dimensions is given by the scalar curvature κ_g of the hermitian connection on TM. In fact, (using [9, (7.25)]) it can be shown that

$$\kappa_g \omega^m = m \operatorname{Ric}(g) \wedge \omega^{m-1}. \tag{4.1}$$

In the following, assume M is compact.

Proposition 4.1. *A compact, homogeneous, hermitian manifold with semi-negative scalar curvature is Ricci-flat and parallelizable. (In particular, it admits no hermitian metric of quasi-negative scalar curvature.)*

Proof. By [24, 4.2], on a homogeneous complex manifold M there exists a hermitian metric h of semi-positive Ricci curvature. (Such an h trivially exists if M is parallelizable.) Let g be a hermitian metric on TM (with fundamental form ω). Define an operator $L : C^\infty(M) \to C^\infty(M)$ by

$$L(u)\omega^m = dd^c u \wedge \omega^{m-1}.$$

There exists a C^∞-function Ψ on M such that

$$\operatorname{Ric}(g) = \operatorname{Ric}(h) + dd^c\Psi. \tag{4.2}$$

If $\kappa_g \leq 0$, then by (4.1) and (4.2), one has

$$L(\Psi)\omega^m \leq -\operatorname{Ric}(h) \wedge \omega^{m-1} \leq 0.$$

Since the mapping L is a Hopf operator, it follows from [18] that Ψ = constant. Thus one has $\mathrm{Ric}(g) = \mathrm{Ric}(h) \geq 0$, $\mathrm{Ric}(h) \wedge \omega^{m-1} \equiv 0$, and $\kappa_g \equiv 0$. On the other hand, for a test form ϕ on M, there exist a constant $N > 0$ with

$$\mathrm{Ric}(h) \wedge (N\omega^{m-1} - \phi) \geq 0 \qquad \text{on } M.$$

Thus $\mathrm{Ric}(TM;\phi) = 0$. Consequently the non-positivity of κ_g implies that M is parallelizable and $\mathrm{Ric}(g) \equiv 0$. Q.E.D.

Remark. The above Proposition and Lemma 3.1 imply that an almost homogeneous hermitian manifold with semi-negative Ricci curvature is Ricci-flat and parallelizable.

Let Y be an m-dimensional compact complex space, $\pi : \hat{Y} \to Y$ a desingularization, and $T\hat{Y}^*$ the holomorphic cotangent bundle of $\hat{Y}$. Let U be the subspace of $H^0(\hat{Y}, T\hat{Y}^*)$ consisting of all pull-backs $\pi^*\Psi$ of holomorphic 1-forms Ψ on Y. Set

$$D_Y = \pi(D(U)) \cup Y_{\mathrm{sing}}.$$

The complex space Y is called (1) *almost ample* if D_Y is thin in Y; and (2) *weakly ample* if D_Y is empty.

Weakly ample manifolds were studied, e.g., in [5] [23] [24] [32]. It was shown, among other things, that a compact Kähler manifold is weakly ample if and only if it admits a holomorphic immersion into a complex torus ([23] [24]). Note that a proper modification of a weakly ample manifold is almost ample. In particular, if the manifold M has Albanese dimension m and has algebraic dimension zero, then by [36.13.7], the Albanese map of M is a modification, hence M is almost ample.

By Lemma 2.2 (or Wu [38, p. 406]), an almost ample manifold admits no hermitian metric of quasi-positive Ricci curvature. If such a manifold carries a hermitian metric of semi-positive Ricci curvature, then it is necessarily Ricci-flat and parallelizable. This is a consequence of Lemma 2.2 and the following:

Proposition 4.2. *A weakly ample hermitian manifold with semi-positive scalar curvature is Ricci-flat and parallelizable. (In particular, it admits no hermitian metric of quasi-positive scalar curvature.)*

Let h be a pseudo-hermitian metric on Y. (Y, h) is said to be *of positive* (resp., *zero*) *total scalar curvature* (with respect to a desingularization $(\hat{Y}, \pi)$ of Y) if and only if the *total scalar curvature*

$$R_{h,\pi}[Y] := m\,\mathrm{Ric}(T\hat{Y}; (\pi^*\omega_h)^{m-1}) \tag{4.3}$$

is positive (resp., zero).

The next Lemma shows that a weakly ample manifold admits no closed pseudo-Kählerian subspace of positive total scalar curvature.

Lemma 4.3. *Let X be a compact complex space of dimension m. Assume X admits a holomorphic map ϕ of rank m into an almost ample complex space Y. If $\phi(X) \not\subseteq D_Y$, then i) X is almost ample; ii) X admits no pseudo-Kähler metric of positive total scalar curvature.*

Proof. Let x be a point of $X_0 =: X \setminus \phi^{-1}(D_Y)$ such that ϕ attains rank m at x. There exist open neighborhoods $U \subseteq X_0$ of x, and $V \subseteq Y$ of $y = \phi(x)$, such that the map $f' = \phi : U \to V$ has pure rank m and the image $Z = \phi(U)$ is a pure m-dimensional analytic subset of V ([2, 1.21]). Let $f = f' : U \to Z$ and D_m be the set of all $q \in U_{\text{reg}} \cap f^{-1}(Z_{\text{reg}})$ such that the Jacobian rank of f at q is less than m. Then the set $E = \overline{D}_m \cup f^{-1}(Z_{\text{sing}}) \cup U_{\text{sing}}$ is thin analytic in U ([33, p. 106]). According to [2, 1.26], the set $T = \{z \in Z | E \cap f^{-1}(z) \neq \emptyset\}$ is almost thin in Z. Take $x_0 \in f^{-1}(Z \setminus T)$. Then $Y_0 = f(x_0) \in Z_{\text{reg}}$ and f has Jacobian rank m at x_0. It follows that the map ϕ is an immersion at x_0. Consequently there exist global holomorphic 1-forms $\eta_1, \dots, \eta_\kappa$ on Y such that the cotangent bundle of X_{reg} is spanned at x_0 by the 1-forms $\phi^*\eta_1, \dots, \phi^*\eta_\kappa$. Thus X is almost ample. Suppose now X admits a pseudo-Kähler metric h. Then it follows from (2.6), (4.3) and [34, 2.1] that, for any desingularization $(\hat{X}, \pi)$ of X, $R_{h,\pi}[X] \leq 0$. Q.E.D.

If M is Kählerian and if either $c_1(M)$ is semi-positive or M is almost homogeneous, then the Albanese map $\alpha : M \to \text{Alb}(M)$ is a holomorphic fiber bundle with connected fibers (see [26], resp. [21]). Therefore, the map α is a biholomorphism provided the irregularity of M is equal to the dimension of M. In the absence of a Kähler metric, one has the following:

Corollary 4.4. *Assume either the canonical bundle of M admits semi-negative Ricci curvature or M is almost homogeneous. Then M is parallelizable if and only if M admits a holomorphic map ϕ of rank m into an almost ample complex space Y with $\phi(M) \not\subseteq D_Y$.*

Proof. The sufficiency part of the conclusion follows immediately from Lemmas 4.3 and 2.2 (resp., 2.1). Q.E.D.

Corollary 4.5. *Assume M admits a holomorphic map ϕ of rank m into an almost ample complex space Y with $\phi(M) \not\subseteq D_Y$. (1) If M is Kählerian with zero total scalar curvature, then M is a torus. (2) If $c_1(M) \geq 0$, then either M is parallelizable or M is not quasi-weakly Kählerian and TM^* is pseudo-trivial. (3) If $c_1(M) \geq 0$ and Y is a Moishezon space, then M is an abelian variety.*

Proof. By Lemma 4.3, M is almost ample. (1) If M is Kählerian with zero total scalar curvature, then by (2.6) and (4.3), TM^* is trivial. Hence Wang [37] implies that M is a torus. (2) This is an immediate consequence of Lemmas 2.4-(2). (3) If Y is a Moishezon space, then so is the image space $\phi(M)$. Since $\phi(M)$ is irreducible ([2, 1.27]), by [24, 5.1],

M is a Moishezon manifold. Therefore, Lemma 3.1 implies that M is a torus, and by Grauert and Remmert [14, Satz 1], M is projective algebraic. Q.E.D.

A proper, surjective, holomorphic map $\pi : X \to Y$ is called a *semi-analytic covering* of the complex space Y, if there exists a thin analytic set S (possibly empty) in Y such that the set $A = \pi^{-1}(S)$ is thin in X and the restriction $\pi_{|X \setminus A}$ defines an analytic covering of $Y \setminus S$.

Corollary 4.6. *Assume M is a semi-analaytic covering manifold of an m-dimensional, almost ample convex space Y, and $c_1(M) \geq 0$. If either $T\hat{Y}^*$ is not pseudo-trivial (for some desingularization $(\hat{Y}, \pi)$ of Y) or Y is non-singular with irregularity $q >$ dim M and admits a holomorphic 1-form with a non-degenerate isolated zero in $Y - D_Y$, then M is an abelian variety.*

Proof. If $T\hat{Y}^*$ is not pseuod-trivial, then by Lemma 2.4-(2) and [36, 3.9], $\hat{Y}$, hence also Y, is a Moishezon space. Supppose now Y is non-singular with irregularity $q >$ dim M, and Y admits a holomorphic 1-form with a non-degenerate isolated zero in $Y - D_Y$. Let $U := H^0(Y, TY^*)$. By Cowen [10, p. 7 6], the classifying map of TY^*, $\phi_U : Y - D_Y \to G_{q-m-1}(U)$, is an immersion at some point. Hence it follows from [24, 5.3] that Y is a Moishezon manifold. Therefore the conclusion follows from Corollary 4.5-(3). Q.E.D.

According to A. Borel [7, 3.5], if the projective space $\mathbf{P}^n(\mathbb{C})$ $(n \geq 2)$ is blown up at one point, the resulting manifold is a symmetric space, and as such is necessarily almost homogeneous ([7, 2.2]). For a compact parallelizable (resp., weakly ample) manifold, the almost homogeneity (resp., weak-ampleness) of the manifold is, however, not preserved by monoidal transformations in codimension ≥ 2:

Corollary 4.7. *Assume Y is a weakly ample manifold of dimension $n \geq 2$. Let M be the manifold obtained by blowing up Y along a closed submanifold Q of codimension ≥ 2. Then i) the degeneracy set $D_M = \phi^{-1}(Q)$, where $\phi : M \to Y$ is a modification defining M; ii) M neither admits a hermitian metric of semi-positive Ricci curvature nor is almost homogeneous.*

Proof. Observe that the fiber $\phi^{-1}(z)$, $z \in Q$, being biholomorphic to $\mathbf{P}^q(\mathbb{C})$ with $q = m -$ dim $Q - 1$, carries a (Fubini-Study) Kähler metric of positive Ricci curvature. Therefore Lemma 4.3 implies that $D_M = \phi^{-1}(Q)$. The assertion ii) is then an immediate consequence of Coro. 4.4 and Lemma 4.3. Q.E.D.

Proposition 4.8. *(1) If the Albanese image of M admits a pseudo-Kähler metric of zero total scalar curvature, then the Albanese map $\alpha : M \to$ Alb(M) is surjective. (2) An almost* *ample, Kähler manifold is a torus if and only if its Albanese image admits a pseudo-Kähler metric of zero total scalar curvature.*

Proof. (1) By [2, 1.27], the image space $Y = \alpha(M)$ is irreducible. As a complex subspace of $\mathrm{Alb}(M)$, Y is almost ample. Assume Y admits a pseudo-Kähler metric h of zero total scalar curvature with respect to a desingularization $(\hat{Y}, \pi)$ of Y. Let ω_h be the associated fundamental form of h. Then $\mathrm{Ric}(K_{\hat{Y}}; (\pi^*\omega_h)^{m-1}) = 0$. Hence it follows from Lemma 2.2-(1) that Y has geometric genus 1. According to Ueno [36, 10.3], Y is non-singular and has Kodaira dimension 0. Hence by [36, 10.6], the Albanese map α is surjective. (2) Assume M is an almost ample, Kähler manifold. By Matsushima [23, p. 312], the Albanese map α of M is an immersion at some point. Consequently, if $\alpha(M)$ admits a pseudo-Kähler metric of zero total scalar curvature, then $m =$ dim $\mathrm{Alb}(M)$, which implies that α is a biholomorphism. Q.E.D.

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Received August 17, 1988 and in revised form July 19, 1990.

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