An alternate proof of “Tame Fréchet spaces are Quasi-Normable”

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Abstract. In [12] Piszczek proved that “tameness always implies quasinnormability” in the setting of Fréchet spaces. In this paper we present an alternate proof of this interesting result.

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Introduction

Let $E$ be a Fréchet space and let $(\|\cdot\|_k)_k$ denote a fundamental system of increasing seminorms defining the topology of $E$ such that the sets $U_k := \{ x \in E | \|x\|_k \leq 1 \}$ form a basis of 0–neighbourhoods in $E$.

A Fréchet space $E$ is called tame if there exists an increasing function $S: \mathbb{N} \to \mathbb{N}$ such that for every continuous linear operator $T: E \to E$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there is a constant $C_k > 0$ for which

$$\forall x \in X : \|Tx\|_k \leq C_k \|x\|_{S(k)}.$$ 

The definition does not depend on the choice of seminorms. The class of tame Fréchet spaces was introduced and studied by Dubinsky and Vogt in [7]. Tameness condition is related to important questions concerning the structure of Fréchet spaces, in particular of infinite/finite type power series spaces and of Köthe sequence spaces, see [7, 10, 11, 12, 13, 14, 16] and the references therein.

A Fréchet space $E$ is called quasinormable if there exists a bounded subset $B$ of $E$ such that

$$\forall n \exists m > n \forall \varepsilon > 0 \exists \lambda > 0 : U_m \subset \lambda B + \varepsilon U_n.$$ 

The class of quasinormable locally convex spaces was introduced and studied by Grothendieck in [8]. Such a class of spaces has received a lot of attention in
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the setting of Fréchet spaces and of Köthe sequence spaces, see [1, 2, 3, 4, 5, 6, 9, 11, 12, 15] and the references therein.

Piszczek proved that every tame Fréchet space is quasinormable, [12] (see also [11]). The aim of this note it to present an alternate proof of such a result. The proof is different in spirit and relies on some results established in [1, 2].

1 An alternate proof

In the sequel, given a Fréchet space $E$ we denote by $(\|\|_k)$ a fundamental system of increasing seminorms defining the topology of $E$ such that the sets $U_k := \{ x \in E \mid \|x\|_k \leq 1 \}$ form a basis of 0–neighbourhoods in $E$. The dual seminorms are defined by $\|f\|'_k := \sup\{ |f(x)| \mid x \in U_k \}$ for $f \in E'$; hence $\|\|'_k$ is the gauge of $\circ U_k$. We denote by $E'_k := \{ f \in E' \mid \|f\|'_k < \infty \}$ the linear span of $\circ U_k$ endowed with the norm topology defined by $\|\|'_k$. Clearly, $(E'_k, \|\|'_k)$ is a Banach space and $E'_k = (E/\ker \|\|_k, \|\|'_k)$.

If $E$ is a Fréchet space with a continuous norm, we may assume that each $\|\|_k$ is a norm on $E$.

In order to give the proof we recall the following two lemmas. The first one, due to Piszczek [11], gives a necessary condition for the tameness of a Fréchet space $E$.

Lemma 1. ([11, Lemma 3]) In every tame Fréchet space $E$ the following condition holds. There exists $\psi: \mathbb{N} \to \mathbb{N}$ such that for any $\varphi: \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $m \geq k$ there are $n \in \mathbb{N}$ and a constant $C_m > 0$ such that

$$\forall f \in E', \ y \in E : \max_{k \leq l \leq m} \|f\|'_{\psi(l)} \|y\|_l \leq C_m \max_{1 \leq p \leq n} \|f\|'_{\varphi(p)} \|y\|_p. \quad (1.1)$$

An examination of the proofs given in [1, Theorem 1] and in [2, Theorem 3] (respectively, for separable Fréchet spaces and for general Fréchet spaces) shows that the following fact holds.

Lemma 2. Let $E$ be a Fréchet space with a continuous norm. If $E$ is not quasinormable, then there exist sequences $(x_{jk})_{j,k \in \mathbb{N}} \subset E$ and $(f_{jk})_{j,k \in \mathbb{N}} \subset E'$, and there exist a decreasing sequence $(\beta_k)_{k \in \mathbb{N}} \subset [0, 1]$ and an increasing sequence $(\alpha_k)_{k \in \mathbb{N}} \subset ]1, +\infty[,$ satisfying the following properties.

(1) $0 < \alpha \leq \inf_{k \in \mathbb{N}} \alpha_k \beta_k < \sup_{k \in \mathbb{N}} \alpha_k \beta_k \leq \beta < \infty$.

(2) $(x_{jk}, f_{jk})_{j,k \in \mathbb{N}}$ is a biorthogonal system.

(3) $\|f_{jk}\|'_k \leq 1$ for all $j, k \in \mathbb{N}$.

(4) $\sup_{j \in \mathbb{N}} \|x_{jk}\|_k \leq \alpha_k$ for all $k \in \mathbb{N}$.
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\( \inf_{k \in \mathbb{N}} \| f_{jk} \|'_k \geq \beta_k \) for all \( k \in \mathbb{N} \).

\( \| f_{jk} \|'_{k+1} \leq k^{-j} \) for all \( j, k \in \mathbb{N} \).

We can now present an alternate proof of the following result which was already established in [12, §3] (see also [11, Theorem 6]).

**Theorem 1.** Every tame Fréchet space is quasinormable.

**Proof.** Let \( E \) be a tame Fréchet space. Then by [11, Proposition 5] we may assume that \( E \) has a continuous norm. So, if suppose that \( E \) is not quasi-normable, Lemma 2 yields that there exist two sequences \((x_{jk})_{j,k \in \mathbb{N}} \subset E \) and \((f_{jk})_{j,k \in \mathbb{N}} \subset E' \), and there exist a decreasing sequence \((\beta_k)_{k \in \mathbb{N}} \subset ]0,1[\) and an increasing sequence \((\alpha_k)_{k \in \mathbb{N}} \subset ]1, +\infty[\) satisfying all the properties (1) ÷ (6) in Lemma 2.

Since \( E \) is tame, by Lemma 1 we have that there exists \( \psi : \mathbb{N} \to \mathbb{N} \) such that for any \( \varphi : \mathbb{N} \to \mathbb{N} \) there exists \( k \in \mathbb{N} \) such that for all \( m \geq k \) there are \( n \in \mathbb{N} \) and a constant \( C_m > 0 \) such that inequality (1.1) holds. So, setting \( m = k \), the following holds.

\[ \forall j, h \in \mathbb{N} : \quad \| f_{h\varphi(k-1)} \|'_{\psi(k)} \| x_{jk-1} \|_k \leq C_k \max_{1 \leq p \leq n} \| f_{h\varphi(k-1)} \|'_{\varphi(p)} \| x_{jk-1} \|_p. \quad (1.2) \]

Without loss of generality we may assume that \( n \geq k \) in (1.2).

As \( \varphi \) is arbitrary, we may choose the function \( \varphi \) such that \( \varphi(k-1) = \psi(k) \) for all \( k \geq 1 \). Therefore, by Lemma 2(5) we have, for every \( h \in \mathbb{N} \), that

\[ \| f_{h\varphi(k-1)} \|'_{\varphi(k-1)} \geq \beta_{\varphi(k-1)}. \]

and so

\[ \frac{1}{\| f_{h\varphi(k-1)} \|'_{\varphi(k-1)}} \leq \beta_{\varphi(k-1)}^{-1}. \quad (1.3) \]

On the other hand, we have that

\[ \lim_{j \to \infty} \| x_{jk-1} \|_k = +\infty. \quad (1.4) \]

Indeed, Lemma 2(2) implies, for every \( j \in \mathbb{N} \), that

\[ 1 = \langle x_{jk-1}, f_{jk-1} \rangle \leq \| x_{jk-1} \|_k \| f_{jk-1} \|_k. \]

Combining this inequality with property (6) in Lemma 2, we obtain, for every \( j \in \mathbb{N} \), that

\[ k^j \leq \frac{1}{\| f_{jk-1} \|'_k} \leq \| x_{jk-1} \|_k \]

\[ \Rightarrow k^j \leq \frac{1}{\| f_{jk-1} \|'_k} \leq \| x_{jk-1} \|_k. \]
from which (1.4) clearly follows.

To estimate the right hand side of (1.2) we proceed as it follows.

If $1 \leq p \leq k - 1$, then by Lemma 2, (3)-(4), we have that

$$\|x_{jk-1}\|_p \leq \|x_{jk-1}\|_{k-1} \leq \alpha_{k-1} \quad (1.5)$$

and that

$$\|f_{h\varphi(k-1)}\|'_{\varphi(p)} \leq \|f_{h\varphi(k-1)}\|'_{1} \leq 1 \quad (1.6)$$

for all $j, h \in \mathbb{N}$.

If $k \leq p \leq n$, then Lemma 2(6) implies that

$$\|f_{h\varphi(k-1)}\|'_{\varphi(p)} \leq \|f_{h\varphi(k-1)}\|'_{\varphi(k-1)+1} \leq \varphi(k-1)^{-h}$$

for all $h \in \mathbb{N}$. Consequently, there exists an increasing sequence $(h_j)_{j \in \mathbb{N}}$ of positive integers such that

$$\forall j \in \mathbb{N}: \max_{1 \leq p \leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq 1. \quad (1.7)$$

Indeed, set $c_1 = \max_{1 \leq p \leq n} \|x_{jk-1}\|_p$, there exists $h_1 \geq 1$ such that $c_1 \cdot \varphi(k-1)^{-h_1} \leq 1$ and hence, $\max_{1 \leq p \leq n} \|f_{h_1\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq c_1 \cdot \varphi(k-1)^{-h_1} \leq 1$.

Assume we have determined $h_1 < h_2 < \ldots < h_r$ such that (1.7) is satisfied for $j = 1, \ldots, r$. Next, set $c_{r+1} = \max_{1 \leq p \leq n} \|x_{r+k-1}\|_p$, there exists $h_{r+1} > h_r$ such that $c_{r+1} \cdot \varphi(k-1)^{-h_{r+1}} \leq 1$ and hence, $\max_{1 \leq p \leq n} \|f_{h_{r+1}\varphi(k-1)}\|'_{\varphi(p)}\|x_{r+k-1}\|_p \leq c_{r+1} \cdot \varphi(k-1)^{-h_{r+1}} \leq 1$.

Thus, combining inequalities (1.5), (1.6) and (1.7) we obtain that

$$\forall j \in \mathbb{N}: \max_{1 \leq p \leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq \max\{1, \alpha_{k-1}\}. \quad (1.8)$$

Now, from (1.2), (1.3) and (1.8) it follows that

$$\forall j \in \mathbb{N}: \|x_{jk-1}\|_k \leq \frac{1}{\|f_{h_j\varphi(k-1)}\|'_{\varphi(k-1)} C_k \max_{1 \leq p \leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p} \leq \beta^{-1}_{\varphi(k-1)} C_k \max\{1, \alpha_{k-1}\}.$$ 

But, by (1.4), $\|x_{jk-1}\|_k \to +\infty$ as $j \to \infty$. Thus, we obtain a contradiction. \[Q.E.D\]

References

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