Canonical coordinate systems and exponential maps of $n$-loop

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Abstract. This paper is devoted to the study of canonical coordinate systems and the corresponding exponential maps of $n$-ary differentiable loops and to the discussion of their differentiability properties. Canonical coordinate systems can be determined by the canonical normal form of the power series expansion of the $n$-th power map $x \mapsto x \circ x \cdots \circ x$.

Keywords: loops, $n$-ary systems, local Lie groups

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1 Introduction

The canonical coordinate systems of Lie groups are important tools for the investigation of local properties of group manifolds. They can be generalized for non-associative differentiable loops. The first study of the expansion of analytical loop multiplication in a canonical coordinate system using formal power series was given in the paper [1] by M. A. Akivis in 1969, (cf. [6, Chapter 2]). The convergence conditions of power series expansions of loop multiplications were investigated later in [2] (1986). E. N. Kuzmin in [9] (1971) treated the local Lie theory of analytic Moufang loops using power series expansion in canonical coordinate systems and gave a generalization of the classical Campbell-Hausdorff formula. V. V. Goldberg introduced canonical coordinates using power series expansions in local analytic $n$-ary loops, (cf. [6, Chapter 3]).

As it is well-known differentiable groups are automatically (analytic) Lie groups. But in the case of non-associative loop theory the class of $C^k$-differentiable loops contains the class of $C^l$-differentiable loops for any $k < l; k, l = 0, 1, \ldots, \infty$, as a proper subclass (cf. P. T. Nagy – K. Strambach [10] (2002)).

The theory of normal forms of $C^\infty$-differentiable $n$-ary loop multiplications has been investigated in the paper of J-P. Dufour and P. Jean [4], (1985) by the application of S. Sternberg's linearization theorem to the coordinate representation of $n + 1$-webs, which are the differential geometric structures determined...
by the level manifolds of $n$-ary loop multiplications and its inverse operations. J. Kozma in [8] (1987) defined the canonical coordinates of binary $C^\infty$-loops by the linearizing coordinate systems of the square map $x \to x \circ x$. For Lie groups these canonical coordinate systems coincide with the classical systems defined with help of one-parameter subgroups.

Now, we consider a natural generalization of Kozma’s construction to $n$-ary $C^k$-differentiable loops. According to Sternberg’s linearization theorem the linearizing coordinate system of the $n$-th power map $x \to x \circ x \circ \cdots \circ x \circ x$ has the same differentiability property as the $n$-ary loop multiplication map if $k \geq 2$. Hence in the following we will assume that the differentiability class $C^k$ of the investigated $n$-ary loops satisfies $k \geq 2$. Similar construction for canonical coordinate systems was introduced by V. V. Goldberg in [6, Chapter 3], in the case of analytic $n$-loop multiplications using formal power series expansions.

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2 Canonical coordinate systems of $n$-loops

1 Definition. Let $H$ be a differentiable manifold of class $C^k$, let $e \in H$ be a given element and let $m: H^n \to H$, $\delta_i: H^n \to H$ be differentiable maps of class $C^k$, where $i = 1, \ldots, n$. Then $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ is called a $C^k$-differentiable $n$-ary loop (or shortly $n$-loop) with unit element $e$ if the multiplication $m$ and the $i$-th divisions $\delta_i$, $i = 1, \ldots, n$, satisfy the following identities:

1. $m(e, \ldots, e, a, e, \ldots, e^{(i)} = a$, for all $a \in H$, $(1 \leq i \leq n)$, where $x^{(i)}$ means that the $i$-th argument has the value $x$,

2. $m(a_1, a_2, \ldots, a_{i-1}, \delta_i(b, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), a_{i+1}, \ldots, a_n) = b$
   for all $a_i \in H$, $(1 \leq i \leq n)$, $b \in H$,

3. $\delta_i(m(a_1, a_2, \ldots, a_n), a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = a_i$ for all $a_i \in H$,
   $(1 \leq i \leq n), b \in H$.

2 Definition. If $H$ is a differentiable manifold of class $C^k$, $e \in H$ is a given element and $m: H^n \to H$, $\delta_i: H^n \to H$ are differentiable maps of class $C^k$, $i = 1, \ldots, n$, which are defined in a neighbourhood of $e \in H$, then $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ is called a $C^k$-differentiable local $n$-loop with unit element $e$, provided that the multiplication $m$ and the $i$-th divisions $\delta_i$, $i = 1, \ldots, n$ satisfy the following identities:

1. $m(e, \ldots, e, a, e, \ldots, e^{(i)} = a$, for all $a \in H$, $(1 \leq i \leq n)$, where $x^{(i)}$ means that the $i$-th argument has the value $x$,
Canonical coordinate systems and exponential maps of $n$-loop

(2) $m(a_1, a_2, \ldots, a_{i-1}, \delta_i(b, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), a_{i+1}, \ldots, a_n) = b$
for all $a_i \in H$, $(1 \leq i \leq n), b \in H$,

(3) $\delta_i(m(a_1, a_2, \ldots, a_n), a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = a_i$
for all $a_i \in H$,
$(1 \leq i \leq n), b \in H$

in a neighbourhood of $e \in H$.

3 Definition. Let $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ be a $C^k$-differentiable local $n$-loop. A coordinate map $\varphi: U \to \mathbb{R}^q$ of class $C^k$ of the open neighbourhood $U \subset H$ of $e \in H$ into the coordinate space $\mathbb{R}^q$ is called a canonical coordinate system of $\mathcal{H}$ if $\varphi(e) = 0$ and the coordinate function

$$M = \varphi \circ m \circ (\varphi^{-1} \times \cdots \times \varphi^{-1}): \varphi(U) \times \cdots \times \varphi(U) \to \mathbb{R}^q$$

of the multiplication map $m: H^n \to H$ satisfies

$$M(x, x, \ldots, x) = n x$$

for all $x \in \varphi(U)$.

We will need the following assertions in the investigation of canonical coordinate systems:

4 Lemma. Let be $k \geq 2$ and $\phi$ a local $C^k$-diffeomorphism of $\mathbb{R}^q$ keeping $0 \in \mathbb{R}^q$ fixed which is defined in some neighbourhood of $0 \in \mathbb{R}^q$ and let $\phi_*|_{(0)}$ denote the tangent map of $\phi$ at $0 \in \mathbb{R}^q$. We assume that $\phi$ satisfies $\phi_*|_{(0)} = \lambda \text{id}_{\mathbb{R}^q}$ with $\lambda \neq 0, 1, -1$. Then there exists a unique local $C^k$-diffeomorphism $\rho$ of $\mathbb{R}^q$ keeping $0 \in \mathbb{R}^q$ fixed such that $\rho \cdot \phi \cdot \rho^{-1} = \phi_*|_{(0)}$ and $\rho_*|_{(0)} = \text{id}_{\mathbb{R}^q}$.

Proof. The existence of a local $C^k$-diffeomorphism $\rho$ of $\mathbb{R}^q$ satisfying the conditions of the assertion follows from Sternberg’s Linearization Theorem for local contractions (cf. [11]) since either the map $\phi$ or its inverse $\phi^{-1}$ is a local contraction, the minimum and maximum of eigenvalues of its tangent map coincide, $k \geq 2$ and it satisfies the so called resonance condition $\lambda \neq \lambda^m$ for any $m > 1$. The unicity of the map $\rho$ follows from the ideas of the proof of Sternberg’s Theorem, since the difference of two solutions must be a fixed point of a contractive operator on a linear space of differentiable maps. Hence the difference of these solution is 0.

5 Lemma. Let $\kappa$ be a differentiable map of a star shaped neighbourhood $W \subset \mathbb{R}^p$ into $\mathbb{R}^q$ with $\kappa(0) = 0$. If there exists a real number $0 < r < 1$ such that $\kappa(r x) = r \kappa(x)$ holds for all $x \in W$ then $\kappa$ is the restriction of a linear map.

Proof. Since the map $\kappa: W \to \mathbb{R}^p$ is differentiable one can define the continuous map $\omega: W \to \mathbb{R}^p$ satisfying

$$\kappa(x) = \kappa_*|_{(0)}(x) + \|x\|\omega(x), \quad \omega(0) = 0.$$
Hence
\[ \kappa(r \cdot x) = r (\kappa_r |_0 (x) + \|x\| \omega(r \cdot x)) = r \kappa(x) + r (\kappa_r |_0 (x) + \|x\| \omega(x)). \]
It follows \( \omega(x) = \omega(r^m \cdot x) \) for any natural number \( m \in \mathbb{N} \) and hence
\[ \omega(x) = \lim_{m \to \infty} \omega(r^m) = \omega(0) = 0 \]
for all \( x \in W \).

6 Theorem. For any \( C^k \)-differentiable local \( n \)-loop \( H = (H, c, m, \delta_1, \ldots, \delta_n) \) with \( k \geq 2 \) there exists a canonical coordinate system.

If \( (U, \varphi) \) is a canonical coordinate system of \( H \) then for any linear map \( \lambda : \mathbb{R}^q \to \mathbb{R}^q \) the pair \( (U, \lambda \circ \varphi) \) is a canonical coordinate system of \( H \), too.

If \( \varphi : U \to \mathbb{R}^q \) and \( \psi : U \to \mathbb{R}^q \) are the coordinate maps of canonical coordinate systems of \( H \) defined on the same neighbourhood \( U \) then \( \varphi \circ \psi^{-1} \) is the restriction of a linear map \( \mathbb{R}^q \to \mathbb{R}^q \).

Proof. Let \( (U, \varphi) \) be a coordinate system of \( H \), let \( M \) be the coordinate function of the local \( n \)-loop multiplication \( m \) with respect to \( (U, \varphi) \). Now, we introduce the map \( G : \varphi(U) \to \mathbb{R}^q \) defined by \( G(x) = M(x, x, \ldots, x) \). Clearly one has \( G(0) = 0 \). Since \( M(0, \ldots, 0, x, 0, \ldots, 0) = x \) the tangent map \( G_* |_0 : \mathbb{R}^q \to \mathbb{R}^q \) of \( G \) at the point \( 0 \) satisfies \( G_* |_0 = \lambda \id_{\mathbb{R}^q} \). The map \( G \) is of class \( C^k \) in a neighborhood of \( 0 \) and hence it has an inverse map in a neighborhood of \( 0 \) of the same class \( C^k \). We can apply Lemma 4 for \( G^{-1} \). It follows that there exists a local \( C^k \)-diffeomorphism \( \rho \) keeping \( 0 \in \mathbb{R}^q \) fixed such that \( (\rho \circ G \circ \rho^{-1})_* |_0 = \rho \circ G \circ \rho^{-1} \). We consider the composed map \( \varphi = \rho \circ \varphi \) as the coordinate map of a new coordinate system \( (U, \varphi) \) with a suitable neighborhood \( U \). The coordinate function of the multiplication map \( m : H^n \to H \) satisfies \( M = \rho \circ \bar{M} \circ \rho^{-1} \). Let \( Q \) be the following function
\[ Q : x \mapsto Q(x) = (x, x, \ldots, x) : \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^q \times \cdots \times \mathbb{R}^q. \]
Then we have the equation
\[ G = M \circ Q = (\rho \circ \bar{M} \circ \rho^{-1})(\rho \circ Q \circ \rho^{-1}) = \rho \circ \bar{G} \circ \rho^{-1} = (\rho \circ \bar{G} \circ \rho^{-1})_* |_0 = \rho \id_{\mathbb{R}^q}. \]
Hence \( (U, \varphi) \) is a canonical coordinate system of \( H \).

For a canonical coordinate system \( (U, \varphi) \) of the local \( n \)-loop \( H \) the coordinate function
\[ M = \varphi \circ m \circ (\varphi^{-1} \times \cdots \times \varphi^{-1}) : \varphi(U) \times \cdots \times \varphi(U) \to \mathbb{R}^q \]
of the multiplication map \( m : H^n \to H \) satisfies \( M(x, x, \ldots, x) = nx \) for all \( x \in \varphi(U) \). Hence for arbitrary linear map \( \lambda : \mathbb{R}^n \to \mathbb{R}^n \) one has
\[ \lambda \circ M(\lambda^{-1} y, \ldots, \lambda^{-1} y) = \lambda(\lambda^{-1} y) = n y, \quad y \in \lambda \circ \varphi(U). \]
It follows that \((U, \psi = \lambda \circ \varphi)\) is also a canonical coordinate system of \(\mathcal{H}\).

Let \((U, \varphi)\) and \((U, \psi)\) be canonical coordinate systems of \(\mathcal{H}\) given on the same neighbourhood \(U\) and let \(M_\varphi\) and \(M_\psi\) be the coordinate functions of the multiplication map \(m: H^n \to H\). We denote \(\kappa = \varphi \circ \psi^{-1}: \psi(U) \to \varphi(U)\). For all \(x \in \varphi(U)\) and \(y \in \psi(U)\) we have

\[M_\varphi(x, x, \ldots, x) = nx \quad \text{and} \quad M_\psi(y, y, \ldots, y) = ny.\]

Since

\[M_\varphi(\kappa(y), \kappa(y), \ldots, \kappa(y)) = \kappa(M_\psi(y, y, \ldots, y))\]

we obtain \(n \kappa(y) = \kappa(n y)\). Putting \(z = ny\) we get \(\kappa(r z) = r \kappa(z)\) for all \(z \in \psi(U)\), where \(r = \frac{1}{n}\). It follows by Lemma 5 that the map \(\kappa = \psi \circ \varphi^{-1}\) is the restriction of a linear map.

\[\text{QED}\]

7 Example. The local non-associative loop-multiplication \(f(x, y) = x + y + x^2 y(x - y)\) is defined in a canonical coordinate system.

3 Exponential map

There are different natural possibilities for the definition of the exponential map \(W \to H\) with \(0 \in W \subset T_e H\) of \(C^k\)-differentiable local \(n\)-loops. One of them is analogous to the usual construction in Lie group theory, namely the map \(\exp\) could be determined by the integral curves of vector fields defined by the \(i\)-th translations of tangent vectors at the unit element of the \(n\)-loop. In binary Lie groups these curves are 1-parameter subgroups, but for smooth loops it is not always the case (cf. J. Kozma [8]). An other disadvantage of such construction is that one can expect only \(C^{k-1}\)-differentiability of the the map \(W \to H\) with \(0 \in W \subset T_e H\) which is determined by integral curves of \(C^{k-1}\)-differentiable vector fields defined by the \(i\)-th translations of tangent vectors.

An alternative natural possibility for the definition of the exponential map is given by using the construction of canonical coordinate systems studied in the previous section.

8 Theorem. Let \(\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)\) be a \(C^k\)-differentiable local \(n\)-loop with \(k \geq 2\). There exists a unique local \(C^k\)-diffeomorphism \(\exp: W \to H\), where \(W\) is a neighbourhood of \(0 \in T_e H\), such that the following conditions hold:

(i) \(\exp(0) = e\) and \(\exp(n x) = m(\exp(x), \ldots, \exp(x))\),

(ii) \(\exp|_0 = id_{T_e H}\).

Proof. Let \(\varphi: U \to \mathbb{R}^q\) be the coordinate map of a canonical coordinate system \((U, \varphi)\) of the local \(n\)-loop \(\mathcal{H}\). According to Theorem 6 \((U, \varphi|_0^{-1} \circ \varphi)\) is also
a canonical coordinate system of $\mathcal{H}$ where the vector space $T_e H$ is the coordinate space and $\varphi_*|_0^{-1} \circ \varphi: U \to T_e H$ is the coordinate map. Let $W \subset \varphi_*|_0^{-1} \circ \varphi(U)$ be a neighbourhood of $0 \in T_e H$. Then the coordinate function

$$M = \varphi_*|_0^{-1} \circ \varphi \circ m \circ ((\varphi_*|_0^{-1} \circ \varphi)^{-1} \times \cdots \times (\varphi_*|_0^{-1} \circ \varphi)^{-1}): W \times \cdots \times W \to T_e H$$

of the multiplication map $m: H^n \to H$ satisfies $M(x, \ldots, x) = nx$, or equivalently

$$m(\varphi^{-1} \circ \varphi_*|_0(x), \ldots, \varphi^{-1} \circ \varphi_*|_0(x)) = \varphi^{-1} \circ \varphi_*|_0(nx)$$

for any $x \in W$. Moreover one has $(\varphi^{-1} \circ \varphi_*|_0)_*|_0 = \text{id}_{T_e H}$. Hence we can define $\exp = \varphi^{-1} \circ \varphi_*|_0$ and this map satisfies the conditions given in the assertion.

Let us assume that the map $\widetilde{\exp}: W \to H$ fulfills the conditions (i) and (ii). Then $(\exp(W), \exp^{-1})$ is a canonical coordinate system of the $n$-loop $\mathcal{H}$ and according to the previous theorem the map $\widetilde{\exp}^{-1} \circ \exp: W \to T_e H$ is the restriction of a linear map $\alpha: T_e H \to T_e H$. Since both of the maps $\exp$ and $\exp$ satisfy the condition (ii) the linear map $\alpha: T_e H \to T_e H$ must be the identity map. Hence $\widetilde{\exp} = \exp: W \to H$ which proves that the map $\exp: W \to H$ is determined uniquely.

9 Theorem. Let $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ and $\mathcal{H'} = (H', e', m', \delta'_1, \ldots, \delta'_n)$ be $C^k$-differentiable local $n$-loops and let $\exp: W \to H$, $\exp': W' \to H'$ be the corresponding exponential maps, where $W \subset T_e H$ and $W' \subset T_e H'$.

If $\alpha: \mathcal{H} \to \mathcal{H'}$ is a continuous local homomorphism then the composed map $\exp'^{-1} \circ \alpha \circ \exp: W \to T_e H'$ is locally linear.

Proof. Let us consider the $C^k$-differentiable binary local loops $\mathcal{H}$ and $\mathcal{H}'$ which are determined by the multiplication and division maps of $\mathcal{H}$ and $\mathcal{H}'$ in such a way that in the multiplication and division functions the $j$-th variable ($j \geq 3$) is replaced by the identity element $e \in H$ and $e' \in H'$ respectively. The map $\alpha: H \to H'$ is clearly a continuous local loop homomorphism. According to the result of R. Bödi and L. Kramer [3] the map $\alpha: H \to H'$ is $C^k$-differentiable. Hence according to Lemma 5 the identity

$$\exp'^{-1} \circ \alpha \circ \exp(nx) = n \exp'^{-1} \circ \alpha \circ \exp(x),$$

or equivalently

$$\exp'^{-1} \circ \alpha \circ \exp(r y) = r \exp'^{-1} \circ \alpha \circ \exp(y)$$

with $y = nx$ and $r = \frac{1}{n}$, implies the assertion.
References


