Common Fixed Point Theorems in Uniform Spaces

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Abstract. In this paper we prove some fixed point theorems for weakly compatible mappings with the notation of $A$-distance and $E$-distance in uniform space.

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1 Introduction and Preliminaries

The concept of weakly compatible is defined by Jungck and Rhoades [3]. In this paper we take weakly compatible to prove common fixed point theorems. Recently, Aamri and Moutawakil [1] introduce the concept of $A$-distance and $E$-distance in uniform space. With the help of these $A$-distance and $E$-distance we prove common fixed point for weakly compatible.

Definition 1. Two self maps $T$ and $S$ of a metric space $X$ are said to be weakly compatible if they commute at there coincidence points, i.e. if $Tu = Su$ for $u$ in $X$, then $TSu = STu$.

By Bourbaki [2], we call uniform space $(X, \vartheta)$ a non empty set $X$ endowed of an uniformity $\vartheta$, the latter being a special kind of filter on $X \times X$, for all whose elements contain the diagonal $\Delta = \{(x,x) | x \in X\}$. If $V \in \vartheta$ and $(x,y) \in V$, $(y,x) \in V$, $x$ and $y$ are said to be $V$-close and a sequence $(x^n)$ in $X$ is a Cauchy sequence for $\vartheta$ if for any $V \in \vartheta$, there exists $N \geq 1$ such that $x^n$ and $x^m$ are $V$-close for $n,m \geq N$. An uniformly $\vartheta$ defines a unique topology $T(\vartheta)$ on $X$ for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X | (x,y) \in V\}$ when $V$ runs over $\vartheta$.

A uniform space $(X, \vartheta)$ is said to be Hausdorff if and only if the intersection of all $V \in \vartheta$ reduces to the diagonal $\Delta$ of $X$ i.e. if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y, x) | (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x,y) \in W$ then $x$ and $y$ are both $W$ and $V$-close, then for our purpose, we assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space $(X, \vartheta)$, they always refer to the topological space $(X, T(\vartheta))$.

Definition 2. Let $(X, \vartheta)$ be a uniform space. A function $p : X \times X \rightarrow \mathbb{R}^+$ is said to be an $A$-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$.

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Definition 3. Let \((X, \vartheta)\) be uniform space. A function \(p : X \times X \rightarrow \mathbb{R}^+\) is said to be an \(E\)-distance if \(p\) is an \(A\)-distance and \(p(x, y) \leq p(x, z) + p(z, y)\), for every \(x, y, z \in X\).

Definition 4. Let \((X, \vartheta)\) be uniform space and \(p\) be an \(A\)-distance on \(X\).
(I) \(X\) in \(S\) complete if for every \(p\)-Cauchy sequences \(\{x_n\}\) there exists \(x \in X\) such that \(\lim x_n = x\).
(II) \(X\) is \(p\)-Cauchy complete if for every \(p\)-Cauchy sequences \(\{x_n\}\) there exists \(x \in X\) such that \(\lim x_n = x\) with respect to \(\tau(\vartheta)\).
(III) \(f : X \rightarrow X\) is \(p\)-continuous if \(\lim x_n = x\) implies \(\lim f(x_n) = f(x)\).
(IV) \(f : X \rightarrow X\) is \(T(\vartheta)\)-continuous if \(\lim x_n = x\) with respect to \(T(\vartheta)\) implies \(\lim f(x_n) = f(x)\) with respect to \(\tau(\vartheta)\).
(V) \(X\) is said to be \(p\)-bounded if \(\delta_p(X) = \sup \{p(x, y)| x, y \in X\} < \infty\).

Lemma 1. Let \((X, \vartheta)\) be uniform space and \(p\) be an \(A\)-distance on \(X\). Let \(\{x_n\}, \{y_n\}\) be arbitrary sequences in \(X\) and \(\{\alpha_n\}, \{\beta_n\}\) be sequences in \(\mathbb{R}^+\) and converging to 0. Then, for \(x, y, z \in X\), the following holds
(a) If \(p(x_n, y) \leq \alpha_n\) and \(p(x_n, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(y = z\). In particular, if \(p(x, y) = 0\) and \(p(x, z) = 0\), then \(y = z\).
(b) If \(p(x, y) \leq \alpha_n\) and \(p(x, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(\{y_n\}\) converges to \(z\).
(c) If \(p(x_n, x_m) \leq \alpha_n\) for all \(m > n\), then \(\{x_n\}\) is a Cauchy sequences in \((X, \vartheta)\).

Let \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be continuous and satisfying the conditions
(i) \(\psi\) is nondecreasing on \(\mathbb{R}^+\).
(ii) \(0 < \psi(t) < t\), for each \(t \in (0, \infty)\).

Theorem 1. Let \((X, \vartheta)\) be a Hausdorff uniform space and \(p\) be an \(A\)-distance on \(X\). Let \(f\) and \(g\) are two weakly compatible defined on \(X\) such that
(I) \(f(X) \subseteq g(X)\)
(II) \[p(f(x), f(y)) \leq \psi\left[\max\{p(g(x), g(y))\}, \frac{1}{2}[p(g(x), f(x)) + p(g(y), f(y))], \frac{1}{2}[p(g(y), f(y)) + p(g(u), f(z))]\]\[= \psi\left[\max\{p(z, f(z))\}, \frac{1}{2}[p(g(u), f(g(u))) + p(g(f(u), f(z)))], \frac{1}{2}[p(g(f(u), f(g(u))) + p(g(u), f(z))]]\] If \(f(X)\) or \(g(X)\) is a \(S\)-complete subspaces of \(X\), then \(f\) and \(g\) have a common fixed point.

Proof. Let \(x_0 \in X\) and choose \(x_1 \in X\) such that \(f(x_0) = g(x_1)\). Choose \(x_2 \in X\) such that \(f(x_1) = g(x_2)\). In general \(f(x_n) = g(x_{n+1})\). Then let the sequence \(y_{n+1}\) such that
\[y_{n+1} = f(x_n) = g(x_{n+1})\ldots\]
Now there arise two cases:
Case 1 If \(y_n = y_{n+p}\) for \(n \in \mathbb{N}\), we have \(z = y_n = g(x_n) = f(x_n) = g(x_{n+1}) = y_{n+1}\). Now taking \(u = x_n\), then \(f(u) = g(u)\) and by weakly compatibility \(g(u) = g(f(u))\). Now
\[
d(f(z), z) = d(f(z), f(u)) \\ \leq \psi\left[\max\{p(g(z), g(u))\}, \frac{1}{2}[p(g(z), f(z)) + p(g(z), f(u))], \frac{1}{2}[p(g(u), f(u)) + p(g(u), f(z))]\}\]
\[
\leq \psi\left[\max\{p(z, f(z))\}, \frac{1}{2}[p(g(u), f(g(u))) + p(g(f(u), f(z)))], \frac{1}{2}[p(g(f(u), f(g(u))) + p(g(u), f(z)))]\]\n\[
\leq \psi\left(p(z, f(z)) < p(z, f(z))\right)\]
Hence $g_0$, lemma 1(a) gives $p$ which is a contradiction. Thus $p$ which is a contradiction. Thus $p$ for this $z \in g(X)$ there exist $\omega$ in $X$ such that $z = g(\omega)$. Now by condition (II) of theorem we have

$$p(f(\omega), g(\omega)) \leq p(f(\omega), f(x_n)) + p(f(x_n), g(\omega))$$

$$\leq \psi(\max\{p(g(\omega), g(x_n)), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), f(x_n))], 1/2[p(g(x_n), f(x_n)) + p(g(x_n), f(\omega))]) + p(f(x_n), g(\omega))$$

$$\leq \psi(\max\{p(z, z), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), g(\omega))], 1/2[p(x, z) + p(g(\omega), f(\omega))]) + p(z, z), \text{ as } n \to \infty$$

$$\leq \psi(p(g(\omega), f(\omega))) \leq p(g(\omega), f(\omega))$$

It implies that $f_\omega = g_\omega$. The assumption that $f$ and $g$ are weakly compatible implies $f(\omega) = g(\omega)$. Also $f(\omega) = g(\omega) = g(\omega) = g(\omega)$. Suppose that $p(f(\omega), f(\omega)) \neq 0$. From (II), it follows

$$p(f(\omega), f(\omega)) \leq \psi \max\{[p(g(\omega), g(\omega)), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), f(\omega))], 1/2[p(g(f(\omega), f(\omega)) + p(g(\omega), f(\omega)))]\}$$

$$\leq \psi(p(f(\omega), f(\omega))) < p(f(\omega), f(\omega))$$

which is a contradiction. Thus $p(f(\omega), f(\omega)) = 0$.

Suppose that $p(f(\omega), f(\omega)) \neq 0$, then also by (II)

$$p(f(\omega), f(\omega)) \leq \psi \max\{[p(g(\omega), g(\omega)), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), f(\omega))], 1/2[p(g(f(\omega), f(\omega)) + p(g(\omega), f(\omega)))]\}$$

$$\leq \psi(p(f(\omega), f(\omega))) < p(f(\omega), f(\omega))$$

which is a contradiction. Thus $p(f(\omega), f(\omega)) = 0$. Since $p(f(\omega), f(\omega)) = 0$ and $p(f(\omega), f(\omega)) = 0$, lemma 1(a) gives $f(\omega) = f(\omega)$.

Hence $g(\omega) = f(\omega) = f(\omega)$ and $z = f(\omega)$ is common fixed point of $f$ and $g$.

\[ \therefore \]

**Theorem 2.** Let $(X, \varrho)$ be a Hausdorff uniform space and $p$ be an $E$-distance on $X$. Let $f$ and $g$ are two weakly compatible defined on $X$ such that

(I) $f(X) \subseteq g(X)$

(II) $p(f(x), f(y))) \leq \psi(\max\{p(g(x), g(y))\}, 1/2[p(g(x), f(x)) + p(g(x), f(y))], 1/2[p(g(y), f(y)) + p(g(y), f(x))])$

If $f(X)$ or $g(X)$ is a $S$ complete subspaces of $X$, then $f$ and $g$ have a unique common fixed point.

**Proof.** Since $E$ distance in $A$ distance therefore $f$ and $g$ have a common point. Suppose $z_1$ and $z_2$ are common fixed points of $f$ and $g$, then $f(z_1) = g(z_1) = z_1$ and $f(z_2) = g(z_2) = z_2$. 


If $p(z_1, z_2) \neq 0$, then by (II)
\[
\begin{align*}
p(z_1, z_2) &= p(f(z_1), f(z_2)) \leq \psi \max[p(g(z_1), g(z_2)), \\
&\quad 1/2[p(g(z_1), f(z_1)) + p(g(z_1), f(z_2))], \\
&\quad 1/2[p(g(z_2), f(z_2)) + p(g(z_2), f(z_1))]]
\end{align*}
\]
Consequently by (p2) we have $p(z_1, z_1) \leq p(z_1, z_2) + p(z_2, z_1) \Rightarrow p(z_1, z_2) = 0$. Now, we have $p(z_1, z_1) = 0$ and $p(z - 1, z_2) = 0$ therefore $z_1 = z_2$.

\textbf{Theorem 3.} Let $(X, \varnothing)$ be a Hausdorff uniform space and $p$ be an $A$-distance on $X$. Let $f$ and $g$ are two weakly compatible defined on $X$ such that
(I) $f^*(X) \subseteq g^*(X)$
(II) $p(f^*(x), f^*(y))) \leq \psi \max[p(g^*(x), g^*(y)))], \\
&\quad 1/2[p(g^*(x), f^*(x)) + p(g^*(x), f^*(y))], 1/2[p(g^*(y), f^*(y)) + p(g^*(y), f^*(x))]]
\]
where $r$ and $s$ are positive integers. If $f(X)$ or $g(X)$ is a $S$ complete subspaces of $X$, then $f$ and $g$ have a common fixed point.

\textbf{Proof.} Same as theorem 1.

\textbf{Theorem 4.} Let $(X, \varnothing)$ be a Hausdorff uniform space and $p$ be an $A$-distance on $X$. Let $f$ and $g$ are two weakly compatible defined on $X$ such that
(I) $f^*(X) \subseteq g^*(X)$
(II) $p(f^*(x), f^*(y))) \leq \psi \max[p(g^*(x), g^*(y)))], \\
&\quad 1/2[p(g^*(x), f^*(x)) + p(g^*(x), f^*(y))], 1/2[p(g^*(y), f^*(y)) + p(g^*(y), f^*(x))]]
\]
where $r$ and $s$ are positive integers. If $f(X)$ or $g(X)$ is a $S$ complete subspaces of $X$, then $f$ and $g$ have a unique common fixed point.

\textbf{Proof.} Same as theorem 2.

\textbf{Example.} Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Self mappings $f$ and $g$ are defined as $f_x = x^2$ if $x \in [0, 1/2]$ & $= 1/2$ if $x \in [1/2, 1]$ and $g_x = 0$ if $x \in [0, 1/2]$ & $= x$ if $x \in [1/2, 1]$. Now, consider the functions $p$ and $\psi$ as: $\psi(x) = x^2$ and $p(x, y) = 0$ if $y \in [0, 1/2]$ & $= y$ if $y \in [1/2, 1]$. All conditions of Theorem 2 are satisfied and 1/2 is common point of $f$ and $g$.

\textbf{References}