On Harmonious Coloring of Middle Graph of $C(C_n)$, $C(K_{1,n})$ and $C(P_n)$

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Abstract. In this paper, we present the structural properties of middle graph of central graph of cycles $C_n$, star graphs $K_{1,n}$ and paths $P_n$ denoted by $M(C(C_n))$, $M(C(K_{1,n}))$ and $M(C(P_n))$ respectively. We mainly have our discussion on the harmonious chromatic number of $M(C(C_n))$, $M(C(K_{1,n}))$ and $M(C(P_n))$.

Keywords: central graph, middle graph, harmonious coloring

MSC 2000 classification: 05C75, 05C15

Introduction

For a given graph $G = (V, E)$ we do an operation on $G$, by subdividing each edge exactly once and joining all the non-adjacent vertices of $G$. The graph obtained by this process is called central graph [1, 29, 31–35] of $G$ denoted by $C(G)$.

The middle graph [6] of $G$, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of $G$ or one is a vertex and the other is an edge incident with it and it is denoted by $M(G)$. Additional graph theory terminology used in this paper can be seen in [3,15].

A harmonious coloring [2, 7, 8, 10–14, 16–28, 36, 37, 39] of a simple graph $G$ is proper vertex coloring such that each pair of colors appears together on at most one edge. Formally, a harmonious coloring [4, 5, 9] is a function $c$ from a color set $C$ to the set $V(G)$ of vertices of $G$ such that for any edge $e$ of $G$, with end points $x, y$ say $c(x) \neq c(y)$, and for any pair of distinct edges $e, e'$ with end points $x, y$ and $x', y'$ respectively, then $\{c(x), c(y)\} \neq \{c(x'), c(y')\}$. The
harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

The first paper on harmonious graph coloring was published in 1982 by Frank Harary and M. J. Plantholt [16]. However, the proper definition of this notion is due to J. E. Hopcroft and M. S. Krishnamoorthy [18] in 1983. S. Lee and John Mitchum [22], published a paper consisting of the upper bound for the harmonious chromatic number of graphs in 1987.


Zhikang Lu [40], in 1995, published a paper on the harmonious chromatic number of a complete 4-ary tree. Also K. J. Edwards [7] worked and gave results on the harmonious chromatic number of almost all trees. In the next year (1996) he investigated on the harmonious chromatic number of bounded degree trees [8].


In 1997, K. J. Edwards, [9] continued his work on the harmonious chromatic number of bounded degree graphs, and also published papers relating harmonious coloring and achromatic number.

Zhikang Lu [41, 42] published a paper on the exact value of the harmonious chromatic number of a complete binary tree in 1997 and trinary tree in 1998.

In 1998, K. J. Edwards [10] published a work emphasizing a new upper bound for the harmonious chromatic number, and in 1999 on the harmonious
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A work on the harmonious chromatic number of $P(\alpha, K_n)$, $P(\alpha, K_{1,n})$ and $P(\alpha, K_{m,n})$, was published by M. F. Mammana \[24\] in 2003.

D. Campbell and K. J. Edwards \[4\] again gave a new lower bound for the harmonious chromatic number in 2004.

In 2007, Vernold Vivin. J in his Ph.D thesis \[35\] had a detailed study on the harmonious chromatic number of central graph families.

Vernold Vivin. J et al. \[32\] published a paper on the harmonious coloring of central graph in 2008. For some background on this topic, see \[29–35\].

1 Structural properties of $M(C(C_n))$

In $M(C(C_n))$, there are $n$ vertices of degree 2, $n$ vertices of degree $(n-1)$, $2n$ vertices of degree $(n+1)$ and $\frac{n^2-3n}{2}$ vertices of degree 2$(n-1)$.

Therefore

- The number of vertices $p_{M(C(C_n))} = \frac{n^2 + 5n}{2}$.
- The number of edges, $q_{M(C(C_n))} = \frac{n^3 - n^2 + 6n}{2}$.
- $\Delta = 2(n-1)$.

2 Harmonious coloring of $M(C(C_n))$

1 Theorem. The harmonious chromatic number of middle graph of central graph of cycles $C_n$, $\chi_H(M(C(C_n))) = \left\lceil \frac{n^2 + 5}{2} \right\rceil$.

Proof. Let $V(C(C_n)) = \{v_1, v_2, \ldots, v_n\}$. On the process of centralization of $C_n$, let $u_i$ be the vertex of subdivision of the edge $v_iv_{i+1}(1 \leq i \leq n)$. Also let $u_iv_i = e_i(1 \leq i \leq n)$ and $u_iv_{i+1} = e'_i(1 \leq i \leq n-1)$ and $u_nv_1 = e'_n$. Also for non-adjacent vertices $v_i$ and $v_j$ of $C_n$, let $e_{ij} = v_iv_j$. Since we consider only undirected graphs $e_{ij} = e_{ji}$.

Middle graph of $C(C_n)$ has the vertex set $V(C(C_n)) \cup E(C(C_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n, e_{13}, e_{15}, \ldots, e_{24}, e_{25}, \ldots\}$. Each $v_i$ is incident with the edges $e_i, e'_{i-1}, e_{ij}(i \neq j)$ and $(2 \leq i \leq n)$. Also $v_1$ is incident with $e_1, e'_n, e_{13}, e_{15}, \ldots, e_{1(n-1)}$. i.e., Total number of incident edges with
Figure 1. Central Graph of Cycles $C_n$

$v_i$ is $(n - 1)v \ (i = 1, 2, \ldots, n)$. By the definition of middle graph the edges incident with $v_i$ together with the vertex $v_i$ induces a clique of $n$ vertices in $M(C(C_n)) \ (1 \leq i \leq n)$.

Let $K_n^{(i)}$ be the cliques in $M(C(C_n))(i = 1, 2, \ldots, n)$. Since $e_{ij} = e_{ji}$, each $K_n^{(i)}$ shares their exactly $(n - 3)$ vertices with the remaining cliques. Therefore in each clique, the harmonious coloring is performed by distinct $n$ colors. $K_n^{(1)}$ is assigned $n$ colors, whereas since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n - 1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$.

Since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n - 2)$ colors and so on. Now we turn our proof in the direction of induction.

**Case (i)**

If $n$ is odd, for $n = 3, C(C_3)$ is $C_6$ and for its middle graph the harmonious chromatic number is $n^2 + 5 = 7$. Therefore the result is trivial for $n = 3$. Now we assume that the result is valid for $C_n$, when $n$ is odd i.e., $\chi_H(M(C(C_n))) = \frac{n^2 + 5}{2}$. Now consider $C_{n+2}$ by introducing two new vertices $v_{n+1}$ and $v_{n+2}$. Consider the incident edges of $v_{n+1}$ and $v_{n+2}$ in $C(C_{n+2})$. These edges together with the vertices $v_{n+1}$ and $v_{n+2}$ induces two more cliques of order $n + 2$ in $M(C(C_n))$. The vertices $v_{n+1}, v_{n+2}, e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by $(n - 3) + (n - 3) = (2n - 6)$ colors. Therefore $\chi_H(M(C(C_{n+2}))) = \frac{n^2 + 5}{2} + 2n - 6 + 8 = \frac{n^2 + 5}{2} + (2n+2)$. Hence $\chi_H(M(C(C_{n+2}))) = \frac{(n+2)^2+5}{2}$. Therefore by induction hypothesis $\chi_H(M(C(C_n))) = \frac{n^2 + 5}{2}$, if $n$ is odd.

**Case (ii)**
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If \( n \) is even, we prove that, \( \chi_H(M(C(C_n))) = \frac{n^2+6}{2} \). For \( n = 4 \), the harmonious chromatic number of the middle graph of \( C(C_4) \) is \( \frac{n^2+6}{2} = 11 \). Now we assume that the result is valid for \( n \), when \( n \) is even, i.e., \( \chi_H(M(C(C_n))) = \frac{n^2+6}{2} \).

Now consider \( C_{n+2} \) by introducing two new vertices \( v_{n+1} \) and \( v_{n+2} \). Consider the incident edges of \( v_{n+1} \) and \( v_{n+2} \) in \( C(C_{n+2}) \). These edges together with the vertices \( v_{n+1} \) and \( v_{n+2} \) induces two more cliques of order \( n+2 \) in \( M(C(C_n)) \).

The vertices \( v_{n+1}, v_{n+2}, e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2} \) are assigned by 8 colors and the cliques \( K_{n+1} \) and \( K_{n+2} \) are assigned by \( (n-3) + (n-3) = (2n-6) \) colors. Therefore \( \chi_H(M(C(C_{n+2}))) = \frac{n^2+6}{2} + 2n - 6 + 8 = \frac{n^2+6}{2} + 2n + 2 \). Hence \( \chi_H(M(C(C_{n+2}))) = \frac{(n+2)^2+6}{2} \). Therefore by induction hypothesis \( \chi_H(M(C(C_n))) = \frac{n^2+6}{2} \), if \( n \) is even. If \( n \) is odd then \( n^2 = \left\lceil \frac{n^2+5}{2} \right\rceil \), if \( n \) is even then \( \frac{n^2+6}{2} = \left\lceil \frac{n^2+5}{2} \right\rceil \). Therefore in both the cases, \( \chi_H(M(C(C_n))) = \left\lceil \frac{n^2+5}{2} \right\rceil \).

\[ \text{QED} \]

3 Structural properties of \( M(C(K_{1,n})) \) and \( M(C(P_n)) \)

In \( M(C(K_{1,n})) \) and \( M(C(P_n)) \) there are \( n \) vertices of degree 2, \( n+1 \) vertices of degree \( n \), \( 2n \) vertices of degree \( (n+2) \) and \( \frac{n^2-n}{2} \) vertices having degree \( 2n \). Therefore

- The number of vertices, \( p_{M(C(P_n))} = \frac{n^2 + 7n + 2}{2} \).
- The number of edges, \( q_{M(C(P_n))} = \frac{n^3 + 2n^2 + 7n}{2} \).
\[ \chi_H(M(C_5)) = \left\lceil \frac{5^2 + 5}{2} \right\rceil = 15. \]

- \( \Delta = 2n \).
- We infer that \( M(K_{1,n}) \) and \( M(P_n) \) are isomorphic graphs.

4 Harmonious coloring of \( M(K_{1,n}) \) and \( M(P_n) \)

The harmonious chromatic number of \( M(K_{1,n}) \) and \( M(P_n) \) are equal since they are isomorphic graphs.

2 Theorem. The harmonious chromatic number of middle graph of central graph of star graphs \( K_{1,n} \),

\[ \chi_H(M(C(K_{1,n}))) = \left\lceil \frac{n^2 + 2n + 5}{2} \right\rceil. \]

Proof. Let \( V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\} \) where \( \deg v = n \). On the process of centralization of \( K_{1,n} \), let us denote the vertices of subdivision by \( u_1, u_2, \ldots, u_n \).

i.e., \( vu_i \) is subdivided by \( u_i(1 \leq i \leq n) \). Let \( e_i = v_iu_i \) and \( e'_i = vu_i(1 \leq i \leq n) \).

i.e., \( V(C(K_{1,n})) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\} \), \( E(C(K_{1,n})) = \{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n}, e_{12}, e_{13}, \ldots, e_{1n}, e_{23}, \ldots, e_{2n}, \ldots, e_{(n-1)n}\} \). By the definition of middle graph, \( V(M(C(K_{1,n}))) = V(C(K_{1,n})) \cup E(C(K_{1,n})) \). The structure is described below. The vertices \( v_1, e_1, e_2, \ldots, e_n \) induces a clique of order \( (n + 1) \) in its middle graph. The vertices \( u_i \) is adjacent to \( e_i \) and \( e'_i(1 \leq i \leq n) \). Let \( S_i = \{e_{ij} : j = 1, 2, \ldots, i - 1, i + 1, \ldots, n\}, (1 \leq i \leq n) \). Clearly \( S_i \cap S_j = \{e_{ij}\} \) if \( i \neq j \) and let \( S^{(n)} = \bigcup_{i=1}^{n} S_i \). Clearly \( |S^{(n)}| = \binom{n}{2} \). Now the vertices \( v_i \) and \( e'_i \) together with vertices of \( S_i \) induces a clique of order \( (n + 1), (1 \leq i \leq n) \).

Therefore \( M(C(K_{1,n})) \) contains \( n + 1 \) clique of order \( (n + 1) \). Now we prove that the harmonious chromatic number of this graph is \( \left\lceil \frac{n^2 + 2n + 5}{2} \right\rceil \) by induction.
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Figure 4. Central Graph of Star Graphs $K_{1,n}$

method.

Case (i)

If $n$ is odd

We prove $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+5}{2}$. If $n = 3$, then $C(K_{1,3})$ has 7 vertices and its middle graph is shown in figure 6.

Clearly $\chi_H(M(C(K_{1,3}))) = \frac{n^2+2n+5}{2} = 10$. Therefore the result is true for $n = 3$. Assume that the result is true for any integer $n$ and we prove the same for $n + 2$. i.e., $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+5}{2}$. Let $v_{n+1}, v_{n+2}$ be two non-adjacent vertices introduced in $K_{1,n}$ which are adjacent to $v$. Let $u_{n+1}$ and $u_{n+2}$ be the vertices of subdivision in its centralization. Clearly by the structure given in figure 5, the middle graph $C(K_{1,n+2})$ has the following structural property. (i) There are $(n + 3)$ cliques $K_{n+3}^{(1)}, K_{n+3}^{(2)}, \ldots, K_{n+3}^{(n+3)}$ of order $(n + 3)$. (ii) The vertices $u_i$ is adjacent with $e_i$ and $e'_i$ $(1 \leq i \leq n + 2)$. (iii) Each $K_{n+3}^{(i)}$ has exactly one vertex common with $K_{n+3}^{(j)}$ where $(2 \leq i \leq n + 3), (2 \leq j \leq n + 3)$ and $i \neq j$. By induction hypothesis the minimum number of colors for the harmonious coloring in $C(K_{1,n})$ is $\frac{n^2+2n+5}{2}$. By the above said structure of $M(C(K_{1,n})), |S^{(n)}| = |S_1 \cup S_2 \cup \cdots \cup S_n| = \binom{n+2}{2} = \frac{n(n+1)}{2}$. Also in $M(C(K_{1,n+2})), |S^{(n+2)}| = |S_1 \cup S_2 \cup \cdots \cup S_{n+2}| = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$, also we have new vertices $e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}, v_{n+1}, v_{n+2}$ in $M(C(K_{1,n+2}))$. Therefore the total number of added vertices in $M(C(K_{1,n+2})) = \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} + 8 = 2n + 1 + 8 = 2n + 9$. Now we find the minimal harmonious coloring in $M(C(K_{1,n+2}))$ as below. By the induction hypothesis $C(K_{1,n})$ has harmonious coloring with the minimum number of $\frac{n^2+2n+5}{2}$ colors. With this same colors assigned to the vertices of $M(C(K_{1,n+2}))$, we assign some new colors to the remaining vertices as below. The vertices $u_{n+1}$ and $u_{n+2}$ are assigned the same
color as in $u_i (1 \leq i \leq n)$. Then all the $e_{n+1}$ vertices of $S^{(n+2)}$ are assigned $(2n + 1)$ new colors. Also, among the remaining vertices $e_{n+1}, e_{n+2}, e'_{n+1}, e'_{n+2}, v_{n+1}$ and $v_{n+2}, (e_{n+1}, e_{n+2},) (e'_{n+1}, v_{n+1}) (e'_{n+2}, v_{n+2}) \in E(M(C(K_{1,n+2}))),$ we use three distinct colors to color these vertices. Clearly the above said new coloring is a minimal harmonious coloring. Here we use $2n + 1 + 3 = 2n + 4$ colors.

Therefore $\chi_H(M(C(K_{1,n+2}))) = \frac{n^2 + 2n + 5}{2} + 2n + 4 = \frac{(n+2)^2 + 2(n+2) + 5}{2}$. Hence by induction hypothesis the result follows, $\chi_H(M(C(K_{1,n}))) = \frac{n^2 + 2n + 5}{2}.$

Case (ii)

If $n$ is even, we prove $\chi_H(M(C(K_{1,n}))) = \frac{n^2 + 2n + 6}{2}$ by induction method, following the same procedure as above, $\frac{n^2 + 2n + 5}{2}$ and $\frac{n^2 + 2n + 6}{2} = \left\lceil \frac{n^2 + 2n + 5}{2} \right\rceil \forall n$. Therefore $\chi_H(M(C(K_{1,n}))) = \left\lceil \frac{n^2 + 2n + 5}{2} \right\rceil$. 

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![Diagram of a graph with labels 1 to 10, illustrating the middle graph of a central graph of star graphs $K_{1,3}$]

Figure 6. Middle Graph of Central Graph of Star Graphs $K_{1,3}$

$$\chi_H(M(C(K_{1,3}))) = \left\lceil \frac{3^2 + 2(3) + 5}{2} \right\rceil = 10.$$ 


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