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On a new proof of completeness of p. i. Propositional Calculus

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Abstract. In this paper, in order to obtain some interesting properties of the implicational algebras, we prove a completeness theorem on positive implicational Propositional Calculus in which we use the axiom schema $((a \supset b) \supset b) \supset ((b \supset a) \supset a)$ in an essential manner. We demonstrate this theorem without using neither Zorn's Lemma, nor similar properties.

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1 Preliminaries and remarks

If $\mathbf{X}$ is a set of propositional variables (*statement letters*) and C is a set of connectives different from $\supset$ (the *implication*), one can consider the set $\mathbf{\Phi}_C$ (or $\mathbf{\Phi}$, if $C = \emptyset$) of well-formed formulas (*statement forms*) of Propositional Calculus built up from the statement letters by appropriate applications of the connectives in $\{\supset\} \cup C$. Whenever one only uses the connective $\supset$, the statement forms are called p. i. (*positive implicational*) statement forms; hence one speaks of p. i. Propositional Calculus.

Now we consider formal theories on $\mathbf{\Phi}_C$ in which modus ponens (**MP**) is the only inference rule and the axiom set contains at least the statement forms of the type (1), (2) and (3)[1] below:

(1) $a \supset (b \supset a)$;

(2) $(a \supset (b \supset c)) \supset ((a \supset b) \supset (a \supset c))$;

(3) $((a \supset b) \supset b) \supset ((b \supset a) \supset a)$.

If $\mathbf{K}$ is the axiom set of a theory of the above type, then $\mathbf{K}$ shall represent also such a theory and $[\mathbf{K}]$ shall denote the associated set of theorems; in particular, $\mathbf{K}^0$ shall be the axiom set containing only statement forms of the type (1), (2), (3). In these theories $a \supset a$ is a theorem (cf. [6, p. 31, Lemma 1.7]). Henceforth

[1] a, b, c and similar letters shall represent arbitrary elements of $\mathbf{\Phi}_C$.

the symbol $\mathbf{\Theta}$ shall denote the family of subsets of $\mathbf{\Phi}_C$ of type $[\mathbf{K}]$. It is obvious that $\mathbf{\Theta}$ is an algebraical closure system on $\mathbf{\Phi}_C$ having $\left[\mathbf{K}^0\right]$ as the minimum element.

If $\mathbf{H}$ is an arbitrary subset of $\mathbf{\Phi}_C$, then $(\mathbf{H})$ (or (a), if $\mathbf{H}$ is equal to $\{a\}$) will be the associated closed set. Moreover, if $\mathbf{I}$ and $\mathbf{J}$ belong to $\mathbf{\Theta}$, then $\mathbf{I} \Cup \mathbf{J}$ shall represent the closure of the set union $\mathbf{I} \cup \mathbf{J}$.

In these theories the Deduction Theorem (**DT**) holds; i.e.: for any $a, b \in \mathbf{\Phi}_C$, if $b \in \mathbf{I} \Cup (a)$, then $a \supset b \in \mathbf{I}$ (cf. [6, p. 32, Proposition 1.8]) and vice versa.

In the sequel of this section we will show some interesting properties, easy to prove by **DT**, which depend only on the axiom schemas (1) and (2). For example, if $\mathbf{I} \in \mathbf{\Theta}$ and $a, b, c \in \mathbf{\Phi}_C$, then we have:

(4) $(a \supset c) \supset ((c \supset b) \supset (a \supset b)) \in \mathbf{I}$.

1 Theorem. *Let* $\mathbf{J} \in \mathbf{\Theta}$. *For any* $a, b, c \in \mathbf{\Phi}_C$, *if* $(a \supset b) \supset b \in \mathbf{J}$, *then* $(a \supset c) \supset ((c \supset b) \supset b) \in \mathbf{J}$.

Proof. Let $\mathbf{I} = \mathbf{J} \Cup (a \supset c) \Cup (c \supset b)$. Then by **DT** it is sufficient to verify that $b \in \mathbf{I}$.

In fact, since $a \supset c$ and $c \supset b$ belong to $\mathbf{I}$, by **MP** from (4) we get $a \supset b \in \mathbf{I}$. Thus, since we have also $(a \supset b) \supset b \in \mathbf{I}$, then $b \in \mathbf{I}$ by **MP**. $\boxed{QED}$

If $\mathbf{I} \in \mathbf{\Theta}$, then for any $x, y \in \mathbf{\Phi}_C$ we set $x \leq_{\mathbf{I}} y$ whenever $x \supset y \in \mathbf{I}$ (cf. [3, p. 4]). Thus from (4) we get the following properties:

(5) if $a \leq_{\mathbf{I}} c$, then $c \supset b \leq_{\mathbf{I}} a \supset b$;

(6) if $a \leq_{\mathbf{I}} c$ and $c \leq_{\mathbf{I}} b$, then $a \leq_{\mathbf{I}} b$.

Thereby $\leq_{\mathbf{I}}$ represents a reflexive preorder relation on $\mathbf{\Phi}_C$. Moreover (see (8) below) the equivalence relation $\cong_{\mathbf{I}}$ associated with $\leq_{\mathbf{I}}$ has $\mathbf{I}$ as an equivalence class. We easily get the following properties. The first one is a consequence of (1); the other ones depend on an easy application of **MP**.

(1') $a \leq_{\mathbf{I}} b \supset a$.

(7) $a \supset (b \supset c) \leq_{\mathbf{I}} b \supset (a \supset c)$; hence $a \supset (b \supset c) \cong_{\mathbf{I}} b \supset (a \supset c)$.

(8) If $i \in \mathbf{I}$, then $(i \supset a) \supset a \in \mathbf{I}$; therefore $i \supset a \cong_{\mathbf{I}} a$ by (1').

(9) $(a \supset (a \supset b)) \supset (a \supset b) \in \mathbf{I}$; therefore $a \supset (a \supset b) \cong_{\mathbf{I}} a \supset b$ by (1').

(10) $(a \supset b) \supset ((c \supset a) \supset (c \supset b)) \in \mathbf{I}$; thus if $a \leq_{\mathbf{I}} b$, then $c \supset a \leq_{\mathbf{I}} c \supset b$.

By (7) $(a \supset b) \supset (a \supset c) \leq_{\mathbf{I}} a \supset ((a \supset b) \supset c)$. Moreover, by (5), from $b \leq_{\mathbf{I}} a \supset b$ we have $(a \supset b) \supset c \leq_{\mathbf{I}} b \supset c$; hence by (10) we get $a \supset ((a \supset b) \supset c) \leq_{\mathbf{I}} a \supset (b \supset c)$. Thus, by transitivity, $(a \supset b) \supset (a \supset c) \leq_{\mathbf{I}} a \supset (b \supset c)$. Therefore by (2) we have:

(2') $(a \supset b) \supset (a \supset c) \cong_{\mathbf{I}} a \supset (b \supset c)$.

In the sequel we will use $\supset$ also as an operation that associates $a \supset b$ to the ordered pair of statement forms a and b. Then $\mathbf{\Phi}_C$ is also the support of an algebraical structure denoted by $(\mathbf{\Phi}_C, \supset)$.

For any $a, a', b \in \mathbf{\Phi}_C$, if $a \cong_{\mathbf{I}} a'$ (hence $a \leq_{\mathbf{I}} a'$ and $a' \leq_{\mathbf{I}} a$), then we get $a \supset b \cong_{\mathbf{I}} a' \supset b$ from (5) and $b \supset a \cong_{\mathbf{I}} b \supset a'$ from (10). This means that $\cong_{\mathbf{I}}$ is a right and a left congruence with respect to $\supset$; thereby $\cong_{\mathbf{I}}$ is a congruence of $(\mathbf{\Phi}_C, \supset)$ (cf. [3, p. 5]). For simplicity's sake we will represent with $(\mathbf{\Phi}_C/\mathbf{I}, \supset)$ the quotient structure determined by $\cong_{\mathbf{I}}$.

2 Some essential consequences of axiom schema (3)

Let $\mathbf{I} \in \mathbf{\Theta}$. From axiom schema (3) we immediately get the following property:

(3') $(a \supset b) \supset b \cong_{\mathbf{I}} (b \supset a) \supset a$.

Now let us set $a \vee b := (a \supset b) \supset b$. Then, respectively by (9) and by (2') used twice, we get:

(9') $a \vee (a \supset b) \in \mathbf{I}$.

(11) $a \supset (b \vee c) \cong_{\mathbf{I}} (a \supset b) \vee (a \supset c)$.

The symbol $\vee$ represents a binary operation on $\mathbf{\Phi}_C$ such that, by (3'), $a \vee b \cong_{\mathbf{I}} b \vee a$. Moreover for any $a, b, c \in \mathbf{\Phi}_C$ we immediately have:

(12) $a \leq_{\mathbf{I}} a \vee b$ (by (3'), (1')) and $b \leq_{\mathbf{I}} a \vee b$ (by (1')).

(13) $c \vee c = (c \supset c) \supset c \cong_{\mathbf{I}} c$ (since $c \supset c \in \mathbf{I}$; see also property (8)).

If $a \leq_{\mathbf{I}} a'$ and $b \leq_{\mathbf{I}} b'$, then $(a \supset b) \supset b \leq_{\mathbf{I}} (a' \supset b) \supset b \cong_{\mathbf{I}} (b \supset a') \supset a'$ by (5) and (3'). Similarly, $(b \supset a') \supset a' \leq_{\mathbf{I}} (b' \supset a') \supset a' \cong_{\mathbf{I}} (a' \supset b') \supset b'$. Therefore by transitivity we have:

(14) If $a \leq_{\mathbf{I}} a'$ and $b \leq_{\mathbf{I}} b'$, then $a \vee b \leq_{\mathbf{I}} a' \vee b'$.

In particular, if $a \cong_{\mathbf{I}} a'$ and $b \cong_{\mathbf{I}} b'$, then $a \vee b \cong_{\mathbf{I}} a' \vee b'$. This means that $\cong_{\mathbf{I}}$ is a congruence with respect to $\vee$ too.

We point out that for any $a, b, c \in \mathbf{\Phi}_C$, if $a \leq_{\mathbf{I}} c$ and $b \leq_{\mathbf{I}} c$, then by (14) and (13) $a \vee b \leq_{\mathbf{I}} c \vee c \cong_{\mathbf{I}} c$. By (12) this means that the equivalence class $[a \vee b]_{\mathbf{I}}$ is the least upper bound of $[a]_{\mathbf{I}}$ and $[b]_{\mathbf{I}}$ with respect to the order relation on $\mathbf{\Phi}_C/\mathbf{I}$ associated with $\leq_{\mathbf{I}}$. Thus $\mathbf{\Phi}_C/\mathbf{I}$ becomes an upper-semilattice having the quotient operation of $\vee$ as the corresponding semilattice operation.

Now we recall that an element $\mathbf{I} \in \mathbf{\Theta}$ is said to be *prime* whenever for any $a, b \in \mathbf{\Phi}_C$ such that $a \vee b \in \mathbf{I}$, either a or b belongs to $\mathbf{I}$. Moreover $\mathbf{I}$ is said to be *maximal* whenever it is maximal in the set of elements of $\mathbf{\Theta}$ different from $\mathbf{\Phi}_C$.

If $\mathbf{I}$ is a prime element of $\mathbf{\Theta}$ different from $\mathbf{\Phi}_C$, then one can see that $\mathbf{I}$ is maximal. To this purpose it is sufficient to verify that for any $a \in \mathbf{\Phi}_C$, with a not belonging to $\mathbf{I}$, $\mathbf{I} \Cup (a)$ includes $\mathbf{\Phi}_C$. In fact, for any $b \in \mathbf{\Phi}_C$, $a \vee (a \supset b) \in \mathbf{I}$ by (9'). Therefore, since $\mathbf{I}$ is prime and a does not belong to $\mathbf{I}$, $a \supset b \in \mathbf{I}$; hence b belongs to $\mathbf{I} \Cup (a)$. Conversely, let $\mathbf{I}$ be maximal and $a \vee b \in \mathbf{I}$, with a not belonging to $\mathbf{I}$. Thus $b \in \mathbf{\Phi}_C = \mathbf{I} \Cup (a)$, hence $a \supset b \in \mathbf{I}$ by **DT**. As a consequence, since $(a \supset b) \supset \ b \ = \ a \vee b \in \mathbf{I}$, then $b \in \mathbf{I}$ by **MP**. Therefore $\mathbf{I}$ is prime.

2 Theorem. *Let* $\mathbf{I}$ *be a maximal element of* $\mathbf{\Theta}$*. Then the set* $\mathbf{0}$ *of elements in* $\mathbf{\Phi}_C$ *not belonging to* $\mathbf{I}$ *is an equivalence class of* $\cong_{\mathbf{I}}$*. Furthermore* $\mathbf{I} = \mathbf{0} \supset \mathbf{0} = \mathbf{0} \supset \mathbf{I} = \mathbf{I} \supset \mathbf{I}$ *and* $\mathbf{I} \supset \mathbf{0} = \mathbf{0}$*, where the symbol* $\supset$ *represents the quotient operation on* $\mathbf{\Phi}_C/\mathbf{I}$ *(hence this operation acts on* $\mathbf{0}$ *and* $\mathbf{1}$ *as well as the usual implication acts on* 0 *and* 1*).*

Proof. For any $a, b \in \mathbf{0}$, being $\mathbf{I}$ maximal, b belongs to $\mathbf{I} \Cup (a)$ and a belongs to $\mathbf{I} \Cup (b)$; hence $a \supset b$ and $b \supset a$ belong to $\mathbf{I}$ by **DT**. Thus $a \cong_{\mathbf{I}} b$.

The second part of the theorem is obvious. $\boxed{QED}$

3 The completeness theorem

Henceforth we will consider only the case in which $C = \emptyset$. Then we have the following

3 Lemma. *Let* $\mathbf{I}$ *be a maximal element of* $\mathbf{\Theta}$*. Then all the tautologies in* $\mathbf{\Phi}$ *belong to* $\mathbf{I}$*.*

Proof. Let $a \in \mathbf{\Phi}$. Then, being $\cong_{\mathbf{I}}$ a congruence of $(\mathbf{\Phi}, \supset)$, the equivalence class $[a]_{\mathbf{I}}$ is the result of a formula a^* obtained from a by replacing in it any statement letter A with its class $[A]_{\mathbf{I}}$ and $\supset$ with the corresponding quotient operation. Thus, independently from the fact that the equivalence classes of the various statement letters are $\mathbf{0}$ or $\mathbf{I}$, if a is a tautology, then by Theorem 2 we get $[a]_{\mathbf{I}} = \mathbf{I}$. Therefore $a \in \mathbf{I}$. $\boxed{QED}$

Thus it is easy to prove the following completeness theorem.

4 Theorem. *Let* t *be a tautology belonging to* $\mathbf{\Phi}$*. Then* t *is a theorem of* $\mathbf{K}^0$ *(see the first part of Section 1).*

Proof. Let us assume, without loss of generality, that we have only the statement letters of t. Thereby the statement forms in $\mathbf{\Phi}$ have a natural enumeration $a_1, a_2, \ldots, a_h, \ldots$. If t does not belong to $(\mathbf{K}^0)$, let us define a chain

$\mathbf{I}_0, \ldots, \mathbf{I}_k, \ldots$ of elements of $\mathbf{\Theta}$ in the following way (cf. [6, p. 64, proof of Lindenbaum's Lemma]): $\mathbf{I}_0 = (\mathbf{K}^0)$; then, assuming $\mathbf{I}_k$ to be given for $k \geq 0$, let $\mathbf{I}_{k+1} = \mathbf{I}_k \Cup (a_{k+1})$ if t does not belong to $\mathbf{I}_k \Cup (a_{k+1})$; otherwise let $\mathbf{I}_{k+1} = \mathbf{I}_k$.

Clearly the tautology t does not belong to the set union $\mathbf{I}$ of the above $\mathbf{I}_0, \ldots, \mathbf{I}_k, \ldots$; moreover $\mathbf{I}$ is maximal in the set $\mathbf{\Theta}^{-t}$ of elements $\mathbf{J}$ of $\mathbf{\Theta}$ such that t does not belong to $\mathbf{J}$. We will prove that $\mathbf{I}$ is a prime element of $\mathbf{\Theta}$. Indeed let $a, b \in \mathbf{\Phi}$, with $a \vee b = (a \supset b) \supset b \in \mathbf{I}$; therefore $(a \supset t) \supset ((b \supset t) \supset t) \cong_{\mathbf{I}} (a \supset t) \supset ((t \supset b) \supset b) \in \mathbf{I}$ (see Theorem 1, with the symbol c replaced by t). Moreover let a and b do not belong to $\mathbf{I}$. Then, since $\mathbf{I}$ is maximal in $\mathbf{\Theta}^{-t}$, t belongs to $\mathbf{I} \Cup (a)$; hence $a \supset t \in \mathbf{I}$ by **DT**. Analogously $b \supset t \in \mathbf{I}$.

As a consequence, by **MP**, from $(a \supset t) \supset ((b \supset t) \supset t) \in \mathbf{I}$ we get $t \in \mathbf{I}$. This is absurd, since t does not belong to $\mathbf{I}$. Therefore $\mathbf{I}$ is prime and hence it is maximal in $\mathbf{\Theta}$, since $\mathbf{I}$ is different from $\mathbf{\Phi}$. This is absurd as well, by Lemma 3, since t is a tautology that does not belong to $\mathbf{I}$. $\boxed{QED}$

We remark that we proved Theorem 4 without using neither Zorn's Lemma, nor similar properties. Indeed at each step of the construction of the chain $\mathbf{I}_0, \ldots, \mathbf{I}_k, \ldots$ we had a well defined rule in order to decide whether to adjoin or not the element a_{k+1} to $\mathbf{I}_k$ in order to obtain $\mathbf{I}_{k+1}$.

Recently we became aware of an analogous completeness theorem, proved without the use of Zorn's Lemma, due to K. Segerberg ([9]). However we point out that in this paper we have used a different set of axioms. In particular, the previous axiom schema (3) is essential for our applications in implicational algebras (see the following section and [5]).

4 A simple consequence for implicational algebras

Now let $\mathbf{A}$ be a set with a fixed element 1 and a binary operation $\supset$. Then $(\mathbf{A}, 1, \supset)$ is called an *implicational algebra* (cf. [1], [2] and [5]) whenever, for any $a, b, c \in \mathbf{A}$, one has the following properties:

(j_0) if $a \supset b = 1 = b \supset a$, then $a = b$;

(j_1) $a \supset (b \supset a) = 1$;

(j_2) $[a \supset (b \supset c)] \supset [(a \supset b) \supset (a \supset c)] = 1$;

(j_3) $[(a \supset b) \supset b] \supset [(b \supset a) \supset a] = 1$.

Obviously, if $\supset$ is the usual implication on $\{0, 1\}$, then $(\{0, 1\}, 1, \supset)$ is an implicational algebra.

It is easy to prove that in an implicational algebra $(\mathbf{A}, 1, \supset)$ for any $b \in \mathbf{A}$ the following property holds (see [5]):

($\circ$) $1 \supset b = b$.

Now let $(\mathbf{A}, 1, \supset)$ and b be respectively an implicational algebra and a p. i. statement form. Thus one can interpret the implication of b as the operation $\supset$ of $(\mathbf{A}, 1, \supset)$; then, for any assignment to the statement letters in b of an element of $\mathbf{A}$, one obtains a corresponding element of $\mathbf{A}$. Thereby b defines a function from $\mathbf{A}^n$ into $\mathbf{A}$ (where n is the number of different statement letters of b). Obviously, by the above (j_1), (j_2) and (j_3), any statement form obtained from one of the above axiom schemas (1), (2), (3) defines a constant function taking only the value 1 in any implicational algebra.

Clearly if a and $a \supset b$ takes only the value 1 in a fixed implicational algebra, then by ($\circ$) also b takes only the value 1 in the same implicational algebra.

5 Remark. If b takes only the value 1 in any implicational algebra, then b is a tautology. In fact one can refer to the implicational algebra $(\{0, 1\}, 1, \supset)$. Conversely, we can see that if b is a tautology, then b takes only the value 1 in any implicational algebra. Indeed, by Theorem 4, let b_1, $b_2, \ldots, b_m = b$ be a formal proof of b in the theory $\mathbf{K}^0$. Then b_1 must be an axiom from one of the above axiom schemas (1), (2), (3), hence in any implicational algebra it takes only the value 1. Therefore by induction we immediately get that any statement form $b_1, b_2, \ldots, b_m = b$ takes only the value 1.

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