

COLLINEATION GROUPS OF THE SMALLEST BOL 3-NETS¹

GÁBOR NAGY

Abstract. *Some associativity properties of a loop can be interpreted as certain closure configuration of the corresponding 3-net. It was known that the smallest non-associative loops with the so called left Bol property have order 8. In this paper, we determine the direction preserving collineation groups of the 3-nets belonging to these smallest Bol loops. For that, we prove some new results concerning collineations of 3-nets and autotopisms of loops.*

1. INTRODUCTION

The theory of 3-nets and loops is a discipline which takes its roots from geometry, algebra and combinatorics. In geometry, it arises from the analysis of web structures; in algebra, from non-associative products; and in combinatorics, from Latin Squares. The subject has grown by relating these aspects to an increasing variety of new fields, and is now developing into a thriving branch of mathematics.

The reader of this paper is referred to H.O. PFLUGFELDER'S book [10] for the definition of the *loop*, the *nuclei* and the *left and right translations* of a given loop ([10], p. 3., 5., 16.). The concept of a 3-net can be formulated in many equivalent ways, for us, if a 3-net is coordinatized by the loop (L, \cdot) , then the transversal lines are sets of points of the form $\{(x, y) \in L \times L : x \cdot y = \text{constant}\}$. We define the *section* $S(L)$ of the loop (L, \cdot) by $S(L) = \{\lambda_x : x \in L\}$, and the *group generated by the left translations* of (L, \cdot) is $G(L) = \langle S(L) \rangle$.

There was not much work made in the direction of collineation groups of 3-nets, few exceptions are [1], [2], [7]. On the other hand, the class of Bol loops are also connected with configurations of the projective plane, cf. [6], [5].

We have the

Definition 1.1. *A bijection γ on L is called a right (left) pseudo-automorphism of a loop (L, \cdot) if there exists at least one element $c \in L$ such that*

$$x^\gamma \cdot (y^\gamma \cdot c) = (xy)^\gamma \cdot c$$

$((c \cdot x^\gamma) \cdot y^\gamma = c \cdot (xy)^\gamma)$ for all $x, y \in L$. The element c is then called a companion of γ .

This concept will play a significant role in the examination of the collineations of a 3-net. The next proposition gives a list of the most important properties of pseudo-automorphisms.

Proposition 1.2. *Let γ be a right pseudo-automorphism of a loop (L, \cdot) with a companion c . Then it has the following properties:*

- (i) $1^\gamma = 1$. *The map γ^{-1} is a right pseudo-automorphism with a companion d such that $c^{\gamma^{-1}} \cdot d = 1$. If γ' is another pseudo-automorphism of L with a companion c' then $\gamma\gamma'$ is*

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a pseudo-automorphism of L with the companion $c^\gamma \cdot c'$. The right pseudo-automorphism of a given loop form a group.

- (ii) It holds $n^\gamma \cdot x^\gamma = (nx)^\gamma$ for any $n \in N_\lambda$ and $x \in L$. The permutation γ induces an automorphism of the left nucleus of (L, \cdot) .
- (iii) It holds $x^\gamma \cdot n^\gamma = (xn)^\gamma$ for any $n \in N_\mu$ and $x \in L$. The permutation γ induces an automorphism of the middle nucleus of (L, \cdot) .

Analogous statements are valid for left pseudo-automorphisms.

Proof. The statements of (i) are easy to prove with a few calculation (see [20], p. 74-76.). In [10], p. 77. we find the points (ii) and (iii), but they are stated in an incorrect way, and also the proof contains some technical faults, so we give now a complete proof.

Let γ be a right pseudo-automorphism with a companion c of (L, \cdot) with unit element 1. Then

$$(nx)y = n(xy) \quad \text{for all } x, y \in L \quad \text{and} \quad n \in N_\lambda.$$

With $\langle \gamma, \gamma\rho_c, \gamma\rho_c \rangle$ applied to this, we have

$$(nx)^\gamma \cdot (y^\gamma \cdot c) = n^\gamma \cdot (x^\gamma(y^\gamma \cdot c)).$$

Replace $y^\gamma \cdot c$ by z to obtain

$$(nx)^\gamma \cdot z = n^\gamma(x^\gamma \cdot z),$$

or with $z = 1$,

$$(nx)^\gamma = n^\gamma \cdot x^\gamma.$$

Comparing the two last equations, one has

$$(n^\gamma \cdot x^\gamma) \cdot z = n^\gamma \cdot (x^\gamma \cdot z),$$

which implies that $n^\gamma \in N_\lambda$, thus (ii) is shown. Similarly, we can prove for (iii) as well.

A *collineation* of a 3-net is a one-to-one map on the point set, such that two points are collinear if and only if their images are collinear. We speak about a *direction preserving collineation* if for any line of the 3-net, the line and its image belong to the same parallel class. If we want to study the collineation group of a 3-net, it could be useful to be able to write down a collineation map with algebraic tools, using the coordination of the 3-net.

Proposition 1.3. *Let N be a 3-net coordinatized by the loop (L, \cdot) and γ be a permutation of the point set of N such that $(x, y)^\gamma = (x^{\alpha_y}, y^{\beta_x})$, with α_y, β_x permutations of L . Then γ is a direction preserving collineation if and only if*

- (i) α_y and β_x do not depend on y and x , respectively, this means that we can write $\alpha = \alpha_y$ and $\beta = \beta_x$.
- (ii) For every $x, y \in L$ holds

$$x^\alpha \cdot y^\beta = 1^\alpha \cdot (xy)^\beta.$$

Proof. Suppose that γ is a direction preserving collineation. Then it preserves the vertical lines, so the first coordinate of the image of a point depends only on the first coordinate of the point. The same idea holds for the horizontal lines with the second coordinates, so (i)

follows. Moreover, because of the definition of the 3-net, the points (x, y) and $(1, xy)$ lie on the same transversal line. Then so do (x^α, y^β) and $(1^\alpha, (xy)^\beta)$, i.e. (ii) holds.

Conversely, let γ be such that $(x, y)^\gamma = (x^\alpha, y^\beta)$ and α and β satisfies (ii). It is easy to see that γ maps vertical and horizontal lines to vertical and horizontal lines, respectively. And it preserves the third direction as well, because the points (x, y) and (x', y') are on a transversal line iff $x \cdot y = x' \cdot y'$. Then

$$x^\alpha \cdot y^\beta = 1^\alpha \cdot (xy)^\beta = 1^\alpha \cdot (x'y')^\beta = x'^\alpha \cdot y'^\beta,$$

thus the images of the two points lay on a transversal line, too.

As an application of this Proposition, we can determine certain stabiliser subgroups of the direction preserving collineation groups.

Firstly, let us suppose that (α, β) is a collineation fixing the horizontal line through the origin, that means that $1^\alpha = a$ and $1^\beta = 1$. Applying (α, β) for the point $(x, 1)$, from (ii), we get $x^\alpha = x^\alpha \cdot 1^\beta = 1^\alpha \cdot x^\beta = c \cdot x^\beta$. Now, for every $x, y \in L$ we have

$$(c \cdot x^\beta) \cdot y^\beta = x^\alpha \cdot y^\beta = c \cdot (xy)^\beta,$$

thus, a collineation which fixes the horizontal line through the origin determines a left pseudo-automorphism with the companion $c = 1^\alpha$. Similarly, we can prove the link between stabilisers of the vertical line and right pseudo-automorphism. Finally, if (α, β) fixes the origin, $1^\alpha = 1^\beta = 1$, then $\alpha = \beta$ are automorphisms of the coordinate loop.

Moreover, let P denote the orbit of the origin in the action of the group of the direction preserving collineations, and l_h and l_v the horizontal and vertical lines through the origin, respectively. Then we have the natural bijections $l_h \cap P \rightarrow C_\lambda$ and $l_v \cap P \rightarrow C_\rho$ are the sets of those elements of L which can occur as left and right companions, respectively.

It is known that in a given 3-net, the coordinate loop depends on the choice of the origin point. The following proposition gives a sufficient and necessary conditions for that, that the coordinate loops with the origins P and Q be isomorphic.

Proposition 1.4. *Let P and Q be points of the 3-net N and let (L, \cdot) and (L, \circ) be the coordinate loops with the origin P and Q , respectively. These two loops are isomorphic if and only if there is a direction preserving collineation γ of the net, mapping P onto $Q = P^\gamma$.*

Proof. Clearly, if the collineation γ exists, then it induces a one-to-one map between the point sets of the horizontal lines through P and Q , which leads to a permutation of L , which is going to be an isomorphism. Conversely, let U be an isomorphism and R be a point of N with coordinates (x, y) in the coordinate system belonging to P . Define the image of R by the point R' , which has the coordinates (x^U, y^U) in the coordinate system belonging to Q . Obviously, the map $\gamma : R \rightarrow R'$ is a direction preserving collineation of N .

2. SPECIAL LOOP CLASSES

If we have an algebraic structure with a non-associative multiplication, then one can ask questions about the necessary and sufficient conditions for special algebraic identities which interpret weak associative properties. In the most cases, such identities lead to geometrical configurations in the coordinatized 3-net.

We say that a loop satisfies the *left inverse property* (L.I.P.), if the inverse of any left translation is contained in the section. With other words, this means that for any element x of the loop there is a unique element x^{-1} such that $\lambda_x^{-1} = \lambda_{x^{-1}}$. Let us define the map $J : L \rightarrow L$ by $J : x \mapsto x^{-1}$.

Proposition 2.1. *Let (L, \cdot) be a loop satisfying the left inverse property, and let N_λ and N_μ be its left and middle nuclei, respectively. Then we have $N_\lambda = N_\mu$.*

Proof. Suppose $n \in N_\lambda$, then also $n^{-1} \in N_\lambda$. This is equivalent to the fact that for all $x \in L$, we have $\lambda_x \lambda_{n^{-1}} \in S(L)$, or, we can write $\lambda_{x^{-1}} \lambda_{n^{-1}} \in S(L)$. Using the left inverse property, we obtain that for all $x \in L$

$$\lambda_n \lambda_x = \lambda_{n^{-1}}^{-1} \lambda_{x^{-1}}^{-1} = (\lambda_{x^{-1}} \lambda_{n^{-1}})^{-1} \in S(L).$$

And this holds if and only if $n \in N_\mu$ as well.

An important class of loops having the L.I.P. is the so called *Bol loops*, which can be characterized by an algebraic identity, or, equivalently, by a closure configuration in the 3-net. G. BOL'S configuration and the corresponding algebraic identity have always played an important role in the development of the geometric theory of 3-nets and in the algebraic theory of loops.

Definition 2.2. *A loop (L, \cdot) is said to be a (left) Bol loop if the identity*

$$x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z$$

holds true for all $x, y, z \in L$.

Or, we can formulate this, saying that (L, \cdot) is a (left) Bol loop if for all $x, y \in L$ we have

$$\lambda_x \lambda_y \lambda_x \in S(L).$$

In a Bol 3-net, the Bol configuration defines a collineation which is a reflection with a vertical axis. In the paper [4], the subgroup of the direction preserving collinaetion group $\Gamma(N)$ containing the products of these Bol reflections with an even number of factors was considered. Let us denote this subgroup by N .

Proposition 2.3. *The set of Bol reflections of a Bol 3-net is a normal set in the full group of collineations.*

Proof. Let us observe that the paralell class of the vertical lines is special, because the Bol reflection by them is equivalent with the left Bol condition, hence the full group of collineations of the 3-net leaves the set of vertical lines fixed. Let σ_m be the Bol reflection with the vertical line $(x = m)$ as axis. Then, the unique collineation which fixes the axis pointwise and interchanges the horizontal and transversal directions is σ_m . Hence, if we conjugate σ_m with an arbitrary collineation γ , we get the Bol reflection $\sigma_{m'}$, where the vertical line $(x = m')$ is the image of the axis $(x = m)$ by γ .

It follows that N is normal in the group of direction preserving collineations. In [4], it is shown that there is an endomorphism $\Phi : N \rightarrow G(L)$, where L is a coordinate loop of the 3-net

and $G(L)$ is the group generated by the left translations of L . Moreover, the kernel of Φ is isomorphic to a subgroup of the left nucleus of L . We also know that N acts transitively on the set of the horizontal lines. In the paper, a new coordinatization of the 3-net was introduced. The origin was the same, and if a point had coordinates (x, y) in the old system, then in the new one it is (ξ, η) , with $\xi = y$ and $\eta = xy$. It was shown, that this coordinate system gives the same multiplication on the point set of the horizontal line through the origin. Then, N is generated by the collineations of the form

$$(\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta}) = (\alpha^{-1}\xi, a\eta),$$

with $\alpha \in L$. Let us translate this in the old coordinate system. Define (\bar{x}, \bar{y}) as the image of the point (x, y) by the above collineation. Then we have

$$\bar{y} = \bar{\xi} = a^{-1}\xi = a^{-1}y = y^{\lambda_a^{-1}},$$

and

$$\bar{x} = \bar{\eta} / \bar{\xi} = (a\eta) / (a^{-1}\xi) = (a \cdot xy) / (a^{-1}y) = a \cdot xa = x^{\rho_a \lambda_a}.$$



The last step follows from the fact, this is a direction preserving collineation, thus the x -coordinate of the image point does not depend on the y -coordinate, and so we can choose $y = a$. We got that the generators of $N(N)$ are of the form $(\rho_a \lambda_a, \lambda_a^{-1})$, where $a \in L$. The map Φ is a projection of the second permutation.

A useful characterization of the Bol loops are provided by the following.

Proposition 2.4. *A loop (L, \cdot) is a Bol loop if and only if for any $x \in L$, the pair of permutations $(\rho_x \lambda_x, \lambda_x^{-1})$ is a direction preserving collineation of the 3-net, coordinatized by the loop (L, \cdot) .*

Proof. By the above, it is obvious that a Bol 3-net has collineations of the form $(\rho_x \lambda_x, \lambda_x^{-1})$. Conversely, if we assume that $(\rho_x \lambda_x, \lambda_x^{-1})$ is a collineation, than using 1.3 for the point (u, xv) , we have

$$(x \cdot ux) \cdot v = u^{\rho_x \lambda_x} \cdot v(xv)^{\lambda_x^{-1}} = 1^{\rho_x \lambda_x} \cdot (u \cdot xv)^{\lambda_x^{-1}} = x(u \cdot xv),$$

thus, the Bol identity holds for every $u, v \in L$.

Concerning the kernel of Φ we have the

Proposition 2.5. *Let a 3-net be coordinatized by the Bol loop (L, \cdot) . Suppose that J is a right pseudo-automorphism of L . Then the kernel of Φ is trivial.*

Proof. The generators of $-N$ are in the form $(\rho_x \lambda_x, \lambda_x^{-1})$. Then one can easily prove by induction on k that the elements of $-N$ are in the form

$$(\rho_{m_1} \lambda_{m_1} \dots \rho_{m_k} \lambda_{m_k}, \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}) = (\rho_u \lambda_{m_1} \dots \lambda_{m_k}, \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}),$$

with $u = m_1(\dots(m_{k-2} \cdot m_{k-1} m_k) \dots) = 1^{\lambda_{m_k} \dots \lambda_{m_1}}$. Obviously,

$$\Phi : (\rho_{m_1} \lambda_{m_1} \dots \rho_{m_k} \lambda_{m_k}, \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}) \mapsto \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}.$$

Suppose that $(\alpha, \beta) \in \text{Ker}\Phi$, then $\beta = id$. By 1.3, α must be of the form λ_a with $a = 1^\alpha$. Since $id = \beta = \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}$, we have $m_k^{-1}(\dots(m_3^{-1} \cdot m_2^{-1} m_1^{-1}) \dots) = 1$. Let c be a companion element of J . Then $u = 1^{\beta^{-1}} = 1$ and

$$\begin{aligned} ac &= c^\alpha = m_k(\dots(m_3(m_2 \cdot m_1 c)) \dots) = \\ &= m_k(\dots(m_3((m_2^{-1} m_1^{-1})^{-1} \cdot c)) \dots) = m_k(\dots((m_3^{-1} \cdot m_2^{-1} m_1^{-1})^{-1} c) \dots) = \\ &= \dots = (m_k^{-1}(\dots(m_3^{-1} \cdot m_2^{-1} m_1^{-1})) \dots)^{-1} \cdot c = 1^{-1} \cdot c = c, \end{aligned}$$

thus $a = 1$ and $\alpha = id$.

Remark. Actually, we have proven now a little more than the theorem. We have just shown that if $(\alpha, \beta) \in -N$ and $1^\beta = 1$, then α fixes those elements of the loop which are companions of the right pseudo-automorphism J .

We are interested in another class of loops as well, namely the class of loops (L, \cdot) for which $S(L)$ is closed under conjugation in the symmetric group on L (cf. [9]). More precisely, we make the following

Definition 2.6. We say that the loop (L, \cdot) is a left conjugacy closed loop if

$$\lambda_x^{-1} \lambda_y \lambda_x \in S(L)$$

holds for all $x, y \in L$.

This class of loops can be characterized by right pseudo-automorphism of a given form.

Proposition 2.7. A loop (L, \cdot) is left conjugacy closed if and only if for all $x \in L$, the permutation $\lambda_x \rho_x^{-1}$ is a right pseudo-automorphism with the companion element x .

Proof. We have $u^{\lambda_x \rho_x^{-1}} \cdot (v^{\lambda_x \rho_x^{-1}} \cdot x) = (xu) / x \cdot (xv)$ and $(uv)^{\lambda_x \rho_x^{-1}} \cdot x = x \cdot uv = x \cdot (u \cdot (x^{-1} \cdot xv))$. Setting $\bar{v} = xv$, we got that the permutation $\lambda_x \rho_x^{-1}$ is a right pseudo-automorphism with companion x exactly if $(xu) / x \cdot \bar{v} = x \cdot (u \cdot (x^{-1} \bar{v}))$ for all $u, \bar{v}, x \in L$, i.e. if $\lambda_x^{-1} \lambda_u \lambda_x \in S(L)$ for all $u, x \in L$.

Corollary 2.8. Let a 3-net be coordinatized by a left conjugacy closed loop and consider the action of the collineation group on the point set. Then the orbit of the origin is a union of vertical lines.

Proof. By 2.4, the orbit contains the vertical line through the origin. Now, suppose that it contains a point P outside of this line. Then we coordinatize with this point, and by 1.4, the coordinate loop is going to be isomorphic to the original one, hence it is left conjugacy closed, and so the orbit contains again the whole vertical line through P .

3. AUTOMORPHISM OF THE SMALLEST BOL LOOPS

The smallest Bol loop which is not a group must have eight elements. These loops form two isotopy classes; representatives of the isotopy classes are given by the following (cf. [3]).

1.) The lool B_1 has as elements $e, f, f^2, f^3, g, fg, f^2g, f^3g$ such that the multiplication is defined by the following rules

- (i) $f^i(f^j g) = f^{i+j} g$,
- (ii) $(f^i g)f^j = f^{i+j} g$,
- (iii) $(f^i g)(f^j g) = f^{2-i+j}$, $i, j \in \mathbb{Z}_4$, with $i, j \in \mathbb{Z}_4$.

2.) The loop B_2 has as elements $e, f, f^2, f^3, g, fg, f^2 g, f^3 g$ such that the multiplication is defined by the following rules

- (i) $f^i(f^j g) = f^{i+j} g$,
- (ii) $(f^i g)f^j = f^{i+j+2ij} g$,
- (iii) $(f^i g)(f^j g) = f^{i+j+2ij}$, with $i, j \in \mathbb{Z}_4$.

The unit element is e , and for both cases, the left nucleus contains the elements e and f^2 , is has order 2. The right nuclei have order 4, but in the first case, $N_\rho = \{e, f, f^2, f^3\} \cong C_4$, and in the second case $N_\rho = \{e, g, f^2, gf^2\} \cong C_2 \times C_2$.

In this chapter, we are going to examine the automorphism group of the loop B_1 . 3.3 will state that almost the same statements hold for B_2 , as well. For this theorem, we will not give the whole proof, just mention the most important differences, compared to the proofs of the first part.

Proposition 3.1. *The maps $\phi_1 : f \mapsto f, g \mapsto fg$ and $\phi_2 : f \mapsto f^{-1}, g \mapsto g$ can be extended to an automorphism of B_1 . The permutation ϕ_1 has order 4, the permutation ϕ_2 has order 2. Let $\phi_3 = \phi_1 \circ \phi_2$. Then ϕ_3 has order 2.*

Proof. Let us define \hat{g} by $\hat{g} = fg$. We will show that the same multiplication rules hold as above if we write \hat{g} in the place of g .

- (i) $f^i(f^j \hat{g}) = f^i(f^j(fg)) = f^i(f^{j+1} g) = f^{i+j+1} g = f^{i+j}(fg) = f^{i+j} \hat{g}$;
- (ii) $(f^i \hat{g})f^j = (f^{i+1} g)f^j = f^{i+j+1} g = f^{i+j} \hat{g}$;
- (iii) $(f^i \hat{g})(f^j \hat{g}) = (f^{i+1} g)(f^{j+1} g) = f^{2-i+j}$.

By this we have proven that ϕ_1 can be extended to an automorphism of B_1 . From now on, we will call the automorphism itself ϕ_1 . In a very similar way we can prove for ϕ_2 , and we will use this notion also for the automorphism.

The order of ϕ_1 follows from the first multiplication rule, namely $g^{\phi_1^2} = f_1^\phi \cdot g_1^\phi = f(fg) = f^2 g$, and by induction, $g^{\phi_1^i} = f^i g$. Clearly, ϕ_2 is an involution. The action of ϕ_3 on the generators is $f \mapsto \hat{f} = f^{-1}, g \mapsto \hat{g} = fg$. Applying ϕ_3 once again, we get $\hat{f} \mapsto \hat{f}^{-1} = (f^{-1})^{-1} = f, \hat{g} \mapsto \hat{f} \hat{g} = f^{-1}(fg) = g$, so $\phi_3^2 = id$.

Theorem 3.2. *In the loop B_1 , each left pseudo-automorphism is an automorphism, only the elements of the left nucleus occur as left companions. And the group of automorphisms of B_1 is isomorphic to the dihedral group D_8 of order 8.*

Proof. By 3.1, the subgroup $\langle \phi_1, \phi_2 \rangle \cong D_8$ generated by the permutations ϕ_1 and ϕ_2 has the relations $\phi_1^4 = id, \phi_2^2 = id$ and $(\phi_1 \circ \phi_2)^2 = id$ on its generators, thus this subgroup is isomorphic to the dihedral group of order 8.

On the other hand, we have an upper bound for the order of the group of left pseudo-automorphisms. By 1.2 (ii), a left pseudo-automorphism induces an automorphism on $N_\mu = N_\lambda$, so it must fix e and f^2 . Hence we have two choices for the image of f , itself or f^3 . For the image of g we get at most 4 possibilities, since g is not contained in N_ρ , so neither its image. Now, we know by 1.2 (ii) as well, that the images of f and g determine uniquely the pseudo-automorphism. For their images, we have altogether at most $2 \cdot 4 = 8$ choices, hence B_1 has at least 8 automorphisms. These are also pseudo-automorphisms (one can have the

unit element as companion), so there is no more pseudo-automorphism. This also means that there are exactly 8 automorphisms, hence their group is the dihedral group D_8 .

Now, let us consider the case of the loop B_2 .

Theorem 3.3.

- (i) *The maps $\psi_1 : f \mapsto fg, g \mapsto g$ and $\psi_2 : f \mapsto f, g \mapsto f^2g$ can be extended to an automorphism of B_2 . The permutations ψ_1 and ψ_2 have order 2. Let be $\psi_3 = \psi_1 \circ \psi_2$. Then $\psi_3 \neq \psi_2 \circ \psi_1$ has order 4.*
- (ii) *In the loop B_2 , each left pseudo-automorphisms are automorphisms, only the elements of the left nucleus occur as left companions, and the group of automorphisms of B_2 is isomorphic to D_8 .*

Proof. As in the first part of chapter, we could show (i) using the multiplication rules of B_2 . In the prove of (ii), we use that the left nucleus has 2 elements, the right nucleus has 4 elements. In this case, it is not true that N_ρ contains the powers of f , but $N_\rho = \{e, g, f^2, f^2g\}$ and $N_\lambda = \{e, f^2\}$. Hence, we can show (ii), as well.

In the rest of this chapter, we prove two propositions which give some extra information on the non-associative multiplication structure of these smallest Bol loops.

Proposition 3.4. *Let be $a, b, c, \in L$ such that $1^{\lambda_a \lambda_b \lambda_c} = 1$, where $L = B_1$ or $L = B_2$. Then, $\lambda_a \lambda_b \lambda_c = id$ iff at least one of the elements a, b, c , belong to the left nucleus N_λ .*

Proof. It is obvious that for any Bol loop, if the product of 2 left translations fixes the unit element 1 of the loop, then the product is the identity permutation. Consider the product of permutations $\lambda_a \lambda_b \lambda_c$. If one of a, b or c belongs to $N_\lambda = N_\mu$ then the $\lambda_a \lambda_b \lambda_c$ can be written as the product of 2 left translations, and so $\lambda_a \lambda_b \lambda_c = id$.

By $c \cdot ba = 1$, at least one of these elements belongs to N_ρ , since N_ρ has index 2 in L . Assume that at least 2 of them belong to N_ρ . Then all of them are in N_ρ , and at least one is in $N_\lambda \leq N_\rho$, since $[N_\rho : N_\lambda] = 2$. Thus, in this case, $\lambda_a \lambda_b \lambda_c = id$. This also means that for any $n, m \in N_\rho, \lambda_n \lambda_m \lambda_{mn}^{-1} = id$, i.e. $\lambda_n \lambda_m = \lambda_{mn}$.

Notice that there exist elements $a', b', c' \in L$, such that $c' \cdot b'a' = 1$ and $\lambda_{a'} \lambda_{b'} \lambda_{c'} \neq id$, since L is not a group. By the paragraph above, exactly one element of a', b', c' must be in N_ρ . Now, suppose that exactly one of the elements a, b, c is in N_ρ , but none of them is in N_λ . The permutation $\lambda_a \lambda_b \lambda_c$ is always a left pseudo-automorphism (cf. [4]), thus for L it is an automorphism, we denote it by τ . For any automorphism α and $x \in L$ we have $\alpha^{-1} \lambda_x \alpha = \lambda_{x^\alpha}$, and so

$$\lambda_a \lambda_b \lambda_c = \tau = \lambda_a^{-1} \tau \lambda_{a^\tau} = \lambda_b \lambda_c \lambda_{a^\tau}$$

This means that we may assume w.l.o.g. that $a, a' \in N_\rho$. We claim that there exists exactly one automorphism α of L such that $\alpha^\alpha = a', b^\alpha = b'$ and $c^\alpha = c'$. We know that $N_\rho \setminus N_\lambda$ and $L \setminus N_\rho$ are orbits in the action of the automorphisms on the carrier set L . Moreover, we have exactly $8 = |N_\rho \setminus N_\lambda| \cdot |L \setminus N_\rho|$ automorphisms, so there is exactly one α with $\alpha^\alpha = a'$ and $b^\alpha = b'$, and than $c^\alpha = c'$ since $c \cdot ba = c' \cdot b'a' = 1$ and α is automorphism. We got

$$id \neq \lambda_{a'} \lambda_{b'} \lambda_{c'} = \lambda_{\alpha^\alpha} \lambda_{b^\alpha} \lambda_{c^\alpha} = \alpha^{-1} \lambda_a \lambda_b \lambda_c \alpha,$$

thus $\lambda_a \lambda_b \lambda_c \neq id$.

Remark. If $c \cdot ba = 1$ then the automorphism $\lambda_a\lambda_b\lambda_c$ fixes the elements of the right nucleus or with other words, it commutes with the left translations by a right nucleus element.

Proposition 3.5. *Let the loop L be equal to the loop B_1 or B_2 . Then the subgroup $[\text{Aut}(L), G(L)]$ of the symmetric group on the elements of L is isomorphic to the right nucleus of L .*

Proof. Take an element $\alpha \in \text{Aut}(L)$ and a generating element λ_x^{-1} of $G(L)$. We have

$$[\alpha, \lambda_x^{-1}] = \alpha^{-1}\lambda_x\alpha\lambda_x^{-1} = \lambda_u\lambda_u^{-1}\lambda_x\alpha\lambda_x^{-1} = \lambda_u\tau(x, \alpha),$$

where $u \in L$ such that $x^\alpha u^{-1} = x$ and $\tau(x, \alpha) = \lambda_u^{-1}\lambda_x\alpha\lambda_x^{-1}$.

Using again $[L : N_\rho] = 2$, we obtain that $u \in N_\rho$. Furthermore, we know that any element of N_ρ can occur as u , because the group of automorphisms acts transitively on the set $L \setminus N_\rho$. By the remark after the 3.4, λ_u commutes with the involution $\tau(x, \alpha)$. 3.4 says that τ depends only on u , since $\tau(x, \alpha) = id$ if and only if among u, x, x^α there is a left nucleus element iff u is a left nucleus element. But the left nucleus N_λ has index 2 in the right nucleus N_ρ , and the nontrivial $\tau(u)$ has order 2, thus the map $[\alpha, \lambda_x^{-1}] = \lambda_u^{-1}\tau(u) \mapsto u$ define an isomorphism between the set of generators of $[\text{Aut}(L), G(L)]$ and the group N_ρ .

4. DIRECTION PRESERVING COLLINEATIONS

In this and the next sections, we are going to determine the group of direction preserving collineations in the smallest Bol 3-nets. For that, let us fix some notations. In the following, $\Gamma(N)$ will always denote the direction preserving collineation group of the Bol 3-net N . If it does not cause misunderstanding, we only write Γ . If the net is coordinatized by the loop (L, \cdot) , then $\Gamma_0(N)$ is going to be the subgroup stabilizing the origin. By l_h and l_v we denote the horizontal and vertical lines through the origin, respectively, and $\Gamma_H(N)$ and $\Gamma_V(N)$ are the stabilizers of l_h and l_v in $\Gamma(N)$, respectively. We use the notation P for the orbit of the origin in the action of $\Gamma(N)$ on the point set of N . Furthermore, we introduce the notation $H = N(N)\Gamma_0(N) \leq \Gamma(N)$.

Now, we state our Main Theorem, whose proof needs this last chapter.

Main Theorem. *Let the 3-net N coordinatized by B_1 or B_2 . Then, using the above introduced notations, we can state the following*

(i) *In both cases B_1 and B_2 , we have $H' = \Phi(H)$ and these groups are Abelian*

(ii) *For B_1 ,*

$$\Gamma = H\Gamma_0 \times_\varphi N \cong \text{Aut}(B_1) \times_\varphi G(B_1).$$

(iii) *For B_2 ,*

$$\Gamma = H \times C_2,$$

moreover there exists an Abelian normal subgroup Λ of $G(B_2)$, which operates sharply transitively on the set B_2 , and holds

$$H \cong \text{Aut}(B_2) \times_\varphi \Lambda.$$

First, consider the kernel of the map $\Phi : N(N) \rightarrow G(L)$, where the loop (L, \cdot) is equal to B_1 or B_2 and the 3-net N is coordinatized by (L, \cdot) . With some calculation, one can obtain that in the loop B_1 , the map $J : x \mapsto x^{-1}$ is a right pseudo-automorphism but not an automorphism. On the other hand in the loop B_2 , J is an automorphism, more precisely $J = \psi_1^2 \circ \psi_2$. By 2.5, this means, that in the smallest Bol 3-nets, the kernel of Φ is trivial. The elements of H are collineations, this means they are pairs of two permutations, and the projection of the second permutation of these collineations gives us an isomorphism from H onto the subgroup

$$\bar{H} = G(L)Aut(L)$$

of the symmetric group on the carrier set L . From now on, we will consider the group \bar{H} .

It is easy to see that the commutator subgroup \bar{H}' is generated by $Aut(L)'$ and $[G(L), Aut(L)], G(L)'$.

In the paper [9], the generated groups $G(B_1)$ and $G(B_2)$ are determined. Their result was that they are non-Abelian groups of order 16, such that their commutator subgroups have order 2 and contain the square elements of the group. Actually, also the dihedral group D_8 of order 8 has this last property, its commutator subgroup contains the squares of the group.

Theorem 4.1. *Let the loop L be equal to B_1 or B_2 . Then the commutator subgroup \bar{H}' is isomorphic to $C_2 \times C_4$ or $C_2 \times C_2 \times C_2$, respectively.*

Proof. Firstly, we show that $G(L)' \subseteq [Aut(L), G(L)]$. By [9] we know that the group $G(L)'$ is a group of order 2, and its non-trivial element is the left translation by the non-trivial element n_0 of the left nucleus. On the other hand, using the notation of 3.5, we can choose $u = n_0$, and so we have $u \in N_\lambda$, so $\tau(u) = id$, thus $\lambda_{n_0} \in [Aut(L), G(L)]$.

In the second step, we show that $[Aut(L), G(L)] \cap Aut(L)' = \{id\}$. As we know, the elements of $[Aut(L), G(L)]$ are in the form $\lambda_u^{-1}\tau(u)$, with $u \in N_\rho$. This permutation fixes the unit element 1 of the loop L only if $u = 1$, but any element of $Aut(L)'$ must fix 1.

Finally, we show that elements of $Aut(L)'$ commute with elements of the group $[Aut(L), G(L)]$. In both cases, the group $Aut(L)$ of automorphisms is isomorphic to the dihedral group D_8 of order 8. The commutator of this group contains that unique involution which commutes with any other element. On the other hand, we saw already, that $\lambda_a\lambda_b\lambda_c$, with $c \cdot ba = 1$ is an involutory automorphism. It can take two different values, the identity or the non-trivial $\tau(u)$. One can easily check that, for both cases, it commutes with any other automorphism of the loop L . Thus, the non-trivial $\tau(u)$ is the only non-trivial element of the commutator group $Aut(L)'$. But we had already noticed that this element fixes the elements of the right nucleus, which means that it commutes with an element of the form $\lambda_u^{-1}\tau(u)$.

So, the theorem is proven, hence we have shown that

$$\bar{H}' \cong C_2 \times N_\rho,$$

and $N_\rho \cong C_4$ in the loop B_1 and $N_\rho \cong C_2 \times C_2$ in the loop B_2 .

The nice structure of these smallest Bol loops gives us another interesting result.

Theorem 4.2. *If the loop L is equal to B_1 or B_2 and \bar{H} is the group $\bar{H} = Aut(L)G(L)$ as before, then the Frattini subgroup of \bar{H} is the commutator subgroup \bar{H}' .*

Proof. It is known that for a p -group, the Frattini subgroup $\Phi(\bar{H})$ is the smallest normal subgroup K such that \bar{H}/K is elementary Abelian, and so it must contain the commutator group. Thus, we only have to show that the square of any element of \bar{H} is contained in the commutator subgroup \bar{H}' .

Any element of $G(L)$ is in the form $\tau\lambda_x$, where $\tau \in G(L)$ stabilizes the unit element 1 and so it is an automorphism of L . This means that an element of $\text{Aut}(L)G(L)$ is of the form $\beta\tau\lambda_x = \alpha\lambda_x$, where $\beta \in \text{Aut}(L)$ and $\alpha = \beta\tau \in \text{Aut}(L)$. Then,

$$\begin{aligned} \alpha\lambda_x\alpha\lambda_x &= \alpha^2\lambda_{x^\alpha}\lambda_x = \alpha^2\lambda_u^{-1}\lambda_{x^\alpha}\lambda_x^{-1}\lambda_{x^2} = \\ &= \alpha^2\lambda_u^{-1}\tau(u)\lambda_{x^2} \in (\text{Aut}(L))'[G(L), \text{Aut}(L)]G(L)' = (\text{Aut}(L)G(L))'. \end{aligned}$$

Thus, we obtain that the Frattini subgroup $\Phi(\bar{H})$ is contained in the commutator subgroup \bar{H}' , and so they must be equal.

In the following two subsections, we will consider the 3-nets determined by the loops B_1 and B_2 . We will collect information about their collineation group, and for that we use two methods which are different but rather similar. Anyway, both methods can be a useful tool in the examination of collineation groups of more general Bol 3-nets.

4.1 Collineations of the 3-net B_1

Proposition 4.3. *Let M be a Bol 3-net coordinatized by the Bol loop (L, \cdot) . Assume that J is a right pseudo-automorphism of L but not an automorphism and that the right nucleus has index 2 in the loop L . Then $N = N(M)$ acts semi-regularly on P , the orbit of the origin in M .*

Proof. Suppose that $(\alpha, \beta) \in M$ and $1^\alpha = 1^\beta = 1$. The permutation $\beta \in G(L)$ fixes the unit element 1, thus it is generated by elements of the form $\lambda_a\lambda_b\lambda_c$, with $c \cdot ba = 1$. Then β fixes the elements of the right nucleus N_ρ , since they are fixed by the elements of the type $\lambda_a\lambda_b\lambda_c$, as we have seen it in the remark after 3.4. Another remark, after 2.5, says that if β fixes the unit element 1 then α fixes those elements which can occur as the right companion of J . By the conditions of this proposition, this means that α fixes any element outside of N_ρ . So (α, β) fixes the origin, hence $\alpha = \beta$, thus $\alpha = \beta = id$.

The loop B_1 is left conjugacy closed (cf. [3]), so the orbit P is a union of vertical lines. Moreover, we have a bijection from $l_h \cap P$ onto N_λ , and so $|P| = 16$. Comparing this with 4.3, the result is that N acts regularly on P . This means $\Gamma = \Gamma_0N$ and $\Gamma_0 \cap N = \{1\}$. Applying the isomorphism theorem of groups,

$$\Gamma/N = \Gamma_0N/N \cong \Gamma_0/\Gamma_0 \cap N = \Gamma_0.$$

So we got that Γ splits over N , with complement Γ_0 . From the group theory it is known that in this case we have the semidirect product $\Gamma = \Gamma_0 \times_\varphi N$, where the action φ of Γ_0 on N is defined by restriction of the action of Γ on N by conjugation.

Theorem 4.4. *Let N be the Bol 3-net coordinatized by the Bol loop B_1 , and let Γ be the group of the direction preserving collineations of N . Moreover, denote by φ the action of $\text{Aut}(B_1)$ on $G(B_1)$, defined by $\alpha : \lambda_x \mapsto \lambda_{x^\alpha}$. Then*

$$\Gamma \cong \text{Aut}(B_1) \times_\varphi G(B_1).$$

Proof. We know already that Γ is a semi-direct product of this form, $\alpha^{-1}\lambda_x\alpha = \lambda_{x\alpha}$. If we conjugate inside of the collineation group, the action will be the same, since the isomorphism from the subgroups of Γ onto $G(B_1)$ and $\text{Aut}(B_1)$ are projections in fact.

Proposition 4.5. *The commutator subgroup Γ' is equal to the Frattini subgroup $\Phi(\Gamma)$ and they are isomorphic to the Abelian group $C_2 \times C_4$.*

Proof. If we denote $H = \Gamma_0 N$, then by the last theorem we have $\Gamma = H \cong \bar{H}$. We have already proven that $\Gamma' \cong \bar{H}' = \Phi(\bar{H}) \cong \Phi(\Gamma)$ is isomorphic to $C_2 \times C_4$.

4.2 Collineations of the 3-net B_2

Unfortunately, in the case of B_2 we can not apply the above method, because the map J is an automorphism, and so $N(M)$ does not act semi-regularly on P , the orbit of the origin.

Proposition 4.6. *Let M be a Bol 3-net coordinatized by the Bol loop (L, \cdot) and suppose that J is an automorphism of L . Then*

$$N_0 \cong G_1 < \text{Aut}(L),$$

where N_0 is the stabilizer of the origin in the normal subgroup $N(M)$ and $G_1 < G(L)$ contains the elements fixing the unit element 1 of the loop.

Proof. We saw already, that if we coordinatize the 3-net M by (L, \cdot) , then the elements of $N(M)$ are in the form

$$(\rho_u \lambda_{m_1} \dots \lambda_{m_k}, \lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}) = (\alpha, \beta),$$

with $u = m_1(\dots(m_{k-2} \dots_{k-1} m_k) \dots) = 1^{\beta^{-1}}$. If the map J is an automorphism, then

$$1^{\lambda_{m_1} \dots \lambda_{m_k}} = (1^{\lambda_{m_1}^{-1} \dots \lambda_{m_k}^{-1}})^{-1},$$

That means that in our case,

$$\beta \in G_1 \Leftrightarrow u = 1 \text{ and } 1^\alpha = (1^\beta)^{-1} \Leftrightarrow (\alpha, \beta) \text{ fixes } 1,$$

and so we have a bijection from G_1 onto N_0 , which is the restriction of Φ^{-1} , thus, it is an isomorphism.

By [4], we have $|G_1| \cdot |B_2| = |G(B_2)| = 16$, hence $G_1 = \{id, \tau\}$, where $\tau = \lambda_a \lambda_b \lambda_c$ with $c \cdot ba = 1$. By 4.6,

$$N_0 = \{(id, id), (\tau, \tau)\} = N \cap \Gamma_0 / \Gamma_0.$$

Now, let us consider the collineation group of the 3-net as a permutation group on the horizontal lines. From the paper [4], we know that the subgroup $N = N(M)$ acts transitively on this set of lines. By 1.3, an element of the stabiliser of such a horizontal line, say l_h , determines a left pseudo-automorphism of the loop B_2 , whit a companion element. However, any left pseudo-automorphism of B_2 is an automorphism and the left companion elements form the group N_λ . Thus, the stabiliser of l_h is isomorphic to the semidirect product $\text{Aut}(B_2) \times_\varphi N_\lambda(B_2)$, since an automorphism of the loop induces an automorphism on the left nucleus (see 1.2).

Moreover, the left nucleus of B_2 has 2 elements, so its only automorphism is the identity, hence we have

Proposition 4.7. *Let N denote the 3-net coordinatized by the Bol loop B_2 . Then for the stabiliser Γ_H of the vertical line l_h through the origin in the direction preserving collineation group of the net, we have the direct product*

$$\Gamma_H \cong D_8 \times C_2.$$

Proposition 4.8. *For the 3-net N , coordinatized by the Bol loop B_2 , we have*

$$\Gamma/N \cong (Aut(B_2)/G_1) \times N_\lambda \cong C_2 \times C_2 \times C_2.$$

Proof. By the isomorphism theorem of groups,

$$\Gamma/N = \Gamma_H N / N \cong \Gamma_H / (\Gamma_H \cap N).$$

Obviously, $N_0 \subseteq \Gamma_H \cap N$, since if a collineation fixes the origin, it fixes the line through it, as well. On the other hand, if $(\alpha, \beta) \in \Gamma_H \cap N$, then $1^\beta = 1$, and by 4.6, $1^\alpha = 1$, thus $N_0 = \Gamma_H \cap N$. But N_0 is normal in Γ_0 which is normal in $\Gamma_H = \Gamma_0 \times C_2$, so

$$\Gamma_H / (\Gamma_H \cap N) = (\Gamma_0 \times C_2) / N_0 = (\Gamma_0 / N_0) \times C_2 \cong (Aut(B_2) / G_1) \times N_\lambda.$$

We still have to show that $Aut(B_2) / G_1 \cong C_2 \times C_2$. The group $Aut(B_2)$ is isomorphic to D_8 , which contains 5 involutions, but only one of them commutes with all the others. In the case of $Aut(B_2)$ this is $\tau = \lambda_a \lambda_b \lambda_c \in G(L)$, with $c \cdot ba = 1$. Now, if we factorize by this special involution, we obtain exactly the factor group $C_2 \times C_2$.

Corollary 4.9. *Denote the subgroup $\Gamma_0 N$ of the collineation group Γ by H and by σ the collineation (λ_0, id) where λ_0 is the left translation with the nontrivial element of the left nucleus of B_2 . Then*

- (i) $\Gamma = H \times \{(id, id), \sigma\}$;
- (ii) the commutator subgroup Γ' is isomorphic to the Abelian group $C_2 \times C_2 \times C_2$;
- (iii) the Frattini subgroup of Γ is equal to the commutator subgroup, thus, Γ is generated by 4 elements.

Proof. Obviously, σ is an involutory collineation of the 3-net. From the definition of the left nucleus elements follows that λ_0 commutes with all right translations. By [3], λ_0 is contained in the center of the group $G(B_2)$, and so λ_0 commutes with any element of N . The left nucleus has order 2, an automorphism of the loop induces an automorphism of the nuclei, hence the elements of the left nucleus are fixed by every automorphism. With other words, $\sigma = (\lambda_0, id)$ commutes with each element of Γ_0 .

Moreover, if an element of N fixes the horizontal line l_h through the origin, then it fixes the origin, too. So does any element $\gamma_0 \nu \in H$, where $\gamma_0 \in \Gamma_0$ and $\nu \in N$, hence σ is not contained in H . And the subgroup H has index 2, since for the factor groups $|H/N| = 4$ and $|\Gamma/N| = 8$, thus (i) is proven.



From this it follows that $\Gamma' = H' \cong C_2 \times C_2 \times C_2$.

Clearly, the Frattini subgroup $\Phi(\Gamma)$ is generated by the squares in Γ , this means by (i) that $\Phi(\Gamma) = \Phi(H) = H'$, see Theorem 5.5. Finally, we have

$$\Gamma/\Gamma' \cong C_2 \times C_2 \times C_2 \times C_2,$$

the factor group has order 2^4 , but than the minimal number of element which generate Γ is 4 (cf [8], p. 58.).

Theorem 4.10. *There exists an Abelian normal subgroup Λ of $G(B_2)$, which operates sharply transitively on the set B_2 , and holds $H \cong \text{Aut}(B_2) \times_{\varphi} \Lambda$.*

Proof. By [9], we know that there is a unique Abelian normal subgroup Λ of $G(B_2)$, which operates sharply transitively on B_2 . To this Λ , an Abelian normal subgroup N_{Λ} of N corresponds, which acts sharply transitively on the set of vertical lines. Its uniqueness means that it is a normal subgroup in H as well. Then we have that H/N_{Λ} is isomorphic to the stabilizer of the vertical line through the origin in H , which is equal to the stabilizer of the origin in Γ , thus we have the desired semidirect product $H \cong \text{Aut}(B_2) \times_{\varphi} \Lambda$.

And by this, the proof of the Main Theorem is complete. Finally, let us turn back to the full collineation group. We saw already, that any collineation of a non-Moufang Bol 3-net leaves the set of vertical lines fixed. This means that a collineation which does not preserve the directions must interchange the horizontal and the trasversal directions, the product of two such ones is a direction preserving collineation. Thus, the $\Gamma(M)$ is a normal subgroup of index 2 in the full group of collineations of the 3-net M .

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Gábor Nagy

Bolyai Institute

József Attila University

Aradi vértanúk tere 1.

6720 Szeged - HUNGARY