THE 2-SUMMING NORM OF l_p^n COMPUTED WITH n VECTORS

A. HINRICHS

Abstract. *The 2-summing norm of an n-dimensional Banach space computed with n vectors is known to belong between $n^{1/2}/\sqrt{2}$ and $n^{1/2}$. It is shown that the 2-summing norm of real l_1^3 computed with three vectors is 5/3. Some lower estimates for 2-summing norms of l_p^n computed with n vectors are stated, which are considerably better than the universal ones and are based on the existence of certain block designs or Hadamard matrices.*

1. INTRODUCTION

Let X be a finite dimensional (real or complex) Banach space and let X' be its dual. The 2-summing norm $\pi_2(X)$ of X is defined as the smallest constant c such that

$$\left(\sum_{k=1}^{m}\|x_k\|^2\right)^{1/2} \leq c \sup\left\{\left(\sum_{k=1}^{m}|\langle x_k, a\rangle|^2\right)^{1/2} \quad : \quad a \in X', \|a\| \leq 1\right\}$$

for all positive integers m and all families $x_1, \dots, x_m \in X$. If the number of elements in the above inequality is restricted to n we usually get another constant which we shall denote by $\pi_2^{(n)}(X)$. It is known (see for instance [5]), that $n^{1/2}/\sqrt{2} \leq \pi_2^{(n)}(X) \leq \pi_2(X) = n^{1/2}$ if $\dim(X) = n$. For information on summing norms we also refer to [3].

As usual, for $1 \leq p \leq \infty$, we denote by l_p^n the n-dimensional Banach space of all vectors $x = (x(k))_{k=1}^n$ of (real or complex) scalars with norm

$$\|x\|_p = \left(\sum_{k=1}^{n}|x(k)|^p\right)^{1/p}.$$

We sometimes write $l_p^n(R)$ or $l_p^n(C)$ to indicate the field of scalars.

The paper deals with the quantities $\pi_2^{(n)}(l_p^n)$. We will show that the general estimates stated above can be improved in certain cases of this special situation.
It can be derived from the geometry of the maximal volume ellipsoid contained in the unit ball of l_p^n (which is actually a certain multiple of the standard euclidian ball), that $\pi_2^{(n)}(l_p^n) = \sqrt{n}$ in the complex case and for $2 \leq p \leq \infty$ in the real case. The same holds in the real case for $1 \leq p < 2$ iff n is a Hadamard number, i.e. there exists a matrix A of order n with entries ± 1 that satisfies $AA' = nI_n$. Here I_n is the unit matrix of order n. For convenience, we give an analytical prove of this fact in section 2.

Theorem 3.1 gives a lower estimate for the remaining case of real scalars, $1 \leq p < 2$ and n no Hadamard number. It is obtained from assumptions on the existence of certain designs. Theorem 4.1 states that $\pi_2^{(3)}(l_1^3) = 5/3$.

2. CASE OF MAXIMALITY

Proposition 2.1. *Let n be a positive integer. Then*

(i) $\pi_2^{(n)}(l_p^n(\mathbb{C})) = \sqrt{n}$ *for* $1 \leq p \leq \infty$.

(ii) $\pi_2^{(n)}(l_p^n(\mathbb{R})) = \sqrt{n}$ *for* $2 \leq p \leq \infty$.

(iii) For $1 \leq p < 2, \pi_2^{(n)}(l_p^n(\mathbb{R})) = \sqrt{n}$ *if and only if n is a Hadamard number.*

Proof. Clearly, $\pi_2^{(n)}(l_p^n) \leq \sqrt{n}$ in both the real and the complex case. Let $2 \leq p \leq \infty$ and choose $x_1, \ldots, x_n$ as the standard bases of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. Then we have

$$\left(\sum_{k=1}^{n} \|x_k\|_p^2\right)^{1/2} = \sqrt{n}$$

and for arbitrary $a \in l_{p'}^n$,

$$\left(\sum_{k=1}^{n} |\langle x_k, a\rangle|^2\right)^{1/2} = \left(\sum_{k=1}^{n} |a(k)|^2\right)^{1/2} \leq \|a\|_{p'}.$$

Hence, $\pi_2^{(n)}(l_p^n) \geq \sqrt{n}$.

Now, let us assume that $1 \leq p < 2$. In the case of complex scalars, we define vectors $x_1, \ldots, x_n$ by

$$x_k(\ell) := \exp(\frac{2\pi i}{n} k\ell).$$

Then

$$\left(\sum_{k=1}^{n} \|x_k\|_p^2\right)^{1/2} = n^{1/2+1/p}$$

and, since the matrix $(x_k(\ell))_{k,\ell}$ is unitary,

$$\begin{aligned}\left(\sum_{k=1}^{n} |\langle x_k, a\rangle|^2\right)^{1/2} &= \left(\sum_{k=1}^{n} \left(\sum_{\ell=1}^{n} x_k(\ell) a(\ell)\right)^2\right)^{1/2} \\ &= n^{1/2} \left(\sum_{k=1}^{n} |a(k)|^2\right)^{1/2} = n^{1/2}\|a\|_2 \leq n^{1/p}\|a\|_{p'}.\end{aligned}$$

Thus, $\pi_2^{(n)}(l_p^n) \geq \sqrt{n}$.

The same computation applies for real scalars, if we choose the vectors $x_1, \ldots, x_n$ such that $(x_k(\ell))$ is a Hadamard matrix.

Finally, we have to show that $\pi_2^{(n)}(l_p^n) = \sqrt{n}$ for real scalars and $1 \leq p < 2$ implies that n is a Hadamard number. The assumption enables us to find vectors $x_1, \ldots, x_n \in l_p^n$ such that

$$\left(\sum_{k=1}^{n} \|x_k\|_p^2\right)^{1/2} = n^{1/2+1/p} \tag{1}$$

and

$$\left(\sum_{k=1}^{n} |\langle x_k, a\rangle|^2\right)^{1/2} \leq n^{1/p}\|a\|_{p'} \tag{2}$$

for $a \in l_{p'}^n$. For fixed ℓ, we choose $a \in l_{p'}^n$ with $\|a\|_{p'} = 1$ and $\|x_\ell\|_p = \langle x_\ell, a\rangle$. Therefore,

$$\|x_\ell\|_p \leq \left(\sum_{k=1}^{n} |\langle x_k, a\rangle|^2\right)^{1/2} \leq n^{1/p}.$$

Now, (1) implies $\|x_\ell\|_p = n^{1/p}$ for $\ell = 1, \ldots, n$.

Let us denote the set $\{-1, 1\}^n$ by D_n. It is a well-known fact that

$$\left(\frac{1}{2^n} \sum_{d \in D_n} \langle x, d\rangle^2\right)^{1/2} = \|x\|_2 \tag{3}$$

for any n-dimensional vector x. Since $\|d\|_{p'} = n^{1/p'}$ for $d \in D_n$, the following chain of inequalities holds:

$$\begin{aligned}
n^{1/p+1/2} &= \left(\sum_{k=1}^{n} \|x_k\|_p^2\right)^{1/2} \leq n^{1/p-1/2}\left(\sum_{k=1}^{n} \|x_k\|_2^2\right)^{1/2} = \\
&= n^{1/p-1/2}\left(\frac{1}{2^n} \sum_{d \in D_n} \sum_{k=1}^{n} \langle x_k, d\rangle^2\right)^{1/2} \leq \\
&\leq n^{1/p-1/2}\left(\frac{1}{2^n} \sum_{d \in D_n} \sum_{k=1}^{n} n^{2/p} n^{2/p'}\right)^{1/2} \leq n^{1/p+1/2}.
\end{aligned}$$

Hence, we actually have equalities. This implies

$$\|x_k\|_p = n^{1/p-1/2}\|x_k\|_2 \quad \textit{for } k = 1, \ldots, n$$

and

$$\left(\sum_{k=1}^{n} \langle x_k, d\rangle^2\right)^{1/2} = n \quad \textit{for } d \in D_n.$$

By ([2], Theorem 16), we conclude from the first equation $|x_k(\ell)| = 1$ for $k, \ell = 1, \ldots, n$.

Now, let $1 \leq i < j \leq n$ and $x \in \mathbb{R}^n$. Let $y_\pm \in \mathbb{R}^{n-1}$ be the vector

$$(x(1), \ldots, x(i-1), x(i) \pm x(j), x(i+1), \ldots, x(j-1), x(j+1), \ldots, x(n)).$$

By (3), we have

$$\frac{1}{2^{n-1}} \sum_{\substack{d \in D_n \\ d(i)=d(j)}} \langle x, d\rangle^2 = \frac{1}{2^{n-1}} \sum_{d \in D_{n-1}} \langle y_+, d\rangle^2 = \|y_+\|_2^2 = \|x\|_2^2 + 2x(i)x(j).$$

Analogously we get with y_-

$$\frac{1}{2^{n-1}} \sum_{\substack{d \in D_n \\ d(i) \neq d(j)}} \langle x, d\rangle^2 = \|x\|_2^2 - 2x(i)x(j).$$

Then

$$4\sum_{k=1}^{n} x_k(i)x_k(j) = \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}} \sum_{\substack{d \in D_n \\ d(i) = d(j)}} \langle x_k, d\rangle^2 - \frac{1}{2^{n-1}} \sum_{\substack{d \in D_n \\ d(i) \neq d(j)}} \langle x_k, d\rangle^2 \right) = 0.$$

Thus, the matrix $(x_k(\ell))$ is Hadamard.

Remark. This proof follows similar lines as a proof in [4].

3. LOWER ESTIMATES

From now we only deal with real scalars, $1 \leq p < 2$ and non-Hadamard numbers n. We describe a method to obtain a lower estimate of $\pi_2^{(n)}(l_p^n)$ using certain designs.

We shall look for orthonormal systems $x_1, \dots, x_n \in \mathbb{R}^n$ such that all x_k are 'close' to vectors with coordinates $\pm n^{-1/2}$. Let us assume that the coordinates of $x_1, \dots, x_n$ take only two values, say x and y. Furthermore, x and y shall occur k and $n-k$ times, respectively. To handle the orthogonality conditions, we additionally require the following relations for every pair of vectors x_i and x_j with $i \neq j$:

$$\begin{aligned} \mathrm{card}\{\ell : x_i(\ell) = x_j(\ell) = x\} &= \lambda \\ \mathrm{card}\{\ell : x_i(\ell) = x_j(\ell) = y\} &= \mu \\ \mathrm{card}\{\ell : x_i(\ell) \neq x_j(\ell)\} &= \nu, \end{aligned}$$

where λ, μ and $\nu > 0$ are certain nonnegative integers. Let us regard the matrix $A = (\alpha_{ij})$, $i = 1, \dots, n, j = 1, \dots, n$ defined by

$$\alpha_{ij} = \begin{cases} 1 \text{ if } x_i(j) = x \\ 0 \text{ if } x_i(j) = y. \end{cases}$$

Then we meet a well-known combinatorial concept: A is the incidence matrix of a symmetric balanced incomplete block design. For this concept see e.g. [1] or [6]. We only need the fact that λ, μ and ν are determined by n and k:

$$\lambda = \frac{k(k-1)}{n-1}, \mu = 2(k-\lambda), \nu = n - \lambda - \mu.$$

Of course, $n-1$ must be a factor of $k(k-1)$ ([1] p. 126). To formulate the next theorem let us call such such a design an (n, k)-design.

Theorem 3.1. *Let n be a positive integer and $1 \leq p < 2$. Assume that there exists an (n,k)-design. Let x, y be solutions of the system*

$$\begin{aligned} kx^2 + (n-k)y^2 &= 1 \\ \lambda x^2 + \nu xy + \mu y^2 &= 0, \end{aligned}$$

where $\lambda = \frac{k(k-1)}{n-1}$, $\mu = 2(k-\lambda), \nu = n - \lambda - \mu$. Then

$$\pi_2^{(n)}(l_p^n) \geq n^{1/2}\left(\frac{k|x|^p + (n-k)|y|^p}{n^{1-p/2}}\right)^{1/p}.$$

Proof. Let us regard the incidence matrix $A = (a_{ij})$ of the given (n,k)-design. We define vectors $x_1, \ldots, x_n \in l_p^n$ by

$$x_i(j) := \begin{cases} x \text{ if } a_{ij} == 1 \\ y \text{ if } a_{ij} == 0 \end{cases}$$

From the assumptions on x and y one easily derives that the matrix $(x_i(j))$ is orthogonal. Therefore, we compute for arbitrary $a \in l_{p'}^n$

$$\left(\sum_{i=1}^{n} |\langle x_i, a\rangle|^2\right)^{1/2} = \left(\sum_{j=1}^{n} |a(j)|^2\right)^{1/2} = \|a\|_2 \leq n^{1/p-1/2}\|a\|_{p'}.$$

Furthermore,

$$\left(\sum_{i=1}^{n} \|x_i\|_p^2\right)^{1/2} = n^{1/2}(k|x|^p + (n-k)|y|^p)^{1/p}.$$

This proves the theorem.

To illustrate the theorem let us state some examples. It is easy to show (see again [1] p. 239) that $n+1$ is a Hadamard number iff there exists an $\left(n, \frac{n-1}{2}\right)$-design. Then

$$\begin{aligned} x &= \frac{1}{n}\sqrt{n+2-2\sqrt{n+1}} \\ y &= -\frac{1}{n}\sqrt{\frac{n^2-n+2+2(n-1)\sqrt{n+1}}{n+1}} \end{aligned}$$

satisfy the equations of the theorem. Thus,

$$\pi_2^{(n)}(l_p^n) \geq n^{1/2}\left(\frac{(n-1)|x|^p + (n+1)|y|^p}{2n^{1-p/2}}\right)^{1/p}.$$

The case $n = 3$ is a special example for this situation. Then $k = 1$ and we have to consider the matrix

$$\begin{pmatrix} x\ y\ y \\ y\ x\ y \\ y\ y\ x \end{pmatrix}.$$

The equations

$$x^2 + 2y^2 = 1 \quad \text{and} \quad 2xy + y^2 = 0$$

are satisfied for $x = 1/3$ and $y = -2/3$. This yields $\pi_2^{(3)}(l_1^3) \geq 5/3$. We prove in the next section that this is the exact value. A good estimate for the next interesting case $n = 5$ is provided by the trivial (5,1)-design, which gives the matrix

$$\begin{pmatrix} x\,y\,y\,y\,y \\ y\,x\,y\,y\,y \\ y\,y\,x\,y\,y \\ y\,y\,y\,x\,y \\ y\,y\,y\,y\,x \end{pmatrix}.$$

Hence $x = 3/5, y = -2/5$ and

$$\pi_2^{(5)}(l_p^5) \geq \left(\frac{3^p + 4 * 2^p}{5}\right)^{1/p} \geq \frac{11}{5}.$$

In view of section 2, it is rather clear that the behaviour of the sequence $\pi_2^{(n)}(l_p^n)$ is significantly influenced by the frequency of Hadamard numbers.
If we choose the positive integer k such that $2^k \leq n < 2^{k+1}$ then already the Hadamard-Walsh matrices for powers of 2 provide us with the Tomczak-Jaegermann bound for l_p^n:

$$\pi_2(l_p^n) = n^{1/2} < 2^{(k+1)/2} = \sqrt{2}\pi_2^{(2^k)}(l_p^{2^k}) \leq \sqrt{2}\pi_2^{(n)}(l_p^n).$$

If even the Hadamard conjecture were true, i.e. apart from 1 and 2 exactly the multiplies of 4 are Hadamard, then we would even have that $\sqrt{n} - \pi_2^{(n)}(l_p^n)$ tends to zero (with order $n^{-1/2}$).

4. THE CASE OF REAL l_1^3

Theorem 4.1. *For real scalars* $\pi_2^{(3)}(l_1^3) = 5/3$.

Proof. We already obtained $\pi_2^{(3)}(l_1^3) \geq 5/3$. The reverse inequality is proved using a definite asymmetry which we produce at first. Let $x_1, x_2, x_3 \in l_1^3$. It remains to show that

$$\sum_{i=1}^{3} \|x_i\|^2 \leq \frac{25}{9} \sup_{\|a\|_\infty \leq 1} \sum_{i=1}^{3} \langle x_i, a\rangle^2.$$

To this end, we arrange these vectors as a matrix

$$\begin{pmatrix} x_1(1)\; x_1(2)\; x_1(3) \\ x_2(1)\; x_2(2)\; x_2(3) \\ x_3(1)\; x_3(2)\; x_3(3) \end{pmatrix}.$$

Observe that both sides of the above inequality are invariant under the following manipulations:

- permutation of rows or columns
- multiplication of a row or a column by -1.

Therefore, it is enough to verify the inequality for the cases

$$\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix} \quad \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & + \end{pmatrix},$$

where + and - means that $x_i(j) \geq 0$ and $x_i(j) \leq 0$, respectively.

Now, let

$$a_1 = (1,1,1), a_2 = (1,1,-1), a_3 = (1,-1,1), a_0 = (1,-1,-1) \in l_\infty^3$$

and

$$f(x) = 3\sum_{j=1}^{3} \langle x, a_j \rangle^2 + \langle x, a_0 \rangle^2$$

for $x \in l_1^3$. We show that

$$f(x) \geq \frac{18}{5} \|x\|^2$$

for every vector $x = (x(1), x(2), x(3))$ which satisfies one of the following conditions:

$$\begin{aligned} (i)\ & x(1) \geq 0 \quad x(2) \geq 0 \quad x(3) \geq 0 \\ (ii)\ & x(1) \geq 0 \quad x(2) \geq 0 \quad x(3) < 0 \\ (iii)\ & x(1) \geq 0 \quad x(2) < 0 \quad x(3) \geq 0. \end{aligned}$$

Since $f(x') = f(x)$ for $x' = (x(1), x(3), x(2))$, case (iii) reduces to case (ii). Let us treat case (i). By the Cauchy-Schwarz inequality,

$$\begin{aligned} \|x\| &= \langle x, a_1 \rangle = \langle x, a_2 \rangle + \langle x, a_3 \rangle - \langle x, a_0 \rangle \\ &\leq \left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3}\right)^{1/2} \left((3\langle x, a_2 \rangle)^2 + (3\langle x, a_3 \rangle)^2 + 3\langle x, a_0 \rangle^2\right)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= 3\|x\|^2 + \frac{1}{3}\left((3\langle x, a_2 \rangle)^2 + (3\langle x, a_3 \rangle)^2 + 3\langle x, a_0 \rangle^2\right) \\ &\geq \frac{18}{5}\|x\|^2. \end{aligned}$$

Now, let us assume (ii). Again by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|x\| &= \langle x, a_2 \rangle = \langle x, a_1 \rangle - \langle x, a_3 \rangle + \langle x, a_0 \rangle \\ &\leq \left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3}\right)^{1/2} \left((3\langle x, a_1 \rangle)^2 + (3\langle x, a_3 \rangle)^2 + 3\langle x, a_0 \rangle^2\right)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= 3\|x\|^2 + \frac{1}{3}\left((3\langle x, a_1\rangle)^2 + (3\langle x, a_3\rangle)^2 + 3\langle x, a_0\rangle^2\right) \\ &\geq \frac{18}{5}\|x\|^2. \end{aligned}$$

We define the function g in l_∞^3 by

$$g(a) := \sum_{i=1}^{3} \langle x_i, a\rangle^2.$$

Clearly, we have

$$\sup_{\|a\|_\infty \leq 1} g(a) \geq \frac{1}{10}\left(3\sum_{j=1}^{3} g(a_j) + g(a_0)\right) = \frac{1}{10}\sum_{i=1}^{3} f(x_i).$$

But now we are done:

$$\sum_{i=1}^{3} \|x_i\|^2 \leq \frac{5}{18}\sum_{i=1}^{3} f(x_i) \leq \frac{25}{9} \sup_{\|a\|_\infty \leq 1} \sum_{i=1}^{3} \langle x_i, a\rangle^2.$$

REFERENCES

[1] M. HALL, *Combinatorial Theory*, New York: Wiley 1986.
[2] G.H. HARDY, J.E. LITTLEWOOD and G. POLYA, *Inequalities*, Cambridge Univ. Press 1934.
[3] A. PIETSCH, *Operator ideals*, Amsterdam, New York, Oxford: North Holland 1980.
[4] A. PIETSCH, *Sequences of ideal norms*, Note di Mat. 2 (1990), 411-441.
[5] N. TOMCZAK-KAEGERMANN, *Banach-Mazur distances and finite-dimensional operator ideals*, Harlow: Pitman 1989.
[6] W.D. WALLIS, A.P. STREET and J.S. WALLIS, *Combinatorics: Room squares, sum-free sets, Hadamard matrices*, Lecture Notes in Math. 292, Springer 1972.

Received March 11, 1994
Aicke Hinrichs
FSU Jena
Mathematisches Institut
07740 Jena
e-mail: nah@hpux.rz.uni-jena.de