A QUANTIZATION OF DIFFUSION PROCESSES ON MANIFOLDS

Y.G. LU

Abstract. *Via the well known chaos decomposition with respect to the multidimensional Brownian motion, we give a possible quantization of a diffusion process (in particular, Brownian motion) on a Riemannian manifold. The quantized diffusion process on Riemannian manifold is a quantum Brownian motion on a Hilbert space determined by the momentum algebra.*

1. INTRODUCTION

In the theory of stochastic processes (both in the classical and the quantum case), Brownian motion plays a fundamental role. From a theoretical point of view, Brownian motion is the milestone of stochastic calculus; from a practical point of view, Brownian motion and the related stochastic calculus gives a good description of many interesting phenomena.

Quantum Brownian motion in a certain sense is a pair of classical Brownian motions. But one can't imagine this pair of Brownian motions as a usual 2-dimensional Brownian motion since the two do not commute. A typical example of quantum Brownian motion is the pair: momentum and position processes $\{P(t)\}_{t\leq 0}$, $\{Q(t)\}_{t\geq 0}$ in the Fock space over $L^2(\mathbb{R}_+)$. Each of these processes is a family of operators on the Fock space, and they do not commute but satisfy the canonical (or Boson) commutation relation:

$$[P(s), Q(t)] = i\min(s,t) \tag{1.1}$$

The fundamental tool to establish a relation between the process $\{P(t)\}_{t\leq 0}$ (resp. $\{Q(t)\}_{t\leq 0}$) and the classical Brownian motion is the so called chaos decomposition. In the 1-dimensional case (the multidimensional case is similar), if $\{B(t)\}_{t\leq 0}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, any element in the space $L^2(\Omega)$, say F, has the representation:

$$E(F) + \int_0^\infty f_1(t)dB(t) + \int_{\Delta_2} f_2(t_1,t_2)dB(t_1)dB(t_2) + \dots \tag{1.2}$$

where, for any $n\in\mathbb{N}$, Δ_n denotes the n-simplex (i.e. the set $\{(t_1,\dots,t_n) : t_1 \leq t_2 \leq \dots t_n\}$); the family of functions $\{f_n\}_{n=1}^\infty$ is determined uniquely by the function F and for any $n\in\mathbb{N}$, $f_n \in L^2(\mathbb{R}^n)$.

It is well known that there is an isomorphism π_P (resp. π_Q) between the Fock space $\Gamma(L^2(\mathbb{R}_+))$ and $L^2(\Omega)$, such that if $B(t)$ is considered as an unbounded, self-adjoint operator on $L^2(\Omega)$:

$$(B(t,\cdot)F)(\omega) \;:=\; B(t,\omega)F(\omega) \tag{1.3}$$

the induced map on operators sends $P(t)$ (resp. $Q(t)$) to $B(t)$ for any $t\in\mathbb{R}_+$. Now a natural question arises: how can one quantize a classical Brownian motion, or generally, a diffusion process, on a Riemannian manifold?

A simple and clear way to look a classical diffusion process on a manifold M is to consider a stochastic differential equation:

$$dX(t) = L_\alpha(X(t)) \bullet dB^\alpha(t) + L_0(X(t))dt \tag{1.4}$$

where,

$-\{X(t)\}_{t\geq 0}$ is a stochastic process on M;

$-\{B^1(t), \ldots, B^d(t)\}_{t\geq 0}$ is a classical d-dimensional Brownian motion on $\mathbb{R}^d$;

$-\{L_0, L_1, \ldots, L_d\}$ is a family of (smooth) vector fields on M;

$- \bullet\, dB(t)$ means the Stratonovich stochastic differential with respect to the standard Brownian motion.

The solution of the equation (1.4) is called the L-diffusion process on the manifold.

A standard argument shows that the equation (1.4) can be rewritten in the form:

$$dX(t) = L_\alpha(X(t))dB^\alpha(t) + \left(L_0(X(t)) + \frac{1}{2}L_\alpha(X(t)))\right) df \tag{1.4'}$$

where, the Stratonovich stochastic differential has been replaced by the Ito stochastic differential.

The Brownian motion on the manifold is the solution of the stochastic differential equation (1.4) (or equivalently (1.4)′) by taking:

$-L_0 = 0$

$-L_\alpha$ the canonical horizontal vector field on the orthonormal frame bundle $O(M)$:

$$L_\alpha := L_\alpha(e) := e^\beta_\alpha \underset{\beta}{\partial} - \Gamma^q_{i,j} e^i_\alpha e^j_\beta \frac{\partial}{\partial e^q_\beta} \tag{1.5}$$

That is one has, first of all, a Brownian motion $\{r(t)\}_{t\geq 0}$ on $O(M)$, then the process $\{X(t)\}_{t\geq 0}$ is obtained as the projection (form the orthonormal frame bundle $O(M)$ to the base manifold M) of $\{r(t)\}_{t\geq 0}$.

An important fact is that the equation (1.4) is just a notation. Precisely, one must interpret it as follows: for any $f \in C_0^\infty(M)$,

$$df(X(t)) = (L_\alpha f)(X(t)) \bullet dB^\alpha(t) + (L_0 f)(X(t))dt \tag{1.6}$$

In particular, the Brownian motion on the manifold M is determined by the equation

$$df(X(t)) = e^i_j(\underset{x_i}{\partial} f)(X(t))dB^j(t) + \frac{1}{2}(\Delta f)(X(t))df \tag{1.7}$$

where, Δ is the Laplace-Beltrami operator on functions.

In the present note, we shall work with the stochastic differential equation (1.6) forgetting the explicit form of vector fields. For the existence and uniqueness of the solution of the stochastic differential equation (1.6), see e.g. [1]. Moreover, thanks to the Whitney embedding theorem, one can suppose (if necessary) that our manifold M is a subset of multidimensional Euclidean space and includes the origin.

2. SOME PRELIMINARY DISCUSSIONS

Now let us do some preparations in order to quantize the diffusion process $\{X(t)\}_{t\geq 0}$ on the given manifold M, which is the solution of the stochastic differential equation (1.6) with certain initial condition $X(0) = x \in M$.

Clearly, the main problem now is how to interpret the following objects:

$$dX(t), \qquad dX(t_1)dX(t_2), \qquad dX(t_1)dX(t_2)dX(t_3), \dots$$

The L-diffusion process (on the manifold M) $\{X(t)\}_{t\geq 0}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}_t, P)$ takes values on M. On the same probability space, we have the classical d-dimensional ($d \geq 1$) Brownian motion $\{B(t)\}_{t\geq 0}$. The chaos decomposition establishes an isomorphism between $L^2(\Omega)$ and

$$\mathbb{C} \oplus L^2(\mathbb{R}_+) \otimes \mathbb{C}^d \oplus L^2(\Delta_2) \otimes \mathbb{C}^d \otimes \mathbb{C}^d \oplus L^2(\Delta_3) \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \oplus \dots$$

by the formula

$$F = f_0 + \sum_{n=1}^{\infty} \sum_{1\leq j_1,\dots,j_n\leq d} \int_{\Delta_n} f_n^{(j_1,\dots,j_n)}(t_1,\dots,t_n)dB^{j_1}(t_1)\dots dB^{j_n}(t_n) \tag{2. 1}$$

Moreover, the system of the functions $\{f_n^{(j_1,\dots,j_n)}\}_{n=0}^{\infty}$ is determined uniquely by $F \in L^2(\Omega)$. This shows that one can identify F to the system $\{f_n^{(j_1,\dots,j_n)}\}_{n=0}^{\infty}$. In order to underline the fact, we shall write $\{f_n^{(j_1,\dots,j_n)}{}_n\}_{n=0}^{\infty} \in L^2(\Omega)$

For each $\{f_n^{j_1,\dots,j_n)}\}_{n=0}^{\infty} \in L^2(\Omega)$, let consider a new object

$$f_0 + \sum_{1\leq j\leq d} \int_0^{\infty} f_1^{(j)}(t_1) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) +$$

$$+ \sum_{1\leq j_1,j_2\leq d} \int_{\Delta_2} (f_2^{(j_1,j_2)}(t_1,t_2)$$

$$\left(L_{j_1}(X(t_1))dB^{j_1}(t_1) + \frac{1}{2}L_{j_1}^2(X(t_1))dt_1 \right) * \left(L_{j_2}(X(t_2))dB^{j_2}(t_2) + \frac{1}{2}L_{j_2}^2(X(t_2))dt_2 \right) +$$

$$+ \sum_{1\leq j_1,j_2,j_3\leq d} \int_{\Delta_3} f_2^{(j_1,j_2,j_3)}(t_1,t_2,t_3) \left(L_{j_1}(X(t_1))dB^{j_1}(t_1) + \frac{1}{2}L_{j_1}^2(X(t_1))dt_1 \right) *$$

$$* \left(L_{j_2}(X(t_2))dB^{j_2}(t_2) + \frac{1}{2}L_{j_2}^2(X(t_2))dt_2 \right) *$$

$$* \left(L_{j_3}(X(t_3))dB^{j_3}(t_3) + \frac{1}{2}L_{j_3}^2(X(t_3))dt_3 \right) + \dots =$$

$$= f_0 + \sum_{n=1}^{\infty} \sum_{1\leq j_1,\dots,j_n\leq d} \int_{\Delta_n} f_n^{(j_1,\dots,j_n)}(t_1,\dots,t_n)$$

$$*_{k=1}^{n}\left(L_{j_k}(X(t_k))dB^{j_k}(t_k)+\frac{1}{2}L_{j_k}^2(X(t_k))dt_k\right) \tag{2.2}$$

where, for $t_1\neq t_2$,

$$\left(L_{j_1}(X(t_1))dB^{j_1}(t_1)+\frac{1}{2}L_{j_1}^2(X(t_1))dt_1\right)*$$

$$*\left(L_{j_2}(X(t_2))dB^{j_2}(t_2)+\frac{1}{2}L_{j_2}^2(X(t_2))dt_2\right):=$$

$$:=L_{j_1}(X(t_1))\otimes L_{j_2}(X(t_2))dB^{j_1}(t_1)dB^{j_2}(t_2)+L_{j_1}^2(X(t_1))\otimes L_{j_2}(X(t_2))dt_1dB^{j_2}(t_2)+$$

$$+L_{j_1}(X(t_1))\otimes L_{j_2}^2(X(t_2))dB^{j_1}(t_1)dt_2+L_{j_1}^2(X(t_1))\otimes L_{j_2}^2(X(t_2))dt_1dt_2 \tag{2.3}$$

and in general, for any $(t_1,\dots,t_n)\in\Delta_n$ and $\sigma\in\mathcal{S}_n$,

$$*_{k=1}^{n}\left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)})+\frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)}\right):=\sum_{p=0}^{n}\sum_{1\le l_1\le l_2<\dots<l_p\le n}$$

$$L_{j_1}(X(t_{\sigma(1)}))\otimes\dots\otimes L_{j_{l_1}}^2(X(t_{\sigma(l_1)}))\otimes\dots\otimes L_{j_{l_p}}^2(X(t_{\sigma(l_p)}))\otimes\dots\otimes$$

$$\otimes L_{j_n}(X(t_{\sigma(n)}))dB^{j_1}(t_{\sigma(1)})\dots dt_{\sigma(l_1)}\dots dt_{\sigma_{(l_p)}}\dots dB^{j_n}(t_{\sigma(n)}) \tag{2.4}$$

i.e. corresponding to the l_1-th, $\dots,l_p$-th positions we have the operator L^2 and the usual differential operation df and corresponding to other positions we have the operator L and the Ito stochastical differential operation $dB(t)$. The quantity (2.2) (in which f_0 is a constant) will be defined as an operator from $\bigoplus_{n=1}^{\infty}[C_0^\infty(M)]^{\otimes n}$ to $\bigoplus_{k=0}^{\infty}[L^2(M)\otimes L^2(\Omega)]^{\otimes k}$: for any $c\in\mathbb{C}, g_{j,k}\in C_0^\infty(M)(k,j\in\mathbb{N},k\le j)$,

$$f_0(c,g_{1,1},g_{2,1}\otimes g_{2,2},g_{3,1}\otimes g_{3,2}\otimes g_{3,3},\dots):=(cf_0,g_{1,1},g_{2,1}\otimes g_{2,2},g_{3,1}\otimes g_{3,2}\otimes g_{3,3},\dots) \tag{2.5a}$$

and for any $n\ge 1,\sigma\in\mathcal{S}_n$

$$\int_{\Delta_n}f_n^{(j_1,\dots,j_n)}(t_1,\dots,t_n)$$

$$*_{k=1}^{n}\left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)})+\frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)}\right)\bigoplus_{k=0}^{\infty}[C_0^\infty(M)]^{\otimes k}:=$$

$$:=\bigoplus_{k=0}^{n-1}[C_0^\infty(M)]^{\otimes k}\oplus$$

$$\oplus\int_{\Delta_n}f_n^{(j_1,\dots,j_n)}(t_1,\dots,t_n)$$

$$\left\{*_{k=1}^{n}\left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)})+\frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)}\right)[C_0^\infty(M)]^{\otimes n}\right\}\oplus$$

$$\oplus \bigoplus_{k=n+1}^{\infty} [C_0^\infty(M)]^{\otimes k} \tag{2.5b}$$

where, on the subspace $[C_0^\infty(M)]^{\otimes n}$,

$$\int_{\Delta_n} f_n^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n)$$

$$*_{k=1}^n \left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)}) + \frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)} \right) (g_{n,1} \otimes \ldots \otimes g_{n,n}) :=$$

$$:= \sum_{p=0}^{n} \sum_{1\le l_1 < l_1 < l_2 < \ldots < l_p \le n} \int_{\Delta_n} f_n^{(j_1,j_n)}(t_1,\ldots,t_n)$$

$$(L_{j_1}g_{n,1})(X(t_{\sigma(1)})) \otimes \ldots \otimes (L_{j_{l_1}}(L_{j_{l_1}}g_{n,l_1}))(X(t_{\sigma(l_1)})) \otimes \ldots \otimes$$

$$\otimes (L_{j_{l_p}}(L_{j_{l_p}}g_{n,l_p}))(X(t_{\sigma(l_p)})) \otimes \ldots \otimes (L_{j_n}g_{n,n})(X(t_{\sigma(n)}))$$

$$dB^{j_1}(t_{\sigma(1)}) \ldots dt_{\sigma(l_1)} \ldots dt_{\sigma(l_p)} \ldots dB^{j_n}(t_{\sigma(n)}) \tag{2.6}$$

Denote by $\mathcal{H}_n$ the linear span of the set

$$\{ \sum_{1\le j_1,\ldots,j_n\le d} \int_{\Delta_n} f_n^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n)$$

$$*_{k=1}^n \left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)}) + \frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)} \right),$$

$$\{f_n\}_{n=0}^\infty \in L^2(\Omega), \sigma \in \mathcal{S}_n\} \tag{2.7}$$

and introduce inner products:

$$<f_0, g_0> := \bar{f}_0 g_0 \tag{2.8a}$$

Moreover, for any pair of measurable sets $U_n \subset \mathbb{R}^n, V_m \subset \mathbb{R}^m$ which satisfy the condition: $t_i \neq t_j$ if $i \neq j$; $s_p \neq s_q$ if $p \neq q$, we define

$$< \int_{U_n} f_n^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n) *_{k=1}^n \left(L_{j_k}(X(t_k))dB^{j_k}(t_k) + \frac{1}{2}L_{j_k}^2(X(t_k))dt_k \right),$$

$$\int_{V_m} g_n^{(h_1,\ldots,h_m)}(t_1,\ldots,t_n) *_{k=1}^m \left(L_{h_k}(X(t_k))dB^{h_k}(t_k) + \frac{1}{2}L_{h_k}^2(X(t_k))dt_k \right) > :=$$

$$:= \delta_m^n \int_{U_n} \int_{V_n} \mathbb{E}\left(dB^{j_1}(t_1)\ldots dB^{j_n}(t_n)dB^{h_1}(s_1)\ldots dB^{h_n}(s_n)\right)$$

$$\overline{f_n^{j_1,\ldots,j_n}(t_1,\ldots,t_n)}g_n^{(h_1,\ldots,h_n)}(s_1,\ldots,s_n) \tag{2.8b}$$

where, $\mathbb{E}\left(dB^{j_1}(t_1)\dots dB^{j_n}(t_n)dB^{h_1}(s_1)\dots dB^{h_n}(s_n)\right)$ is understood in the following way: if the U_n, V_n are such sets that there exists a variable t_j which is never equal to a s_k, the right hand side of (2.8b) is defined as zero; otherwise which is defined as

$$\delta^n_m \dots \int_{V_n} \delta^{h_{\tau(1)},\dots,h_{\tau(n)}}_{j_1,\dots,j_n} ds_1 \dots ds_n \overline{f_n^{(j_1,\dots,j_n)}(s_{\tau(1)},\dots,s_{\tau(n)})} g_n^{(h_1,\dots,h_n)}(s_1,\dots,s_n) \tag{2.8c}$$

where, τ is a n-permutation determinated by the relation: $t_j = s_{\tau(j)} \forall_j = 1,2,\dots,n$. In particular, it follows from (2.8b), by a trivial calculation, that

$$< \int_{\Delta_n} f_n^{(j_1,\dots,j_n)}(t_1,\dots,t_n) *_{k=1}^n \left(L_{j_k}(X(t_k))dB^{j_k}(t_k) + \frac{1}{2}L^2_{j_k}(X(t_k))dt_k\right),$$

$$\int_{\Delta_n} g_n^{(h_1,\dots,h_n)}(t_1,\dots,t_n) *_{k=1}^n \left(L_{h_k}(X(t_k))dB^{h_k}(t_k) + \frac{1}{2}L^2_{h_k}(X(t_k))dt_k\right) > =$$

$$= \delta^{h_1,\dots,h_n}_{j_1,\dots,j_n} \int_{\Delta_n} \overline{f_n^{j_1,\dots,j_n}(t_1,\dots,t_n)} g_n^{(h_1,\dots,h_n)}(t_1,\dots,t_n)dt_1\dots dt_n \tag{2.8d}$$

Thus, $\{\mathcal{H}_n, <\cdot,\cdot>\}$ becomes a pre-Hilbert space.

Now we try to understand the pre-Hilbert space $\mathcal{H}_n$ as a symmetric n-fold tensor product of a certain pre-Hilbert space. In fact, let consider the tensor pre-Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_1$ by introducing the following inner product on the algebraic tensor product:

$$< \int_0^\infty f_1^{(j_1)}(t)\left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2}L^2_{j_1}(X(t))dt\right) \otimes$$

$$\otimes \int_0^\infty f_2^{(j_2)}(t)\left(L_{j_2}(X(t))dB^{j_2}(t) + \frac{1}{2}L^2_{j_2}(X(t))dt\right),$$

$$\int_0^\infty g_1^{(k_1)}(t)\left(L_{k_1}(X(t))dB^{k_1}(t) + \frac{1}{2}L^2_{k_1}(X(t))dt\right) \otimes$$

$$\otimes \int_0^\infty g_2^{(k_2)}(t)\left(L_{k_2}(X(t))dB^{k_2}(t) + \frac{1}{2}L^2_{k_2}(X(t))dt\right) > :=$$

$$:= < \int_0^\infty f_1^{(j_1)}(t)\left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2}L^2_{j_1}(X(t))dt\right),$$

$$\int_0^\infty g_1^{(k_1)}(t)\left(L_{k_1}(X(t))dB^{k_1}(t) + \frac{1}{2}L^2_{k_1}(X(t))dt\right) >.$$

$$\cdot < \int_0^\infty f_2^{(j_2)}(t)\left(L_{j_2}(X(t))dB^{j_2}(t) + \frac{1}{2}L^2_{j_2}(X(t))dt\right),$$

$$\int_0^\infty g_2^{(k_2)}(t)\left(L_{k_2}(X(t))dB^{k_2}(t) + \frac{1}{2}L^2_{k_2}(X(t))dt\right) > =$$

$$= \delta_{j_1,j_2}^{k_1,k_2} \int_0^\infty \overline{f_1^{(j_1)}(t)} g_1^{(k_1)}(t)dt \quad \cdot \quad \int_0^\infty \overline{f_2^{(j_2)}}(t) g_2^{(k_2)}(t)dt \tag{2.9}$$

On the other hand, the inner product between

$$\int_0^\infty \int_0^{t_1} f_1^{(j_1)}(t_1) f_2^{(j_2)}(t_2)$$

$$\left(L_{j_1}(X(t_1))dB^{j_1}(t_1) + \frac{1}{2}L_{j_1}^2(X(t_1))dt_1\right) * \left(L_{j_2}(X(t_2))dB^{j_2}(t_2) + \frac{1}{2}L_{j_2}^2(X(t_2))dt_2\right) +$$

$$+ \int_0^\infty \int_{t_1}^\infty f_1^{(j_1)}(t_1) f_2^{(j_2)}(t_2)$$

$$\left(L_{j_1}(X(t_1))dB^{j_1}(t_1) + \frac{1}{2}L_{j_1}^2(X(t_1))dt_1\right) * \left(L_{j_2}(X(t_2))dB^{j_2}(t_2) + \frac{1}{2}L_{j_2}^2(X(t_2))dt_2\right)$$

and

$$\int_0^\infty \int_0^{t_1} g_1^{(k_1)}(t_1) g_2^{(k_2)}(t_2)$$

$$\left(L_{j_1}(X(t_1))dB^{k_1}(t_1) + \frac{1}{2}L_{k_1}^2(X(t_1))dt_1\right) * \left(L_{k_2}(X(t_2))dB^{k_2}(t_2) + \frac{1}{2}L_{k_2}^2(X(t_2))dt_2\right) +$$

$$+ \int_0^\infty \int_{t_1}^\infty g_1^{(k_1)}(t_1) g_2^{(k_2)}(t_2)$$

$$\left(L_{k_1}(X(t_1))dB^{k_1}(t_1) + \frac{1}{2}L_{k_1}^2(X(t_1))dt_1\right) * \left(L_{k_2}(X(t_2))dB^{k_2}(t_2) + \frac{1}{2}L_{k_2}^2(X(t_2))dt_2\right)$$

is equal to, by (2.8b),

$$\delta_{j_1,j_2}^{k_1,k_2} \int_0^\infty \int_0^{t_1} \overline{f_1^{(j_1)}(t_1)} g_1^{(k_1)}(t_1) \overline{f_2^{(j_2)}(t_2)} g_2^{(k_2)}(t_2) dt_1 dt_2 +$$

$$+\delta_{j_1,j_2}^{k_2,k_1} \int_0^\infty \int_0^{t_1} \overline{f_2^{(j_2)}(t_1)} g_1^{(k_1)}(t_1) \overline{f_1^{(j_1)}(t_2)} g_1^{(k_1)}(t_2) dt_1 dt_2 +$$

$$+\delta_{j_1,j_2}^{k_2,k_1} \int_0^\infty \int_{t_1}^\infty \overline{f_1^{(j_1)}(t_1)} g_2^{(k_2)}(t_1) \overline{f_2^{(j_2)}(t_2)} g_2^{(k_2)}(t_2) dt_1 dt_2 =$$

$$= \delta_{j_1,j_2}^{k_1,k_2} \int_0^\infty \int_0^\infty \overline{f_1^{(j_1)}(t_1)} g_1^{(k_1)}(t_1) \overline{f_2^{(j_2)}(t_2)} g_2^{(k_2)}(t_2) dt_1 dt_2 +$$

$$+\delta_{j_1,j_2}^{k_2,k_1} \int_0^\infty \int_0^\infty \overline{f_1^{(j_1)}(t_1)} g_2^{(k_2)}(t_1) \overline{f_2^{(j_2)}(t_2)} g_1^{(k_1)}(t_2) dt_1 dt_2 \tag{2.10}$$

which is nothing but

$$< \int_{\Delta_2} f_1^{(j_1)}(t_1) f_2^{(j_2)}(t_2) \sum_{\sigma \in S_n} *_{k=1}^2 \left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)}) + \frac{1}{2}L_{j_k}^2(X(t_{\sigma(k)}))dt_{\sigma(k)}\right),$$

$$\int_{\Delta_2} g_1^{(h_1)}(t_1) g_2^{(h_2)}(t_2) \sum_{\sigma\in S_n} *_{k=1}^{2} \left(L_{j_k}(X(t_{\sigma(k)}))dB^{j_k}(t_{\sigma(k)}) + \frac{1}{2} L_{j_k}^2 (X(t_{\sigma(k)}))dt_{\sigma(k)} \right) > \qquad (2.11)$$

This gives the following:

Lemma 2.1. *Under an isometry,*

$$\mathcal{H}_2 = \mathcal{H}_1 \circ \mathcal{H}_1$$

and for any $f, g \in L^2(\mathbb{R}_+)$,

$$\int_0^\infty f(t) \left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2} L_{j_1}^2 (X(t))dt \right) \circ$$

$$\circ \int_0^\infty g(t) \left(L_{j_2}(X(t))dB^{j_2}(t) + \frac{1}{2} L_{j_2}^2 (X(t))dt \right) =$$

$$= \int_{\Delta_2} f(t_1) g(t_2) \sum_{\sigma\in S_2} *_{k=1}^{2} \left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)}) + \frac{1}{2} L_{\sigma(k)}^2 (X(t_{\sigma(k)}))dt_{\sigma(k)} \right) \qquad (2.12)$$

where, as usual "$\circ$" means the symmetric tensor product.

In general, we have

Theorem 2.2. *For any* $n \in \mathbb{N}$, *under an isometry*

$$\mathcal{H}_n = \overbrace{\mathcal{H}_1 \circ \ldots \circ \mathcal{H}_1}^{n \text{ times}} \qquad (2.13)$$

and for any $f_1, \ldots, f_n \in L^2(\mathbb{R}_+)$,

$$\int_0^\infty f_1(t) \left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2} L_{j_1}^2 (X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_n(t) \left(L_{j_n}(X(t))dB^{j_n}(t) + \frac{1}{2} L_{j_n}^2 (X(t))dt \right) =$$

$$= \int_{\Delta_n} f_1(t_1) \ldots f_n(t_n) \sum_{\sigma\in S_n} *_{k=1}^{n} \left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)}) + \frac{1}{2} L_{\sigma(k)}^2 (X(t_{\sigma(k)}))dt_{\sigma(k)} \right) \qquad (2.14)$$

Moreover, the Hilbert space $\mathcal{H}(:= \bigoplus_{n=0}^{\infty} \mathcal{H}_n)$ is isomorphic to the Fock Hilbert space over the Hilbert space $\mathcal{H}_1$ and which will be denoted, as usual, by $\Gamma(\mathcal{H}_1)$.

Remark. *The proof of the theorem (2.2) could be given by the same method as from (2.9) to (2.11).*

3. A POSSIBLE QUANTIZATION OF DIFFUSION PROCESS ON MANIFOLD

Now we are ready to give a possible quantization of diffusion process on manifold: For a diffusion process $\{X(t)\}_{t\geq 0}$, a possible quantization is

$$\int_0^t dX(t_1) := \int_0^t \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) =$$

$$= \int_0^\infty \chi_{[0,t)}(t_1) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) \tag{3.1}$$

where, the right hand side of (3.1) is considered as an operator on the Hilbert space $\mathcal{H}$. In order to insist the fact that the right hand side of (3.1) is an operator, we shall rewrite it as

$$Q_{\int_0^\infty \chi_{[0,t)}(t_1)\left(L_j(X(t_1))dB^j(t_1)+\frac{1}{2}L_j^2(X(t_1))dt_1\right)} \tag{3.2}$$

Suggested by the Ito formula, we define the operator by

$$Q_{\int_0^\infty \chi_{[0,t)}(t_1)\left(L_j(X(t_1))dB^j(t_1)+\frac{1}{2}L_j^2(X(t_1))dt_1\right)} c :=$$

$$:= c \cdot \int_0^\infty \chi_{[0,t)}(t_1) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) \in \mathcal{H}_1 \tag{3.3}$$

$$Q_{\int_0^\infty \chi_{[0,t)}(t_1)\left(L_j(X(t_1))dB^j(t_1)+\frac{1}{2}L_j^2(X(t_1))dt_1\right)} \left[\int_0^\infty f(s) \left(L_k(X(s))dB^k(s) + \frac{1}{2}L_k^2(X(s))ds \right) \right]$$

$$:= \int_0^\infty \int_0^{t_1} \chi_{[0,t)}(t_1) f(t_2) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) *$$

$$* \left(L_k(X(t_2))dB^k(t_2) + \frac{1}{2}L_k^2(X(t_2))dt_2 \right) +$$

$$+ \int_0^\infty \int_{t_1}^\infty \chi_{[0,t)}(t_1) f(t_2) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) *$$

$$* \left(L_k(X(t_2))dB^k(t_2) + \frac{1}{2}L_k^2(X(t_2))dt_2 \right) +$$

$$+ \int_0^\infty \chi_{[0,t)}(t_1) f(t_1) \mathbb{E}\left(dB^j(t_1) dB^k(t_1) \right) \tag{3.4}$$

By (2.12) and (2.8b), this is equal to

$$\int_0^\infty \chi_{[0,t)}(t_1) \left(L_j(X(t_1))dB^j(t_1) + \frac{1}{2}L_j^2(X(t_1))dt_1 \right) \circ$$

$$\circ \int_0^\infty f(s) \left(L_k(X(s))dB^k(s) + \frac{1}{2}L_k^2(X(s))ds \right) +$$

$$+<\int_0^\infty \chi_{[0,t)}(t_1)\left(L_j(X(t_1))dB^j(t_1)+\frac{1}{2}L_j^2(X(t_1))dt_1\right),$$
$$\int_0^\infty f(t_1)\left(L_k(X(t_1))dB^k(t_1)+\frac{1}{2}L_k^2(X(t_1))dt_1\right)> \tag{3.5}$$

In general, with the idea of the Ito calculus, we define the action of the operator (3.2) on the quantity

$$\int_{\Delta_n} f_1(t_1)\dots f_n(t_n)\sum_{\sigma\in S_n} *_{k=1}^n\left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)})+\frac{1}{2}L^2_{\sigma(k)}(X(t_{\sigma(k)}))dt_{\sigma(k)}\right) \tag{3.6a}$$

as

$$\int_{\Delta_{n+1}} \chi_{[0,t)}(t_1)f_1(t_2)\dots f_n(t_{n+1})\sum_{\sigma\in S_{n+1}}$$
$$*_{k=1}^{n+1}\left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)})+\frac{1}{2}L^2_{\sigma(k)}(X(t_{\sigma(k)}))dt_{\sigma(k)}\right)+$$
$$+\sum_{\sigma\in S_n}\sum_{h=1}^n\int_0^\infty \chi_{[0,t)}(s)f_h(s)\mathbb{E}(dB^j(s)dB^{j_h}(s))$$
$$\int_{\Delta_{n-1}}\prod_{\substack{1\le k\le n\\ k\neq h}} f_k(t_k) *_{k=1}^{h-1}\left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)})+\frac{1}{2}L^2_{\sigma(k)}(X(t_{\sigma(k)}))dt_{\sigma(k)}\right)*$$
$$* *_{k=h+1}^n\left(L_{j_{\sigma(k)}}(X(t_{\sigma(k)}))dB^{j_{\sigma(k)}}(t_{\sigma(k)})+\frac{1}{2}L^2_{\sigma(k)}(X(t_{\sigma(k)}))dt_{\sigma(k)}\right) \tag{3.6b}$$

Thus, it follows from a quite simple calculation that

Theorem 3.1. *The action of the operator (3.2) on the quantity like (3.6a) is equal to*

$$\int_0^\infty \chi_{[0,t)}(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)\circ$$
$$\circ\int_0^\infty f_1(t)\left(L_{j_1}(X(t))dB^{j_1}(t)+\frac{1}{2}L_{j_1}^2(X(t))dt\right)\circ\dots\circ$$
$$\circ\int_0^\infty f_n(t)\left(L_{j_n}(X(t))dB^{j_n}(t)+\frac{1}{2}L_{j_n}^2(X(t))dt\right)+$$
$$+\sum_{h=1}^n\delta_j^{j_h}\int_0^\infty \chi_{[0,t)}(s)f_h(s)ds\int_0^\infty f_1(t)\left(L_{j_1}(X(t))dB^{j_1}(t)+\frac{1}{2}L_{j_1}^2(X(t))dt\right)\circ\dots\circ$$
$$\circ\int_0^\infty f_{h-1}(t)\left(L_{j_{h-1}}(X(t))dB^{j_{h-1}}(t)+\frac{1}{2}L_{j_{h-1}}^2(X(t))dt\right)\circ\dots\circ$$

$$\circ \int_0^\infty f_{h+1}(t) \left(L_{j_{h+1}}(X(t))dB^{j_{h+1}}(t) + \frac{1}{2}L^2_{j_{h+1}}(X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_n(t) \left(L_{j_n}(X(t))dB^{j_n}(t) + \frac{1}{2}L^2_{j_n}(X(t))dt \right) \tag{3.7}$$

Definition 3.2. *The operator Q will be called* **position operator** *and the family*

$$\{Q_{\int_0^\infty \chi_{[0,t]}(s)\left(L_j(X(s))db^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}\}_{t\geq 0}$$

will be called **position process** *on the Fock space* $\mathcal{H}$.

Definition 3.3. *For any* $f \in L^2(\mathbb{R}_+)$, *the operator which maps the quantity (3.6a) to*

$$\int_0^\infty f(s) \left(L_j(X(s))dB^j(s) + \frac{1}{2}L_j^2(X(s))ds \right) \circ$$

$$\circ \int_0^\infty f_1(t) \left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2}L^2_{j_1}(X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_n(t) \left(L_{j_n}(X(t))dB^{j_n}(t) + \frac{1}{2}L^2_{j_n}(X(t))dt \right) \tag{3.8}$$

will be called the **creation operator** (or **creator**) with respect to

$$\int_0^\infty f(s) \left(L_j(X(s))dB^j(s) + \frac{1}{2}L_j^2(X(s))ds \right)$$

and denoted by

$$A^+_{\int_0^\infty f(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}$$

The operator which maps the quantity (3.6a) to

$$\sum_{h=1}^n \delta_j^{j_h} \int_0^\infty \bar{f}(s)f_h(s)ds \int_0^\infty f_1(t) \left(L_{j_1}(X(t))dB^{j_1}(t) + \frac{1}{2}L^2_{j_1}(X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_{h-1}(t) \left(L_{j_{h-1}}(X(t))dB^{j_{h-1}}(t) + \frac{1}{2}L^2_{j_{h-1}}(X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_{h+1}(t) \left(L_{j_{h+1}}(X(t))dB^{j_{h+1}}(t) + \frac{1}{2}L^2_{j_{h+1}}(X(t))dt \right) \circ \ldots \circ$$

$$\circ \int_0^\infty f_n(t) \left(L_{j_n}(X(t))dB^{j_n}(t) + \frac{1}{2}L^2_{j_n}(X(t))dt \right) \tag{3.9}$$

for $n \geq 1$ and maps a constant to zero will be called the **annihilation operator** (or **annihilator**) with respect to

$$\int_0^\infty f(s) \left(L_j(X(s))dB^j(s) + \frac{1}{2}L_j^2(X(s))ds \right)$$

and denoted

$$A_{\int_0^\infty f(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}$$

The family

$$\{A^+_{\int_0^\infty \chi_{[0,t)}(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}\}_{t\geq 0}$$

will be called **creation process** on the Fock space $\mathcal{H}$, and the family

$$\{A_{\int_0^\infty \chi_{[0,t)}(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}\}_{t\geq 0}$$

will be called **annihilation process** on the Fock space $\mathcal{H}$.

Remark. *It is clear that* $Q = A^+ + A$.

With the same argument used in [2], one can easily obtain

Lemma 3.4. *On the Hilbert space* $\mathcal{H}$, *creators and annihilators are well defined; they have a common dense domain and with respect to the same element in* $\mathcal{H}_1$ *the annihilator is the essential conjugate of the creator.*

Lemma 3.5.

$$[A_{\int_0^\infty f(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}, A^+_{\int_0^\infty g(s)\left(L_j(X(s))dB^j(s)+\frac{1}{2}L_j^2(X(s))ds\right)}] =$$

$$= < \int_0^\infty f(s)\left(L_j(X(s))dB^j(s) + \frac{1}{2}L_j^2(X(s))ds\right), \int_0^\infty g(s)\left(L_j(X(s))dB^j(s) + \frac{1}{2}L_j^2(X(s))ds\right) > \tag{3.10}$$

REFERENCES

[1] N. IKEDA and S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, 1981.
[2] P.A. MEYER, *Quantum Probability for Probabilists*, Lecture Notes in Math. No. 1538, 1993.

Received January 10, 1994
Y.G. LU
Dipartimento di Matematica
Università di Bari
Centro V. Volterra
Università di Roma II
ITALIA