ON VECTOR-VALUED SEQUENCE SPACES

L. FRERICK

Abstract. In this paper we investigate the topological properties of vector valued sequence spaces. After an introduction of normal Banach sequence spaces \( A \) we consider the vector valued sequence spaces \( \lambda(E) \), \( E \) a locally convex Hausdorff space and we prove some basic facts concerning these spaces. We give complete characterizations for barrelled vector valued DF spaces and distinguished vector valued Fréchet spaces. At the end we give sufficient conditions guaranteeing that \( \lambda(E) \) is bornological.

1. INTRODUCTION

Given a locally convex Hausdorff space \( E \) and denote with \( cs(E) \) the system of all continuous seminorms on \( E \). The vector space \( h(E) \) of all bounded sequences \( (x_n)_{n \in \mathbb{N}} \) in \( E \) is naturally equipped with the locally convex Hausdorff topology induced by the seminorms

\[
(x_n)_{n \in \mathbb{N}} \mapsto \sup_{n \in \mathbb{N}} p(x_n), \quad p \in cs(E).
\]

If \( E \) is metrizable (or normed), then the same holds for \( l_\infty(E) \). The space \( c_0(E) \) of all zero sequences in \( E \) is a (closed) subspace of \( l_\infty(E) \). Both spaces are part of a more general concept of constructing vector valued sequence spaces:

Consider a normal Banach sequence space \( (\lambda, \| . \|_\lambda) \), i.e. a Banach space containing \( \varphi \) which is included in \( \omega \) such that its closed unit ball is normal. For every locally convex Hausdorff space \( E \) this condition required for \( A \) guarantees that

\[
\lambda(E) := \{ (x_n)_{n \in \mathbb{N}} \in E^\mathbb{N} : (p(x_n))_{n \in \mathbb{N}} \in A \text{ for all } p \in cs(E) \}
\]

is a vector space. The seminorms

\[
(x_n)_{n \in \mathbb{N}} \mapsto \| (p(x_n))_{n \in \mathbb{N}} \|_\lambda, \quad p \in cs(E),
\]

define a (quite natural) Hausdorff locally convex topology on \( \lambda(E) \). Using this principle of construction, one obtains a large class of vector valued sequence spaces containing (besides the above mentioned examples) e.g. the spaces \( l_p(E) \) of the \( p \)-summable sequences, \( p \in [1,\infty) \).

In [17] we investigated completeness properties of this vector valued sequence spaces.

These notes are mainly devoted to the examination of the (topological) properties of the spaces \( \lambda(E) \) in dependence on properties of the spaces \( \lambda \) and \( E \). Since \( \lambda(E) \) contains \( E \) as a complemented subspace, it is clear that the behavior of \( \lambda(E) \) is i.e. not ”better” than the behavior of \( E \).

At the end of this introduction we recall some notations and the terminology which will be used in the chapters 2-6.

The second chapter is devoted to the study of normal Banach sequence spaces. We introduce some additional properties of normal Banach sequence spaces (see[4],§1), which
will be important in the following chapters. Moreover, a connection to perfect sequence spaces is obtained: A norinal Banach sequence space, whose closed unit ball is even closed in \( \omega \), is already a perfect space.

In the third chapter first we prove some fundamental properties of vector valued sequence spaces \( \lambda(E) \), using to some extent the results given in the previous chapter. At the end we show (analogously as it is done in [15]) that \( \lambda(E) \) contains a complemented copy of \( l_\infty(E) \) whenever \( \varphi \) is not dense in \( \lambda \) and \( E \) is an arbitrary locally convex Hausdorff space.

Chapter four comprises a description of the bounded subsets of Frechet- and \( gDF \)-space-valued sequence spaces. In this context we prove the following characterization of \( gDF \) spaces:

A locally convex Hausdorff space \( E \) is a \( gDF \) space if and only if it contains a sequence \((B_n)_{n \in \mathbb{N}}\) of bounded and absolutely convex subsets satisfying \( 2B_n \subset B_{n+1} \), \( n \in \mathbb{N} \), such that for every sequence \((U_n)_{n \in \mathbb{N}}\) of zero neighbourhoods in \( E \) the set \( \bigcap_{n \in \mathbb{N}} (U_n + B_n) \) is again a zero neighbourhood. In this case, the closures of the \( B_n \)'s define a fundamental sequence of bounded sets in \( E \).

Using this result, one obtains that \( \lambda(E) \) is a \( gDF \) space (or a \( DF \) space), if so is \( E \). Moreover, we characterize the strong dual of \( \lambda(E) \) in the case that \( \varphi \) is dense in \( \lambda \) and \( E \) is either metrizable or a (quasi)barrelled \( DF \) space. This result is essentially contained in [34] (with a different approach). In the last part of chapter four we prove some permanence properties of vector valued sequence spaces. For example, we show that the bounded subsets of \( \lambda(E) \) are inertrizable, whenever \( E \) is a \( gDF \) space satisfying the same property.

Barrelledness conditions concerning vector valued sequence spaces are investigated in the fifth chapter. It is well-known (see [16], that \( \lambda(E) \) is a (quasi)barrelled \( DF \) space, whenever \( E \) is a (quasi)barrelled \( DF \) space and \( \varphi \) is dense in the normal Banach sequence space \( \lambda \). It is also well-known that for a given \( DF \) space \( E \) the space \( l_\infty(E) \) is quasi-barrelled \( \varphi \) if and only if \( E \) satisfies the dual density condition, cf. [2]. If, in addition, \( E \) is barrelled then \( l_\infty(E) \) is also barrelled, cf. [2]. We can show that this remains true if one replaces \( l_\infty \) by any normal Banach sequence space in which \( \varphi \) is not dense. Furthermore, a characterization of the distinguishedness of Frechet valued sequence spaces is obtained:

Let \( E \) be a metrizable locally convex space and let \( \lambda \) be a normal Banach sequence. If \( \varphi \) is dense in \( \lambda \) and dense in the \( \alpha \)-dual \( \lambda^\times \) of \( \lambda \), then \( \lambda(E) \) is distinguished if and only if \( E \) is distinguished (cf. [16]).

If \( \varphi \) is not dense in \( \lambda \) or \( \lambda^\times \) the space \( \lambda(E) \) is distinguished if and only if \( E \) satisfies the density condition.

This last part of the described result contains clearly the cases \( \lambda = l_1 \), cf. [1], and \( \lambda = l_\infty \).

Sufficient conditions guaranteeing that \( \lambda(E) \) is (ultra)bornological are given in the last chapter. For example we prove that \( \lambda(E) \) is bornological for every DFM space \( E \) whenever the normal Banach sequence space \( \lambda \) satisfies the following property:

\[
\gamma \quad \| (\alpha_k)_{k \in \mathbb{N}} \|_{\lambda} = \sup_{n \in \mathbb{N}} \| (\alpha_k)_{k \leq n}, (0)_{k > n} \|_{\lambda} \quad \text{for every} \ (\alpha_k)_{k \in \mathbb{N}} \in \lambda.
\]

In particular, \( c_0(E) \) is bornological for every DFM space \( E \) (a result of S. Dierolf, unpublished).

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Notations. For a locally convex Hausdorff space $E := (E, \mathcal{T})$ over the scalar field $R$ or $C$ we denote by $U_0(E)$ the zero neighbourhood filter of $E$ or, equivalently,

$$U_0(E) := \{ U \subseteq E : U \supseteq p^{-1}([0, i]) \text{ for some } p \in \text{cs}(E) \}.$$  

We write $E'$ for the topological dual of $E$ and we write $E''$ for the bidual of $E$, i.e. $E'' = (E')' \beta(E', E))'$ where we denote for a dual pair $(E, F)$ by $\sigma(E, F)$ the weak and by $\beta(E, F)$ the strong topology defined on $E$, respectively. A closed linear subspace $L$ of $E$ defines a quotient $E / L$ which we equip with the quotient topology. If $\| \cdot \|$ is a norm on $E$, we denote this normed space with $(E, \| \cdot \|)$.

Complete normed spaces are called Banach spaces and complete metrizable locally convex spaces are called Frechet spaces.

Let $B$ and $C$ be subsets of the locally convex Hausdorff space $E$ and let $\rho$ be a scalar. Then

$$B + C := \{ x + y : x \in B, y \in C \} \text{ and } \rho B := \{ \rho x : x \in B \}.$$ 

We denote by $[B]$ the linear span of $B$ and in a fixed dual system $(E, F)$ by $B^¥$ its polar:

$$B^¥ := \{ f \in F : \sup_{x \in [B]} |f(x)| \leq 1 \}.$$ 

Moreover, we write $\Gamma(B)$ for the absolutely convex hull of $B$. If $B$ is already absolutely convex, we may equip $[B]$ with the Minkowski functional $p_B$ of $B$, which is defined by

$$p_B(x) := \inf \{ \rho > 0 : x \in \rho B \}.$$ 

$([B], p_B)$ is then a seminormed space. If $B$ is bounded and absolutely convex and $([B], p_B)$ is a Banach space, we will call $B$ a Banach disc. $B$ is called a barrel if it is an absorbent, absolutely convex and closed subset of $E$. We call $B$ bornivorous if it absorbs every bounded subset of $E$.

For a given sequence $(E_n)_{n \in \mathbb{N}}$ of locally convex Hausdorff spaces we denote by $\prod_{n \in \mathbb{N}} E_n$, its cartesian product equipped with the product topology and by $\bigoplus_{n \in \mathbb{N}} E_n$, its locally convex direct sum. We write $E^N := \prod_{n \in \mathbb{N}} E$ and $E^{(N)} := \bigoplus_{n \in \mathbb{N}} E$. For $K \in \{ R, C \}$ let $\omega := K^N$ and $\varphi := K^{(N)}$.

Now let us introduce some well-known topological properties for locally convex Hausdorff spaces $E$. 


$E$ is called barrelled, if every barrel in $E$ is a zero neighbourhood. It is called quasibarrelled, if every bornivorous barrel in $E$ is a zero neighbourhood. We call $E$ bornological if every absolutely convex and bornivorous subset of $E$ is a zero neighbourhood. $E$ is ultrabornological if and only if every absolutely convex subset of $E$ which absorbs the Banach discs is already a zero neighbourhood.

2. NORMAL BANACH SEQUENCE SPACES

For sequences $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ and $\beta = (\beta_n)_{n \in \mathbb{N}}$ of real numbers we define:

$$\alpha \leq \beta \text{ if and only if } \alpha_n \leq \beta_n \text{ for every } n \in \mathbb{N},$$

$$\alpha < \beta \text{ if and only if } \alpha_n < \beta_n \text{ for every } n \in \mathbb{N},$$

and for $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in \omega$ let

$$|\alpha| := (|\alpha_n|)_{n \in \mathbb{N}}.$$

**Definition 2.1.** We call a set $A \subset \omega$ normal if

$$A = \bigcup_{\alpha \in A} \{ \beta \in \omega : |\beta| \leq |\alpha| \}.$$

Every normal set is clearly circled and the span of an (absolutely) convex and normal set is again normal. A linear subspace of $\omega$ is normal if and only if it is normal in the sense of G. Kothe ([22], §30, 1.). Now we introduce the notion of normal Banach sequences spaces (cf. J. Bonet & S. Dierolf [4], §1):

**Definition 2.2** We call a Banach space $(\lambda, \| \cdot \|_\lambda)$ normal Banach sequence space if it satisfies the following two properties:

1. $\emptyset \subset A \subset \omega$.
2. The closed unit ball $B_\lambda$ of $(\lambda, \| \cdot \|_\lambda)$ is normal.

Property (2) is equivalent to:

For all $\alpha \in \lambda$, $\beta \in \omega$ satisfying $|\beta| \leq |\alpha|$ it follows $\beta \in A$ and $\|\beta\|_\lambda \leq \|\alpha\|_\lambda$.

For $p \in [1, \infty)$ let

$$l_p := \{ \alpha \in \omega : \sum_{n=1}^{\infty} |\alpha_n|^p < \infty \}$$

be the space of all $p$-summable sequences equipped with the usual norm

$$\| \cdot \|_p : l_p \to \mathbb{R}, \alpha \mapsto \left( \sum_{n=1}^{\infty} |\alpha_n|^p \right)^{\frac{1}{p}}.$$

Moreover, let

$$l_\infty := \{ \alpha \in \omega : \sup_{n \in \mathbb{N}} |\alpha_n| < \infty \}$$
be the space of all bounded sequences equipped with the norm
\[
\| \cdot \|_\infty : l_\infty \to R, \alpha \mapsto \sup_{n \in N} |\alpha_n|
\]
and
\[
c_0 := \{ a \in W : \lim_{n \to \infty} a_n \neq 0 \}
\]
be the space of all sequences converging to 0 normed by \(\| \cdot \|_\infty\). Then \(c_0\) and \(l_\infty\) are normal Banach sequence spaces \((p \in [1, \infty])\).

**Remark 2.3.**

i) Let \(A\) be a normal Banach sequence space. Then the projection
\[
P_n : \omega \to \lambda, \alpha \mapsto (\alpha_k)_{k \leq n}, (0)_{k > n}
\]
is for every \(n \in N\) welldefined and satisfies (because of property \(B\)) \(P_n(B_\lambda) \subset P_{n+1}(B_\lambda) \subset B_\lambda, n \in N\). For every \(a \in A\) and \(k \in N\) we have \(\|P_k(\alpha)\|_\lambda \leq \lim_{n \to \infty} \|P_n(\alpha)\|_\lambda = \sup_{n \in N} \|P_n(\alpha)\|_\lambda \leq \|\alpha\|_\lambda\).

ii) If \(A\) is a normal Banach sequence space then the inclusions \(\varphi \hookrightarrow A \hookrightarrow W\) are continuous. Indeed, the first one is continuous, because \(\varphi\) carries the finest locally convex topology on \(\varphi\). Property \(B\) implies \(\|\alpha_n\| = \|(\delta_n)_{k \in N}\|_{\lambda}^{-1} \|(\alpha_n)_{k \in N}\|_\lambda \leq \|(\delta_n)_{k \in N}\|_{\lambda}^{-1} \|\alpha\|_\lambda\) for every \(a \in A\) and every \(n \in N\). So the second inclusion \(A \hookrightarrow W\) is also continuous.

For a normal Banach sequence space \((A, \| \cdot \|_\lambda)\) and \(0 < a = (\alpha_n)_{n \in N} \in W\) let
\[
\alpha \cdot \lambda := \{ (\alpha_n^a)_{n \in N} : \beta \in \lambda \},
\]
\[
\|(\alpha_n^a)_{n \in N}\|_{\alpha \cdot \lambda} := \|\beta\|_\lambda, \beta \in \lambda,
\]
be the diagonal transform of \(A\) with respect to \(\alpha\). Than \((\alpha \cdot A, \| \cdot \|_{\alpha \cdot \lambda})\) is also a normal Banach sequence space. The next result, due to S. Dierolf & C. Fernández [13], is a sharper version of remark 2.3 ii):

**Proposition 2.4.** Let \(\lambda\) be a normal Banach sequence space with closed unit ball \(B_\lambda\) and let \(\alpha := (\|(\delta_n)_{k \in N}\|_{\alpha}^{-1})_{n \in N}\). Then
\[
B_{\alpha \cdot l_1} \subset B_\lambda \subset B_{\alpha \cdot l_\infty}
\]
and hence all inclusions
\[
\varphi \hookrightarrow \alpha \cdot l_1 \hookrightarrow \lambda \hookrightarrow \alpha \cdot l_\infty \hookrightarrow \omega
\]
are continuous.

**Proof.** If we define \(\alpha^{-1} := (\alpha_n^{-1})_{n \in N}\) then the assertion is equivalent to
\[
B_{l_1} \subset B_{\alpha^{-1} \cdot \lambda} \subset B_{l_\infty} \quad (\ast).
\]
\(\alpha^{-1}\) is a normal Banach sequence space with \(\|(\delta_n)_{k \in N}\|_{\alpha^{-1} \cdot \lambda} = 1\) for every \(n \in N\).
To prove the first part of (3) let \( x = (x_n)_{n \in \mathbb{N}} \in B_{l_1} \) be given. Then for every natural numbers \( m > k \) the triangle inequality yields
\[
\|P_m(x) - P_k(x)\|_{\alpha^{-1}, \lambda} \leq \sum_{n=k+1}^{m} |x_n| = \|P_m(x) - P_k(x)\|_1 \quad \text{and}
\]
\[
\|P_m(x)\|_{\alpha^{-1}, \lambda} \leq \sum_{n=1}^{m} |x_n| = \|P_m(x)\|_1 \leq 1.
\]

Therefore \((P_m(x))_{m \in \mathbb{N}}\) is a Cauchy sequence in \( \alpha^{-1}, \lambda \) which converges to some \( y \in B_{\alpha^{-1}, \lambda} \). Because the spaces \( l_1 \) and \( \alpha^{-1}, \lambda \) are both continuously included in \( w \) we get that \( y \) coincides with \( x \), the limit of \((P_m(x))_{m \in \mathbb{N}}\) in \( l_1 \). This implies \( x \in B_{\alpha^{-1}, \lambda} \).

It remains to show \( B_{(\alpha^{-1}, \lambda)} \subset B_{l_\infty} \). But this is clear because property \( \beta \) guarantees for every \( x = (x_n)_{n \in \mathbb{N}} \in B_{\alpha^{-1}, \lambda} \) and every \( n \in \mathbb{N} \) that
\[
|x_n| = \|(x_n \delta_{nk})_{k \in \mathbb{N}}\|_{\alpha^{-1}, \lambda} \leq \|(x_n)_{n \in \mathbb{N}}\|_{\alpha^{-1}, \lambda} \leq 1.
\]

The proof is complete.

Of interest are the following properties which can be satisfied by normal Banach sequence spaces \( \lambda \).

**Definition 2.5.** We say that a normal Banach sequence space \((\lambda, \| \cdot \|_\lambda)\) satisfies

\( \gamma \) if \( \| \alpha \|_\lambda = \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \) for every \( \alpha \in \lambda \).

\( \gamma_w \) if there exists \( \rho \geq 1 \) such that \( \| \alpha \|_\lambda \leq \rho \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \) for every \( \alpha \in l_1 \).

\( \beta \) if for every \( \alpha \in w \) with \( \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda < \infty \) it follows that \( \alpha \in \lambda \) and \( \| \alpha \|_\lambda = \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \).

\( \delta_w \) if there is \( \rho \geq 1 \) such that for every \( \alpha \in w \) with \( \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda < \infty \) it follows that \( \alpha \in \lambda \) and \( \| \alpha \|_\lambda = \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \).

\( \epsilon \) if \( \lim_{n \to \infty} \| P_n(\alpha) - \alpha \|_\lambda = 0 \) for every \( \alpha \in \lambda \).

We may remark that according to remark 2.3 i) one can replace in the above definition "\( \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \)" always by "\( \lim_{n \to \infty} \| P_n(\alpha) \|_\lambda \)".

Every normal Banach sequence space with property \( \epsilon \) satisfies also property \( \gamma \). For \( p \in [1, \infty) \) the space \( l_p \) satisfies the properties \( \beta \) and \( \epsilon \), the space \( l_\infty \) satisfies \( \beta \) but not \( \epsilon \), and the space \( c_0 \) satisfies \( \epsilon \) but not \( \beta \). Later we give an example of a normal Banach sequence space, which does not satisfy even condition \( \gamma_w \).

Connected with the properties introduced in the previous definition are the following spaces associated to a normal Banach sequence space:

**Definition 2.6.** Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space and let
\[
\lambda^{(\delta)} := \{ \alpha \in w : \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda < \infty \}
\]

equipped with the norm
\[
\| \cdot \|_{\lambda^{(\delta)}} : \lambda^{(\delta)} \to R, \alpha \mapsto \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda.
\]
Moreover, define \( \lambda^{(s)} \) to be the closure of \( A \) in \((\lambda^{(s)}, ||\cdot||_{\lambda^{(s)}})\).

First we prove that these associated spaces are also normal Banach sequence spaces:

**Proposition 2.7.** Let \((A, ||\cdot||_{\lambda})\) be a normal Banach sequence space.

i) Then \((\lambda^{(s)}, ||\cdot||_{\lambda^{(s)}})\) is also a normal Banach sequence space.

ii) If \( \varphi \subset [\ell^p] = L \subset A \) is normal, then the closure \( \hat{L}^{\lambda} \) equipped with the induced norm \( ||\cdot||_\lambda \) is a normal Banach sequence space.

Especially \((\hat{\varphi}^{(s)}, ||\cdot||_{\lambda})\) and \((\lambda^{(s)}), ||\cdot||_{\lambda^{(s)}})\) are normal Banach sequence spaces.

**Proof.** i) Property \( a \) is clearly satisfied and it is easy to see that \( B_{\lambda^{(s)}} \) is bounded in \( w \). Let

\[
\delta_n : \omega \rightarrow R, \alpha \mapsto ||P_n(\alpha)||_{\lambda}, n \in N.
\]

Then the mappings \( \delta_n \) are continuous for every \( n \in N \) and \( B_{\lambda^{(s)}} = \bigcap_{n \in N} \hat{\delta}_n^{-1}([0, 1]) \). Hence, \( B_{\lambda^{(s)}} \) is a closed and bounded subset of \( \mathcal{W} \) and therefore \( \lambda^{(s)} \) is a Banach space. To prove property \( \beta \) let \( \alpha \in B_{\lambda^{(s)}} \) and \( \beta \in \mathcal{W} \) with \( ||\beta|| \leq ||\alpha|| \) be given. Because \( A \) satisfies property \( \beta \) we get \( \sup_{n \in N} ||P_n(\beta)||_\lambda \leq \sup_{n \in N} ||P_n(\alpha)||_\lambda \) and this implies \( \beta \in B_{\lambda^{(s)}} \).

ii) First we remark, that \( \lambda^{(s)} \) is a Banach space, which satisfies property \( a \).

To prove that \( B_{\lambda^{(s)}} = B_{\mathcal{W}} \cap \lambda^{(s)} \) is normal we note that intersections of normal sets are also normal. So it remains to show that \( \lambda^{(s)} \) is a normal subset of \( \mathcal{W} \). Therefore let \( \alpha \in \lambda^{(s)} \) and \( \beta \in \mathcal{W} \) with \( ||\beta|| \leq ||\alpha|| \) be given. Then there exists a sequence \((\alpha^{(s)})_{n \in N} = ((\alpha^{(s)}))_{n \in N} \subset \mathcal{W} \setminus L \) with \( ||\alpha^{(s)} - \alpha||_\lambda \rightarrow 0 \) for \( k \rightarrow \infty \). We define for \( n, k \in N \): \((\beta^{(k)})_{n \in N} = ((\beta^{(k)}))_{n \in N} \subset \mathcal{W} \)

\[
\beta^{(k)} := \begin{cases} 
\beta_n & \text{if } ||\beta_n|| \leq ||\alpha^{(k)}|| \\
\frac{||\alpha^{(k)}||}{||\beta_n||} \beta_n & \text{if } ||\beta_n|| > ||\alpha^{(k)}||
\end{cases}
\]

Since \( L \) is normal and \( ||\beta^{(k)}|| \leq ||\alpha^{(k)}|| \) we have \( \beta^{(k)} \in L \) for all \( k \in N \).

Let \( n, k \in N \).

If \( ||\beta_n|| \leq ||\alpha^{(k)}|| \) then \( \beta^{(k)} - \beta_n = 0 \leq ||\alpha^{(k)}|| - ||\alpha_n|| \).

If \( ||\beta_n|| > ||\alpha^{(k)}|| \) then \( \beta^{(k)} - \beta_n = 1 - \frac{||\alpha^{(k)}||}{||\beta_n||} \). Hence \( ||\beta_n - \alpha^{(k)}|| \leq ||\alpha_n - \alpha^{(k)}|| \leq ||\alpha_n - \alpha^{(k)}|| \).

This implies \( ||\beta^{(k)} - \beta|| \leq ||\alpha^{(k)} - \alpha|| \) for every \( k \in N \). Using that \( \lambda \) is a normal Banach sequence space we obtain:

\[
||\beta^{(k)} - \beta||_\lambda \leq ||\alpha^{(k)} - \alpha||_\lambda \rightarrow 0 \text{ for } k \rightarrow \infty.
\]

Remembering \( \beta^{(k)} \in L \), this yields \( \beta \in \lambda^{(s)} \) and we are done.

**Remark 2.8.** The proof of i) shows that for every normal Banach sequence space \((A, ||\cdot||_{\lambda})\) the closed unit ball \( B_{\lambda^{(s)}} \) is even closed in \( \mathcal{W} \).

**Theorem 2.9.** (cf. [4], 1.1) Let \((\lambda, ||\cdot||_{\lambda})\) be a normal Banach sequence space.

i) T.f.a.e. (The following are equivalent)

1) \((A, ||\cdot||_{\lambda})\) satisfies property \( \gamma \).

2) \((\lambda, ||\cdot||_{\lambda}) = (\lambda^{(s)}, ||\cdot||_{\lambda^{(s)}})\).

3) \( B_{\lambda} \) is closed in \( \mathcal{W} \) equipped with the relative topology induced by \( \mathcal{W} \).
**ii) T.f.a.e.**

1) \( (A, \| \cdot \|_\lambda) \) satisfies property \( \gamma_w \).

2) \( A = \lambda^{(\gamma)} \) algebraically and the corresponding norms are equivalent.

3) There exists \( \rho \geq 1 \) such that the closure of \( B_\lambda \) in \( A \) equipped with the relative topology induced by \( w \) is contained in \( \rho B_\lambda \).

**iii) T.f.a.e.**

1) \( (A, \| \cdot \|_\lambda) \) satisfies property \( \delta_w \).

2) \( A = \lambda^{(\delta)} \) algebraically and the corresponding norms are equivalent.

3) \( B_\lambda \) is closed in \( A \).

**iv) T.f.a.e.**

1) \( (A, \| \cdot \|_\lambda) \) satisfies property \( \delta_w \).

2) \( A = \lambda^{(\delta)} \) algebraically and the corresponding norms are equivalent.

3) There exists \( \rho \geq 1 \) such that the closure of \( B_\lambda \) in \( w \) is contained in \( \rho B_\lambda \).

**v) T.f.a.e.**

1) \( (A, \| \cdot \|_\lambda) \) satisfies property \( \varepsilon \).

2) \( \varphi \) is dense in \( (A, \| \cdot \|_\lambda) \).

**Proof.** We only prove iii), iv) and v). i) and ii) can be proved similarly as iii) and iv).

iii) If \( (A, \| \cdot \|_\lambda) \) satisfies property \( \delta_w \) then clearly \( (A, \| \cdot \|_\lambda) = (\lambda^{(\delta)}, \| \cdot \|_{\lambda^{(\delta)}}) \).

If \( (A, \| \cdot \|_\lambda) = (\lambda^{(\delta)}, \| \cdot \|_{\lambda^{(\delta)}}) \) we have \( B_\lambda = B_{\lambda^{(\delta)}} \) and according to remark 2.8 this set is closed in \( w \).

If \( B_\lambda \) is closed in \( \omega \), we have for \( \alpha \in \mathbb{E} w \) with \( \lim_{n \to \infty} \| P_n(\alpha) \|_\lambda = a < \infty \) that \( \alpha \in \overline{aB_\lambda} = aB_\lambda \), therefore \( \alpha \in \mathbb{E} A \) and \( \alpha = \lim_{n \to \infty} \| P_n(\alpha) \|_\lambda \leq \| \alpha \|_\lambda \leq \varepsilon \). So \( A, \| \cdot \|_\lambda \) satisfies property \( \delta \).

iv) If \( (A, \| \cdot \|_\lambda) \) satisfies property \( \delta_w \) then clearly \( A = \lambda^{(\delta)} \) algebraically and there exists \( \rho \geq 1 \) such that for every \( \alpha \in \mathbb{E} A \)

\[
\sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda \leq \| \alpha \|_\lambda \leq \rho \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda.
\]

Hence \( \| \cdot \|_\lambda \) and \( \| \cdot \|_{\lambda^{(\delta)}} \) are equivalent.

If \( A = \lambda^{(\delta)} \) algebraically and the corresponding norms are equivalent then there is \( \rho \geq 1 \) such that \( \overline{B_\lambda} \subset B_{\lambda^{(\delta)}} \subset \rho B_\lambda \).

Hence, for \( \alpha \in \mathbb{E} w \) with \( \lim_{n \to \infty} \| P_n(\alpha) \|_\lambda = a < \infty \) we have \( \alpha \in \overline{aB_\lambda} \subset a\rho B_\lambda \), therefore \( \alpha \in \mathbb{E} A \) and \( \| \alpha \|_\lambda \leq \varepsilon \). So \( (A, \| \cdot \|_\lambda) \) satisfies property \( \delta_w \).

v) If \( (A, \| \cdot \|_\lambda) \) satisfies property \( \varepsilon \), then \( \varphi \) is dense in \( (A, \| \cdot \|_\lambda) \), because \( P_n(\alpha) \in \varphi \) for every \( \alpha \in \mathbb{E} A \).

For the converse, let \( \alpha \in A \) and \( \varepsilon > 0 \). Then there is \( \beta \in \varphi \) with \( \| \alpha - \beta \|_\lambda < \varepsilon \). Choose \( m \in \mathbb{N} \) such that \( P_m(\beta) = \beta \). Then \( \| P_n(\alpha) - \alpha \|_\lambda \leq \| P_n(\beta) - \alpha \|_\lambda \) implies

\[
\| P_n(\alpha) - \alpha \|_\lambda \leq \| \beta - \alpha \|_\lambda < \varepsilon, m \leq n \in \mathbb{N}.
\]

**Remark 2.10.** Let \( (\lambda, \| \cdot \|_\lambda) \) be a normal Banach sequence space. Since \( \lambda^{(\gamma)} = (\lambda^{(\gamma)})^{(\gamma)} \) and \( \lambda^{(\delta)} = (\lambda^{(\delta)})^{(\delta)} \) always we have that \( \lambda^{(\gamma)} \) satisfies property \( \gamma \), \( \lambda^{(\delta)} \) satisfies property \( \delta \) and \( (\varphi, \| \cdot \|_\lambda) \) satisfies property \( \varepsilon \).
For a given normal Banach sequence space \( A \) we have shown that \( \lambda^{(e)}, \lambda^{(e)} \) and \( \varphi^{\lambda} \) are again normal Banach sequence spaces. Now we prove that \( \lambda \cap \mu \) and \( A + \mu \) are normal Banach sequence spaces, if both \( A \) and \( \mu \) are of that type.

**Proposition 2.11.** Let \( A \) and \( \mu \) be normal Banach sequence spaces with closed unit balls \( B_A \) and \( B_\mu \) respectively. Let \( \| \cdot \|_{\lambda+\mu} \) be the Minkowski functional of \( B_A \cap B_\mu \) and \( \| \cdot \|_{\lambda+\mu} \) be the Minkowski functional of \( B_A + B_\mu \). Then \( (\lambda \cap \mu, \| \cdot \|_{\lambda+\mu}) \) and \( (A + \mu, \| \cdot \|_{\lambda+\mu}) \) are normal Banach sequence spaces.

**Proof.** In both cases property \( a) \) is clearly satisfied.

It is easy to see that \( B_A \cap B_\mu \) is normal and that its Minkowski functional defines a complete norm on \( A \cap \mu \). Hence \( \lambda \cap \mu \) is a normal Banach sequence space.

We define

\[
q : \lambda \times \mu \to \lambda + \mu, (\alpha, \beta) \mapsto \alpha + \beta.
\]

This map \( q \) is linear, continuous and open. Hence \( (A + \mu, \| \cdot \|_{\lambda+\mu}) \) is a separated quotient of a Banach space and therefore itself a Banach space.

It remains to show that \( B_{\lambda+\mu} \) is normal.

Let \( \alpha^{(1)} \in B_\lambda, \alpha^{(2)} \in B_\mu \), and \( \beta \in \omega \) with \( \| \beta \| \leq \| \alpha^{(1)} + \alpha^{(2)} \| \) be given. For all \( n \in \mathbb{N} \) there exists a scalar \( z_n \), \( |z_n| \leq 1 \), such that \( \beta_n = z_n |\alpha^{(1)}_n| + z_n |\alpha^{(2)}_n| \).

Then \((z_n |\alpha^{(1)}_n|)_{n \in \mathbb{N}} \in B_\lambda \) and \((z_n |\alpha^{(2)}_n|)_{n \in \mathbb{N}} \in B_\mu \), because \( B_\lambda \) and \( B_\mu \) are normal. Therefore

\[
\beta = (z_n |\alpha^{(1)}_n|)_{n \in \mathbb{N}} + (z_n |\alpha^{(2)}_n|)_{n \in \mathbb{N}} \in B_\lambda + B_\mu,
\]

and this implies that \( B_\lambda + B_\mu \) is normal.

Since the intersection of normal sets is also normal, we get that

\[
B_{\lambda+\mu} = \bigcap_{\rho > 1} \rho (B_\lambda + B_\mu)
\]

is normal and we are done.

**Example 2.12.** Let \( \alpha_{2n-1} := 1, \alpha_{2n} := n, n \in \mathbb{N} \), and \( \alpha := (\alpha_n)_{n \in \mathbb{N}} \). Then \((\alpha \cdot c_0) \cap l_\infty \) is a normal Banach sequence space satisfying property \( y \) but neither property \( b \) nor property \( \varepsilon \).

Let for \( k \in \mathbb{N} \) be \( I_k := I_{k+1} \cup I_k \cup N \) such that \( \| I_k \| = \| I_k \setminus I_{k+1} \| = \infty \).

We define

\[
B_k := \{ (\alpha_n)_{n \in \mathbb{N}} \in l_\infty : \| (\alpha_n)_{n \in \mathbb{N}} \|_\infty + k \limsup_{n \to \infty, n \in I_k} |\alpha_n| \leq 1 \}, k \in \mathbb{N}.
\]

\( B_k \) is normal and it is the closed unit ball of a norm on \( l_\infty \) which is equivalent to \( \| \cdot \|_\infty \). Hence \( B := \bigcap_{k \in \mathbb{N}} B_k \) is normal and closed in \( l_\infty \) if we denote with \( \lambda \) the linear span of \( B \) (which contains \( \varphi \)) and with \( \| \cdot \|_{\lambda} \) the Minkowski functional of \( B \), then \( (A, \| \cdot \|_{\lambda}) \) is a normal Banach sequence space.

**Example 2.13.** \((A, \| \cdot \|_{\lambda}) \) does not satisfy property \( \gamma_w \).
Proof. For \( k \in \mathbb{N} \) let \( \alpha^{(k)} = (\alpha_n^{(k)})_{n \in \mathbb{N}} \) be defined by:

\[
\alpha_n^{(k)} := \begin{cases} 
1 & : \ n \in I_k \setminus I_{k+1} \\
0 & : \ n \notin I_k \setminus I_{k+1}.
\end{cases}
\]

For \( m, k \in \mathbb{N} \) we have

\[
\left\| \alpha^{(k)} \right\|_\infty + m \limsup_{n \to \infty, n \notin I_m} |\alpha_n^{(k)}| = \begin{cases} 
m + 1 & : \ m \leq k \\
1 & : \ m > k.
\end{cases}
\]

This implies \( \alpha^{(k)} \in \lambda \) and \( \left\| \alpha^{(k)} \right\|_\lambda = k + 1, k \in \mathbb{N} \). On the other side we compute \( \left\| P_n(\alpha^{(k)}) \right\|_\lambda = 1 \) for every \( k, n \in \mathbb{N} \) and therefore \( (\lambda, \left\| \cdot \right\|_\lambda) \) does not satisfy property \( \gamma_w \).

We remark that the first example of such type was constructed by S. Dineen, see [13].

In the remaining part of this chapter we examine the connection between normal Banach sequence spaces and perfect spaces.

Definition 2.14. For a linear space \( \lambda \subset \mathcal{W} \) we define (the so-called \( \alpha \)-dual)

\[
\lambda^\times := \{ \beta \in \omega : \sum_{n \in \mathbb{N}} |\alpha_n \beta_n| < \infty \text{ for all } \alpha \in \lambda \}.
\]

\( \lambda^\times \) is called perfect if \( \lambda = \lambda^{\times\times} \).

The following elementary properties are taken from [24], §30, 2.

Remark 2.15. Let \( \lambda \subset \mathcal{W} \) be a linear space. Then \( \varphi \subset \lambda^\times = \lambda^{\times\times} \) and \( \lambda^\times \) is a normal subset of \( \mathcal{W} \).

If one assumes, in addition, that \( \lambda \) contains \( \varphi \) then \( \lambda \) and \( \lambda^\times \) form in a canonical way a dual system \( \langle \lambda, \lambda^\times \rangle \) with bilinear form \( (\alpha, \beta) \mapsto \sum_{n=1}^\infty \alpha_n \beta_n \). So, for example, the strong topology \( \beta(\lambda, \lambda^\times) \) and the weak topology \( \sigma(\lambda, \lambda^\times) \) on \( \lambda \) are uniquely determined.

Now we prove a connection between normal Banach sequence spaces and perfect spaces:

Theorem 2.16. Let \( (\lambda, \left\| \cdot \right\|_\lambda) \) be a normal Banach sequence space satisfying property \( \gamma_w \). Then \( \lambda \) is perfect and \( \left\| \cdot \right\|_\lambda \) induces the strong topology \( \beta(\lambda, \lambda^\times) \) on \( \lambda \).

Proof.

1) For \( A \subset \lambda \) let

\[
A^\circ := \{ \beta \in \lambda^\times : \sum_{n \in \mathbb{N}} |\alpha_n \beta_n| \leq i \text{ for all } \alpha \in A \}.
\]

If \( B_\lambda \) denotes (as usual) the closed unit ball of \( \lambda \), we obtain (since \( B_\lambda \) is normal)

\[
B_\lambda^\circ = \{ \beta \in \omega : \sum_{n \in \mathbb{N}} |\alpha_n \beta_n| \leq \text{ for all } \alpha \in B_\lambda \}.
\]

\( B_\lambda^\circ \) is an absolutely convex, bounded and normal subset of \( \mathcal{W} \) and its span \([B_\lambda^\circ]\) contains \( \varphi \). Moreover, \( B_\lambda^\circ \) is closed in \( \omega \):
Let $\beta = (\beta_n)_{n \in \mathbb{N}} \in B^\infty_\lambda$. Then there is a sequence $((\beta^{(k)}_n)_{n \in \mathbb{N}})_{k \in \mathbb{N}} = (\beta^{(k)})_{k \in \mathbb{N}}$ in $B^\infty_\lambda$ such that $\lim_{k \to \infty} \beta^{(k)}_n = \beta_n$ for all $n \in \mathbb{N}$.

Because $B_\lambda$ is normal, we have for every $N \in \mathbb{N}, a \in B_\lambda$ that $P_N(\alpha) \in B_\lambda$ and hence for every $k \in \mathbb{N}$: \[ \sum_{n=1}^{N} |\alpha_n \beta_n^{(k)}| \leq 1 \]

. and it follows $\beta \in B^\infty_\lambda$.

If we denote by $\| \cdot \|_\infty$ the Minkowski functional of $B^\infty_\lambda$, we get that $(B^\infty_\lambda, \| \cdot \|_\infty)$ is a normal Banach sequence space. Theorem 2.9 implies that this space satisfies even property 6.

It is clear that $[B^\infty_\lambda] \subset \lambda^\infty$. In the next step we prove that even $[B^\infty_\lambda] = \lambda^\infty$ is true. Assume this is wrong. Then there exist $(\beta_n)_{n \in \mathbb{N}} = a \in \lambda^\infty$ and a sequence $((\beta_n^{(k)})_{n \in \mathbb{N}})_{k \in \mathbb{N}} = (\alpha_n^{(k)})_{k \in \mathbb{N}} \in B_\lambda$ with $\sum_{n=1}^{\infty} |\alpha_n^{(k)}| \beta_n^{(k)} | \geq 2^k$ for every $k \in \mathbb{N}$. The completeness of $(\lambda, \| \cdot \|_\infty)$ guarantees that $(\alpha_n)_{n \in \mathbb{N}} = \alpha := \sum_{k=1}^{\infty} 2^{-k} \alpha^{(k)} \in B_\lambda$. Then

\[
\sum_{n=1}^{\infty} |\alpha_n \beta_n| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^{-k} |\alpha_n^{(k)}| \beta_n^{(k)} | = \sum_{k=1}^{\infty} 2^{-k} \sum_{n=1}^{\infty} |\alpha_n^{(k)}| \beta_n^{(k)} | \geq \sum_{k=1}^{\infty} 1 = \infty,
\]

a contradiction to $\beta \in \lambda$.

So we proved up to now that $(\lambda', \| \cdot \|_\infty)$ is a normal Banach sequence space with closed unit ball $B^\infty_\lambda$ which satisfies property 6. If we denote with $\| \cdot \|_{\times \times}$ the Minkowski functional of $B^\infty_\lambda$ (the second polar taken in $\lambda^{\times \times}$) we get that $(\lambda^{\times \times}, \| \cdot \|_{\times \times})$ is also a normal Banach sequence space with closed unit ball $B^\infty_\lambda$ which satisfies property 6).

2) For $N \in \mathbb{N}$ let $L_N := \{ a \in \omega : \alpha_n = 0 \text{ for all } n > N \}$. and define for $A \subset \omega$, $L_N^A$:

$A^{\circ L_N^A} := \{ \beta \in L_N : \sum_{n=1}^{N} |\alpha_n \beta_n| \leq 1 \text{ for all } \alpha \in A \}$.

Since $B^\infty_\lambda$ is normal, we have $B^\infty_\lambda \cap L_N = (B_\lambda \cap L_N)^{\circ L_N^A}$ and the same argument also yields $B^\infty_\lambda \cap L_N = (B_\lambda \cap L_N)^{\circ L_N^A}$. The bipolar theorem implies $(B_\lambda \cap L_N)^{\circ L_N^A} = B_\lambda \cap L_N$ and therefore the norms $\| \cdot \|_\lambda$ and $\| \cdot \|_{\times \times}$ coincide on $L_N$. This is true for every $N \in \mathbb{N}$ and hence both norms coincide on $\mathfrak{P}(\{ P_n(\alpha) : a \in \omega, n \in \mathbb{N} \})$. With 1) we conclude that the normal Banach sequence spaces $(\lambda, \| \cdot \|_\lambda)$ and $(\lambda^{\times \times}, \| \cdot \|_{\times \times})$ both satisfy property $\delta_w$. It follows

$\lambda = \{ \alpha \in \omega : \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda < \infty \} = \{ \alpha \in \omega : \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_{\times \times} < \infty \} = \lambda^{\times \times}$

and with theorem 2.9 we obtain that the iiorins $\| \cdot \|_\lambda$ and $\| \cdot \|_{\times \times}$ are equivalent.

To prove that $\| \cdot \|_\lambda$ generates the strong topology $\beta(\lambda, A')$ on $A$ it suffices to show that $B^\infty_\lambda$ (the closed unit ball of $\| \cdot \|_{\times \times}$ is a bounded barrel in $(\lambda, \sigma(\lambda, A'))$.

$B^\infty_\lambda$ is clearly absolutely convex and absorbant. In 1) we have shown that $[B^\infty_\lambda] = \lambda$ and therefore $\{ \sum_{n=1}^{\infty} \alpha_n \beta_n : a \in B_\lambda \}$ is bounded for every $\beta \in \lambda$. Hence $B^\infty_\lambda$ is $\sigma(\lambda, \lambda^\times)$-bounded. Again from 1) we deduce that $B^\infty_\lambda$ is closed in $\omega$ and this implies that it is $\sigma(\lambda, \lambda^\times)$-closed.
Remark 2.17.  i) The completeness of \((A, \| \cdot \|_\lambda)\) is essential for the proof of theorem 2.16. It is not sufficient to take \(\lambda\) as a normal and dense subspace of a normal Banach sequence space.

Consider \(\varphi\) equipped with \(\| \cdot \|_1\). Then this space is dense in \((l_1, \| \cdot \|_1)\) and \(\varphi\) is not normal but \(\beta(\varphi, \varphi^\times) = \beta(\varphi, w)\) is the finest locally convex topology on \(\varphi\).

ii) From the proof of theorem 2.16 follows:

For every normal Banach sequence space \(\lambda\) the \(\alpha\)-dual \(\lambda^\times\) is a normal Banach sequence space satisfying property \(\delta\), if we equip \(\lambda^\times\) with the Minkowski functional of \(B_{\lambda^\times}^\times\).

iii) Analysing the proof of the above theorem one can give the following sharper result:

Let \((A, \| \cdot \|_\lambda)\) be a normal Banach sequence space. Then \(\lambda^{\times\times} = \lambda^{(6)}\) and \(\| \cdot \|_{\lambda^{(6)}}\) generates the strong topology \(\beta(\lambda^{\times\times}, \lambda^\times)\) on \(\lambda^{\times\times}\).

iv) Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space satisfying property \(\varepsilon\). With theorem 2.9 and the previous remark we obtain that \(\lambda\) is the regular subspace (in the sense of T. Kômura and Y. Kômura [23]) of the perfect space \(\lambda^{(6)}\). Confer also [36], p. 195.

3. VECTOR VALUED SEQUENCE SPACES

Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space with closed unit ball \(B_\lambda\). For an absolutely convex subset \(C\) of a locally convex Hausdorff space \(E = (E, \mathcal{V})\) denote by \(p_C\) its Minkowski functional (defined on the linear span \([C]\) of \(C\)). So we have that the system \(cs(E)\) of all continuous seminorms on \(E\) is equal to \(\{p_U : U \in \mathcal{U}_0(E)\text{ absolutely convex}\}\). We define the vector valued sequence space

\[\lambda(E) := \lambda((E, \mathcal{T})) := \{x^N : (p(x_n))_{n \in \mathbb{N}} \in \lambda \text{ for every } p \in cs(E)\}\]

and for an absolutely convex subset \(C\) of \(E\)

\[\lambda(C) := \lambda(C, (E, \mathcal{T})) := \{x \in \lambda(E) \cap [C]^N : (p_C(x_n))_{n \in \mathbb{N}} \in B_\lambda\}\].

Then \(\lambda(E)\) is for every normal Banach sequence space and every locally convex Hausdorff space \((E, \mathcal{T})\) a vector space which we equip with the locally convex Hausdorff topology admitting

\[\{\lambda(U) : U \in \mathcal{U}_0(E)\text{ absolutely convex}\}\]

as a zero basis. Equivalently, the topology on \(\lambda(E)\) is induced by the seminorms \(x = (x_n)_{n \in \mathbb{N}} \longmapsto \|p(x_n))_{n \in \mathbb{N}\|_\lambda, p \in cs(E)\).

We note that \(\lambda(E)\) contains \(E\) as a complemented subspace.

If \(E\) is metrizable, then so is \(\lambda(E)\). Moreover, if \((E, \| \cdot \|_E)\) is a normed space, then the topology of \(\lambda(E)\) is induced by the norm

\[\lambda = (x_n)_{n \in \mathbb{N}} \|\|x_n\||_{n \in \mathbb{N}}\|_\lambda\].

We will denote this normed space by \(\lambda((E, \| \cdot \|_E))\) or by \((\lambda(E), \| \cdot \|_{\lambda(E)})\).

Let the locally convex space \((E, \mathcal{T})\) be continuously included in the locally convex Hausdorff space \((F, \mathcal{S})\). Then \(\lambda((E, \mathcal{T}))\) is continuously included in \(\lambda((F, \mathcal{S}))\). Hence, if \(B\) is a bounded and absolutely convex subset of \((E, \mathcal{T})\) we get \(\lambda([B], p_B) \subset \lambda(E, \mathcal{T})\) and therefore (algebraically) \(\lambda(B, [B], p_B) = \lambda(B, (E, \mathcal{T}))\).
We now prove that for every absolutely convex $C \subset E$ the set $\lambda(C)$ depends only on the relative topology on $C$ induced by $(E, \mathcal{T})$. We denote by $\mathcal{T}_C$ the finest locally convex topology on the spair $[C]$ of $C$ which coincides with $\mathcal{T}$ on $C$.

**Proposition 3.1.** Let $(A, \|\cdot\|_\lambda)$ be a normal Banach sequence space, $(E, \mathcal{T})$ be a locally convex Hausdorff space and $C \subset E$ be absolutely convex. Then

$$\lambda(C, (E, \mathcal{T})) = \lambda(C, ([C], \mathcal{T}_C))$$

and the relative topology induced by $\lambda((E, \mathcal{T}))$ and $\lambda(([C], \mathcal{T}_C))$ coincide on this set.

**Proof.** 1) Let $V \in \mathcal{U}_0(([C], \mathcal{T}_C))$ be absolutely convex. Then there is $U \in \mathcal{U}_0((E, \mathcal{T}))$ absolutely convex such that $\mathcal{U} \cap C \subset \frac{1}{4} V$.

For $\mathcal{S} = (x_n)_{n \in \mathbb{N}} \in [C]^N$ with $(p_U(x_n))_{n \in \mathbb{N}} \in B_\lambda$ and $(p_C(x_n))_{n \in \mathbb{N}} \in B_\lambda$ we get for $n \in \mathbb{N}$:

$$x_n \in (2p_U(x_n)U) \cap (2p_C(x_n)C) \subset 2(p_U(x_n) + p_C(x_n)U \cap C$$

$$\subset 2(p_U(x_n) + p_C(x_n)) - \frac{1}{4} V \quad \text{if } p_U(x_n), p_C(x_n) > 0,$$

$$x_n \in p_C(x_n) - \frac{1}{4} V \quad \text{if } p_U(x_n) = 0 < p_C(x_n),$$

$$x_n \in p_U(x_n) - \frac{1}{4} V \quad \text{if } p_C(x_n) = 0 < p_U(x_n),$$

$$x_n \in \frac{1}{4} V \text{ for every } \rho > 0 \quad \text{if } p_C(x_n) = p_U(x_n) = 0.$$

For every $n \in \mathbb{N}$, this implies $0 \leq p_V(x_n) \leq 2(p_U(x_n) + p_C(x_n)) - \frac{1}{4}$ and therefore $(p_V(x_n))_{n \in \mathbb{N}} \in B_\lambda$ since $B_\lambda$ is normal. Therefore we obtain

$$\{x \in [C]^N : (p_U(x_n))_{n \in \mathbb{N}}, (p_C(x_n))_{n \in \mathbb{N}} \in B_\lambda\}$$

$$\subset \{x \in [C]^N : (p_V(x_n))_{n \in \mathbb{N}} \in B_\lambda\}.$$

2) Clearly one has $\lambda(([C], \mathcal{T}_C)) \subset \lambda((E, \mathcal{T})) \cap [C]^N$, this shows $\lambda(C, ([C], \mathcal{T}_C)) \subset \lambda(C, (E, \mathcal{T}))$.

Let $V \in \mathcal{U}_0(([C], \mathcal{T}_C))$ be absolutely convex. Because of 1) there is an absolutely convex $U \in \mathcal{U}_0((E, \mathcal{T}))$ such that

$$\lambda(C, (E, \mathcal{T})) \subset \{x \in [C]^N : (p_U(x_n))_{n \in \mathbb{N}}, (p_C(x_n))_{n \in \mathbb{N}} \in \lambda\}$$

$$\subset \{x \in [C]^N : (p_V(x_n))_{n \in \mathbb{N}} \in \lambda\}.$$

Since $V$ is arbitrary we get $\lambda(C, (E, \mathcal{T})) \subset \lambda(([C], \mathcal{T}_C))$ and therefore $\lambda(C, (E, \mathcal{T})) \subset \lambda(C, ([C], \mathcal{T}_C))$. To prove the remaining topological identity we first note that $\lambda((E, \mathcal{T}))$ induces on $\lambda(C, (E, \mathcal{T}))$ a coarser topology than $\lambda(([C], \mathcal{T}_C))$. For the converse let an absolutely convex $V \in \mathcal{U}_0(([C], \mathcal{T}_C))$ be given. With 1) and $\lambda(([C], \mathcal{T}_C)) \supset \lambda(C, (E, \mathcal{T}))$ we get the existence of an absolutely convex $U \in \mathcal{U}_0((E, \mathcal{T}))$ such that

$$\lambda(C, (E, \mathcal{T})) \cap \{x \in \lambda((E, \mathcal{T})) : (p_U(x_n))_{n \in \mathbb{N}} \in B_\lambda\}$$

$$\subset \{x \in \lambda(([C], \mathcal{T}_C)) : (p_U(x_n))_{n \in \mathbb{N}}, (p_C(x_n))_{n \in \mathbb{N}} \in B_\lambda\}$$

$$\subset \{x \in \lambda(([C], \mathcal{T}_C)) : (p_V(x_n))_{n \in \mathbb{N}} \in B_\lambda\}.$$
and this yields the conclusion.

**Corollary 3.2.** Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space, \(E\) be a vector space and \(C \subseteq E\) be absolutely convex. Moreover, let \(F_1, F_2\) be algebraic subspaces of \(E\) containing \(C\), equipped with locally convex Hausdorff topology \(\mathcal{T}_1, \mathcal{T}_2\), resp.). If both topologies coincide on \(C\) we have

\[ \lambda(C, (F_1, \mathcal{T}_1)) = \lambda(C, (F_2, \mathcal{T}_2)) \]

and the topologies induced by \(\lambda((F_1, \mathcal{T}_1))\) and \(\lambda((F_2, \mathcal{T}_2))\) coincide on this set.

In the next result we consider properties of the sets \(\lambda(B)\) if \(B\) is a bounded, closed and absolutely convex subset of a locally convex space \(E\). It is clear that then \(\lambda(B)\) is again bounded in \(\lambda(E)\). We prove implications of additional properties required for \(\lambda\).

**Theorem 3.3.** Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space and let \(E\) be a locally convex Hausdorff space. Moreover, let \(B \subseteq E\) be absolutely convex, bounded and closed.

i) Then \(\overline{\lambda(B)}^{E^N} \subseteq [B]^N\) and for every \(x = (x_n)_{n \in \mathbb{N}} \in \overline{\lambda(B)}^{E^N}\) and every \(m \in \mathbb{N}\) we have

\[ (p_B(x_n))_{n \leq m}, (0)_{n > m} = p_m((p_B(x_n))_{n \in \mathbb{N}}) \in B_\lambda. \]

ii) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\gamma\) then \(\lambda(B)\) is closed in its span \([\lambda(B)]\) equipped with the relative topology induced by \(E^N\).

iii) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\gamma_w\) then there is \(\rho > 0\) such that the closure of \(\lambda(B)\) taken in its span \([\lambda(B)]\) equipped with the relative topology induced by \(E^N\) is contained in \(\rho \lambda(B)\).

iv) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\delta\) then \(\lambda(B)\) is closed in \(E^N\).

v) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\delta_w\) then there is \(\rho > 0\) such that the closure of \(\lambda(B)\) taken in \(E^N\) is contained in \(\rho \lambda(B)\).

vi) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\varepsilon\) then \(E^N\) is large in \(\lambda(E)\), i.e. for every bounded set \(B_1\) in \(\lambda(E)\) there is another bounded set \(B_2\) in \(\lambda(E)\) such that \(B_1\) is contained in the closure of \(B_2 \cap E^N\). In particular \(E^N\) is dense in \(\lambda(E)\).

**Proof.** i) Since \(B\) is closed, we get \(\overline{\lambda(B)}^{E^N} \subseteq \prod_{n \in \mathbb{N}} \| (\delta_{n_k})_{k \in \mathbb{N}} \| \lambda B \subseteq [B]^N\). Let \(x = (x_n)_{n \in \mathbb{N}} \in \overline{\lambda(B)}^{E^N}\) and \(m \in \mathbb{N}\) be given. For arbitrary \(\varepsilon > 0\) there is \(x^{(\varepsilon)} = (x_n^{(\varepsilon)})_{n \in \mathbb{N}} \in \lambda(B)\) such that

\[ p_B(x_n) \leq p_B(x_n^{(\varepsilon)}) + \varepsilon \text{ for every } n \in \{1, 2, \ldots, m\}. \]

Hence

\[ \| p_m((p_B(x_n))_{n \in \mathbb{N}}) \| \leq \| p_m((p_B(x_n^{(\varepsilon)})) + \varepsilon)_{n \in \mathbb{N}}) \| \leq 1 + \varepsilon \sum_{n=1}^{m} \| (\delta_{n_k})_{k \in \mathbb{N}} \| \lambda. \]

ii) Let \(x = (x_n)_{n \in \mathbb{N}} \in \overline{\lambda(B)}^{E^N} \cap [\lambda(B)]\) be given. This implies \(x \in \lambda(E)\) and \((p_B(x_n))_{n \in \mathbb{N}} \in \lambda\). Using i) we get

\[ \sup_{m \in \mathbb{N}} \| p_m((p_B(x_n))_{n \in \mathbb{N}}) \| \leq 1 \]

and property \(\gamma\) implies \((p_B(x_n))_{n \in \mathbb{N}} \in B_\lambda\).

iii) If \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\gamma_w\) then Theorem 2.9 guarantees the existence of an equivalent norm on \(\lambda\) which satisfies property \(\gamma\). The assertion follows with ii).

iv) Let \(x = (x_n)_{n \in \mathbb{N}} \in \overline{\lambda(B)}^{E^N}\). With i) we get

\[ \sup_{m \in \mathbb{N}} \| p_m((p_B(x_n))_{n \in \mathbb{N}}) \| \leq 1 \]
and property 6) implies \((p_B(x_n))_{n \in \mathbb{N}} \in B_{\lambda}\). It remains to show that \(x \in \lambda(E)\). But this is implied by the boundedness of \(B\).

v) Can be proved in the same arguments used for iii).

vi) Let \(B_1\) be a bounded subset of \(\lambda(E)\). Then we have for every \(p \in cs(E)\) that 
\[
\rho_p := \max \{1, \sup_{x \in B_1} \|p(x_n)\|_{\lambda}\}
\]
is finite. Thus, 
\[
B_2 := \{x \in \lambda(E) : \rho_p(x_n)_{n \in \mathbb{N}} \in \rho_p B_{\lambda} \text{ for every } p \in cs(E)\}
\]
is bounded in \(\lambda(E)\) and contains \(B_1\).

Let \(x = (x_n)_{n \in \mathbb{N}} \in B_1\) be given. The construction of \(B_2\) guarantees that it contains the sections \((x_n)_{n \leq m}, (0)_n)\) for every \(m \in \mathbb{N}\). Since \((\lambda, \| \cdot \|_{\lambda})\) satisfies property \(\varepsilon\) we have 
\[
\lim_{m \to \infty} \|((0)_{n \leq m}, (p(x_n))_{n > m})\|_{\lambda} = \lim_{m \to \infty} \|P_m((p(x_n))_{n \in \mathbb{N}} - (p(x_n))_{n \in \mathbb{N}}\|_{\lambda} = 0
\]
for every \(p \in cs(E)\). This implies that the sections \((x_n)_{n \leq m}, (0)_n)_{n > m}\) converge to \(x(m \to \infty)\) and we are done.

**Remark 3.4.** In the previous theorem the boundedness of the set \(B\) is essential as the trivial example \(E := R, B := R\) shows. In this case we have \(\lambda(B)^{E^{\infty}_{\lambda}} = E^\infty_{\lambda} \not\subset \lambda(E)\) for every normal Banach sequence space \(\lambda\).

In contrast to part ii) of the previous theorem we present a simple example of a normal Banach sequence space \((\lambda, \| \cdot \|_{\lambda})\) satisfying property \(\varepsilon\), a locally convex space \(E\) containing an absolutely convex, bounded, and closed set \(B\) such that \(\lambda(B)\) is not closed in \(\lambda(E)\) (hence not closed in \(\lambda(E)\) equipped with the relative topology induced by \(E^{\infty} \)).

**Example 3.5.** \(B_{l_{\infty}}\) (the closed unit ball of \(l_{\infty}\)) is an absolutely convex, bounded and closed subset of \(w\), but \(c_0(B_{l_{\infty}})\) is not closed in \(c_0(w)\).

Indeed, for \(n \in \mathbb{N}\) let \(\alpha_n := ((0)_{n \leq n}, (1)_{n > n}) \in B_{l_{\infty}}\). Then \(\alpha := (\alpha_n)_{n \in \mathbb{N}} \in c_0(\omega) \setminus c_0(B_{l_{\infty}})\). But for every \(m \in \mathbb{N}\) the sections \(\alpha^{(m)} := ((\alpha_n)_{n \leq m}, (0)_n)_{n > m}\) are contained in \(c_0(B_{l_{\infty}})\) and \(\alpha^{(m)} \to \alpha(m \to \infty)\) in \(c_0(\omega)\).

Instead of \(\omega\) one can choose in this example diagonal transforms \(\beta \cdot l_{\infty}\) of \(l_{\infty}\) (if \(0 < \beta_n \to \infty (n \to \infty)\)).

The following result is an application of proposition 2.4.

**Proposition 3.6.** Let \((\lambda, \| \cdot \|_{\lambda})\) be a normal Banach sequence space and let \(E\) be a locally convex Hausdorff space. For \(\alpha := ((\delta_{nk})_{k \in \mathbb{N}} \|_{\lambda}^{-1})_{n \in \mathbb{N}}\) the inclusions 
\[
E^{(N)} \hookrightarrow (\alpha \cdot l_{\infty})_{(E) \hookrightarrow (\alpha \cdot l_{\infty})_{(E) \hookrightarrow E_{\lambda}^N}}
\]
are continuous.

In contrast to the previous result we will prove that for every locally convex Hausdorff space \(E\) the space \(\lambda(E)\) contains \(l_{\infty}(E)\) as a coiuiipleineited subspace whichever the normal Banach sequence space \(\lambda\) satisfies not the property \(\varepsilon\). A similar result, assuming in addition that \(\lambda\) satisfies property \(\gamma\) and \(E\) is a Banach space, was already given by C. Fernández [15]. Our proof is mainly adapted from her paper. The technical part of the proof is done in the following lemma.
Lemma 3.7. Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space with closed unit ball $B_\lambda$ and let $E$ be a locally convex Hausdorff space. Let $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in E$ with $\|\alpha\|_\lambda = 1$. Then there is a continuous linear map $T : \lambda(E) \to E$ which satisfies the following three conditions:

i) $p(T(x)) \leq \|p(x_n)\|_\lambda$ for every $x = (x_n)_{n \in \mathbb{N}} \in \lambda(E)$ and every $p \in \text{cs}(E)$.

ii) $T((\alpha_n x_0)_{n \in \mathbb{N}}) = x_0$ for all $x_0 \in E$.

iii) $\{(x_n)_{n \in \mathbb{N}} \in \lambda(E) : x_n = 0 \text{ if } \alpha_n \neq 0, n \in \mathbb{N}\}$ is contained in the kernel of $T$.

**Proof.** Let $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in E$ with $\|\alpha\|_\lambda = 1$ be given. We denote $S := \{n \in \mathbb{N} : \alpha_n \neq 0\}$.

Then the space $L_S := \{(\beta_n)_{n \in \mathbb{N}} \in \lambda : \beta_n = 0 \text{ if } n \notin S\}$ contains $\alpha$ and it is of finite dimension. We equip $L_S$ with the norm induced by $\| \cdot \|_\lambda$. The dual of this space is isomorphic to $L_S$. Using basic duality theory we get the existence of $(\alpha_n)_{n \in \mathbb{N}} \in L_S$ with the following two properties:

$$\sum_{n=1}^{\infty} a_n \beta_n \leq 1 \text{ for every } (\beta_n)_{n \in \mathbb{N}} \in L_S \cap B_\lambda \text{ and } \sum_{n=1}^{\infty} a_n \alpha_n = 1.$$ 

Since $B_\lambda$ is normal we even get

$$\sum_{n=1}^{\infty} |a_n \beta_n| \leq 1 \text{ for every } (\beta_n)_{n \in \mathbb{N}} \in B_\lambda.$$ 

We define the linear map $T : \lambda(E) \to E$ by $T((x_n)_{n \in \mathbb{N}}) := \sum_{n=1}^{\infty} a_n x_n$.

For every $x_0 \in E$ we have $\sum_{n=1}^{\infty} a_n \alpha_n x_0 = x_0$. This implies ii) and (iii). To prove i) let $x = (\alpha_n)_{n \in \mathbb{N}} \in \lambda(E)$ and $p \in \text{cs}(E)$ be given. Using the properties of $(a_n)_{n \in \mathbb{N}}$ proved before we get:

$$p(T(x)) = p \left( \sum_{n=1}^{\infty} a_n x_n \right) \leq \sum_{n=1}^{\infty} |a_n| p(x_n) \leq \|p(x_n)\|_{\lambda}.$$ 

This shows property i) and hence the continuity of $T$.

**Theorem 3.8.** (cf. [15], 2.3) Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space not satisfying property $\varepsilon$ and let $E$ be a locally convex Hausdorff space. Then $\lambda(E)$ contains a complemented copy of $l_\infty(E)$. 

**Proof.** $\lambda$ does not satisfy property $\varepsilon$ and hence there is $\mu \in E \setminus \lambda$ with $\lim \sup_{n \to \infty} \|P_n(\mu) - \mu\|_\lambda > 0$. Hence $(P_n(\mu))_{n \in \mathbb{N}}$ is not a Cauchy sequence in $\lambda$. This implies the existence of a strictly increasing sequence $(m_k)_{k \in \mathbb{N}}$ of natural numbers and of $6 > 0$ with $\|P_{m_k+1}(\mu) - P_{m_k}(\mu)\|_\lambda > 6 \text{ for every } k \in \mathbb{N}$. We define $(\alpha_n^{(k)})_{n \in \mathbb{N}} := \alpha^{(k)} - P_{m_{k+1}}(\mu) - P_{m_k}(\mu)_{\lambda} = P_{m_{k+1}}(\mu)$ and $(\alpha_n^{(0)})_{n \in \mathbb{N}} := \alpha := \sum_{k=1}^{\infty} \alpha^{(k)}$, the limit taken in $\omega$. (By construction, $\sum_{k=1}^{\infty} \alpha^{(k)}$ cannot converge in $\lambda$. For every $n \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\alpha_n = \sum_{k=1}^{n} \alpha_n^{(k)}$. 

For $x = (x_k)_{k \in \mathbb{N}} \in E$, we define $i(x)_n := i(x) := \sum_{k=1}^{n} (\alpha_n x_k)_{n \in \mathbb{N}}$ the limit taken in $E^N$. 

$$i(x)_n := i(x) := \sum_{k=1}^{n} (\alpha_n x_k)_{n \in \mathbb{N}} \text{ the limit taken in } E^N.$$
For \( p \in \text{cs}(E), (x_k)_{k \in N} \in l_\infty(E) \) and \( n \in N \) we get \( p(\lambda(x)_n) \leq \|p(x_k)\|_\infty |\alpha_n| \), therefore \( (p(\lambda(x)_n))_{n \in N} \in \lambda \) and \( \|p(\lambda(x)_n)\|_\lambda \leq \|\alpha\|_\lambda \|p(x_k)\|_\infty \). This implies that

\[
i : l_\infty(E) \to \lambda(E), x \longmapsto i(x)
\]

is a welldefined, linear and continuous map. It is easy to see that \( i \) is also injective.

Now we construct a continuous linear mapping \( T : \lambda(E) \to l_\infty(E) \) such that \( T \circ i \) is the identity on \( l_\infty(E) \).

For each \( k \in N \) we have that \( (\alpha_{\nu}^{(k)})_{n \in N} = \alpha^{(k)} \in \varphi \) and \( \|\alpha^{(k)}\|_\lambda = 1 \). For \( k \in N \) let \( T_k : \lambda(E) \to E \) be the continuous linear mapping associated to \( \alpha^{(k)} \) constructed in lemma 3.7. We define:

\[
T : \lambda(E) \to l_\infty(E), x \longmapsto (T_k(x))_{k \in N}.
\]

\( T \) is welldefined, linear and continuous, because for every \( k \in N \), \( x = (x_n)_{n \in N} \in \lambda(E) \) and \( p \in \text{cs}(E) \) we have \( p(T_k(x)) \leq \|p(x_n)\|_\lambda \) (by condition i) of lemma 3.7).

By condition iii) of lemma 3.7 we have

\[
T_k((x_n)_{n \in N} \in \lambda(E) : x_n = 0 \text{ if } \alpha^{(k)} \neq 0) = \{0\}
\]

and therefore \( T_k(\sum_{m=1}^{\infty} (\alpha_{\nu}^{(m)} x_m)_{n \in N}) = 0 \) for every \( k \in N \). This and condition ii) of lemma 3.7 imply for every \( x = (x_k)_{k \in N} \in \lambda(E) \):

\[
T(i(x)) = T(\sum_{m=1}^{\infty} (\alpha_{\nu}^{(m)} x_m)_{n \in N}) = (T_k(\sum_{m=1}^{\infty} (\alpha_{\nu}^{(m)} x_m)_{n \in N}))_{k \in N} = (T_k(x))_{k \in N} = x.
\]

So, \( T \circ i \) is the identity on \( l_\infty(E) \). Recalling that \( i : l_\infty(E) \to \lambda(E) \) is a continuous inclusion and \( T : \lambda(E) \to l_\infty(E) \) is linear and continuous, this proves the assertion.

**Remark 3.9.** Let \( \langle h, || \cdot ||_\lambda \rangle \) be a normal Banach sequence space satisfying not property \( \varepsilon \) and let \( E \) be a locally convex Hausdorff space. Then \( \Phi^\lambda(E) \) contains a complemented copy of \( c_0(E) \).

To see this, we rise the maps \( i : \lambda(E) \to \lambda(E) \) and \( T : \lambda(E) \to l_\infty(E) \) constructed in the proof of 3.8. From \( i(\Theta(E)^{(N)}) \subseteq E^{**} \) and \( T \circ i_{\Theta(E)^{(N)}} = id_{\Theta(E)^{(N)}} \) we obtain that \( i(c_0(E)) \subseteq \Phi^\lambda(E) \) and \( T(\Phi^\lambda(E)) = c_0(E) \). With the same argument used at the end of the proof of 3.8 we get the desired statement.

### 4. BOUNDED SETS AND DUALITY IN VECTOR VALUED SEQUENCE SPACES

Let \( (\lambda, || \cdot ||, \lambda) \) be a normal Banach sequence space and let \( E \) be a locally convex Hausdorff space. In this chapter we will consider descriptions for bounded subsets of the space \( \lambda(E) \) especially if \( E \) is metrizable or a gDF space. It will turn out that there is a coiiiijectioii betweeii these descriptionis aid the topological dual of \( \lambda(E) \) provided \( \lambda \) satisfies property \( \varepsilon \).

**Definition 4.1.** Let \( E \) be a locally convex Hausdorff space and let \( \mathcal{B} \) be a subset of the powerset \( \mathcal{P}(E) \) of \( E \). We call \( \mathcal{B} \) a fundaimental systein of bounded sets (in \( E \)) if it contains only absolutely convex and bounded sets and if every bounded set is contained in some element of \( \mathcal{B} \).
A sequence \((B_n)_{n \in \mathbb{N}}\) of subsets of \(E\) is called a fundamental sequence of bounded sets if \(\{B_n : n \in \mathbb{N}\}\) is a fundamental system of bounded sets in \(E\).

With this definition \(\mathcal{B}(E) := \{B \subset E : B \text{ bounded and absolutely convex}\}\) is a fundamental system of bounded sets. If \(E\) possesses a fundamental sequence of bounded sets \((B_n)_{n \in \mathbb{N}}\), then \((C_n)_{n \in \mathbb{N}} := (2^n \sum_{k = 1}^{n} B_k)_{n \in \mathbb{N}}\) is a fundamental sequence of bounded sets in \(E\) satisfying \(2C_n \subset C_{n+1}\) for every \(n \in \mathbb{N}\).

It is easy to see that for every locally convex Hausdorff space \(E\) the sets \(\{1, \infty\}(B) : B \subset \mathcal{B}(E)\) and \(\{B^N : B \subset \mathcal{B}(E)\}\) are fundamental systems of bounded sets in \(l_\infty(E)\) (cf. M. Florencio & P.J. Paul [16], 2.5). Now we give an example that for general normal Banach sequence spaces \(h\) such a description of bounded sets in \(\lambda(E)\) is not possible. We may even choose \(\lambda = l_1\).

**Example 4.2.** Consider \(c_0\) equipped with the weak topology. Then every bounded set in \(E := (c_0, \sigma(c_0, l_1))\) is absorbed by the bounded and closed set \(B_{c_0}\) (the closed unit ball in \(c_0\)), hence \((nB_{c_0})_{n \in \mathbb{N}}\) is a fundamental sequence of bounded sets in \(E = (c_0, \sigma(c_0, l_1))\). We have \(l_1(B_{c_0}) \neq l_1(E)\) and therefore \((n' l_1(B_{c_0}))_{n \in \mathbb{N}}\) is not a fundamental sequence of bounded sets in \(l_1(E)\).

Indeed, \(x := (x_n)_{n \in \mathbb{N}} := (((0)_{n < n}, 1, (0)_{n > n}))_{n \in \mathbb{N}}\) is contained in \(l_1(E)\), because for every \(f \in l_1 = c_0'\) we have \(\sum_{n \in \mathbb{N}} f(x_n) \leq ||f||_1\). On the other hand \(\sum_{n \in \mathbb{N}} ||x_n||_\infty = \infty\), so \(x\) is not contained in the span of \(l_1(B_{c_0})\). Using that \(l_1\) satisfies property 6 we can apply 3.3 to conclude that \(l_1(B_{c_0})\) is closed in \(E^N\) and therefore closed in \(l_1(E)\). Hence \(x \notin l_1(B_{c_0})\).

If \(E\) is a metrizable locally convex space, the situation looks much nicer. We prove the following lemma to get a canonical description for a fundamental system of bounded sets in \(\lambda(E)\).

**Lemma 4.3.** Let \((\lambda, ||\cdot||, \lambda)\) be a normal Banach sequence space and let \(E\) be a locally convex Hausdorff space. If \((C_n)_{n \in \mathbb{N}}\) is a sequence of absolutely convex and closed subsets of \(E\) and \(0 < (\alpha_n)_{n \in \mathbb{N}} = \alpha \in l_1\), then

\[
\bigcap_{k \in \mathbb{N}} \lambda(C_k) \subset \lambda \left( \bigcap_{k \in \mathbb{N}} \alpha_k^{-1} C_k \right).
\]

**Proof.** Set \(\nu := \bigcap_{k \in \mathbb{N}} \alpha_k^{-1} C_k\). Let \((x_n)_{n \in \mathbb{N}} \in \bigcap_{k \in \mathbb{N}} \lambda(C_k)\).

For every \(n \in \mathbb{N}\) and every \(k \in \mathbb{N}\) with \(p_{C_k}(x_n) \geq 0\) we have

\[
x_n \in \alpha_k p_{C_k}(x_n) \alpha_k^{-1} C_k.
\]

Hence, for \(n \in \mathbb{N}\):

\[
x_n \in \left( \sum_{j = 1}^{\infty} \alpha_j p_{C_j}(x_n) \right) \alpha_k^{-1} C_k, k \in \mathbb{N}, \quad \text{if there is } j \in \mathbb{N} \text{ with } p_{C_j}(x_n) \geq 0
\]

and

\[
x_n \in \rho \bigcap_{k \in \mathbb{N}} \alpha_k^{-1} C_k, \rho > 0, \quad \text{if } p_{C_j}(x_n) = 0 \text{ for every } j \in \mathbb{N}.
\]
This implies $p_V(x_n) \leq \sum_{j=1}^{\infty} \alpha_j p_C(x_n)$ for every $n \in \mathbb{N}$ and therefore $(p_V(x_n))_{n \in \mathbb{N}} \leq (\sum_{j=1}^{\infty} \alpha_j p_C(x_n))_{n \in \mathbb{N}} = \sum_{j=1}^{\infty} \alpha_j (p_C(x_n))_{n \in \mathbb{N}}$. The completeness of $\lambda$ and $(p_C(x_n))_{n \in \mathbb{N}} \in \mathcal{B}_\lambda$, $j \in \mathbb{N}$, guarantees $\sum_{j=1}^{\infty} \alpha_j (p_C(x_n))_{n \in \mathbb{N}} \in \|\alpha\|_\lambda \mathcal{B}_\lambda$ and so we get $(p_V(x_n))_{n \in \mathbb{N}} \in \|\alpha\|_\lambda \mathcal{B}_\lambda$.

The following result is due to C. Fernandez and can be found in [28], 2.1.

**Corollary 4.4.** Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space and let $E$ be a metrizable locally convex space. Then $\{\lambda(B) : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$.

**Proof.** $E$ possesses a zero basis $(U_n)_{n \in \mathbb{N}}$ consisting of absolutely convex and closed sets. Let $B$ be a bounded subset of $\lambda(E)$. Then there exist $\rho_n > 0$ such that $p \in \rho_n \lambda(U_n)$ for every $n \in \mathbb{N}$. If $C : \cap_{n \in \mathbb{N}} 2^n \rho_n U_n$, then $C$ is bounded in $E$ and lemma 4.3 implies $B \subseteq \lambda(C)$.

**Remark 4.5.** For $\lambda = l_1$ the above corollary has been proved by A. Pietsch (see [32]) and it means that the space $E$ satisfies the property (B) of Pietsch. If $\lambda$ is perfect, the previous corollary is covered by a result of R.C. Rosier [34] and he called locally convex Hausdorff spaces $E$ fundamentally $\lambda$-bounded if $\{\lambda(B) : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$.

As the example 4.2 shows, an analogous description as proved in corollary 4.4 for the bounded sets in $\lambda(E)$ is not possible in general. Even in the case of DFM spaces (the strong duals of Fréchet-Montel spaces) there are counter examples:

For the DFM space $E$ constructed by Köthe [24], §31, 5. we have $c_0(E) \not\subseteq \cap_{B \in \mathcal{B}(E)} c_0(B)$. (But later we will see that in this case one has to use the closures of the sets $c_0(B)$ to get a fundamental system of bounded sets in $c_0(E)$.)

Now we consider the situation on the dual side of metrizable locally convex spaces. We recall some definitions:

**Definition 4.6.** Let $(E, I)$ be a locally convex Hausdorff space possessing a fundamental sequence of bounded sets $(B_n)_{n \in \mathbb{N}}$. $E$ is called $\alpha$

i) gDF space if $T$ is the finest locally convex topology on $E$ whose restriction to each $B_n$ coincides with $I$.

ii) DF space if $(E, I)$ is $\aleph_0$-quasibarrelled, i.e. whenever $U$ is the countable intersection of absolutely convex and closed zero neighbourhoods and $U$ is bornivorous then $U$ is again a zero neighbourhood.

**Remark 4.7.** i) Let $E$ be a locally convex Hausdorff space which possesses a fundamental sequence of bounded sets $(B_n)_{n \in \mathbb{N}}$ satisfying $2B_n \subseteq B_{n+1}$, $n \in \mathbb{N}$. Then $E$ (cf. e.g. [29], 8.1.12) is a gDF space if and only if for every sequence $(U_n)_{n \in \mathbb{N}}$ of zero neighbourhoods the set $\cap_{n \in \mathbb{N}} (U_n + B_n)$ is again a zero neighbourhood.

ii) Every DF space is a gDF space (cf. e.g. [29], 8.3.3) and every gDF space is quasi-normable (cf. e.g. [29], 8.3.37), i.e. for every absolutely convex zero neighbourhood $U$ in $E$ there exists an absolutely convex zero neighbourhood $V$ in $E$ such that for every $\rho > 0$ there is a bounded subset $B$ in $E$ with $V \subseteq \rho U + B$.

iii) Every gDF space satisfies the countable neighbourhood property, i.e., given any sequence $(U_n)_{n \in \mathbb{N}}$ of zero neighbourhoods in $E$, there are $\rho_n > 0$ for every $n \in \mathbb{N}$ such that $\cap_{n \in \mathbb{N}} \rho_n U_n$ is a zero neighbourhood in $E$ (see e.g. [29] 8.3.5).
iv) The strong dual of a metrizable locally convex space is a complete DF space (cf. e.g. [29], 8.3.9) and the strong dual of a gDF space is a Fréchet space (cf. e.g. [29], 8.3.7).

Now we improve Remark 4.7 i). The idea is due to J. Bonet and A. Peris (personal communication). Compare part ii) with [3], 5.A, p. 579.

Theorem 4.8. Let \( E \) be a locally convex Hausdorff space.

i) \( E \) is a gDF space if and only if \( E \) contains a sequence \((B_n)_{n \in \N}\) of bounded and absolutely convex subsets satisfying \( 2B_n \subset B_{n+1} \) for each \( n \) such that for every sequence \((U_n)_{n \in \N}\) of zero neighbourhoods the set \( \bigcap_{n \in \N}(U_n + B_n) \) is again a zero neighbourhood. In this case, \( \overline{B_n}^E \) is a fundamental sequence of bounded sets in \( E \).

ii) \( E \) is a DF space if and only if \( E \) contains a sequence \((B_n)_{n \in \N}\) of bounded and absolutely convex subsets satisfying \( 2B_n \subset B_{n+1} \) for each \( n \) such that for every sequence \((U_n)_{n \in \N}\) of zero neighbourhoods and every \((\rho_n)_{n \in \N} \in (0, \infty)^\N\) the set \( \bigcap_{n \in \N}(U_n + \sum_{k=1}^n \rho_k B_k) \) is again a zero neighbourhood.

Proof. i) If \( E \) is a gDF space, then Remark 4.7 i) implies the existence of the required sequence \((B_n)_{n \in \N}\). For the converse, let \((B_n)_{n \in \N}\) a sequence of bounded and absolutely convex subsets, satisfying \( 2B_n \subset B_{n+1} \) for each \( n \), be given such that for every sequence \((U_n)_{n \in \N}\) of zero neighbourhoods the set \( \bigcap_{n \in \N}(U_n + B_n) \) is again a zero neighbourhood. Using again Remark 4.7 i) it remains to show that \( \overline{B_n}^E \) is a fundamental sequence of bounded sets in \( E \).

Assume the existence of a bounded subset \( B \) of \( E \) which is not contained in \( \overline{B_n}^E \) for every \( n \in \N \). Since \( 2B_n \subset B_{n+1} \) there exists a bounded sequence \((x_n)_{n \in \N}\) in \( E \) with \( x_n \not\in \overline{B_n}^E \) for every \( n \in \N \). Hence, for every \( n \in \N \) there are zero neighbourhoods \( U_n \in E \) with \( x_n \not\in U_n + nB_n \). This implies \( x_k \not\in k \bigcap_{n \in \N}(U_n + B_n) \) for every \( k \in \N \), a contradiction, because \((x_k)_{k \in \N}\) is bounded and and \( \bigcap_{n \in \N}(U_n + B_n) \) is a zero neighbourhood.

ii) Let \( E \) be a DF space. By definition, \( E \) possesses a fundamental sequence of bounded sets \((B_n)_{n \in \N}\) (satisfying \( 2B_n \subset B_{n+1} \), \( n \in \N \)). Let a sequence \((U_n)_{n \in \N}\) of zero neighbourhoods in \( E \) and \((\rho_n)_{n \in \N} \in (0, \infty)^\N\) be given. We have to show that \( U := \bigcap_{n \in \N}(U_n + \sum_{k=1}^n \rho_k B_k) \) is a zero neighbourhood. There exists absolutely convex zero neighbourhoods \( V_n \in E \) such that \( 2V_n \subset U_n \) for every \( n \in \N \). Heice

\[
V := \bigcap_{n \in \N} (V_n + \sum_{k=1}^n \rho_k B_k) \subset U.
\]

Then \( V \) absorbs each \( B_n \), and therefore it is bornivorous. By definition, \( E \) is \( \aleph_0 \)-quasibarrelled and this yields that \( V \) is a zero neighbourhood in \( E \) (which is contained in \( U \)).

For the converse, let a sequence \((B_n)_{n \in \N}\) of bounded and absolutely convex subsets of \( E \), satisfying \( 2B_n \subset B_{n+1} \) for each \( n \), be given such that for every sequence \((U_n)_{n \in \N}\) of zero neighbourhoods and every \((\rho_n)_{n \in \N} \in (0, \infty)^\N\) the set \( \bigcap_{n \in \N}(U_n + \sum_{k=1}^n \rho_k B_k) \) is again a zero neighbourhood. If \( \rho_n := \frac{1}{2^n}, n \in \N \), we get with i) that \( E \) is a gDF space and that \( \overline{B_n}^E \) is a fundamental sequence of bounded sets in \( E \).

It remains to prove that \( E \) is \( \aleph_0 \)-quasibarrelled. To see this, let a sequence \((V_n)_{n \in \N}\) of closed and absolutely convex zero neighborhoods in \( E \) be given such that \( V := \bigcap_{n \in \N} V_n \) is bornivorous. There are \( \rho_n > 0 \) with \( 2^{n+1} \rho_n B_n \subset V \) for every \( n \in \N \) and therefore
\[\sum_{k=1}^{n} \rho_k B_k \subset \frac{1}{3} V_n \quad \text{for every } n \in \mathbb{N}. \] This implies that \( V \) contains the zero neighbourhood 
\[\bigcap_{n \in \mathbb{N}} \left( \frac{1}{3} V_n + \sum_{k=1}^{n} \rho_k B_k \right) \] and we are done.

To apply the previous theorem to vector valued sequence spaces we need the following lemma which is based on an idea of R. Hollstein [21], lemma 1. (A similar version is due to F. Mayoral [27], 2.8.1.)

**Lemma 4.9.** Let \((\lambda, \| \cdot \|_{\lambda})\) be a normal Banach sequence space and let \( E \) be a locally convex Hausdorff space. Let \( B_1, \ldots, B_m \subset E \) be bounded and absolutely convex and let \( U \) be an absolutely convex zero neighbourhood in \( E \). Then for every \( \varepsilon > 0 \) we have

\[ \lambda(U + \sum_{k=1}^{m} B_k) \subset (1 + \varepsilon)\lambda(U) + \sum_{k=1}^{m} \lambda(B_k). \]

**Proof.** Let \( \varepsilon > 0 \) be given. By 2.4 there is some \( 0 < (\alpha_n)_{n \in \mathbb{N}} \subset E \varepsilon B_{\lambda}. \)

Since \( \lambda(U) \) is a zero neighbourhood which absorbs the bounded sets \( \lambda(B_k) \) we can assume without loss of generality that \( m = 1 \). The full assertion follows by induction.

Let \((x_n)_{n \in \mathbb{N}} \in \lambda(V)\) with \( V := U + B_1 \). The boundedness of the set \( B_1 \) then implies for every \( n \in \mathbb{N} \):

\[ x_n \in \lambda(\alpha_n)U + \lambda(V)B_1. \]

Hence there exist \( y_n \in \lambda(V)(x_n) \) and \( z_n \in \lambda(V)B_1 \) such that \( x_n = y_n + z_n \) for every \( n \in \mathbb{N} \). By the normality of the closed unit ball \( B_{\lambda} \) of \( \lambda \) we get

\[ (p_{V}(y_n))_{n \in \mathbb{N}} \in \lambda(B_{\lambda}) \subset (1 + \varepsilon)B_{\lambda} \]

and

\[ (p_{B_1}(z_n))_{n \in \mathbb{N}} \in \lambda(B_{\lambda}) \subset B_{\lambda}. \]

Then \((y_n)_{n \in \mathbb{N}} \in (1 + \varepsilon)\lambda(U) \) and \((z_n)_{n \in \mathbb{N}} \in \lambda(B_1) \) by the boundedness of \( B_1 \).

**Theorem 4.10.** Let \((\lambda, \| \cdot \|_{\lambda})\) be a normal Banach sequence space and let \( E \) be a DF space (gDF space). Then \( \lambda(E) \) is a DF space (gDF space, resp.). If \((B_n)_{n \in \mathbb{N}} \) is a fundamental sequence of bounded sets in \( E \) satisfying \( 2B_n \subset B_{n+1} \) for every \( n \in \mathbb{N} \), then \( \lambda(B_n)_{n \in \mathbb{N}} \) is a fundamental sequence of bounded sets in \( \lambda(E) \).

**Proof.** In both cases there exists a fundamental sequence of bounded sets \((B_n)_{n \in \mathbb{N}} \) in \( E \) satisfying \( 2B_n \subset B_{n+1} \) for every \( n \in \mathbb{N} \).

Let \( E \) be a gDF space and let \((W_n)_{n \in \mathbb{N}} \) be an arbitrary sequence of zero neighbourhoods in \( \lambda(E) \). Then there are absolutely convex zero neighbourhoods \( U_n \) in \( E \) with \( 2\lambda(U_n) \subset W_n \) for every \( n \in \mathbb{N} \). Using lemma 4.9 we get

\[
\lambda\left(\bigcap_{n \in \mathbb{N}} (U_n + B_n)\right) \subset \bigcap_{n \in \mathbb{N}} \lambda(U_n + B_n) \subset \bigcap_{n \in \mathbb{N}} (2\lambda(U_n) + \lambda(B_n)) \\
\subset \bigcap_{n \in \mathbb{N}} (W_n + \lambda(B_n)).
\]
Since $E$ is a gDF space we get that $\bigcap_{n\in\mathbb{N}}(U_n + B_n)$ is a zero neighbourhood in $E$ and this implies that $\bigcap_{n\in\mathbb{N}}(W_n + \lambda(B_n))$ is a zero neighbourhood in $\lambda(E)$. Applying theorem 4.8 i) we see that $\lambda(E)$ is a gDF space with fundamental sequence of bounded sets $(\lambda(B_n)_{n\in\mathbb{N}}).

The proof in the DF case runs along the same lines: Let $E$ be a DF space and let $(W_n)_{n\in\mathbb{N}}$ be a sequence of zero neighbourhoods in $\lambda(E)$ and let $(\rho_n)_{n\in\mathbb{N}} \in (0, \infty)^\mathbb{N}$ be arbitrary. Then there are absolutely convex zero neighbourhoods $U_n$ in $E$ with $\lambda(2U_n) \subset W_n$ for every $n \in \mathbb{N}$. Using lemma 4.9 we get

\[
\lambda\left(\bigcap_{n\in\mathbb{N}}(U_n + \sum_{k=1}^{n} \rho_k B_k)\right) \subset \bigcap_{n\in\mathbb{N}}\lambda(U_n + \sum_{k=1}^{n} \rho_k B_k) \\
\subset \bigcap_{n\in\mathbb{N}}(2\lambda(U_n) + \sum_{k=1}^{n} \rho_k \lambda(B_k)).
\]

Since $E$ is a DF space we get with theorem 4.8 ii) that $\bigcap_{n\in\mathbb{N}}(U_n + \sum_{k=1}^{n} \rho_k B_k)$ is a zero neighbourhood in $E$ and this implies that $\bigcap_{n\in\mathbb{N}}(W_n + \sum_{k=1}^{n} \rho_k B_k)$ is a zero neighbourhood in $\lambda(E)$. Applying again theorem 5.8 ii) we arrive at the conclusion.

**Corollary 4.11.** Let $(\lambda, || \cdot ||_{\lambda})$ be a normal Banach sequence space and let $E$ be a locally convex space. If $E$ is a gDF space, then $\{h(B) : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$. If either $E$ is a gDF space and $\lambda$ satisfies property 6) or $E$ is metrizable, then even $\{\lambda(B) : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$.

**Proof.** Because of 4.10 and 4.4 only the second assertion in the case that $E$ is a gDF space still needs a proof.

But this follows from the fact that $\lambda(B)$ is closed in $\lambda(E)$ whenever $B$ is a bounded, closed absolutely convex subset of $E$ and $\lambda$ satisfies property 6) (cf. 3.3).

**Remark 4.12.** If $E$ is a gDF space and $\lambda$ satisfies property $\gamma)$, then $\lambda(E)$ is a topological subspace of $\lambda^{(\gamma)}(E)$ and for a given fundamental sequence of bounded sets $(B_n)_{n\in\mathbb{N}}$ satisfying $2B_n \subset B_{n+1}$ for every $n \in \mathbb{N}$ we get that $\lambda^{(\gamma)}(B_n) \cap \lambda(E)$ is a fundamental sequence of bounded sets in $\lambda(E)$.

Now we consider the topological dual $\lambda(E)'$ of $\lambda(E)$ if $\lambda$ satisfies property $\varepsilon)$. It will turn out that the properties proved in corollary 4.11 are important for a canonical description of the strong dual of $\lambda(E)$. The results concerning this topic were essentially proved by R.C. Rosier [34] in the more general context of $\mathcal{M}$-topologies. We give here short proofs for the case we consider. Let $(\lambda, || \cdot ||_{\lambda})$ be a normal Banach sequence space satisfying property $\varepsilon)$ (i.e. $\varphi$ is dense in $\lambda$) and let $E$ be a locally convex Hausdorff space. In this case $E^{(\varepsilon)}$ is dense in $\lambda(E)$ and therefore we can identify a continuous linear form $f \in \lambda(E)'$ via

\[
f_n(x) := f(((0)_k < n, x, (0)_k > n)), n \in \mathbb{N}, x \in E,
\]

with the sequence $(f_n)_{n\in\mathbb{N}}$ in $E'$. 


Lemma 4.13. Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$) and let $E$ be a locally convex Hausdorff space. If $(f_n)_{n \in \mathbb{N}} = f \in \lambda(E)'$ and if $C \subset E$ is absolutely convex, then we have for every $0 \neq (\alpha_n)_{n \in \mathbb{N}} = \alpha \in \lambda$:

$$
\|((\alpha_n p_C \cdot (f_n))_{n \in \mathbb{N}}\|_1 \leq \|\alpha\|_{\lambda} \sup_{x \in \lambda(C)} |f(x)|.
$$

Proof. Without loss of generality we may assume that the right hand side is finite, $\|\alpha\|_{\lambda} = 1$ and $|\alpha| = \alpha \in \varphi$.

First we note $f(x) = \sum_{n \in \mathbb{N}} f_n(x_n)$ for $x = (x_n)_{n \in \mathbb{N}} \in E^{(N)}$ and $\oplus_{n \in \mathbb{N}} \alpha_n C \subset \lambda(C)$. It follows

$$
\|((\alpha_n p_C \cdot (f_n))_{n \in \mathbb{N}}\|_1 = \sum_{n \in \mathbb{N}} \alpha_n p_C \cdot (f_n) = \sum_{n \in \mathbb{N}} \sup_{z \in \lambda(C)} |f(z)|
\leq \sup\left\{ \sum_{n \in \mathbb{N}} |f_n(x_n)| : (x_n)_{n \in \mathbb{N}} \in \oplus_{n \in \mathbb{N}} \alpha_n C \right\}
= \sup\left\{ \sum_{n \in \mathbb{N}} f_n(x_n) : (x_n)_{n \in \mathbb{N}} \in \oplus_{n \in \mathbb{N}} \alpha_n C \right\}
\leq \sup\{ |f(x)| : x \in \lambda(C) \}.
$$

For a normal Banach sequence space $\lambda$ we considered in remark 2.17 ii) its a-dual $\lambda^\times$. We had equipped this space with the norm $\| \cdot \|_{\lambda^\times}$ admitting the polar $B_{\lambda^\times}^\circ$ of $B_\lambda$ as its closed unit ball. We could show that $(\lambda^\times, \| \cdot \|_{\lambda^\times})$ is a normal Beataich sequence space satisfying property 6). With the help of the previous lemma we can prove (cf. [34], p. 492-494) the following representation of the dual of $\lambda(E)$.

Theorem 4.14. Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$) and let $E$ be a locally convex Hausdorff space. Then

i) $\lambda(E)' \subset \lambda^\times((E', \beta(E', E)))$.

ii) For every absolutely convex and closed subset $C$ of $E$ we have

$$
\lambda(C)^{\circ (\lambda(E)')} = \lambda^\times(C^\circ) \cap \lambda(E)'.
$$

iii) The strong topology on the dual of $\lambda(E)$ is finer than the relative topology induced by $\lambda^\times((E', \beta(E', E)))$. They coincide if and only if $\{ \lambda(B) : B \in \mathcal{B}(E) \}$ is a fundamental system of bounded sets in $\lambda(E)$.

Proof. i) Let $f = (f_n)_{n \in \mathbb{N}}$ be a continuous linear form on $\lambda(E)$. Then there is an absolutely convex zero neighbourhood $U$ in $E$ such that $\sup_{x \in \lambda(U)} |f(x)| \leq 1$. The previous lemma implies $(p_U \cdot (f_n))_{n \in \mathbb{N}} \in \lambda^\times$. Hence $f \in \lambda^\times(((U^o)_0, p_U^o)). U^o$ is equicontinuous and therefore strongly bounded and we get $f \in \lambda^\times(((U^o)_0, p_U^o)) \subset \lambda^\times((E', \beta(E', E)))$.

ii) Let $C \subset E$ be absolutely convex and closed. Let $f = (f_n)_{n \in \mathbb{N}} \in \lambda(C)^{\circ \lambda(E)'}$. Using i) we see $f \in \lambda^\times((E', \beta(E', E)))$ and with lemma 4.13 we get $(p_C \cdot (f_n))_{n \in \mathbb{N}} \in B_{\lambda^\times}$. Hence $f \in \lambda^\times(C^\circ)$.

For the converse inclusion let $f = (f_n)_{n \in \mathbb{N}} \in \lambda^\times(C^\circ) \cap \lambda(E)'$ be given. Then we have for $x \in \lambda(C)$ that

$$
|f(x)| \leq \sum_n |f_n(x_n)| \leq \sum_n p_C(x_n)p_C^\circ(f_n) \leq 1,
$$

where $p_C(x_n)$ is the projection of $x_n$ onto $C$. By the choice of $C$ we have $p_C^\circ(f_n) \leq 1$ for all $n \in \mathbb{N}$ and hence $f \in \lambda((C, \beta(C, E)))$. This completes the proof.
because $p_c(x_n) \in B_\lambda$ and $p_{c^*}(f_n) \in B_{\lambda^*} (= B_{\lambda^0})$.

Applying corollary 4.11 we get:

**Corollary 4.15.** Let $(A, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$ and $E$ be a gDF space or a metrizable locally convex space. Then the strong dual of $\lambda(E)$ is a topological subspace of $\lambda^\infty((E', \beta(E', E)))$.

**Remark 4.16.** Let $(A, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$. Then $A' = \lambda^\infty$ and it is easy to see that dual norm $\| \cdot \|_{\lambda'}$ is $\| \cdot \|_{\lambda^\infty}$.

We now consider the general case (the proof is implicitly contained in [34] and [16] where algebraic and topological identities for more complicated vector valued sequence spaces were proved). The second part is proved (under much stronger assumptions) in [28], 2.3.

**Theorem 4.17.** Let $(A, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$ and let $E$ be a locally convex Hausdorff space. Then $\lambda(E)'$ coincides algebraically with $\lambda^\infty((E', \beta(E', E)))$ if and only if for every $f = (f_n)_{n \in N} \in \lambda^\infty((E', \beta(E', E)))$ there is $U \in U_0(E)$ such that $E \in \lambda^\infty(U^\circ)$.

If one of this is true and if, in addition, $\{\ell(B)^\lambda_E : B \in B(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$, then the strong dual of $\lambda(E)$ is $\lambda^\infty((E', \beta(E', E)))$.

**Proof.** Let $U$ be an absolutely convex zero neighbourhood in $E$ and $f = (f_n)_{n \in N} \in \lambda^\infty(U^\circ)$. Then we have for every $x = (x_n)_{n \in N} \in \lambda(U)$

$$\sum_{n \in N} |f_n(x_n)| \leq \sum_{n \in N} p_{U^\circ}(f_n) p_U(x_n) \leq 1$$

and hence $f \in \lambda(E)'$.

For the converse, let $f \in \lambda(E)'$ be given. Then there is an absolutely convex and closed zero neighbourhood $U$ in $E$ such that $f \in \Lambda(U)^{\lambda_{E'}}$. Theorem 4.14 part ii) guarantees $f \in \lambda^\infty(U^\circ)$.

To prove the remaining topological identity we apply theorem 4.14 part iii).

**Remark 4.18.** Let $(A, \| \cdot \|_\lambda)$ be a normed Banach sequence space satisfying property $\varepsilon$ and let $(E, \| \cdot \|_E)$ be a normal space. If we denote the dual norm on $E'$ by $\| \cdot \|_{E'}$, then the theorems 7.17 and 4.14 part ii) imply that $\lambda((E, \| \cdot \|_E))' = \lambda^\infty((E', \| \cdot \|_{E'}))$ (isometrically).

**Corollary 4.19.** Let $(A, \| \cdot \|_\lambda)$ be a normal Banach sequence space satisfying property $\varepsilon$ and let $E$ be a quasiabarrelled locally convex Hausdorff space. If $\{\lambda^\infty(B) : B \in B((E', \beta(E', E)))\}$ is a fundamental system of bounded sets in $\lambda^\infty((E', \beta(E', E)))$, then the dual of $\lambda(E)$ is (algebraically) $\lambda^\infty((E', \beta(E', E)))$. If, in addition, $\{\ell(B)^\lambda_E : B \in B(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$, then the strong dual of $\lambda(E)$ is $\lambda^\infty((E', \beta(E', E)))$.

**Proof.** Using theorem 4.14 it remains to show the algebraic identity. Let $f = (f_n)_{n \in N} \in \lambda^\infty((E', \beta(E', E)))$ be arbitrary. Then there is an absolutely convex and bounded set $B$ in $(E', \beta(E', E))$ with $f \in \lambda^\infty(B)$. $E$ is quasiabarrelled, hence $B \subset U^\circ$ for some zero neighbourhood $U$ in $E$. Applying theorem 4.17 we have $f \in \lambda(E)'$.

Using the corollaries 4.11 and 4.19 and having in mind that $\lambda^\infty$ satisfies property 6) (for every normal Banach sequence space $A$) we get:
Corollary 4.20. Let \((\lambda, \| \cdot \|_\lambda)\) be a normal Banach sequence space satisfying property \(\delta\). If \(E\) is a metrizable locally convex space or if \(E\) is a quasibarrelled DF space, then the strong dual of \(\lambda(E)\) is \(\lambda^\times((E', \beta(E', E)))\).

In the last part of this chapter we wait to consider properties of bounded subsets of vector valued sequence spaces. We use essentially lemma 4.3 and lemma 4.9 to get the results. But first let us prove the following lemma:

Lemma 4.21. Let \(E\) be a locally convex Hausdorff space. Let \(B\) a bounded and absolutely convex subset of \(E\) and let \(U\) be an absolutely convex zero neighbourhood in \(E\). For \(\varepsilon > 0\) the following holds:

i) for every \(x \in \overline{B}^E\) we have \(\overline{B}^E \cap (x + U) \subset 2\overline{B}^E \cap (1 + \varepsilon)U^E + x\),

ii) \(\overline{B}^E \cap U \subset \overline{B}^E \cap (1 + \varepsilon)U^E\).

Proof. i) Let \(x \in \overline{B}^E\) be given. If \(V\) is an absolutely convex zero neighbourhood which is contained in \(\frac{1}{2}U\), then

\[
\overline{B}^E \cap (x + U) \subset \overline{B}^E \cap (x + (1 + \varepsilon)U + x) \\
\subset ((2B) \cap (2V + U)) + 2V + x \subset 2B \cap (1 + \varepsilon)U + 2V + x.
\]

This implies i).

ii) Let \(V\) be as in the proof of part i). Then

\[
\overline{B}^E \cap U \subset (B + V) \cap U \subset (B \cap (V + U)) + V \subset (B \cap (1 + \varepsilon)U) + V.
\]

This proves ii).

Proposition 4.22. Let \((A, \| \cdot \|_A)\) be a normal Banach sequence space and let \((E, \mathcal{T})\) be a locally convex Hausdorff space such that \(\{\lambda(B)^{ME} : B \in \mathcal{B}(E)\}\) is a fundamental system of bounded sets in \(\lambda(E)\) (e.g. let \(E\) be a gDF space). If every bounded subset of \(E\) is metrizable, then the same holds for every bounded subset of \(\lambda(E)\).

Proof. Every bounded subset of \(\lambda(E)\) is contained in \(\lambda(B)\) for some \(B \in \mathcal{B}(E)\), so we may restrict ourself to such sets. Let \(B \in \mathcal{B}(E)\) be given and consider a metrizable locally convex topology \(\mathcal{S}\) on the span of \(B\) which coincides on \(B\) with \(\mathcal{T}\). Thei proposition 3.1 yields that on \(\lambda(B)\) the topologies induced by \(\lambda((E, \mathcal{T}))\) and \(\lambda((B, \mathcal{S}))\) coincide. \(\lambda((B, \mathcal{S}))\) is metrizable and we apply lemma 4.21 to get the assertion.

The following result is of the same type.

Proposition 4.23. Let \((A, \| \cdot \|_A)\) be a normal Banach sequence space and suppose \((E, \mathcal{T})\) is a locally convex Hausdorff space such that \(\{\lambda(B)^{ME} : B \in \mathcal{B}(E)\}\) is a fundamental system of bounded sets in \(\lambda(E)\) (e.g. \(E\) is a gDF space). If \(E\) satisfies the strict Mackey condition (i.e. for every \(B \in \mathcal{B}(E)\) there is \(C \subset B\) in \(\mathcal{B}(E)\) such that the topologies on \(B\) induced by \(\mathcal{T}\) and \(\rho_C\) coincide), then \(A(E)\) satisfies the strict Mackey condition.

Proof. Every bounded set of \(\lambda(E)\) is contained in \(\lambda(B)\) for some \(B \in \mathcal{B}(E)\), so we may again restrict ourself to such sets. Fix \(B \in \mathcal{B}(E)\). Then it exists some \(C \in \mathcal{B}(E)\) such that for
every $\varepsilon > 0$ there is an absolutely convex zero neighbourhood $U$ with $B \cap U \subset \varepsilon C$. Using the lemmata 4.21 and 4.3 we get:

$$\lambda(B)^{\lambda(E)} \cap \lambda(U) \subset \frac{2\lambda(B) \cap \lambda(U)}{4\lambda(B) \cap U} \subset \frac{\lambda(B) \cap U}{2\lambda(B)} \subset 4\varepsilon \frac{\lambda(C)}{\lambda(E)}$$

and it follows that the topologies induced on $\lambda(B)^{\lambda(E)}$ by $\lambda(E)$ and $\rho_{\lambda(E)}$ coincide.

We noted already that for every normed (metrizable locally convex) space $E$ the space $\lambda(E)$ is again normed (metrizable, resp.). Now, we consider quasinormable spaces $E$, i.e. locally convex Hausdorff spaces satisfying the following property:

For every absolutely convex zero neighbourhood $U$ in $E$ there exists an absolutely convex zero neighbourhood $V$ in $E$ such that for every $\rho > 0$ there is a bounded subset $B$ in $E$ with $V \subset \rho U + B$.

**Proposition 4.24.** Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space and assume $E$ is a quasinormable space. Then $\lambda(E)$ is quasinormable.

**Proof.** Let $W$ be an absolutely convex zero neighbourhood in $\lambda(E)$. Then there is an absolutely convex zero neighbourhood $U$ in $E$ with $2\lambda(U) \subset W$. By assumption is $E$ quasinormable. Hence, there exists another absolutely convex zero neighbourhood $V \subset U$ in $E$ such that for every $\rho > 0$ there is a bounded subset $B$ in $E$ with $V \subset \rho U + B$. Applying lemma 4.9 we get:

$$\lambda(V) \subset \lambda(\rho U + B) \subset 2\rho \lambda(U) + \lambda(B) \subset \rho W + \lambda(B).$$

This implies that $\lambda(E)$ is quasinormable.

5. **BARRELLEDNESS CONDITIONS FOR VECTOR-VALUED SEQUENCE SPACES**

Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space and let $E$ be a locally convex Hausdorff space. In this chapter we will consider descriptions for bounded subsets of the space $\lambda(E)$ especially if $E$ is metrizable or a DF space. As already noted the space $\lambda(E)$ is metrizable (or normed) whenever $E$ has this property. In [17] the completeness of the space $\lambda(E)$ was characterized: $\lambda(E)$ is complete if and only if $E$ is complete.

We are now interested in barrelledness conditions for Fréchet- and DF space-valued sequence spaces. Precisely, we want to investigate the barrelledness of $\lambda(E)$ for DF spaces $E$ and the distinguishedness of $\lambda(E)$ for Fréchet spaces $E$.

If $\varphi$ is dense in $\lambda$ we noted in remark 2.17 iv) that $\lambda$ is the regular subspace (in the sense of T. Kōmura & Kōmura, [23]) of the perfect space $\lambda(\beta)$. In this case M. Florencio & P.J. Paúl [16], 3.18, proved that $\lambda(E)$ is a (barrelled) DF space if and only if $E$ is a (barrelled) DF space. If, in addition, $\varphi$ is dense in $(\lambda^\times, \beta(\lambda^\times, \lambda))$ we can again apply the result of M. Florencio & P.J. Paúl [16], 3.18, to ensure for every Fréchet space $E$ that $\lambda(E)$ is distinguished if anti only if so is $E$.

For $\lambda = l$, S. Dierolf ([12], (5.13)) proved that $l_\infty(E)$ is a DF space whenever $E$ is a DF space. K.D. Bierstedt & J. Bonet ([2], 1.5) could show that $l_\infty(E)$ is a quasibarrelled DF space if and only if $E$ is a DF space satisfying the dual density condition. in a previous paper
K.D. Bierstedt & J. Bonet [11] proved for metrizable spaces \( E \) that \( l_1(E) \) is distinguished if and only if \( E \) satisfies the density condition.

First we consider barreled metrizable spaces \( E \) and we prove that in this case \( \lambda(E) \) is barreled. To do this we need a lemma, which is based on an idea of A. Defant & W. Govaerts [8], it appears in a weaker form in [2] and it is contained in an abstract Banach-Steinhaus theorem due to S. Díaz, A. Fernández, M. Ferrer, & P.J. Peláez (see [9], theorem 1). But for the sake of completeness we will prove here version we need.

**Lemma 5.1.** Let \( \lambda(\| \cdot \|_E) \) be a normal Banach sequence space and let \( E \) be a locally convex Hausdorff space. If \( \lambda(E) \) is quasibarreled and \( E \) is barreled, then \( \lambda(E) \) is also barreled.

**Proof.** Let \( T \) be a barrel in \( \lambda(E) \). We have to show that \( T \) absorbs the bounded sets. Let \( B \in B(\lambda(E)) \) be arbitrary. Without loss of generality we assume that \( B = \bigcap_{p \in c_0(E)} \rho_p \lambda(p^{-1}(\{0,1\})) \) with positive \( \rho_p, \rho \in c_0(E) \). To abbreviate we define for \( n \in \mathbb{N} \):

\[
L_n := \prod_{k \leq n} E \times \prod_{k > n} \{0\} \quad \text{and} \quad M_n := \prod_{k \leq n} \{0\} \times \prod_{k > n} E.
\]

For \( n \in \mathbb{N} \) it is immediate that \( \lambda(E) \) is topologically isomorphic to the direct sum \( \lambda(E) \cap L_n \oplus (\lambda(E) \cap M_n) \) and \( \lambda(E) \cap L_n \) is barreled (it is isoiiorphic to \( E^n \)). Hence, \( T \) absorbs the set \( B \cap L_n \) for every \( n \in \mathbb{N} \). From \( B = \bigcap_{p \in c_0(E)} \rho_p \lambda(p^{-1}(\{0,1\})) \) we obtain \( B \supseteq B \cap L_n + B \cap M_n \). Since the proof will be complete if we can show that \( T \) absorbs \( B \cap M_n \) for some \( n \in \mathbb{N} \).

Assume this is not true. Then for every \( n \in \mathbb{N} \), there exists \( \chi^{(n)} := (\chi_k^{(n)}) \in \mathbb{E} \setminus (B \cap M_n) \setminus 2^{2^n}T \). If \( \tilde{E} \) denotes the completion of \( E \), then the space \( \lambda(\tilde{E}) \) is complete (see [17]) and we may define a map \( f : l_1 \to \lambda(\tilde{E}) \) by

\[
f(\alpha) := \sum_{n=1}^{\infty} \alpha_n 2^{-n} \chi^{(n)}.
\]

A standard argument (cf. [29], 3.2.13) shows that \( f(B_k) \) is the closure of the absolutely convex hull of \( \{2^{-n} \chi^{(n)} : n \in \mathbb{N} \} \). So, \( f(B_k) \) is a compact (and absolutely convex) subset of \( \lambda(\tilde{E}) \), hence a Banach disc. But we compute for every \( \alpha \in l_1 \) that

\[
\sum_{n=1}^{\infty} \alpha_n 2^{-n} \chi^{(n)} = \left( \sum_{k=1}^{\infty} \alpha_n 2^{-n} \chi_k^{(n)} \right)_{k \in \mathbb{N}}.
\]

Hence, \( f(B_k) \) is even contained in \( \lambda(E) \). Since every barrel absorbs the Banach discs, \( f(B_k) \) is absorbed by \( T \). In particular, \( T \) absorbs \( \{2^{-n} \chi^{(n)} : n \in \mathbb{N} \} \), a contradiction to \( \chi_n \notin 2^{2^n}T, n \in \mathbb{N} \).

**Remark 5.2.** i) Let \( E \) a locally convex Hausdorff space. Let \( T \) be an absolutely convex subset of \( \lambda(E) \), which absorbs every Banach disc in \( \lambda(E) \). If \( T \) absorbs, in addition, every bounded subset of \( \prod_{k \leq n} E \times \prod_{k \geq n} \{0\} \) for every \( n \in \mathbb{N} \), then the proof of lemma 5.1 shows that \( T \) absorbs every bounded subset of \( \lambda(E) \).

ii) According to i), we can replace in lemma 5.1 (quasi)barreled by \( \mathcal{S}_0 \)- (quasi)barreled or by (ultra)borealological.
If $E$ is a inetrizable locally convex space, then $\lambda(E)$ is quasibarrelled. Using this we get with the help of lemma 5.1:

**Corollary 5.3.** Let $(\mathbf{A}, \| \cdot \|_{\lambda})$ be a normal Banach sequence space and let $E$ be a metrizable locally convex space. If $E$ is barrelled, then $\lambda(E)$ is barrelled.

It is much harder to obtain a result similar as 5.3 if $E$ is a (quasi)barrelled DF space. It will turn out that in this case the property $\varepsilon_j$ for normal Banach sequence spaces plays an important role. We first consider the situation if $\mathbf{A}$ satisfies $\varepsilon_j$. The following theorem is contained in a result of M. Florencio & P.J. Paul (cf. [16], 3.8 (i)).

**Theorem 5.4.** Let $(\mathbf{A}, \| \cdot \|_{\lambda})$ be a normal Banach sequence space satisfying property $\varepsilon_j$ and let $E$ be a quasibarrelled locally convex Hausdorff space such that $\{ \lambda^X(B) : B \in \mathcal{B}((E', \beta(E', E))) \}$ is a fundamental system of bounded sets in $\lambda^X((E', \beta(E', E)))$. Then $\lambda(E)$ is quasibarrelled.

**Proof.** Fix a bounded subset $B$ of the strong dual of $\lambda(E)$. Using theorem 4.14 (ii) we conclude that $B$ is bounded in $\lambda^X((E', \beta(E', E)))$. Now, there exists some $C \subseteq \mathcal{B}((E', \beta(E', E)))$ with $B \subseteq \lambda^X(C)$. Using that $E$ is quasibarrelled, we find a closed and absolutely convex zero neighbourhood $U$ in $E$ whose polar $U^0$ contains $C$. Applying theorem 4.14 (ii) we get $B \subseteq \lambda(U)^{\circ \lambda(E)^j}$ and therefore $B$ is equicontinuous.

**Corollary 5.5.** Let $(\mathbf{A}, \| \cdot \|_{\lambda})$ be a normal Banach sequence space satisfying property $\varepsilon_j$ and let $E$ be a quasibarrelled DF space. Then $\lambda(E)$ is quasibarrelled.

**Proof.** $(E', \beta(E', E))$ is inetrizable and corollary 4.4 implies that $\{ \lambda^X(B) : B \in \mathcal{B}((E', \beta(E', E))) \}$ is a fundamental system of bounded sets in $\lambda^X((E', \beta(E', E)))$. An application of 5.4 yields the desired statement.

**Corollary 5.6.** Let $(\mathbf{A}, \| \cdot \|_{\lambda})$ be a normal Banach sequence space satisfying property $\varepsilon_j$ and let $E$ be a barrelled DF space. Then $\lambda(E)$ is barrelled.

**Proof.** The above corollary implies that $\lambda(E)$ is quasibarrelled. Applying lemma 5.1 completes the proof.

Let us consider the situation in the case that $E$ is a DF space and $\mathbf{A}$ does not satisfy property $\varepsilon_j$.

For example this situation appears, if $E$ is the strong dual of a inetrizable space $F$ and $\lambda$ is the space $l^1$ then $l^\infty(E)$ is the strong dual of the space $l^1(F)$. Hence, $l^\infty(E)$ is (quasi)barrelled if and only if $l^1(F)$ is distinguished. K.D. Bierstedt & J. Bonet [1] proved that $l^1(F)$ is distinguished if and only if $F$ satisfies Stefan Heinrich’s density condition (which will be introduced later). They obtained that this density condition is equivalent to the fact that every bounded subset of $E$ (the strong dual of $F$) is inetrizable. In a subsequent paper [2] K.D. Bierstedt & J. Bonet introduced the so-called dual density condition for locally convex spaces $E$ and they could show, whenever $E$ is the strong dual of an inetrizable space $F$, that $E$ satisfies the dual density condition if and only if $F$ has the density condition. Moreover, whenever $E$ is a DF space, they proved that the space $l^\infty(E)$ is quasibarrelled if and only if $E$ satisfies the dual density condition. We will prove that the same condition is sufficient and necessary even in the general case. Using results from the previous chapter we will see that the dual density condition is sufficient. That it is even necessary will follow by a reduction to the case which
was proved by K.D. Bierstedt & J. Bonet.

Now, we introduce Heinrich’s density condition and the dual density condition. S. Heinrich introduced the density condition in the context of ultrapowers of locally convex spaces.

**Definition 5.7.** (cf. [19], 1.4) A locally convex Hausdorff space $E$ with zero neighbourhood filter $\mathcal{U}_0(E)$ satisfies the density condition if, given any function $p : \mathcal{U}_0(E) \to (0, \infty)$ and an arbitrary $V \in \mathcal{U}_0(E)$, there always exist a finite subset $\mathcal{F}$ of $\mathcal{U}_0(E)$ and a bounded set $B$ in $E$ such that

$$\bigcap_{U \in \mathcal{F}} p(U) U \subset B + V.$$

It is easy to see and already noted by S. Heinrich that every quasinormable locally convex Hausdorff space satisfies the density condition.

**Remark 5.8.** (cf. [19], 1.4) In the definition of the density condition we may replace $\mathcal{U}_0(E)$ by any basis $\mathcal{U}$ of the zero neighbourhood filter of $E$.

Indeed, the resulting condition is not strictly stronger than the density condition:

Given $p : \mathcal{U} \to (0, \infty)$, for every $U \in \mathcal{U}_0(E)$ there exists $V(U) \in \mathcal{U}$ with $V(U) \subset U$. Define $p : \mathcal{U}_0(E) \to (0, \infty)$ by $p(U) := \delta(V(U))$. Hence, for every finite subset $\mathcal{F}$ of $\mathcal{U}_0(E)$ there exists a finite subset $\mathcal{G}$ of $\mathcal{U}$ with $\bigcap_{V \in \mathcal{G}} \delta(V) V \subset \bigcap_{U \in \mathcal{F}} p(U) U$.

For inetrizable spaces we get therefore:

**Proposition 5.9.** Let $E$ be a metrizable locally convex space and let $(U_n)_{n \in \mathbb{N}}$ be a basis of the zero neighbourhood filter in $E$. Then $E$ satisfies the density condition if and only if for every sequence $\left(\rho_n\right)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ and every zero neighbourhood $V$ in $E$ there exist a bounded set $B$ in $E$ and $m \in \mathbb{N}$ such that

$$\bigcap_{n=1}^m \rho_n U_n \subset B + V.$$

Strongly connected with the density condition are the strong dual density condition and the dual density condition which were introduced by K.D. Bierstedt & J. Bonet (cf. [2], 1.1).

**Definition 5.10.** Let $E$ be a locally convex Hausdorff space. $E$ satisfies the (strong) dual density condition if for every function $\alpha : \mathcal{B}(E) \to (0, \infty)$ and every bounded subset $C$ of $E$, these exist always a finite subset $\mathcal{F}$ of $\mathcal{B}(E)$ and a zero neighbourhood $U$ in $E$ such that

$$C \cap U \subset \bigcap_{B \in \mathcal{F}} \alpha(B) B,$$

$$\left( C \cap U \subset \bigcap_{B \in \mathcal{F}} \alpha(B) B \right),$$

respectively.

The following remark is taken from [2], 1.2.

**Remark 5.11.** i) Every locally convex Hausdorff space with the strict Mackey condition satisfies the strong dual density condition.

ii) Using the same technique as in remark 5.8 we obtain that in the definition of the dual density condition one can replace $\mathcal{B}(E)$ by any fundamental system of bounded sets $\mathcal{B}$ in $E$.

Taking polars this yields:
iii) The strong dual of a quasibarrelled locally convex Hausdorff $E$ satisfies the dual density condition if and only if it satisfies the strong dual density condition if and only if $E$ satisfies the density condition. Especially this equivalence holds if $E$ is a metrizable locally convex space.

We now prove a characterization useful for DF spaces (cf. [2], 1.4).

**Proposition 5.12.** Let $E$ be a locally convex Hausdorff space possessing an increasing fundamental sequence $(B_n)_{n \in \mathbb{N}}$ of absolutely convex and closed bounded sets. Then $E$ satisfies the (strong) dual density condition if and only if for every sequence $(\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ and every bounded set $B$ in $E$ there exist a zero neighbourhood $U$ in $E$ and in $\mathbb{N}$ such that

$$B \cap U \subset \sum_{n=1}^{m} \alpha_n B_n$$

$$(B \cap U \subset \sum_{n=1}^{m} \alpha_n B_n)$$

respectively).

**Proof.** For every $(\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ and every $m \in \mathbb{N}$ we get

$$\Gamma(\bigcap_{n=1}^{m} \alpha_n B_n) \subset \sum_{n=1}^{m} \alpha_n B_n \subset \Gamma(\bigcap_{n=1}^{m} 2^n \alpha_n B_n).$$

Using remark 5.11 ii) this implies the assertion.

The proof of the following theorem is essentially contained in [2], 1.5.

**Theorem 5.13.** Let $E$ be a DF space and let $(B_n)_{n \in \mathbb{N}}$ be an increasing fundamental sequence of closed, absolutely convex and bounded sets in $E$. A bounded and absolutely convex subset $B$ of $E$ is metrizable if and only if for every sequence $(\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ there exist a zero neighbourhood $U$ in $E$ and $m \in \mathbb{N}$ such that

$$B \cap U \subset \sum_{n=1}^{m} \alpha_n B_n.$$
the sets \( \sum_{n=1}^{m(\alpha)} \alpha_n B_n \), \( \alpha_n \) positive and rational for \( n \in N \), define a countable system of zero neighbourhoods in \( B \) and it is easy to see that this system is a zero basis in \( B \).

**Corollary 5.14.** (cf. [2], 1.5) Let \( E \) be a DF space. Then the following are equivalent:

i) \( E \) satisfies the dual density condition.

ii) Every bounded subset of \( E \) is metrizable.

iii) There exists a continuous norm \( \| \cdot \| \) on \( E \) such that on every bounded subset of \( E \) the topologies induced by \( E \) and by \( \| \cdot \| \) coincide.

**Proof.** According to proposition 5.12 and theorem 5.13 it remains to show that ii) implies iii).

Therefore let \( (B_n)_{n \in N} \) be a fundamental sequence of bounded sets in \( E \). Assuming ii) we get for every \( n \in N \) a sequence \( (U_{nm})_{m \in N} \) of closed and absolutely convex zero neighbourhoods in \( E \) defining a zero basis in \( B \). \( E \) satisfies the countable neighbourhood property and therefore there exists \( \rho_{nm} > 0 \) such that \( U := \bigcap_{n,m \in N} \rho_{nm} U_{nm} \) is a zero neighbourhood in \( E \). The Minkowski functional \( \| \cdot \| \) of \( U \) has the required property.

**Remark 5.15**

i) Let \( E \) be a DF space in which every bounded subset is metrizable or, equivalently, which satisfies the dual density condition. Then \( E \) is quasibarrelled. (This is due to A. Grothendieck, [18], p. 71, théorème 5).

ii) In every DFM space \( E \) the bounded subsets are precompact and therefore metrizable (see e.g. [31]), so \( E \) satisfies the dual density condition and hence the strong dual density condition.

We are now able to characterize the barrelledness of \( \lambda(E) \) for DF spaces \( E \). We proved in 5.5 and 5.6:

If \( A \) satisfies property \( \varepsilon \), then \( \lambda(E) \) is (quasi)barrelled if and only if \( E \) is (quasi)barrelled.

The remaining part is given in the next result which has been proved by K.D. Bierstedt & J. Bonet (see [2], 1.5) in the case \( \lambda = l_\infty \):

**Theorem 5.16.** Let \( (\lambda, \| \cdot \|_\lambda) \) be a normal Banach sequence space not satisfying property \( \varepsilon \) and let \( E \) be a DF space. Then the following are equivalent:

i) \( E \) satisfies the dual density condition.

ii) Every bounded subset of \( E \) is metrizable.

iii) Every bounded subset of \( (E) \) is metrizable.

iv) \( \lambda(E) \) is quasibarrelled.

Moreover, \( \lambda(E) \) is barrelled if and only if \( E \) is barrelled and one of the conditions i) to iv) is satisfied.

**Proof.** The equivalence of i) and ii) was proved in 5.14. In 4.22 we could show that ii) implies iii). 4.10 yields that \( \lambda(E) \) is a DF space and so iii) implies iv) by remark 5.15 i).

Let \( \lambda(E) \) be quasibarrelled. \( \lambda \) does not satisfy property \( \varepsilon \) and by theorem 3.8 this implies that \( l_\infty(E) \) is a complemented subspace of \( \lambda(E) \). Therefore \( l_\infty(E) \) is also quasibarrelled. (For the following part of the proof compare with [12], (4.5),(5.12) and with [2], 1.5” (3) implies (1)”) Let \( (B_n)_{n \in N} \) be an increasing fundamental sequence of bounded, closed and absolutely convex sets in \( E \).

Let \( (\alpha_n)_{n \in N} \) be \( (0, \infty)^N \) be given.
We define
\[ V := \bigcap_{n \in \mathbb{N}} I_\infty \left( \sum_{k=1}^{n} \alpha_k B_k \right). \]

The sequence \( \left( I_\infty \left( \sum_{k=1}^{n} \alpha_k B_k \right) \right)_{n \in \mathbb{N}} \) is increasing, consists only of closed, absolutely convex and bounded sets and every bounded subset of the DF space \( I_\infty \) is absorbed by some \( I_\infty \left( \sum_{k=1}^{n} \alpha_k B_k \right) \) (cf. 4.11). So, we may apply A. Grothendieck [18], p.72, lemma 4, to get \( V / I_\infty(E) \subset 2V \). Now, \( V / I_\infty(E) \) is a bornivorous barrel in \( I_\infty(E) \) and the quasibarrelledness of \( I_\infty(E) \) implies that \( V \) is a zero neighbourhood. So there is an absolutely convex zero neighbourhood in \( E \) with \( \left( I_\infty(U) \subset V \right. \). We get that for every bounded subset \( B \) in \( E \) there exists \( n \in N \) such that
\[ B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k. \]

(Otherwise there is a bounded sequence \( (x_n)_{n \in \mathbb{N}} \), contained in \( U \), with \( x_n \not\in \sum_{k=1}^{n} \alpha_k B_k \) for each \( n \in N \), a contradiction to \( \left( I_\infty(U) \subset V \right. \).) Because \( (x_n)_{n \in \mathbb{N}} \) is arbitrary we get with proposition 5.12 that \( E \) satisfies the dual density condition. So iv) implies i). The second assertion follows by the first part and lemma 5.1.

**Remark 5.17.** i) Let \( E \) be a DF space with an increasing fundamental sequence \( (B_n)_{n \in \mathbb{N}} \) of bounded, closed and absolutely convex sets. If \( E \) satisfies the dual density condition, then, according to the proof of theorem 5.16, it satisfies the (formally) stronger condition:

For every \( (\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^N \) there is a zero neighbourhood \( U \) such that for every bounded set \( B \) there exists \( n \in N \) with \( B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k \).

ii) It is possible to apply directly the result of K.D. Bierstedt & J. Bonet mentioned above to show that the dual density condition is necessary for the quasibarrelledness of \( l_1 \) (\( E \)).

We now want to consider distinguished (vector valued) metrizable spaces. Grothendieck called a locally convex Hausdorff space \( E \) distinguished if its strong dual \((E', \sigma(E', E)) \) is barrelled. If \( E \) is, in addition, metrizable, then \( E \) is distinguished if and only if \((E', \sigma(E', E)) \) is quasibarrelled (bornological, ultrabornological, an LB space) (see [18], p. 73, theorem 7).

First we note:

**Remark 5.18.** Let \( E \) be a metrizable locally convex space. If \( E \) satisfies the density condition, then \( E \) is distinguished.

Indeed, \( E \) satisfies the density condition if and only if the DF space \((E', \sigma(E', E)) \) satisfies the dual density condition (see 5.11 iii)). But remark 5.15 i) implies that \((E', \sigma(E', E)) \) then has to be quasibarrelled.

We will give a complete characterization of the distinguishedness of \( \lambda(E) \) whenever \( \lambda \) is a normal Banach space and \( E \) is a metrizable locally convex space. Analogously to the previous results on vector valued DF spaces (which we will use) condition \( \varepsilon \) for normal Banach sequence spaces will play an important role. Let us first consider the situation for norinal Banach sequence spaces \( \lambda \) satisfying property \( \varepsilon \):

**Theorem 5.19.** Let \( (\lambda, \| \cdot \|_\lambda) \) be a normal Banach sequence space satisfying property \( \varepsilon \) and let \( E \) be a metrizable locally convex space.
i) If $\langle \lambda^\times, \| \cdot \|_{\lambda^\times} \rangle$ also satisfies property $\varepsilon$, then $\lambda(E)$ is distinguished if and only if $E$ is distinguished.

ii) If $\langle \lambda^\times, \| \cdot \|_{\lambda^\times} \rangle$ does not satisfy property $\varepsilon$, then $\lambda(E)$ is distinguished if and only if $E$ satisfies the density condition.

**Proof.** Because $\lambda$ satisfies property $\varepsilon$ we get with corollary 4.20 that the strong dual of $\lambda(E)$ is the DF space $\lambda^\times((E', \beta(E', E)))$ (topologically).

i) If $\lambda^\times$ satisfies the property $\varepsilon$, then $\lambda^\times((E', \beta(E', E)))$ is quasibarrelled if and only if $(E', \beta(E', E))$ is quasibarrelled (see 5.5) or, equivalently, if $\alpha(E')$ only if $E$ is distinguished.

ii) If $\lambda^\times$ does not satisfy property $\varepsilon$, then $\lambda^\times((E', \beta(E', E)))$ is quasibarrelled if and only if $(E', \beta(E', E))$ satisfies the dual density condition (see 5.16) or, equivalently, if $\alpha(E')$ only if $E$ satisfies the density condition (see 5.1 i iii).

**Remark 5.20.** i) Theorem 5.19 i) describes the situation whenever $\lambda = l_\infty, p \in (1, \infty)$ or $\lambda = c_0$.

ii) Theorem 5.19 part ii) covers the case $\lambda = l_1$, which was considered by K.D. Bierstedt & J. Bonet [1].

In remark 5.15 i) we noted that a DF space $E$ in which every bounded subset is metrizable (i.e. a DF space $E$ which satisfies the dual density condition) has to be quasibarrelled. We used that this property is stable under forming the vector valued sequence spaces $\lambda(E)$ to get that $\lambda(E)$ is quasibarrelled for every DF space $E$ with the dual density condition (and every normal Banach sequence space $\lambda$). If $\lambda$ does not satisfy property $\varepsilon$, then $\lambda(E)$ contains $l_\infty(E)$ as a complemented subspace. We could show that the dual density condition is also necessary for the quasibarrelledness of $l_\infty(E)$ and therefore necessary for the quasibarrelledness of $\lambda(E)$. We will use the same ideas in the examination of the distinguishedness of $\lambda(E)$ whenever $E$ is metrizable.

**Proposition 5.21.** Let $\langle \lambda, \| \cdot \|_{\lambda} \rangle$ be a normal Banach sequence space and let $E$ be a metrizable locally convex space which satisfies the density condition. Then the (metrizable) space $\lambda(E)$ also satisfies the density condition.

**Proof.** We use the characterization 5.9 of the density condition for metrizable locally convex spaces. Let therefore $(U_n)_{n \in \mathbb{N}}$ be a decreasing sequence of closed and absolutely convex zero neighbourhoods in $E$ which defines a zero basis.

Let $(\rho_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ and a zero neighbourhood $W$ in $\lambda(E)$ be given. We have to show that there are $m, n \in \mathbb{N}$ and $C \in B(\lambda(E))$ with $\bigcap_{n=1}^m \rho_n \lambda(U_n) \subset C + W$.

First we get the existence of an absolutely convex zero neighbourhood $U$ in $E$ with $\lambda(U) \subset W$. Since $E$ satisfies the density condition there are $m, n \in \mathbb{N}$ and $B \in B(E)$ with

$$\bigcap_{n=1}^m \rho_n 2^{n+1} U_n \subset B + U.$$ 

Lemma 4.3 then implies

$$\bigcap_{n=1}^m \rho_n \lambda(U_n) \subset \frac{1}{2} \left( \sum_{k=1}^m 2^{-k} \right) \lambda \left( \bigcap_{n=1}^m \rho_n 2^{n+1} U_n \right) \subset \frac{1}{2} \lambda(B + U).$$
Now, we can apply lemma 4.9 to get $\bigcap_{n=1}^M \rho_n(\mathcal{L}(U_n)) \subset \lambda(B) + \lambda(U) \subset \lambda(B) + W$ and this yields the conclusion.

We recall a result of J. Bonet, S. Dierolf & C. Fernandez (sec [7], lemma i), which gives a necessary condition for the distinguishedness of Frechet spaces. Using this result they could give the first example of a distinguished Frechet space whose bidual is not distinguished. With essentially the same proof as given in [7] we prove the result for metrizable spaces:

**Theorem 5.22.** Let $E$ be a metrizable locally convex space and let $F$ be a topological subspace of the bidual $E''$ of $E$ which contains $E$ as a closed subspace (if we consider $E$ as a subspace of $E''$).

Let $q : F \to F / E$ be the quotient map. **Assume** $F$ is distinguished, then

i) $F / E$ is also distinguished and

ii) the strong dual of $F / E$ is a topological subspace of the strong dual of $F$, i.e. every bounded set $B$ in $F / E$ is contained in the closure of $q(C)$ for some $C \in B(F)$.

**Proof.** The transposed map $q' : (F / E)' \to (F / E)'$ is continuous. Because $E''$ is of an LB space and therefore its strong dual is an LB space. We only have to show that $E''$ is an LB space. (Since the open mapping theorem then implies i) and ii), using the fact that the associated bornological topology to $\beta((F / E)'', F / E)$ is also an LB space topology).

Let $j : E \to F$ and $i : F \to E''$ be the canonical inclusions and let $j^* : F' \to E'$ and $i^* : (E'', \beta(E'', E'))' \to F'$ be their transposed mappings (which are continuous with respect to the strong topologies). First we note that $E''$ is the kernel of $j^*$. Using that the strong dual of $F$ is barrelled, we even get that

$$j^* : (F', \beta(F', F)) \to (E', \beta(E', E''))$$

is continuous. Because $E''$ is barrelled (it is a Frechet space) we have the continuity of

$$i^* : (E'', \beta(E'', E'')) \to (F', \beta(F', F)).$$

Moreover, the mapping $j^* \circ i^*$ is the transpose of $i \circ j$ and therefore its restriction to $E'$ is the identity on $E', \beta(E', E'')$. So $(F', \beta(F', F))$ is the topological direct sum of a copy of $(E', \beta(E', E''))$ and $\ker(j^*) = E''$. Thus $E''$ is a complemented subspace of an LB space and therefore itself an LB space.

We remark that J. Bonet & S. Dierolf [6] could show that a quotient inpr between Frechet spaces, which lifts bounded sets with closure, lifts already bounded sets.

We now turn back to the examination of the distinguishedness of our spaces $\lambda(E)$.

**Theorem 5.23.** Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space and let $E$ be a metrizable locally convex space.

If both $\lambda$ and $\lambda^\times$ satisfy property $\varepsilon$, then $\lambda(E)$ is distinguished if and only if $E$ is distinguished.

If $\lambda$ or $\lambda^\times$ does not satisfy property $\varepsilon$, then $\lambda(E)$ is distinguished if and only if $E$ satisfies the density condition.
**Proof.** According to theorem 5.19 we only have to prove the assertion in the case where \( \lambda \) does not satisfy property \( \varepsilon \). \( E \) satisfies the density condition, proposition 5.21 yields that \( \lambda(E) \) satisfies it. Remark 5.18 implies that \( \lambda(E) \) is distinguished.

Let \( \lambda(E) \) be distinguished. Theorem 3.8 implies that \( l_\infty(E) \) is a complemented subspace of \( \lambda(E) \) and therefore \( l_\infty(E) \) is distinguished. Because \( E \) is metrizable and distinguished we get with corollary 4.40 that \( l_\infty(E') \) is the bidual of \( c_0(E) \). Moreover, \( c_0(E) \) is closed in \( l_\infty(E) \) and \( l_\infty(E) \) is a topological subspace of \( l_\infty(E') \). So, we may apply 5.22 ii) and get that the quotient map \( q : l(E) \to l_\infty(E) / c_0(E) \) lifts bounded sets with closure. We choose a decreasing zero basis \( (U_n)_{n \in \mathbb{N}} \) in \( E \) consisting of closed and absolutely convex sets.

Let \( (\rho_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N} \) be arbitrary. Then

\[ B := \bigcap_{n \in \mathbb{N}} \rho_n q(l_\infty(U_n)) \]

is a bounded subset of \( l_\infty(E) / c_0(E) \). Hence, because \( q \) lifts bounded sets with closure, there is a bounded set \( C \) in \( l_\infty(E) \) with \( B \subseteq q(C) \). By corollary 4.4 we may assume that \( C \subseteq l_\infty(D) \) for some closed \( D \subseteq E \) and we get \( B \subseteq q(l_\infty(D)) \).

Assume there is a closed and absolutely convex zero neighbourhood \( U \) in \( E \) such that there exists \( x_m \in (\bigcap_{n=1}^m \rho_n U_n) \setminus D + 2U \) for every \( m \in \mathbb{N} \). Then we get \( (x_m)_{m \in \mathbb{N}} \in l_\infty(E) \) and \( (x_m)_{m \in \mathbb{N}} + c_0(E) \) is bounded (by the definition of \( B \)). By corollary 4.4 there cannot exist a zero sequience \( (x_m)_{m \in \mathbb{N}} \) in \( E \) with \( (x_m + y_m)_{m \in \mathbb{N}} \subseteq D^N + U^N \). Hence, \( (x_m)_{m \in \mathbb{N}} + c_0(E) \) is a contradiction to \( B \subseteq q(l_\infty(D)) \). So, the assumption is wrong and we have shown that for every zero neighbourhood \( U \) in \( E \) there exists \( m \in \mathbb{N} \) with \( \bigcap_{n=1}^m \rho_n U_n \subseteq D + U \). Because \( (\rho_n)_{n \in \mathbb{N}} \) is arbitrary, this means that \( E \) satisfies the density condition.

**Remark 5.24.** i) Let \( E \) be a metrizable locally convex space with a decreasing zero basis \( (U_n)_{n \in \mathbb{N}} \). If \( E \) satisfies the density condition, the proof of theorem 5.23 shows that it satisfies the (formally) stronger condition:

For every \( (\rho_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N} \) there exists a bounded subset \( D \) of \( E \) such that for every zero neighbourhood \( U \) one can find \( m \in \mathbb{N} \) with \( \bigcap_{n=1}^m \rho_n U_n \subseteq D + U \).

ii) Recently got the information, that J.C. Diáz had independently proved the case \( \lambda = 1 \) (unpublished).

Grothendieck [18], Questions non Résolues 5, posed the question, whether the bidual of a distinguished Fréchet space is again distinguished. As noted above the first counter example was given by J. Boìet, S. Dierolf & C. Fernández [7] using Fréchet spaces of Moscatelli type. Using theorem 5.23 we can give an example in the context of vector valued sequence spaces:

**Example 5.25.** Let \( E \) be a reflexive Fréchet space which does not satisfy the density condition (for such a space see [12], (4.7) or (4.8)). Then we get that \( c_0(E) \) is distinguished. Birt the bidual of \( c_0(E) \) is 1, (E) and therefore not distinguished.

By a result of A. Peris (see [30], 3.1.6) a Fréchet space satisfies the density condition whenever its bidual satisfies it. so one may replace in the above example \( E \) by any distinguished Fréchet space without density condition.

**Remark 5.26.** Assume that \( \lambda \) is a normal Banach sequence space and that \( E \) is a reflexive Fréchet space without the density condition such that \( \lambda(E) \) is distinguished but its bidual does not. Then \( c_0(E) \) is complemented in \( \lambda(E) \).
Indeed, under the assumptions above, A and \( \lambda \) have to satisfy property \( e \) (see theorem 5.23). Moreover, \( \lambda \) must not satisfy property \( e \) (otherwise \( \lambda(E) \) would be reflexive). We may apply remark 3.9 to get that \( \lambda(E) \) contains a complemented copy of \( c_0(E) \).

6. BORNOLOGICAL VECTOR VALUED SEQUENCE SPACES

In this chapter we consider sufficient conditions to guarantee that for a given (ultra) bornological space \( E \) the space \( \lambda(E) \) is also (ultra) bornological. First we consider the situation if \( E \) is metrizable. In this case \( \lambda(E) \) is again metrizable, hence bornological. Using remark 5.2 (i) we get:

**Proposition 6.1.** Let \( (A, \| - \|_A) \) be a normal Banach sequence space and let \( E \) be a metrizable locally convex space. Then \( \lambda(E) \) is ultrabornological if and only if \( E \) is ultrabornological.

The situation is more complicated if \( E \) is a bornological DF space or even a DFM space. We will prove for DFM spaces \( E \) and normal Banach sequence spaces \( A \) which satisfy property \( \gamma \) that \( \lambda(E) \) is bornological. This given a partial answer to the question, posed by K.D. Bierstedt and J. Schmets (see [35]), if for every bornological space \( E \) also \( c_0(E) \) is bornological. We first consider the situation when \( A \) satisfies \( e \) and 6). The following proposition is covered by a result due to M. Florencio & P.J. Paul (see [16], 4.9), which is proved in the context of perfect spaces.

**Proposition 6.2.** Let \( (A, \| - \|_A) \) be a normal Banach sequence space satisfying properties \( e \) and 6). Let \( E \) be a DF space. Then \( \lambda(E) \) is bornological if and only if \( E \) is bornological.

**Proof.** Let \( f \) be an arbitrary locally bounded linear form on \( \lambda(E) \). (We will show that \( f \) is continuous). Define locally bounded linear forms \( f_n, n \in N \) on \( E \) by \( f_n(x) := f(((0)_k < n, x, (0)_k > n)) \), \( x \in E \). Since \( E \) is bornological, these mappings \( f_n \) are continuous on \( E \).

Let \( x = (x_n)_{n \in N} \in \lambda(E) \) be arbitrary. Because \( A \) satisfies property 6) we get \( x \in \lambda(B) \) for some \( B \subset B(E) \). Property \( e \) yields \( \lim_{n \to -\infty} \|((0)_k < n, (p_B(x_k))_k > n)\| \lambda = 0 \). Using that \( f \) is locally bounded, this implies \( \lim_{n \to -\infty} (f(x) - \sum_{k=1}^n f_k(x_k)) = \lim_{n \to -\infty} f(((0)_k < n, (x_k)_k > n)) = 0 \). So, we may identify \( f \) with \( (f_n)_{n \in N} \) via \( f(x) = \sum_{n=1}^\infty f_n(x_n), x = (x_n)_{n \in N} \in \lambda(E) \).

Let \( C \) be a bounded subset of \( \lambda(E) \). Without loss of generality we may assume that \( C = \bigcap_{p \in c_0(E)} \rho_p^{-1}([0, 1]) \), \( \rho_p > 0 \). We get

\[
\bigcup_{n \in N} \left\{ \sum_{k=1}^n f_k(x_k) : (x_k)_{k \in N} \in C \right\} \subset f(C)
\]

and therefore \( (f_k)_{k < n}, (0)_k > n \) is a strongly bounded sequence in \( \lambda(E)' \) which is by corollary 5.5 even equicontinuous. Hence \( f = (f_n)_{n \in N} \) is continuous.

We proved that every locally bounded linear form on \( \lambda(E) \) is already continuous and this implies together with corollary 5.5 that \( \lambda(E) \) is bornological.

**Remark 6.3.** Analysing the proof of proposition 6.2 we see that the result is even true for normal Banach sequence spaces \( A \) satisfying property \( e \) and DF spaces \( E \) whenever

\[
\lambda(E) = \bigcup_{B \in B(E)} \lambda(B).
\]
Example 6.4. (cf. [26]) Let $E$ be a bornological $\textbf{DF}$ space in which every zero sequence converges locally, i.e. $c_0(E) = \bigcup_{B \in \mathcal{B}(E)} c_0(B)$. (This means that $E$ is a retractive $\text{LN}$ space). Then $c_0(E)$ is a bornological $\textbf{DF}$ space.

For the iext result we need a reformulation of lemma 4.9:

**Lemma 6.5.** Let $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach sequence space and let $E$ be a locally convex Hausdorff space. Let $B_1, \ldots, B_m$ be bounded and absolutely convex subsets of $E$. For every $\varepsilon > 0$ it follows

$$\lambda(\sum_{n=1}^{m} B_n) \subset (1 + \varepsilon) \sum_{n=1}^{m} \lambda(B_n).$$

**Proof.** Let $\varepsilon > 0$ be arbitrary. Without loss of generality we may assume that $m = 2$. Let $(x_n)_{n \in \mathbb{N}} \in \lambda(B_1 + B_2)$ be given. Using the boundedness of $B_1$ and $B_2$ we get

$$x_n \in (1 + \varepsilon)p_{B_1 + B_2}(x_n)(B_1 + B_2) = (1 + \varepsilon)p_{B_1 + B_2}(x_n)B_1 + (1 + \varepsilon)p_{B_1 + B_2}(x_n)B_2$$

for every $n \in \mathbb{N}$.

This implies the existence of $y_n \in (1 + \varepsilon)p_{B_1 + B_2}(x_n)B_1$ and $z_n \in (1 + \varepsilon)p_{B_1 + B_2}(x_n)B_2$ with $x_n = y_n + z_n$ for every $n \in \mathbb{N}$. We now compute $\|p_{B_1 + B_2}(x_n)\|_\lambda \leq (1 + \varepsilon)$ and using again the boundedness of $B_1$ and $B_2$, we get $(y_n)_{n \in \mathbb{N}} \in \lambda(B_1)$ and $(z_n)_{n \in \mathbb{N}} \in \lambda(B_2)$. This implies the assertion.

**Proposition 6.6.** Let $(A, \| \cdot \|_A)$ be a normal Banach sequence space satisfying property 6) but not property $\varepsilon)$. Let $E$ be a $\textbf{DF}$ space. Then $\lambda(E)$ is bornological if and only if $E$ satisfies the strong dual density condition.

**Proof.** Let $(B_n)_{n \in \mathbb{N}}$ be an increasing fiindamental sequence of bounded, absolutely convex and closed sets in $E$ satisfying $2B_n \subset B_{n+1}, n \in \mathbb{N}$.

If $C$ is a bounded subset of $\lambda(E)$, we may (in view of property $\delta)$, see 4.11) assume that $C \subset \lambda(B_k)$ for some $k \in \mathbb{N}$. Using that $E$ satisfies the strong dual density condition, we get the existence of an absolutely convex and closed zero neighbourhood $U$ in $E$ and $m \in \mathbb{N}$ such that

$$B_k \cap U \subset \frac{1}{4} \sum_{n=1}^{m} \alpha_n B_n.$$

Using the lemmata 4.3 and 6.5 this yields

$$C \cap \lambda(U) \subset \lambda(B_k) \cap \lambda(U) \subset 2\lambda(B_k \cap U) \subset \lambda\left(\frac{1}{2} \sum_{n=1}^{m} \alpha_n B_n\right) \subset \sum_{n=1}^{m} \alpha_n \lambda(B_n).$$

Hence, $\lambda(E)$ satisfies the strong density condition. Since $\lambda(E)$ is a $\text{DF}$ space this yields that $\lambda(E)$ is bornological.

For the converse let $\lambda(E)$ be bornological. Then $l_\infty(E)$ is (as a completed subspace of $\lambda(E)$, see 3.8) also bornological.
Let \((\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}\) be arbitrary. Then

\[ V := \bigcup_{n \in \mathbb{N}} I_n \left( \sum_{k=1}^{n} \alpha_k B_k \right) \]

is a bornivorous and absolutely convex subset of \(I_\infty(E)\), therefore \(a\) zero neighbourhood. Hence there is a zero neighbourhood \(U\) in \(E\) with \(I_\infty(U) \subset V\). Proceeding as in the proof of 5.16 this implies that for every bounded subset \(B\) of \(E\) there is \(n \in \mathbb{N}\) with

\[ B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k. \]

Therefore \(E\) satisfies the strong dual density condition.

We remark that K.D. Bierstedt & J. Bonet (see [2], 1.5) proved the above result in the case \(A = i\) with \(a\) somewhat different proof.

**Remark 6.7.** i) Let \(E\) be a DF space with an increasing fundamental sequence \((B_n)_{n \in \mathbb{N}}\) of bounded, closed and absolutely convex sets. If \(E\) satisfies the strong dual density condition, then, according to the proof of proposition 6.6, it satisfies the (formally) stronger condition:

For every \((\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}\) there is a zero neighbourhood \(U\) such that for every bounded set \(B\) there exists \(n \in \mathbb{N}\) with \(B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k\).

ii) Let \((A, \| \cdot \|_\lambda)\) be a normal Banach sequence space satisfying property 6) and let \(E\) be a DF space. Combining remark 5.2 i) with proposition 6.2 and proposition 6.6 we get necessary and sufficient conditions deciding whether \(A(E)\) is ultrabornological.

iii) In the previous results of this chapter one may replace “property 6)” by “property \(\delta_n\)”, see theorem 2.9 iv).

We must confess that we can give only partial results in the case when \((A, \| \cdot \|_\lambda)\) does not satisfy property \(\delta_n\).

Up to now we have only remark 6.3 to ensure that in this case \(\lambda(E)\) is bornological. But we can give a positive answer whenever \((\lambda, \| \cdot \|_\lambda)\) satisfies property \(\gamma\) and \(E\) is a DFM space.

Let \((A, \| \cdot \|_\lambda)\) be a normal Banach space satisfying property \(\gamma\). We recall that \((A^n, \| \cdot \|_{\lambda^{(n)}})\) is defined by

\[ \lambda^{(n)} := \{ \alpha \in E : \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda < \infty \}, \]

\[ \| \alpha \|_{\lambda^{(n)}} := \sup_{n \in \mathbb{N}} \| P_n(\alpha) \|_\lambda, \]

where \(P_n\) denote the canonical projection onto the first \(n\) coordinates. Then \((A, \| \cdot \|_\lambda)\) is a subspace of \((\lambda^{(n)}, \| \cdot \|_{\lambda^{(n)}})\) (see 2.9 i) and \((\lambda^{(n)}, \| \cdot \|_{\lambda^{(n)}})\) satisfies property 6), i.e., its closed unit ball is even closed in \(\omega\) (see 2.9 iii) and 2.10). We now prove a technical lemma concerning this spaces.

**Lemma 6.8.** Let \((A, \| \cdot \|_\lambda)\) be a normal Banach sequence space satisfying property \(\gamma\) and let an arbitrary \((\alpha_n)_{n \in \mathbb{N}} \in A\) be given. Moreover, let \(\gamma : [0, 1] \rightarrow [0, 1]\) be increasing such that \(\lim_{x \rightarrow 0+} \gamma(x) = 0\).

Then for every \((\delta_n)_{n \in \mathbb{N}} \in [1, \infty)^\mathbb{N}\) satisfying \((\alpha_n \delta_n)_{n \in \mathbb{N}} \in \lambda^{(n)}\) it follows \((\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in \mathbb{N}} \in A\).
Proof. Let for every subset $S$ of $N$ be $P_S : \omega \to \omega$ the canonical projection onto the coordinates belonging to $S$.

Without loss of generality we may assume that $(\alpha_n \delta_n)_{n \in N}$ is contained in the closed unit ball $B_{\lambda^0(\delta)}$ of $\lambda^0(\delta)$.

Let $\varepsilon > 0$ be arbitrary and define $S := S(\varepsilon) := \{ n \in N : \gamma(\delta_n^{-1}) \leq \varepsilon \}$. Using that $y$ is increasing and satisfies $\lim_{n \to 0} \gamma(y) = 0$ we get the existence of $\rho > 0$ such that $|\alpha_n \delta_n| \leq \rho |\alpha_n|$ for every $n \in N \setminus S$. Hence, $P_{N \setminus S}((\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in N}) \in \lambda$. Moreover, we compute $\|P_S((\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in N})\|_{\lambda^0(\delta)} \leq \varepsilon \|((\alpha_n \delta_n))_{n \in N}\|_{\lambda^0(\delta)} \leq \varepsilon$. Therefore we get $(\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in N} = P_{N \setminus S}((\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in N}) + P_S((\alpha_n \delta_n \gamma(\delta_n^{-1}))_{n \in N}) \in A + \varepsilon B_{\lambda^0(\delta)}$. Because $A$ is closed in $\lambda^0(\delta)$ and $\varepsilon$ is arbitrary this implies the assertion.

We need another lemma concerning zero neighbourhoods of compact subsets of normed spaces.

Lemma 6.9. (S. Dierolf, personal communication) Let $(E, \| \cdot \| )$ be a normed space with closed unit ball $B_{\| \cdot \|}$. Let $K_1$ and $K_2$ be absolutely convex and compact subsets of $E$ such that $2(K_1 + K_2) \subset B_{\| \cdot \|}$. Then there is an increasing mapping $\varepsilon : [0, 1] \to [0, 1]$ satisfying $\lim_{n \to 0} \varepsilon(x) = 0$ such that for every $x \in E$ $[0, 1]$ we have

$$(K_1 + K_2) \cap xB_{\| \cdot \|} \subset (2K_1 \cap \varepsilon(x)B_{\| \cdot \|}) + (2K_2 \cap \varepsilon(x)B_{\| \cdot \|}).$$

Proof. We will prove:

(*) For every $\varepsilon > 0$ there is $\delta > 0$ such that

$$(K_1 + K_2) \cap \delta B_{\| \cdot \|} \subset (2K_1 \cap \varepsilon B_{\| \cdot \|}) + (2K_2 \cap \varepsilon B_{\| \cdot \|}).$$

This implies immediately the assertion.

Assume that (*) is wrong. Then there is $\varepsilon > 0$ and a zero sequence $(z_n)_{n \in N}$ in $E$, contained in $K_1 + K_2$, such that $z_n \notin (2K_1 \cap \varepsilon B_{\| \cdot \|}) + (2K_2 \cap \varepsilon B_{\| \cdot \|})$ for every $n \in N$. Choose sequences $(x_n)_{n \in N}$ in $K_1$ and $(y_n)_{n \in N}$ in $K_2$ with $x_n = x_n + y_n$, $n \in N$. The sets $K_1$ and $K_2$ are (sequentially) compact, so there exists a strictly increasing sequence $(n_k)_{k \in N}$ of positive numbers, $x \in K_1$ and $y \in K_2$ such that $x_{n_k} \to x$ and $y_{n_k} \to y(k \to \infty)$. Since $(z_n)_{n \in N}$ is a zero sequence we compute $x + y = 0$. Hence, there is $m \in N$ such that

$$z_{m_k} = (x_{m_k} - x) + (y_{m_k} - y) \in (2K_1 \cap \varepsilon B_{\| \cdot \|}) + (2K_2 \cap \varepsilon B_{\| \cdot \|}),$$

a contradiction.

Proposition 6.10. Let $(A, \| \cdot \|)$ be a normal Banach sequence space satisfying property $\gamma$ and let $E$ be a DFM space. Moreover, let $K_1, \ldots, K_m$ be absolutely convex and compact subsets of $E$. Then we get

$$\lambda^0(\delta) \big( \sum_{n=1}^m K_n \big) \cap \lambda(E) \subset \sum_{n=1}^m 2^n (\lambda^0(\delta)(K_n) \cap \lambda(E)).$$

Proof. We will prove

$$\lambda^0(\delta)(K_1 + K_2) \cap \lambda(E) \subset 2\lambda^0(\delta)(K_1) \cap \lambda(E) + 2\lambda^0(\delta)(K_2) \cap \lambda(E),$$
the assertion follows then by induction.

Because $E$ satisfies the dual density condition (cf. 5.15 ii)), corollary 5.14 yields the existence of a continuous norm $\| \cdot \|$ on $E$, such that the topologies induced on $K := K_1 + K_2$ by $\| \cdot \|$ and $E$ coincide. Without loss of generality $2K = 2(K_1 + K_2)$ is contained in the closed unit ball $B_{\| \cdot \|}$. Using lemma 6.9 we get an increasing mapping $\varepsilon : [0, 1] \to [0, 1]$ with $\lim_{x \to 0} \varepsilon(x) = 0$ such that for every $x \in [0, 1]$

$$K \cap xB_{\| \cdot \|} \subset (2K_1 \cap \varepsilon(x)B_{\| \cdot \|}) + (2K_2 \cap \varepsilon(x)B_{\| \cdot \|}).$$

Let $(z_n)_{n \in \mathbb{N}} \in \lambda(\delta)(K) \cap \lambda(E)$ be arbitrary. We define for the reason of shortness for $n \in \mathbb{N}$:

$$\delta_n := \begin{cases} \frac{p_K(z_n)}{p_B(\|z_n\|)} : z_n \neq 0 \\ 1 : z_n = 0. \end{cases}$$

Then it follows for $n \in \mathbb{N}$:

$$z_n \in p_K(z_n)K \cap p_B(\|z_n\|)B_{\| \cdot \|} = p_K(z_n) \cdot (K \cap \delta_n^{-1}B_{\| \cdot \|}) \subset p_K(z_n)((2K_1 \cap \varepsilon(\delta_n^{-1})B_{\| \cdot \|}) + (2K_2 \cap \varepsilon(\delta_n^{-1})B_{\| \cdot \|})).$$

Hence, for every $n \in \mathbb{N}$ we have $z_n = u_n + v_n$, where

$$u_n \in p_K(z_n)(2K_1 \cap \varepsilon(\delta_n^{-1})B_{\| \cdot \|}) \text{ and } v_n \in p_K(z_n)(2K_2 \cap \varepsilon(\delta_n^{-1})B_{\| \cdot \|}).$$

From this we first obtain $p_{K_1}(u_n) \leq 2p_K(z_n), n \in \mathbb{N}$ and therefore $(u_n)_{n \in \mathbb{N}} \in 2\lambda(\delta)(K_1)$, if we take into account that $K_1$ is bounded. Analogously we show that $(v_n)_{n \in \mathbb{N}}$ is contained in $2\lambda(\delta)(K_2)$.

It remains to show that $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are contained in $\lambda(E)$. We will do this explicitly for $(u_n)_{n \in \mathbb{N}}$, the other case can be proved analogously.

Let an absolutely convex and closed zero neighbourhood $V$ in $E$ be given. We may assume that $2K = 2(K_1 + K_2)$ is contained in $V$ and that $B_{\| \cdot \|}$ absorbs $V$. Because $\| \cdot \|$ induces on $K_1$ the same topology as $E$ there exists an increasing mapping $\varphi : [0, 1] \to [0, 1]$ with $\lim_{x \to 0} \varphi(x) = 0$ such that for every $x \in [0, 1]$

$$2K_1 \cap xB_{\| \cdot \|} \subset \varphi(x)V.$$

We define $\gamma := \varphi \circ \varepsilon$ and we get for every $n \in \mathbb{N}$:

$$u_n \in p_K(z_n)(2K_1 \cap \varepsilon(\delta_n^{-1})B_{\| \cdot \|}) \subset p_K(z_n)(2K_1 \cap \gamma(\delta_n^{-1})V) \subset p_K(z_n)\gamma(\delta_n^{-1})V.$$
Theorem 6.11. Let $(A, \| \cdot \|_A)$ be a normal Banach sequence space satisfying property $\gamma$ and let $E$ be a DFM space. Then $\lambda(E)$ is bornological (and so a complete LB space).

Proof. Let $(K_n)_{n \in \mathbb{N}}$ be a fundamental sequence of closed, absolutely convex and bounded sets in $E$. Let $(\alpha_n)_{n \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$ be arbitrary. We have to show that $V := \bigcup_{m \in \mathbb{N}} \sum_{n=1}^{m} \alpha_n((\lambda^b)(K_n) \cap \lambda(E))$ is a zero neighbourhood in $\lambda(E)$. Since $E$ is a DFM space, the sets $K_n$ are compact, so theorem 3.3 iv) implies that $C_m := \lambda^b(\sum_{n=1}^{m} \frac{1}{2n+1} \alpha_n K_n) \cap \lambda(E)$ is closed in $\lambda(E)$ for each $m \in \mathbb{N}$. Moreover, $(C_m)_{m \in \mathbb{N}}$ is increasing and every bounded set is absolutely by some $C_m$, see 4.12. So, we may apply A. Grothendieck [18], p. 72, lemme 4, and get that $U := \bigcup_{m \in \mathbb{N}} C_m$ is a zero neighbourhood in the quasibarrelled DF space $\lambda(E)$. Proposition 6.10 implies $U \subset V$ and we are done.

Remark 6.12. We note that one may replace in the previous theorem “property $\gamma$” by “property $\gamma_\infty$”, see 2.9 i, iii). $c_0$ satisfies property $\varepsilon$ and therefore property $\gamma_\infty$ and we obtain:

Corollary 6.13. (S. Dierolf, unpublished) Let $E$ be a DFM space. Then $c_0(E)$ is bornological.
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L. FRERICK
Fachbereich 7 - Mathematik
Gausstrasse 20
D-42097 Wuppertal
GERMANY