NON-EUCLIDEAN PLANE BOL LOOPS

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Abstract. *We extract an interesting family of 2-dimensional Bol-loops with the help of the standard notion of parallel translation using a generalised Poincaré model for two-dimensional surfaces with constant curvature.*
AMS(MOS) subject classifications: 20N05, 20N, 20, 51M10, 51M, 50

1. INTRODUCTION

In the theory of differential loops there is an operation on the local geodesic loops which plays a very important role:

$$x \circ y = \exp_y(\tau_{e,y}(\exp_e^{-1}(x))) \qquad (*)$$

where $\tau_{e,y}$ denotes parallel translation along the goedesic from e to y. (Cf. [1], [2]) This operations is even more interesting if the differential geometric space has a classical structure. In symmetric spaces the goedesic loop multiplication (*) satisfies well-known identities like the automorphic inverse property ([1]) and the right Bol property ([2]).

The goal of this paper is to describe the operation defined by (*) in the simplest symmetric spaces, which are the simply connected Riemannian 2-manifolds of dimension 2 of constant curvature (sphere, hyperbolic plane). For this description we will employ elementary tools using the Poincaré models on the complex plane of the non-Euclidean planes.

In [3], the 2-dimensional local proper (non-associative) Bol loops are classified and proved that there are 3 different isotopy classes. In this paper examining the sphere and the hyperbolic plane, we describe in a rather elementary way the elliptic and hyperbolic cases.

Using the complex plane representation, it is easily seen in which case we can talk about global loops and in which one we can not. We also prove that the nonglobal case cannot be made global adding new elements to the loop. It is interesting because it is shown that in the case of Moufang loops the local Moufang loop can be embedded into a global Moufang loop. (C.f. [4])

Because of the rather elementary structure of our formulas we can prove the well known properties more easily without using hard theoretical tools.

The formulas give us the possibility to define new global Bol loops (even discrete ones).

We assume familiarity with the basics of the loop theory (see e.g. [5]).

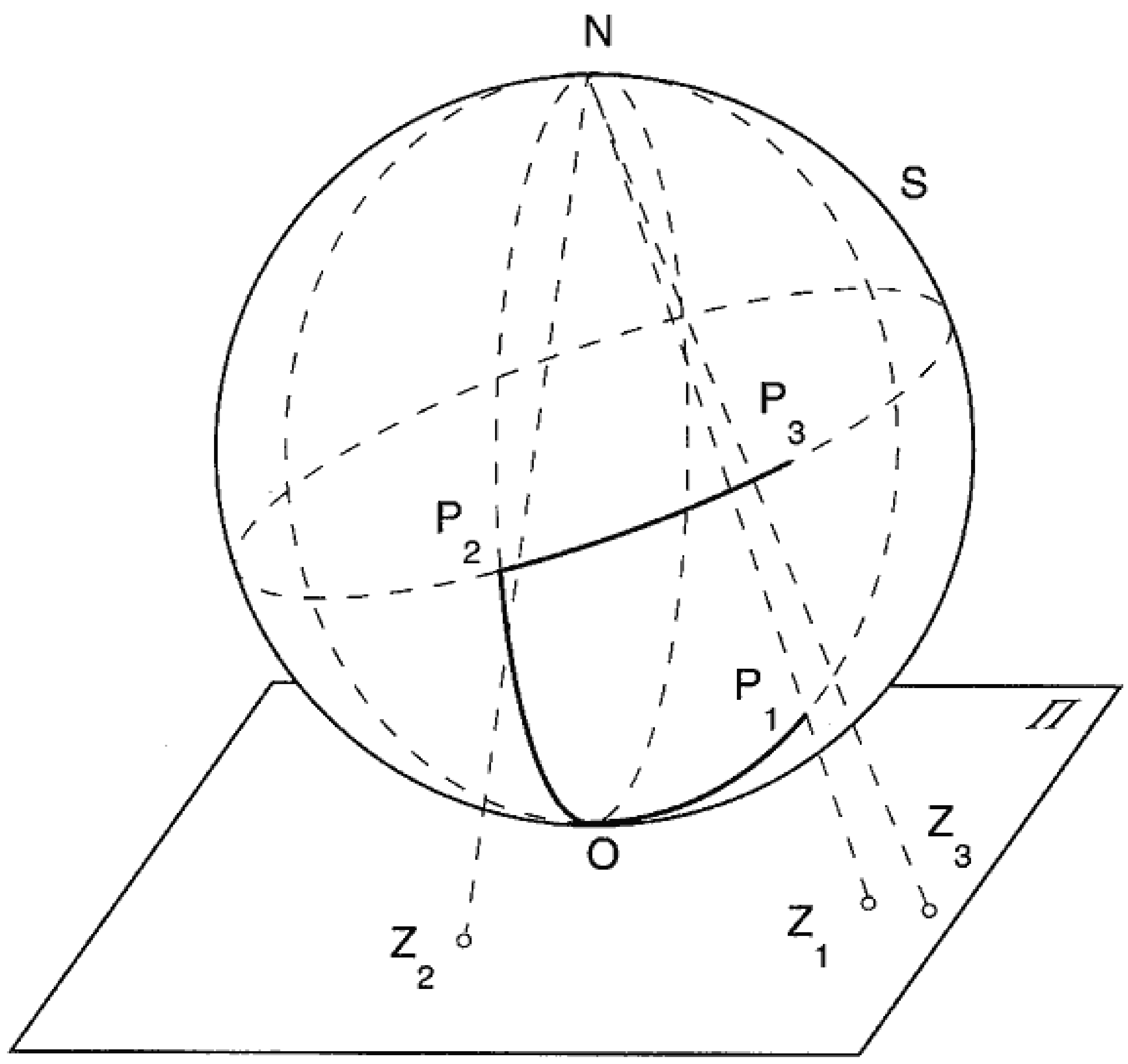
2. PARALLEL TRANSLATION

There is a natural way to introduce the notion of parallel translation of a great circle segment on the sphere with unit diameter along another great circle segment. (Figure 1)

Let $O, N (\in S)$ be *opposite points* of the sphere, which means that the (Euclidean) segment ON is a diameter of the sphere. Let OP_1, OP_2 be the great circle segments between O and P_1 and between O and P_2, respectively. ($P_1, P_2 \neq N$) Let v_1 be the tangent vector to OP_1. The result of the parallel translation of the great circle segment OP_1 along the great circle segment OP_2 is the great circle segment P_2P_3, if the tangent vector of the great circle segment P_2P_3 at the point P_2 is equal to v_1, and the length of OP_1 is equal to the lenght of P_2P_3.

THE SPHERICAL POINCARÉ MODEL.

For further calculations, we introduce the notion of the Poincaré model of the sphere. (Figure 1)

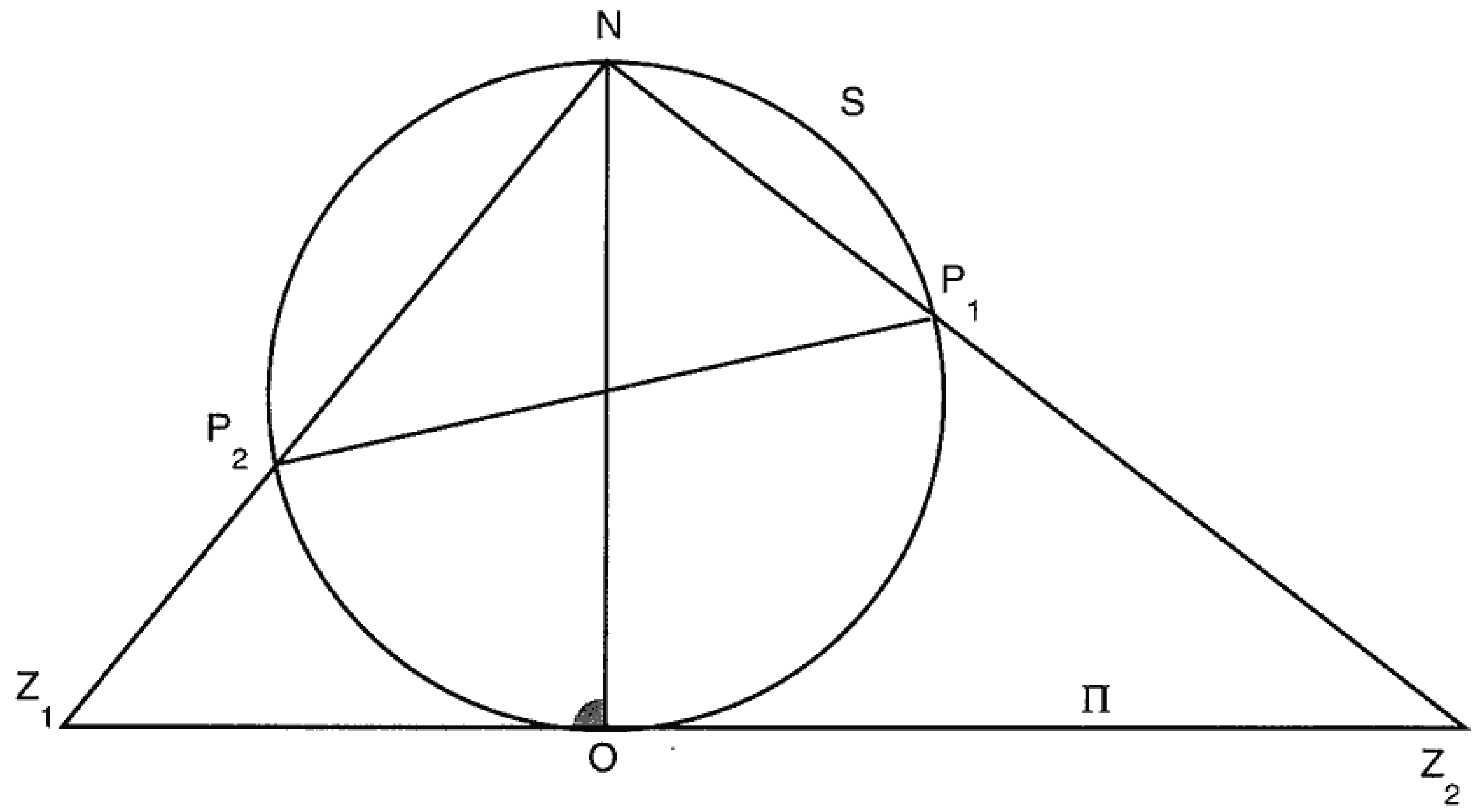


Let S be the sphere of unit diameter and O, P_1, P_2, P_3 points of S. Let Π be the tangent plane of the sphere at the point O, and it will be identified (in a standard way) with the complex plane, O is the origin. The points of the Poincaré model are the points of Π. The correspondance between the sphere and the complex plane is given by the stereographic projection of S from N to Π. The points z_1 and z_2 of Π are called opposite points if z_1 and z_2 are stereographic images of P_1 and P_2 which are opposite points on the sphere. On Figure 2 we can see that $|z_1||z_2| = 1$ and

$$z_2 = -\frac{z_1}{|x_1|^2} = -\frac{1}{\bar{z}_1}.$$

The images of the great circle segments are Euclidean segments, if the segments pass through the point O, otherwise they are circle segments (because the stereographic projection preserves circles). We can see that a circle on Π will be an image of a great circle if and only if it contains a pair of opposite points. The angles are the Euclidean angles, because the stereographic projection preserves angles. We can easily see that the length of the great circle segment OP_1 is equal to the length of the great circle segment P_2P_3 if P_1 and P_3 are on a circle with points contained in a plane parallel to the plane of the great circle OP_2. In this case, we can say that the two circles are *parallel*. It is obvious that two circles are parallel if and only if there are two great circles which are both orthogonal to both of the circles mentioned above.

(Figure 2)



We can summarize the above in the following definition:

Definition 2.1. *The points of the spherical* Poincaré model *are the points of the complex plane.* z_1' *is called the* opposite point *of the point* z_1 *if* $z_1' = -\frac{1}{\bar{z_1}}$. *A* great circle *is a line if it contains O, and is a circle containing pairs of opposite points if it does not contain O. The following formula gives us a very comfortable tool.*

Theorem 2.1. *Let* z_1, z_2 *be the points of the spherical Poincaré model. The parallel translation along the great circle segment* Oz_2 *moves the great circle segment* Oz_1 *to the great circle segment* z_2z_3 *if*

$$z_3 = \frac{z_1 + z_2}{1 - z_1\overline{z_2}}$$

Proof. Its obvious that the great circles through O and z_1 and through O and z_2 are lines. We have to prove that

a) the set $K = \{\frac{kz_1+z_2}{1-kz_1\overline{z_2}}, k \in \mathbb{R}^\infty\}$ is a great circle. ($R^\infty = R \cup \{\infty\}$)

b) Oz_1 is parallel to the tangent vector of K at z_2.

c) the lenght of Oz_1 is equal to the lenght of z_2z_3.

K obviously contains $z_2 (k = 0)$ and its opposite point $-\frac{1}{\bar{z_2}} (k = \infty)$. So if K is a circle then it is a great circle. To prove that it is a circle, it is sufficient to show that the imaginary part of the cross ratio of its four points is zero:

$$Im(z_2, z_3, \frac{kz_1 + z_2}{1 - kz_1\overline{z_2}}, z_2') = 0$$

$$(z_2, z_3, \frac{kz_1 + z_2}{1 - kz_1\overline{z_2}}, \frac{-z_2}{|z_2|^2}) = \frac{-kz_1(1 + |z_2|^2)}{\frac{(k-1)z_1(1+|z_2|^2)}{1-z_1\overline{z_2}}} \frac{\frac{-|z_2|^2(z_1+z_2)+z_2(1-z_1\overline{z_2})}{1-z_1\bar{z_2}}}{z_2(1 + |z_2|^2)} =$$

$$= \frac{k}{k-1} \frac{z_2(|z_2|^2 + 1)}{z_2(|z_2|^2 + 1)} = \frac{k}{k-1},$$

which is real. This completes the proof of a).

$$\lim_{k\to 0} \frac{\frac{kz_1+z_2}{1-kz_1\overline{z_2}} - z_2}{k} = z_1(1+|z_2|)^2 = cz_1 \quad (c \in R)$$

which proves b).

As we have seen earlier, it is enough to prove that the circle $L = \{\frac{z_1+kz_2}{1-kz_1\overline{z_2}}, k \in \mathbb{R}^\infty\}$ is parallel to the great circle Oz_2, i.e. there are two great circles which are both orthogonal to L and the great circle Oz_2.

1st step of c) We have to prove that L is a circle. We choose three arbitrary points: $z_1(k=0), z_3(k=1), -\frac{z_2}{z_1\overline{z_2}}(k=\infty)$, and we show that $Im(z_1, z_3, -\frac{z_2}{z_1\overline{z_2}}, \frac{z_1+kz_2}{1-kz_1\overline{z_2}} = 0$. Using the same computation method as earlier we get

$$(z_1, z_3, -\frac{z_2}{z_1\overline{z_2}}, \frac{(z_1+kz_2)}{1-kz_1\overline{z_2}}) = \frac{k-1}{k},$$

which is real.

2nd step of c) L and Oz_2 are orthogonal to the circle $|z| = 1$. (It is a great circle because the opposite point of z is $-\frac{1}{\bar{z}} = -\frac{z}{|z|^2} = -z$, which is on the circle). For the great circle Oz_2, which is a line on the complex plane, it is obvious. For L we have to prove that L is invariant under the inversion for the circle $|z| = 1$ i.e. for the inversion $z \mapsto \frac{1}{\bar{z}}$.

We have to prove that for an arbitrary $k \in \mathbb{R}$, there exists $\lambda \in \mathbb{R}$ such that:

$$\frac{1}{\overline{\left(\frac{z_1+kz_2}{1-kz_1\overline{z_2}}\right)}} = \frac{z_1+\lambda z_2}{1-\lambda z_1\overline{z_2}},$$

which is equivalent to the equation

$$1 - k\overline{z_1}z_2 - \lambda z_1\overline{z_2} + k\lambda|z_1|^2|z_2|^2 = |z_1|^2 + k\lambda|z_2|^2 + \lambda\overline{z_1}z_2 + kz_1\overline{z_2}$$

Rearranging the equation we get a linear equation for λ:

$$1 - |z_1|^2 - (\bar{z_1}z_2 + z_1\bar{z_2})(\lambda + k) - k\lambda|z_2|^2 + kl|z_1|^2|z_2|^2 = 0,$$

as $(\overline{z_1}z_2 + z_1\overline{z_2})$ is real. (It is its own complex conjugate).

3rd step of c) L and Oz_2 are both orthogonal to the great circle $O(iz_2)$ (which is also a line on the model as Oz_2). For Oz_2 it is obvious. For L we have to prove that L is invariant under the reflexion to the line $O(iz_2)$.

We have to prove that for an arbitrary $l \in R$, there exist a real ν for which

$$\frac{z_1+lz_2}{1-lz_1\overline{z_2}} = (iz_2)^2\overline{\left(\frac{z_1+\nu z_2}{1-\nu z_1\overline{z_2}}\right)}$$

Rearranging this we get

$$\frac{z_1+lz_2}{1-lz_1\overline{z_2}} = \frac{-z_2^2}{|z_2|^2}\frac{(\overline{z_1}+\nu\overline{z_2})}{1-\nu\overline{z_1}z_2}$$

Solving this equation we get

$$\nu = -\frac{z_1\bar{z}_2 + \bar{z}_1 z_2}{|z_2|^2(1-|z_1|^2)} - l,$$

which is real, so the proof is completed.

PARALLEL TRANSLATION OF THE HYPERBOLIC PLANE.

For our observations we use the well-known Poincaré model. We get a very similar result to Theorem 2.1.

The hyperbolic Poincaré model The *points* of the hyperbolic plane are the inner points of n circle of unit radius with center in the origin. Hyperbolic *lines* are lines if they pass through the origin, and are circles which are orthogonal to the limit circle $|z| = 1$ otherwise. The *angles* are the Euclidean angles. The *length* is interpreted with the help of the cross ratio in the well-known way.

The parallel translation of hyperbolic line segment Oz_1 along the hyperbolic line segment Oz_2 is the hyperbolic line segment z_2z_3, if the tangent vector of the hyperbolic line Oz_1 at O is parallel to the tangent vector of the hyperbolic line z_2z_3 at z_2, and the length of z_2z_3 is equal to the length of Oz_2.

We state

Theorem 2.2. *In the Poincaré model of the hyperbolic plane the parallel translations of the hyperbolic line segment Oz_1 along the Oz_2 hyperbolic line segment is the hyperbolic line segment z_2z_3 if*

$$z_3 = \frac{z_1 + z_2}{1 + z_1\bar{z}_2}$$

Proof. We have to prove that

a) the set $K = \{\frac{kz_1+z_2}{1+kz_1\bar{z}_2}, k \in R^{\infty}\}$ is a hyperbolic line i.e. K is a circle and orthogonal to the limit circle

b) Oz_1 is parallel to the tangent vector of K at z_2.

c) the length of Oz_1 is equal to the lenght of z_2z_3.

We know three points of $K : z_2(k = 0), z_2(k = 1)\frac{1}{\bar{z}_2}(k = \infty)K$ is a circle. It is enough to show that the imaginary part of the cross ratio of its four points is zero:

$$Im\left(z_2, z_3, \frac{1}{\bar{z}_2}, \frac{kz_1 + z_2}{1 + kz_1\bar{z}_2}\right) = 0$$

Computing the cross ratio as earlier:

$$(z_2, z_3, \frac{1}{\bar{z}_2}, \frac{kz_1 + z_2}{1 + kz_1\bar{z}_2}) = \frac{k-1}{1},$$

which is real. This completes the proof of a).

$$\lim_{k\to 0} \frac{\frac{kz_1+z_2}{1+kz_1\bar{z}_2} - z_2}{k} = z_1(1 - |z_2|^2) = cz_1(c \in \mathbb{R}),$$

which proves b).

For $k = \frac{1}{|z_1|}$ and $k = -\frac{1}{|z_1|}$ we get two points:

$$\frac{\pm\frac{1}{|z_1|}z_1 + z_2}{1 \pm \frac{z_1}{|z_1|}\bar{z_2}},$$

which are on the limit circle because

$$|\frac{\pm\frac{1}{|z_1|}z_1 + z_2}{1 \pm \frac{z_1}{|z_1|}\bar{z_2}}| = |\frac{1 \pm z_2\frac{|z_1|}{z_1}}{1 \pm \frac{z_1}{|z_1|}\bar{z_1}}|$$

which is true because the numerator and the denominator are complex conjugates.

The length of z_2z_3 is computed from the cross ratio

$$(\frac{\frac{1}{|z_1|}z_1 + z_2}{1 + \frac{z_1}{|z_1|}\bar{z_2}}, \frac{-\frac{1}{|z_1|}z_1 + z_2}{1 - \frac{z_1}{|z_1|}\bar{z_2}}, z_2, \frac{z_1 + z_2}{1 + z_1\bar{z_2}}).$$

The length of Oz_2 is computed from the cross ratio

$$(\frac{z_1}{|z_1|}, -\frac{z_1}{|z_1|}, 0, z_1).$$

Using the simplification method presented above, we get that both the cross ratios are $\frac{1+|z_1|}{1-|z_1|}$, so the lengths are equal.

This completes the proof.

UNIFIED FORMULA.

We can construct a unified formula using Theorem 2.1 and 2.2 for the surface $S : \kappa x^2 + \kappa y^2 + |\kappa|z^2 = 1/4(\kappa \in \mathbb{R}\kappa \neq 0)$ The surface S is equal the unit sphere if $\kappa = 1$ and it is equal to the hyperbolic plane if $\kappa = -1$. If κ is positive, we get spheres with diameter $1/\sqrt{-\kappa}$ if we use the norma $\|(x, y, z)\| = -x^2 - y^2 + z^2$. $\kappa/2$ has a geometric meaning and is called Gaussian curvature. The surfaces S contain all the types of complete surfaces with constant Gaussian curvature. For a surface with positive constant curvature, we can generalize the spherical model, projecting the sphere from its 'north pole' to the complex plain. Substituting κz instead of z on the model for $\kappa = 1$, we get the new model. We can do the same substitution for negative constant Gaussian curvature. In that case the radius of the limit circle is $\frac{1}{\sqrt{-\kappa}}$. The case $\kappa = 0$ can be interpreted as the Euclidean case. The Poincaré model of the Euclidean plane can be itself. We call *lines* the lines in the Euclidean plane, the hyperbolic lines in the hyperbolic plane and the great circles in the 'spherical plane'.

Theorem 2.3. *Let z_1, z_2 be points of the Poincaré model of the surface with its center in the origin with constant κ curvature. The parallel translation of the line segment Oz_1 along the line segment Oz_2 is the line segment z_2z_3 if*

$$z_3 = \frac{z_1 + z_2}{1 - \kappa z_1\bar{z_2}} \tag{2.1}$$

for any κ.

Proof. Substituting κz_i instead of $z_i (i = 1, 2, 3)$, we get the result.

3. BOL LOOPS ON THE COMPLEX PLANE

Definition 3.1. *Let* $\kappa \in R$ *be fixed. Let* z_1, z_2, z_3 *be points of the complex plane. If* κ *is nonnegative, then* z_1, z_2, z_3 *are arbitrary points, if* κ *is negative, then let* z_1, z_2, z_3 *be inside the* $2/\sqrt{-\kappa}$ *circle. We define the operation '*$\circ$*' with the formula (2.1) i.e.:*

$$z_3 = z_1 \circ z_2 = \frac{z_1 + z_2}{1 - \kappa z_1 \bar{z}_2}$$

Theorem 3.1. *If the formula (2.1) is defined, then the multiplication* $z_1 \circ z_2$ *satisfies the following identity:*

$$x \circ ((y \circ z) \circ y) = ((x \circ y) \circ z) \circ y \qquad (3.1)$$

Proof. We have to prove that for any x, y, z when (2.1) is defined, the following equation holds.

$$\frac{x + \frac{\frac{y+z}{1-\kappa y\bar{z}} + y}{1 - \kappa \frac{y+z}{1-\kappa y\bar{z}}\bar{y}}}{1 - \kappa x \frac{\frac{\bar{y}+\bar{z}}{1-\kappa\bar{y}z} + \bar{y}}{1-\kappa\frac{\bar{y}+\bar{z}}{1-\kappa\bar{y}z}y}} = \frac{\frac{\left(\frac{x+y}{1-\kappa x\bar{y}} + z\right)}{1-\kappa\frac{x+y}{1-\kappa x\bar{y}}\bar{z}} + y}{1 - \kappa \frac{\frac{x+y}{1-\kappa x\bar{y}} + z}{1-\kappa\frac{x+y}{1-\kappa x\bar{y}}\bar{z}}\bar{y}}$$

After simplification

$$\frac{x - \kappa \frac{x(y+z)}{1-\kappa y\bar{z}}\bar{y} + \frac{y+z}{1-\kappa y\bar{z}} + y}{1 - \kappa\frac{\bar{y}+\bar{z})}{\kappa\bar{y}z}y - \kappa\frac{x\bar{y}+\bar{z}}{1-\kappa\bar{y}z} - \kappa x\bar{y}} = \frac{\frac{x+y}{1-\kappa x\bar{y}} + z + y - \kappa\frac{y(x+y)\bar{z}}{1-\kappa x\bar{y}}}{1 - \kappa\frac{(x+y)\bar{z}}{1-\kappa x\bar{y}} - \kappa\frac{(x+y)\bar{y}}{1-\kappa x\bar{y}} - \kappa z\bar{y}}$$

Rearranging we have

$$\frac{x + y - \kappa xy\bar{z} - \kappa y^2\bar{z} + yx - \kappa x|y|^2 - \kappa x\bar{y}z}{1 - \kappa\bar{y}z - \kappa x\bar{y} - \kappa^2 x\bar{y}^2 z - 1\kappa(\bar{y} + \bar{z})(x + y)} =$$

$$= \frac{z + y - \kappa x\bar{y}z - \kappa x|y|^2 + x + y - \kappa xy\bar{z} - \kappa y^2\bar{z}}{1 - \kappa x\bar{y} - \kappa z\bar{y} + \kappa^2 x\bar{y}^2 z - \kappa(x + y)(\bar{z} + \bar{y})}$$

This equation obviously holds. We call property (3.1) the right Bol property and we call loops with this property right Bol loops.

Definition 3.2. *If* $z_3 = z_1 \circ z_2$ *where* $z_1 \circ z_2 = \frac{z_1+z_2}{1-\kappa z_1\bar{z}_2}(z_1, z_2 \in Z$ *if* $\kappa \geq 0$ *or* $|z_1|, |z_2| < \frac{1}{\sqrt{(-\kappa)}}$ *if* $\kappa < 0)$ *then the inverse operations can be expressed by:*

a) $z_1 = z_3 / z_2 = z_3 \circ (-z_2)$ (right inverse),

b) $z_2 = z_1 \backslash z_3 = z_3 \circ (\kappa z_3 |z_1|^2) - z_1 \circ (\kappa z_1 |z_3|^2)$ (left inverse)

c) $(z_1 \circ z_2)^{-1} = z_1^{-1} \circ z_2^{-1}$ (automorphic inverse property)

Proof.

a) is easy

b) is more complicated:

$$\frac{z_3 + \kappa z_3|z_1|^2}{1 - z_3\kappa^2\bar{z}_3|z_1|^2} - \frac{z_1 + \kappa z_1|z_3|^2}{1 - z_1\kappa^2\bar{z}_1|z_3|^2} =$$

$$= \frac{1}{1 - \kappa^2|z_1|^2|z_3|^2}(z_3(1 + \kappa|z_1|^2) - z_1(1 + \kappa|z_3|^2)) =$$

$$= \frac{1}{1 - \kappa|z_1|^2|z_3|^2}(z_3 - z_1 + \kappa z_1 z_3(\bar{z}_1 - \bar{z}_3)) =$$

$$= \frac{1}{1 - \kappa^2\frac{|z_1+z_2|^2 z_1|^2}{|1-\kappa z_1\bar{z}_2|^2}}\left(\frac{z_1 + z_2}{1 - \kappa z_1\bar{z}_2} - z_1 + \kappa z_1\frac{z_1 + z_2}{1 - \kappa z_1\bar{z}_2}\left(\bar{z}_1 - \frac{\bar{z}_1 + \bar{z}_2}{1 - \kappa\bar{z}_1 z_2}\right)\right) =$$

$$= \frac{(z_2 + \kappa z_1^2\bar{z}_2)(1 - \kappa\bar{z}_1 z_2) + \kappa z_1(z_1 + z_2)(-\kappa\bar{z_1}^2 z_2 - \bar{z}_2}{|1 - \kappa z_1\bar{z}_2|^2 - \kappa^2|z_1 + z_2|^2|z_1|^2} = z_2,$$

which completes the proof.

c) is obvious using $z^{-1} = -z$.

Remark. *As we have seen, this theorem enables us to interpret the inverses in a geometric way.*

4. COMPLETENESS

It is clear from the Poncaré model that Bol loops with negative κ are complete. In the lack of completeness in the case of the positive κ, we can only talk about partial Bol loops. Can we make these partial Bol loops complete continuously? The answer is in the following theorem.

Theorem 4.1. *Partial Bol loops of positive curvature cannot be made complete continuously adding new elements to the sets.*

Proof. The first case in when we add one element to the partial Bol loop to make it complete. Denote this element with v. $(1/\sqrt{\kappa}) \circ (1/\sqrt{\kappa}) = (i/\sqrt{\kappa}) \circ (i/\sqrt{\kappa}) = v$, because in the partial Bol-loop $(1/\sqrt{\kappa}) \circ (1/\sqrt{\kappa})$ and $(i/\sqrt{\kappa}) \circ (i/\sqrt{\kappa})$ cannot be defined. Using the continuity

$$\lim_{t\to 1}((t/\sqrt{\kappa}) \circ (t/\sqrt{\kappa})) = (1/\sqrt{\kappa}) \circ (1/\sqrt{\kappa}) \text{ and}$$

$$\lim_{t\to 1}((it/\sqrt{\kappa}) \circ (it/\sqrt{\kappa})) = (i/\sqrt{\kappa}) \circ (i/\sqrt{\kappa}).$$

What is the value of $1 \circ ((1/\sqrt{\kappa}) \circ (1/\sqrt{\kappa}))$?

$$1 \circ ((1/\sqrt{\kappa}) \circ (1/\sqrt{\kappa})) = \lim_{t\to 1}(1 \circ ((t/\sqrt{\kappa}) \circ (t/\sqrt{\kappa}))) = \lim_{t\to 1} 1 \circ \left(\frac{2t/\sqrt{\kappa}}{1 - t^2}\right) =$$

$$\lim_{t\to 1}\frac{1 + \frac{2t/\sqrt{\kappa}}{1-t^2}}{1 - \kappa\frac{2t/\sqrt{\kappa}}{1-t^2}} = \lim_{t\to 1}\frac{1 - t^2 + 2t/\sqrt{\kappa}}{1 - t^2 - 2t\sqrt{\kappa}} = -1/\kappa$$

What is the value of $1 \circ ((i / \sqrt{\kappa}) \circ (i / \sqrt{\kappa}))$?

$$1 \circ ((1 / \sqrt{\kappa}) \circ (1 / \sqrt{\kappa})) = \lim_{t \to 1}(1 \circ ((ti / \sqrt{\kappa}) \circ (tu / \sqrt{\kappa}))) =$$

$$= \lim_{t \to 1}\left(1 \circ \left(\frac{2ti / \sqrt{\kappa}}{1-t^2}\right)\right) = \lim_{t \to 1} \frac{1 + \frac{2ti / \sqrt{\kappa}}{1-t^2}}{1 + \kappa \frac{2ti / \sqrt{\kappa}}{1-t^2}} = \lim_{t \to 1} \frac{1 - t^2 + 2ti / \sqrt{\kappa}}{1 - t^2 + 2ti\sqrt{\kappa}} = 1 / \kappa,$$

which is a contradiction. So we cannot make the Bol loop complete with one element.

The second case is when we add at least two elements to the set to make it complete. In this case $(1 / \kappa) \circ (1 / \kappa)$ and $(i / \kappa) \circ (i / \kappa)$ have to be different elements. Denote them by v_1 and v_2. Using the continuity

$$v_1 \circ v_1 = \lim_{t \to 1}(((t / \sqrt{\kappa}) \circ (t / \sqrt{\kappa})) \circ ((t / \sqrt{\kappa}) \circ (t / \sqrt{\kappa}))) =$$

$$= \lim_{t \to 1} \frac{\frac{4t / \sqrt{\kappa}}{1-t^2}}{1 - \kappa \frac{2t / \sqrt{\kappa}}{1-t^2} \frac{2t / \sqrt{\kappa}}{1-t^2}} = 0$$

$$v_1 \circ v_2 = \lim_{t \to 1}(((t / \sqrt{\kappa}) \circ (t / \sqrt{\kappa})) \circ ((ti / \sqrt{\kappa}) \circ (ti / \sqrt{\kappa}))) =$$

$$= \lim_{t \to 1} \frac{\frac{2t / \sqrt{\kappa}}{1-t^2} + \frac{2ti / \sqrt{\kappa}}{1-t^2}}{1 + \kappa \frac{2t / \sqrt{\kappa}}{1-t^2} \frac{2ti / \sqrt{\kappa}}{1-t^2}} = 0$$

Therefore $v_1 \circ v_1 = v_1 \circ v_2 = 0$, which contradicts the cancellativity. So we cannot make the Bol loop complete.

Remark. *In the proof we did not use the Bol loop property so we can say that a partial Bol loop cannot be made complete even as a loop.*

5. FURTHER GENERALIZATION

For $\kappa = -1$, we have a Bol loop on the circle $|z| < 1$. The function $f : z \to \frac{z}{1+|z|}$ is a $1-1$ map of C to the circle $|z| < 1$. Substituting it into z_1 and z_2 in $z_1 \circ z_2 = \frac{(z_1+z_2)}{1+z_1\bar{z}_2}$ we get:

$$z_1 \circ z_2 = f^{-1}\left(\frac{f(z_1) + f(z_2)}{1 + f(z_1) f(\bar{z}_2)}\right)$$

which is defined for the whole C.

From this formula we can get other Bol loops. Let us take $z_1, z_2 \in Q[\sqrt{2}] = \{p + q\sqrt{2} p, q \in Q\}$ and $\bar{z}_1 := p - q\sqrt{2}$ if $z_1 = p\sqrt{2}$. (Previously we only used that $x \to \bar{x}$ is an involution). So we have defined a discrete Bol loop on this set. It is obvious that the operation '$\circ$' preserves the set.

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Received April 25, 1995
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