INFINITE GROUPS SATISFYING A NORMALIZER CONDITION

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Summary. In this article infinite groups $G$ are studied with the property that if $H$ is a non-normal subgroup of $G$ then every normal subgroup of $H$ is normal in the normalizer $N_G(H)$.

1. INTRODUCTION

A subgroup $H$ of a group $G$ is said to satisfy the lower $N/C$-extremal condition if every normal subgroup of $H$ is also normal in the normalizer $N_G(H)$ of $H$. It is clear that a group $G$ is a $T$-group (i.e., a group in which normality is a transitive relation) if and only if all its normal subgroups satisfy the lower $N/C$-extremal condition. In particular, $G$ is a $T$-group (i.e., a group in which all subgroups are $T$-groups) if and only if every subgroup of $G$ satisfies the lower $N/C$-extremal condition.

Let $X$ be the class of groups in which every non-normal subgroup is $N/C$-low. The investigation of the structure of $X$-groups was started in [3] and [4]; the results obtained there mostly concern the case of finite groups. In particular, it was proved that every finite $X$-group is soluble with derived length at most 3. On the other hand, the consideration of Tarski groups shows that arbitrary $X$-group need not be soluble. Here we shall consider infinite soluble $X$-groups, and in particular we shall prove that soluble $X$-groups have derived length at most 4. A well-known result of Robinson [6] states that a finitely generated soluble $T$-group either is finite or abelian. The situation is completely different in the case of soluble $X$-groups: the direct product $Z \times S_3$ is an infinite finitely generated soluble $X$-group. On the other hand, we shall prove that the elements of finite order of a soluble $X$-group form a subgroup, and that the torsion-free soluble $X$-groups are abelian.

For our considerations it will be useful to observe that $X$ is contained in the class $B_2$ of groups in which every subnormal subgroup has defect at most 2. The structure of $B_2$-groups (and more generally of groups in which subnormal subgroups have bounded defect) has been investigated by many authors. In particular, Casolo [1], [2] has proved that finite (respectively: periodic) soluble $B_2$-groups have derived length at most 5 (respectively: at most 10), while Mahdavianary [5] showed that nilpotent $B_2$-groups have class at most 3 (and so they are metabelian).

Most of our notation is standard and can for instance be found in [7].

2. STATEMENTS AND PROOFS

It is clear that subgroups and homomorphic images of $X$-groups are likewise $X$-groups. Our first lemma deals with centralizers of elements of infinite order of an $X$-group.

Lemma 2.1. Let $G$ be an $X$-group, and let $x$ be an element of infinite order of $G$. Then $N_G(\langle x \rangle) = C_G(x)$

Proof. Assume that $G$ contains an element $a$ such that $\langle x \rangle^a = \langle x \rangle$ but $xa \neq ax$. Then $a$ acts as
the inversion on $x$, and so $\langle a, x \rangle$ has a quotient isomorphic to the infinite dihedral group $D_\infty$, a contradiction since $D_\infty$ is not an $X$-group.

**Lemma 2.2.** Let $G$ be a torsion-free nilpotent $X$-group. Then $G$ is abelian.

**Proof.** Assume that $G$ is not abelian, and let $x$ be an element of $G$. Then the normalizer $N_G(\langle x \rangle)$ is subnormal in $G$, and so even normal, since $G$ is an $X$-group. Then $C_G(x)$ is a normal subgroup of $G$ by Lemma 2.1, and hence the identity $[y, x, x] = 1$ holds in $G$. It follows that $G$ has class at most 2 (see [7], 7.14). Without loss of generality it can be assumed that $G = \langle a, b \rangle$, where $[a, b] \neq 1$. Let $m, n$ be coprime integers $> 1$. Since $[a^m, b^n] \neq 1$, it follows from Lemma 2.1 that $b^n$ does not normalize $\langle a^m \rangle$. On the other hand,

$$\langle a^m \rangle \triangleleft \langle a^m, [a^m, b^n] \rangle \triangleleft \langle a^m, b^n \rangle,$$

and hence $\langle a^m, [a^m, b^n] \rangle$ is a normal subgroup of the $X$-group $G$. Similarly $\langle b^n, [a^m, b^n] \rangle$ is normal in $G$, and so also $\langle a^m, b^n \rangle$ is a normal subgroup of $G$. Clearly the factor group $G/\langle a^m, b^n \rangle$ is abelian, so that $[a, b] = a^{mh}b^{nk}[a, b]^{ml}$, where $h, k, l$, are integers. It follows that $a^{mh}b^{nk}$ belongs to the centre of $G$, and hence in particular $1 = [a^m, b^{nk}] = [a, b]^{mnk}$. Then $k = 0$, and similary $h = 0$, so that $[a, b] = [a, b]^{ml}$. Therefore $[a, b] = 1$, and this contradiction proves the lemma.

**Corollary 2.3.** Let $G$ be a locally nilpotent $X$-group. Then the commutator subgroup $G'$ is a periodic abelian group.

**Proof.** Clearly it can be assumed that $G$ is finitely generated, and so nilpotent. Let $T$ be the subgroup of all elements of finite order of $G$. Then $G/T$ is abelian by Lemma 2.2, so that $G' \leq T$, and $G'$ is periodic. Moreover $G'$ is abelian by the quoted result of Mahdavianary [5].

**Lemma 2.4.** Let $G$ be an $X$-group containing an abelian normal subgroup $A$ such that $G/A$ is finite cyclic. Then the commutator subgroup $G'$ of $G$ is periodic.

**Proof.** Without loss of generality it can be assumed that $G$ is finitely generated and has no periodic non-trivial normal subgroups. Then $A$ is a free abelian group of finite rank. Let $G$ be a counterexample with $G/A$ of minimal order, so that in particular $A$ is a maximal abelian normal subgroup of $G$. Let $x$ be an element of $G$ such that $G = \langle x, A \rangle$, and let $p$ be a prime dividing the order of $G/A$. Then $\langle x^p, A \rangle$ is a proper subgroup of $G$, and so $\langle x^p, A \rangle'$ is periodic. Since $\langle x^p, A \rangle'$ is normal in $G$, it follows that $\langle x^p, A \rangle' = 1$. Then $\langle x^p, A \rangle$ is abelian, and hence $\langle x^p, A \rangle = A$. Therefore $x^p \in A$ and $G/A$ has order $p$. For each positive integer $n$, the finite $p$-group $G/A^{p^n}$ belongs to $X$, and so it is a nilpotent $B_2$-group. Then $G/A^{p^n}$ has class at most 3 (see [5]), and so $\gamma_4(G) \subseteq \cap_{A^{p^n}} A^{p^n} = 1$. Then $G$ is a torsion-free nilpotent $X$-group, and Lemma 2.2 yields that $G$ is abelian, a contradiction.

It is now possible to prove that the elements of finite order of a locally soluble $X$-group form a subgroup.

**Proposition 2.5.** Let $G$ be a locally soluble $X$-group. Then the set of all elements of finite order of $G$ is a subgroup.

**Proof.** Let $x$ and $y$ be elements of finite order of $G$. Without loss of generality it can be assumed that $G = \langle x, y \rangle$, so that in particular $G$ is soluble. Let $N$ be the smallest non-trivial
term of the derived series of $G$. By induction on the derived length of $G$ we obtain that $G/N$ is finite, so that also $N$ is finitely generated. Let $a$ be an element of $N$, and consider the subgroups $H = \langle a \rangle^G(x)$ and $K = \langle a \rangle^G(y)$. Then $H'$ and $K'$ are periodic by Lemma 2.4, and there exists a positive integer $m$ such that $[a, x]^m = [a, y]^m = 1$. It follows that $[a^m, x] = [a^m, y] = 1$, so that $a^m \in Z(G)$. Therefore $G/Z(G)$ is periodic, and hence finite, so that also $G'$ is finite (see [7], 4.12). It follows that $G$ is finite.

**Lemma 2.6.** Let $p$ be a prime, and let $(x)$ be a cyclic $p$-group. If $y$ is an automorphism of order $p^k$ of $(x)$ such that the semidirect product $G = \langle y \rangle \rtimes (x)$ is an $X$-group, then $n \leq 1$.

**Proof.** Let $p^m$ be the order of $x$, and assume that $n \geq 2$. Then $x^p = x^{1+p^s}$, where $p$ does not divide $s$ and $t \leq m - 2$. Put $k = m - 1 - t$, and consider the non-normal subgroup $H = \langle x^{p^{n-1}} \rangle \times (y)$ and $x^p$ normalizes $H$, a contradiction, since $G$ is an $X$-group and $[x^p, y] \neq 1$.

**Lemma 2.7.** Let $A$ be a reduced torsion-free abelian group, and let $\sigma$ be a non-trivial automorphism of $A$. Then the semidirect product $G = \langle \sigma \rangle \rtimes A$ is not an $X$-group.

**Proof.** Assume that $G$ is an $X$-group, and let $a$ be an element of $A$ such that $a^\sigma \neq a$. Since $H = \langle a \rangle^G$ is also an $X$-group, and $(a)^H = \langle a \rangle^G$, it can be assumed without loss of generality that $A = \langle a \rangle^G$. The automorphism $\sigma$ has infinite order by Lemma 2.4. For every integer $i$ put $a_1 = a^{i\sigma}$, so that $A = \langle a_1 | i \in kZ \rangle$. Let $k$ be a positive integer, and assume that $A_k = \langle a_1 | i \in kZ \rangle$ is properly contained in $A$. As $a = a_0 \in A_k$, the subgroup $A_k$ is not normal in $G$, and so $A$ is normal in $N_G(A_k)$. Clearly $\sigma^k$ fixes $a$ by Lemma 2.1. Then $\sigma^k = 1$, a contradiction. Therefore $A = A_k$ for every $k \geq 1$. In particular, the set $\{A_i | i \in Z \}$ is dependent, and there exist integers $r$ and $s$, with $r < s$ such that $\{a_r, \ldots, a_s \}$ is independent and $\{a_r, \ldots, a_{s+1} \}$ is dependent. Thus $a_{s+1} = a_{m_1} a_{m_2} \ldots a_{m_r}$, where $m_1, m_2, \ldots, m_r$ are integer and $m \neq 0$. Let $D$ be the divisible hull of $A$, and let $D_0$ be the smallest divisible subgroup of $D$ containing $\{a_r, \ldots, a_s \}$. Then $\sigma$ can be extended to an automorphism $\tau$ of $D$. Since $a_{s+1} \in D_0$, we obtain $\langle a_r, \ldots, a_s \rangle^{\tau} \leq D_0$. Moreover, $D_0$ has the same rank of $\langle a_r, \ldots, a_s \rangle$, so that $D_0 / \langle a_r, \ldots, a_s \rangle$ is periodic and $D^\tau_0 \leq D_0$, since $D / D_0$ is torsion-free. It follows that $D^\tau_0 = D_0$, so that $A_0 = A \cap D_0$ is a subgroup of finite rank of $A$ containing $\langle a_r, \ldots, a_s \rangle$, and $A^\tau_0 = A_0$. Clearly $a = a^\sigma r^{-1} A_0$, so that $A = A_0$ and $A$ has finite rank. Thus the counterexample $G$ can be chosen in such a way that $A$ has minimal rank. As $A$ is reduced, there exists a prime $p$ such that $A^p \neq A$. Let $k$ be the order of the automorphism induced by $\sigma$ on the finite group $A / A^p$. If $i \in kZ$, we obtain that $a_i A^p = A A^p$. Since $A = A_k = \langle a_1 | i \in kZ \rangle$, it follows that $A / A^p$ is cyclic. Then $A / A^p$ is cyclic of order $p^n$ for every $n \geq 0$. Application of Lemma 2.6 yields that the automorphism induced by $\sigma$ on $A / A^p$ has order dividing $p(p - 1)$. Therefore $\sigma^{p(p - 1)}$ acts trivially on $A / A^p$ for each $n \geq 0$. Put $B = \bigcap_{n \geq 0} A^p$, so that $\langle A, A^{p(p - 1)} \rangle \leq B$. Clearly $[A, A^{p(p - 1)}] \neq 1$, and hence it follows from Lemma 2.2 that $\sigma^{p(p - 1)}$ does not act trivially on $B$. Moreover, $A / B$ does not have finite exponent, so that $\langle a \rangle \cap B = 1$, and $B$ has rank less than $A$. By the minimal choice of $A$, we obtain that the subgroup $\langle \sigma^{p(p - 1)} B \rangle$ does not belong to $X$. This contradiction proves the lemma.

We can now prove our main result.

**Theorem 2.8.** Let $G$ be a torsion-free locally soluble $X$-group. Then $G$ is abelian.
Proof. Clearly it can be assumed that $G$ is finitely generated, and hence soluble. Thus by induction the derived length of $G$ we may also suppose that the commutator subgroup $G'$ is abelian. As a finitely generated metabelian group, it is well-know that $G$ is residually finite (see [8], 9.51) and so in particular reduced. Assume that $G$ is not abelian, so that $G$ is not nilpotent by Lemma 2.2, and $C = C_G(G')$ is properly contained in $G$. Let $x$ be an element of $G'$ and $y$ an element of $G \setminus C$ such that $[x, y] \neq 1$, and consider the subgroup $H = \langle x, y \rangle$. Clearly $\langle x \rangle^H$ is contained in $G'$, and so is abelian. If $\langle x \rangle^H \cap \langle y \rangle \neq 1$, then $H / \langle x \rangle^H$ is a finite cyclic group, and $H$ is abelian by Lemma 2.4. This contradiction shows that $\langle x \rangle^H \cap \langle y \rangle = 1$, and hence Lemma 2.7 can be applied to prove that the factor group $H / C_G(\langle x \rangle^H)$ does not belong to $X$. This last contradiction completes the proof.

The above theorem has the following consequence.

Corollary 2.9. Let $G$ be a locally soluble $X$-group. Then the commutator subgroup $G'$ of $G$ is periodic, and $G$ is soluble with derived length at most 4.

Proof. The set $T$ of all elements of finite order of $G$ is a subgroup by Proposition 2.5, and it follows from Theorem 2.8 that the factor group $G / T$ is abelian. Thus $G'$ is periodic, and hence locally finite. Application of Theorem 3.4 of [3] yields now that $G'^4 = 1$.

We leave as an open question whether there exist soluble $X$-groups with derived length 4.
REFERENCES


