AN OSCILLATION THEOREM FOR SCHRÖDINGER EQUATIONS

ADRIAN CONSTANTIN

Abstract. *We consider the problem of characterization of oscillatory semilinear Schrödinger equations in exterior domains.*

KEY WORDS: *Schrödinger equation; Oscillatory solution.*

1. We consider the semilinear Schrödinger equation

$$Lu = \Delta u + f(x,u) = 0, x \in \Omega, \tag{1}$$

in an exterior domain $\Sigma \subset R^n, n \geq 3$, where f is nonnegative and locally Hölder continuous in $\Sigma \times R$ and odd in u, i.e. $f(x,-u) = -f(x,u)$.

Let $|x|$ denote the Euclidean norm of $x = (x_1, x_2, \ldots, x_n) \in R^n$ and for $a > 0$, let

$$\begin{aligned} S_a &= \{x \in R^n : |x| = a\} \\ G_a &= \{x \in R^n : |x| > a\}. \end{aligned}$$

We say that $\Sigma \subset R^n$ is an exterior domain if $G_a \subset \Sigma$ for some $a > 0$.

We introduce the class $\Re$ of nondecreasing functions $w \in C^1(R_+, R_+)$ with $w(t) > 0$ for $t > 0$ satisfying $\int_1^\infty \frac{dt}{w(t)} = \infty$ and $\lim_{t\to\infty} w(t) = \infty$.

Equation (1) is considered in an exterior domain $\Sigma \subset R^n$ subject to the assumptions:

$(A) f \in C^{\lambda}_{loc}(\Sigma \times R)$ for some $\lambda \in (0,1)$ (local Hölder continuous);

$(B) 0 \leq f(x,t) \leq \alpha(|x|) w_0(t)$ for all $x \in \Sigma$ and for all $t > 0$ for some $\alpha \in C(R_+, R_+)$ and $w_0 \in \Re$ with $w_0(0) = 0$;

$(C) p(x)\varphi(t) \leq f(x,t)$ for all $x \in \Sigma$ and for all $t \geq 0$, where p is continuous and nonnegative in Σ; $\varphi \in C^1(R_+)$, $\varphi(t) > 0$, $\varphi'(t) > 0$ for $t > 0$ and $\lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{dt}{\varphi(t)} < \infty$ (the last condition is a sublinear condition on φ).

A solution of (1) in Σ is a function $u \in C^2(\Sigma)$ such that $Lu(x) = 0$ for all $x \in \Sigma$. We say that the operator L given by (1) is *oscillatory* in Σ whenever every solution defined in $G_a \subset \Sigma$ for some $a > 0$ changes sign in G_r for all $r \geq a$. Observe that if $v(x)$ is a solution of (1) then $-v(x)$ is also a solution. Thus L is nonoscillatory in Σ if and only if (1) has a solution $u(x)$ which is positive in G_b for some $b \geq a$.

We intend to give conditions on p and g that guarantee that (1) is an oscillatory equation.

2. In the sequel we will need the following

Lemma 2.1. *[3] Let L be the operator defined by (1) where f is nonnegative for $u \geq 0$ and satisfies assumption (A) in an exterior domain Σ and suppose that $G_a \subset \Sigma$ for some $a > 0$. If there exists a positive solution v_1 and a nonnegative solution v_2 of $Lv_1 \leq 0$ and $Lv_2 \geq 0$, respectively, in G_a such that $v_2(x) \leq v_1(x)$ throughout $G_a \cup S_a$, then equation (1) has at least one solution $u(x)$ satisfying $u(x) = v_1(x)$ on S_a and $v_2(x) \leq u(x) \leq v_1(x)$ throughout G_a.*

Consider now the differential equation

$$u'' + F(t,u) = 0 \tag{2}$$

where $F(t,u)$ is continuous on $\{(t,u) : t \geq 1, u \in R\}$.

Lemma 2.2. *Assume that*

$$F(t,u) = h(t)w_0\left(\frac{u}{t}\right), t \geq 1, u \in R,$$

where $h \in C(R_+, R_+)$ satisfies $\int_1^\infty h(s)ds < \infty$ and $w_0 \in C^1(R,R)$ is odd on R, nonnegative on R_+ and such that $|w_0| \in \Re$.

Then equation (2) has a solution $u(t)$ which is positive in (b,∞) for some $b \geq 1$.

Proof. Under the hypotheses of Lemma 2.2 we know (see [1]) that for every solution $u(t)$ of (2) there are real constants c, d such that $u(t) = ct + d + o(t)$ as $t \to \infty$.

In view of the fact that w_0 is odd on R, it is sufficient to show that (2) has a solution $u(t)$ which is of constant sign in (b,∞) for some $b \geq 1$. We will actually prove that any nontrivial solution $u(t)$ of (2) is positive or negative in (b,∞) for some $b \geq 1$.

Assume that $u(t)$ is a nontrivial solution of (2) which has infinitely many zeros $\{t_n\}_{n\geq 1}$ with $t_n \to \infty$ as $n \to \infty$. Then $c = d = 0$. Taking into account (see [1]) that $c = \lim_{t\to\infty} u'(t)$, this can happen only if $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$. For convenience we consider $\{t_n\}_{n\geq 1}$ strictly increasing.

Let $M = \sup_{t\geq 1}\{|u(t)|\} > 0$. Denote $Q = \sup_{|u|\leq M}\{|w_0'(u)|\} > 0$ and observe by the mean-value theorem ($w_0(0) = 0$) that $|w_0(u)| \leq Q|u|$ for $|u| \leq M$.

Let $t_k > 1$ be a root of $u(t)$ such that $\int_{t_k}^\infty h(s)ds < \frac{1}{Q}$. Since $w_0 \in C^1(R,R)$ we have local uniqueness for the solutions of (2) and so, since $u(t_k) = 0$ and $u(t)$ is nontrivial for $t \geq t_k$, we have $|u'(t_k)| > 0$ ($u'(t_k) = 0$ would imply $u(t) = 0$ for $t \geq t_k$). The relation $\lim_{t\to\infty} u'(t) = 0$ enables us to find a root $t_n > t_k$ of $u(t)$ with $|u'(t)| < \frac{1}{2}|u'(t_k)|$ for $t \geq t_n$. Let T be a point in $[t_k, t_n]$ where $|u'(t)|$ attains its maximal value on this interval. Clearly $|u'(T)| \geq |u'(t_k)| > 0$ and $|u'(t)| \leq |u'(T)|$ for $t_k \leq t$.

For $s \geq T$ observe that, using the mean-value theorem and the fact that $T \geq t_k$,

$$|u(s)| = |u(s) - u(t_k)| \leq (s - t_k)|u'(T)|$$

so that

$$\frac{|u(s)|}{s} \leq \min\{M, |u'(T)|\}, s \geq T.$$

Integrating (2) on $[T,t]$ we get

$$u'(t) - u'(T) + \int_T^t h(s)w_0\left(\frac{u(s)}{s}\right) ds = 0, t \geq T,$$

thus

$$|u'(T)| \leq |u'(t)| + \int_T^\infty h(s)w_0\left(\frac{|u(s)|}{s}\right) ds, t \geq T.$$

Since $\lim_{t\to\infty} u'(t) = \infty$, in view of the previous remarks, we can write

$$|u'(T)| \le \int_T^\infty h(s) w_0 \left(\frac{|u(s)|}{s}\right) ds \le Q \int_T^\infty h(s) \frac{|u(s)|}{s} ds \le$$

$$\le Q|u'(T)| \int_T^\infty h(s) ds \le Q|u'(T)| \int_{t_k}^\infty h(s) ds < |u'(T)|,$$

a contradiction that concludes the proof. □

Lemma 2.3. *Assume that $F \in C(R_+, R_+)$ is such that $F(t) > 0$ for $t > 0$ and*

$$\int_1^\infty \frac{dt}{F(t)} = \infty.$$

If $G \in C(R_+, R_+)$ is such that for some $w \in \Re$ and some constant $M > 1$,

$$G(t) \le F(t) w \left(\int_1^t \frac{dt}{F(t)}\right), t \ge M,$$

then

$$\int_1^\infty \frac{dt}{F(t) + G(t)} = \infty.$$

Proof. Let us denote

$$V(t) = \int_1^t \frac{ds}{F(s)}, W(t) = \int_M^t \frac{ds}{F(s) + G(s)}, t \ge M.$$

We have that

$$W'(t) = \frac{1}{F(t) + G(t)} \ge \frac{\frac{1}{F(t)}}{1 + w\left(\int_1^t \frac{dt}{F(t)}\right)}, t \ge M,$$

and an integration yields

$$W(t) \ge \int_{V(M)}^{V(t)} \frac{ds}{1 + w(s)}, t \ge M. \tag{3}$$

Since w is nondecreasing it is easy to see that $w \in \Re$ implies $\int_1^\infty \frac{ds}{1+w(s)} = \infty$ and since $\lim_{t\to\infty} V(t) = \infty$, by (3) we get $\lim_{t\to\infty} W(t) = \infty$. □

The spherical mean $m(r, u)$ of a continuous function $u : R^n \to R$ over the sphere S_r of radius r is defined by (see [4])

$$m(r, u) = \frac{1}{\omega(S_r)} \int_{S_r} u(x) d\omega$$

where ω denotes the measure on S_r.

Theorem. *Assume that* $(A), (B), (C)$ *hold and that there is an* $M > 1$ *such that for some* $w \in \Re$,

$$m(r,p) \geq \frac{\alpha(r)}{w\left(\int_1^r s\alpha(s)ds\right)}, r \geq M. \tag{4}$$

The necessary and sufficient condition for (1) *to be oscillatory in an exterior domain in* $R^n, n \geq 3$, *is*

$$\int_0^\infty rm(r,p)dr = \infty.$$

Proof of Sufficiency. As noted in Section 1, the operator L is nonoscillatory in Σ whenever (1) has a positive solution $u(x)$ in G_b for some $b \geq a$.

Assume that there is a positive solution $u(x)$ in G_b for some $b \geq a$.

An easy calculation shows that if we denote

$$\Phi(u) = \int_0^u \frac{dt}{\varphi(t)}, u > 0,$$

(well-defined in view of (C)), then

$$\Delta\Phi(u) = \frac{\Delta u}{\varphi(u)} - \varphi'(u)|\nabla\Phi(u)|^2$$

from which, in view of (1) and assumption (C), we get

$$\Delta\Phi(u) \leq -p - \varphi'(u)|\nabla\Phi(u)|^2$$

and so

$$-\Delta\Phi(u(x)) \geq p(x), x \in G_b. \tag{5}$$

The spherical mean of any function $z \in C^2(G_b)$ satisfies (see [2, page 69]),

$$\frac{d}{dr}\left[r^{n-1}\frac{dm(r,z)}{dr}\right] = \frac{r^{n-1}}{\omega(S_1)}\int_{S_1} \Delta z(x)d\omega$$

so that, on the basis of (5),

$$-\frac{d}{dr}\left[r^{n-1}\frac{dm(r,\Phi(u))}{dr}\right] \geq r^{n-1}m(r,p). \tag{6}$$

The change of variables

$$r = \beta(s) = \left(\frac{1}{n-2}\right)^{\frac{1}{n-2}}, h(s) = sm(\beta(s),\Phi(u))$$

transforms (6) into

$$-h''(s) \geq s^{-3}[\beta(s)]^{2n-2}m(\beta(s),p) = \frac{1}{n-2}\beta'(s)\beta(s)m(\beta(s),p). \tag{7}$$

Integration over (B,s) where $B=\beta^{-1}(b), s=\beta^{-1}(r)$, yields

$$-h'(s)+h'(B)\geq \frac{1}{n-2}\int_b^r tm(t,p)dt. \tag{8}$$

Observe that $h(s)>0$ for $s>B$ and $h'(s)$ is nonincreasing on $[B,\infty)$ by (7). This shows that $h'(s)\geq 0$ on $[B,\infty)$ - otherwise, there is a $C\geq B$ with $h'(C)<0$ and we get by the mean-value theorem and the monotonicity of h' that $-h(C)\leq h(s)-h(C)\leq h'(C)(s-C)\to -\infty$ as $s\to\infty$, impossible. By (8) we get

$$\int_b^\infty rm(r,p)dr\leq (n-2)h'(B)<\infty$$

and so

$$\int_0^\infty rm(r,p)dr<\infty$$

if there is a positive solution in G_b for some $b\geq a>0$. This shows that the condition $\int_0^\infty rm(r,p)dr=\infty$ is a sufficient condition for (1) to be oscillatory.

Proof of Necessity. It is enough to prove that if

$$\int_0^\infty rm(r,p)dr<\infty \tag{9}$$

then (1) has a positive solution in G_b for some $b\geq a>0$.

We show that if (9) holds, then

$$\int_0^\infty r\alpha(r)dr<\infty \tag{10}$$

and that (10) implies the existence of a positive solution of (1) in G_b for some $b\geq a>0$.

Let us assume that (9) holds and that $\int_0^\infty r\alpha(r)dr=\infty$.

Observe that there is a constant $K\geq 1$ so that $p(x)\leq K\alpha(|x|)$ for all $x\in\Sigma$ (we can take $K=1+\frac{w(1)}{\varphi(1)}$) and this shows that $m(r,p)\leq K\alpha(r)$ for $|x|=r>0$.

Define

$$F(r)=\frac{1}{Kr\alpha(r)+\frac{Kr}{(r+1)^3}},\ F(r)+G(r)=\frac{1}{rm(r,p)+\frac{Kr}{(r+1)^3}},\ r>a,$$

and extend F,G to $[0,a]$ so as to make them continuous and positive on R_+.

An easy computation shows that (4) implies

$$G(r)\leq KF(r)w\left(\int_1^r\frac{ds}{F(s)}\right),\ r\geq M+a,$$

thus, by Lemma 2.3 (since $Kw\in\Re$), the assumption $\int_1^\infty\frac{ds}{F(s)}=\infty$ implies $\int_1^\infty\frac{ds}{F(s)+G(s)}=\infty$. Since $\int_0^\infty\frac{rdr}{(r+1)^3}<\infty$ we get $\int_0^\infty rm(r,p)dr=\infty$, a contradiction with (9). This proves that if (9) holds, we have

$$\int_0^\infty r\alpha(r)dr<\infty.$$

We consider the ordinary differential equation

$$\frac{d}{dr}\left\{r^{n-1}\frac{dy}{dr}\right\}+r^{n-1}\alpha(r)w_0(y)=0, \tag{11}$$

where we define $w_0(y)=-w_0(-y)$ for $y<0$ (we can do this since $w_0(0)=0$). The so-defined $w_0\in C(R,R)$ is continuously differentiable on R as one can easily check.

The change of variables

$$r=\beta(s)=\left(\frac{1}{n-2}s\right)^{\frac{1}{n-2}},h(s)=sy(\beta(s))$$

transforms (11) into

$$h''(s)+\beta'(s)\frac{\beta(s)}{n-2}\alpha(\beta(s))w_0\left(\frac{h(s)}{s}\right)=0. \tag{12}$$

By Lemma 2.2, (12) has a positive solution in some interval (B,∞) with $b=s^{-1}(B)>a$ if

$$\int_0^\infty \beta'(s)\beta(s)\alpha(\beta(s))ds=\int_0^\infty r\alpha(r)dr<\infty.$$

Returning to (11), we have that if $\int_0^\infty r\alpha(r)dr<\infty$ then there is a positive solution $y(r)$ of (11) for all $r\geq b\geq a>0$. Using Lemma 2.1 we will show that this yields a solution of (1) which is positive in G_b.

Let us define $v_1(x)=y(r),r=|x|\geq b$. We have

$$r^{n-1}Lv_1(x)=\frac{d}{dr}\left\{r^{n-1}\frac{dy}{dr}\right\}+r^{n-1}f(x,v_1(x))\leq$$

$$\leq\frac{d}{dr}\left\{r^{n-1}\frac{dy}{dr}\right\}+r^{n-1}\alpha(r)w_0(y(r))$$

and hence $Lv_1(x)\leq 0$ for all $x\in G_b$. Clearly $v_2(x)=0$ satisfies $Lv_2(x)\geq 0$ in G_b. Lemma 2.1 shows that (1) has a solution $u(x)$ with $0\leq u(x)\leq v_1(x)=y(r)$ for $|x|\geq b$ with $u(x)=v_1(x)>0$ for $|x|=b$. Since $u(x)\geq 0$ for $|x|=c>b$, by the maximum principle ($\Delta u(x)\leq 0$ in $\{x\in R^n: b<|x|<c\}$) we get that $u(x)>0$ for $b<|x|<c$. The arbitrariness of $c>b$ shows that $u(x)$ is a positive solution of (1) in G_b.

3. To compare our theorem with the results of Swanson [5] observe that if we consider (1) with

$$f(t,x)=\frac{1}{(1+|x|^2}tln(1+t),t\geq 0,x\in R^3, \tag{13}$$

we can deduce by our theorem that (1) is oscillatory whereas the results of Swanson [5] are powerless. This shows that our condition (B) allows sometimes a higher degree of liberty than in the case of [5].

The main difference lies however in condition (4). To make this clear, assume that $tg(r,t) = \alpha(r)w_0(t)$ where g is nonincreasing in $t>0$ for every fixed $r>0$. As observed before,

$$p(x) \le K\alpha(r), |x| = r > 0,$$

for some constant $K \ge 1$. In [5] one works under the limited assumption

$$\limsup_{r\to\infty} \frac{\alpha(r)}{m(r,p)} < \infty$$

that is, for r large enough,

$$0 < K_1 \le \frac{m(r,p)}{\alpha(r)} \le K.$$

As we said before, the right-hand side bound is natural. It appears that the left-hand side bound is very restrictive. Observe that in our theorem we allow in the oscillatory case $\lim_{r\to\infty} \frac{m(r,p)}{\alpha(r)} = 0$ controlling the way it goes to zero (slower than $\frac{1}{w\left(\int_1^r s\alpha(s)ds\right)}$) - in the oscillatory case $\int_0^\infty s\alpha(s)ds = \infty$ so that $\frac{1}{w\left(\int_1^r s\alpha(s)ds\right)}$ goes to zero as $r \to \infty$.

This improvement becomes clear when one specializes (1) to the equation

$$\Delta u + p(x)|u|^\gamma sgnu = 0, 0 < \gamma < 1, x \in \Sigma, \tag{14}$$

where $p(x)$ is non negative and locally Hölder continuous in an exterior domain $\Sigma \subset \mathbb{R}^n$, $n \ge 3$. Let

$$P(r) = \sup_{|x|=r} \{p(x)\}.$$

The results of [5] enable us to conclude that if

$$\limsup_{r\to\infty} \frac{P(r)}{m(r,p)} < \infty$$

then the necessary and sufficient condition for (1) to be oscillatory is

$$\int_0^\infty rm(r,p)dr = \infty \tag{15}$$

whereas our theorem works also in cases when

$$\limsup_{r\to\infty} \frac{P(r)}{m(r,p)} = \infty$$

provided that the growth is controlled by $w\left(\int_1^r sP(s)ds\right)$ - take $\alpha(r) = P(r)$ for $r \ge 0$. In such cases we still have that (15) is the necessary and sufficient condition for (1) to be oscillatory.

REFERENCES

[1] A. CONSTANTIN, *On the asymptotic behavior of second order non-linear differential equations*, Rend. Mat. Roma, 13 (1993), 627-634.

[2] E.S. NOUSSAIR and C.A. SWANSON, *Oscillation theory for semilinear Schrödinger equations and inequalities*, Proc. Roy. Soc. Edinburgh, A 75 (1975/1976), 67-81.

[3] E.S. NOUSSAIR and C.A. SWANSON, *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal. Appl., 75 (1980), 121-133.

[4] W.A. STRAUSS, *"Partial Differential Equations"*, Wiley & Sons, New York, 1992.

[5] C.A. SWANSON, *Criteria for oscillatory sublinear Schrödinger equations*, Pacific J. Math., 104 (1983), 483-493.

Received November 9, 1995
A. Constantin
Mathematisches Institut
Universität Basel
Rheinsprung, 21
4051, Basel