THE BERNSTEIN INEQUALITY FOR SOME OPERATORS OF THE SZÁSZ - MIRAKJAN TYPE

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Abstract. *We give the Bernstein inequality for derivatives of the operators of the Szász - Mirakjan type introduced and studied in [3 - 6] for functions of one and several variables, continuous and having the polynomial or exponential growth at infinity*

In §1 we consider these operators for functions of one variable.

In §2 we investigate some analogues of these operators for functions of two variables.

The present inequalities are very important for approximation properties of the considered operators

Key words: *Bernstein inequality, linear positive operator*

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1. THE OPERATORS $L_{n,i}$ FOR FUNCTIONS OF ONE VARIABLE

1. Notation

1.1. Let $R_o := [0,\infty), N := \{1,2,\dots\}, N_0 : N \cup \{0\}$ and let $w_p(\cdot), p \in N_0$, be the weight function defined on R_0 by the formula

$$w_0(x) := 1, \qquad w_p(x) := (1+x^p)^{-1} \qquad \text{for} \quad p \geq 1. \tag{1}$$

Similarly as in [1] and [3, 5] for every $p \in N_0$, we denote by $C_{1,p}$, the space of realvalued functions f defined on R_0 and such that $w_p(\cdot)f(\cdot)$ is uniformly continuous and bounded on R_0. The norm in $C_{1,p}$ is defined by

$$\|f\|_{1,p} := \sup_{x \in R_0} w_p(x)|f(x)|. \tag{2}$$

1.2. Let $q > 0$ be a fixed number and let $v_q(\cdot)$ be function defined by

$$v_q(x) := e^{-qx} \qquad \text{for} \qquad x \in R_0. \tag{3}$$

Similarly as in [2] and [4, 6] for every $q > 0$, we denote by $C_{2,q}$, the space of realvalued functions f defined on R_0 and such that $w_p(\cdot)f(\cdot)$ is uniformly continuous and bounded on R_0. The norm in $C_{2,q}$ is given by

$$\|f\|_{2,p} := \sup_{x \in R_0} v_p(x)|f(x)|. \tag{4}$$

1.2. In the papers [3 - 6] we introduced the following operators $L_{n,i}$, $n \in N$, $i = 1,2,3,4$ ($L_{n,1}$ and $L_{n,2}$ are considered in [3, 4]; $L_{n,3}$ and $L_{n,4}$ were defined in [5, 6]):

$$L_{n,1}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{2k}{n}\right), \tag{5}$$

$$L_{n,2}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x)\frac{n}{2}\int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t)dt, \tag{6}$$

$$L_{n,3}(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{2k+1}{n}\right), \tag{7}$$

$$L_{n,4}(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} b_{n,k}(x)\frac{n}{2}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{2}} f(t)dt, \tag{8}$$

for functions f belonging to $C_{1,p}$ or $C_{2,q}$ for some $p \in N_0$ and $q > 0$, $x \in R_0$, where

$$a_{n,k}(x) := \frac{1}{\cosh nx}\frac{(nx)^{2k}}{(2k)!}, \tag{9}$$

$$b_{n,k}(x) := \frac{1}{1+\sinh nx}\frac{(nx)^{2k+1}}{(2k+1)!}, \quad k \in N_0, \tag{10}$$

and $\sinh x$, $\cosh x$, $\tanh x$ are the elementary hyperbolic functions.

$L_{n,i}$, $n \in N$, $1 \le i \le 4$, are linear positive operators defined on every space $C_{1,p}$, $p \in N_0$, and

$$L_{n,i}(1,x) = 1 \quad \text{for} \quad x \in R_0, n \in N, 1 \le i \le 4. \tag{11}$$

In [3] and [5] it was proved that every $L_{n,i}$ with a fixed $n \in N$ and $1 \le i \le 4$ is an operator from $C_{1,p}$ into $C_{1,p}$ for every fixed $p \in N_0$.

The operators $L_{n,i}$ are well-defined on every space $C_{2,q}$, $q > 0$. In [4] and [6] it was proved that $L_{n,i}$, $1 \le i \le 4$, is an operator from $C_{2,q}$, $q > 0$, into $C_{2,r}$ with every $r > q$ provided that $n > q\left(ln\frac{r}{q}\right)^{-1}$.

In [3] and [5] it was proved that if $f \in C_{1,p}$ with some $p \in N_0$, then there exists a positive constant M_p depending only on p such that

$$w_p(x)|L_{n,i}(f,x) - f(x)| \le M_p\omega\left(f, C_{1,p}, \sqrt{\frac{x+1}{n}}\right),$$

for all $x \in R_0$, $n \in N$ and $1 \le i \le 4$, where $\omega(f, C_{1,p}, \cdot)$ is the modulus of continuity of f, i.e.

$$\omega(f, C_{1,p}, t) = \sup_{|h| \le t} \|f(\cdot + h) - f(\cdot)\|_{1,p}, \quad t \ge 0.$$

Similar estimates hold for $f \in C_{2,q}, q > 0$.

Below we shall denote by $M_{a,b}$ suitable positive constants depending only on indicated parameters a, b.

2. Auxiliary results

The following two lemmas were proved by mathematical induction in the papers [3, 5].

Lemma 1. *([3]). For every $s \in N$ there exist finitely many positive numbers $\xi_{s,k}, \eta_{s,k}, \xi^*_{s,k}$ and $\eta^*_{s,k}$ depending only on s and $0 \leq k \leq s$ such that for all $n \in N$ and $x \in R_0$*

$$L_{n,1}(t^s;x) = \sum_{k=0}^{s} \xi_{s,k} \frac{x^k}{n^{s-k}} + (\tanh nx - 1) \sum_{k=1}^{\left[\frac{s+1}{2}\right]} \eta_{s,k} \frac{x^{2k-1}}{n^{s+1-2k}},$$

$$L_{n,2}(t^s;x) = \sum_{k=0}^{s} \xi^*_{s,k} \frac{x^k}{n^{s-k}} + (\tanh nx - 1) \sum_{k=1}^{\left[\frac{s+1}{2}\right]} \eta^*_{s,k} \frac{x^{2k-1}}{n^{s+1-2k}},$$

and $\xi_{s,s} = 1 = \xi^*_{s,s}$ ([y] is the integral part of $y \in R$).

Lemma 2. *([5]). For every $s \in N$ there exist finitely many positive numbers $\lambda_{s,k}, \rho_{s,k}, \lambda^*_{s,k}$, $\rho^*_{s,k}$ depending only on s and $0 \leq k \leq \left[\frac{s+1}{2}\right]$ such that for all $n \in N$ and $x \in R_0$*

$$L_{n,3}(t^s;x) = S(nx) \sum_{k=1}^{\left[\frac{s}{2}\right]} \lambda_{s,k} \frac{x^k}{n^{s-k}} + T(nx) \sum_{k=1}^{\left[\frac{s+1}{2}\right]} \rho_{s,k} \frac{x^{2k-1}}{n^{s+1-2k}},$$

$$L_{n,4}(t^s;x) = S(nx) \sum_{k=1}^{\left[\frac{s}{2}\right]} \lambda^*_{s,k} \frac{x^k}{n^{s-k}} + T(nx) \sum_{k=1}^{\left[\frac{s+1}{2}\right]} \rho^*_{s,k} \frac{x^{2k-1}}{n^{s+1-2k}},$$

$$S(nx) := \frac{\sinh nx}{1 + \sinh nx}, \qquad T(nx) := \frac{\cosh nx}{1 + \sinh nx} \tag{12}$$

and $\lambda_{2m,m} = 1 = \lambda^*_{2m,m}$ for $m \in N$ and $\rho_{2m+1,m+1}$ for $m \in N$ and $\rho_{2m+1,m+1} = 1 = \rho^*_{2m+1,m+1}$ for $m \in N_0$. (We assume that $\sum_{k=n_1}^{n_2} y_k \equiv 0$ if $n_1 > n_2$).

Using Lemmas 1 and 2, we shall prove two lemmas.

Lemma 3. *For every fixed $p \in N_0$ there exists a positive constant M_p such that for all $n \in N$ and $1 \leq i \leq 4$ one has*

$$\left\| L_{n,i}\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{1,p} \leq M_p. \tag{13}$$

Proof. The inequality (13) is obvious for $p = 0$ by (1), (2) and (11).

Let $i = 1$. Then by (1), (5) and Lemma 1 we get for $n \in N$ and $x \in R_0$

$$w_1(x)L_{n,1}\left(\frac{1}{w_1(t)};x\right) = \frac{1}{1+x}\{1 + L_{n,1}(t;x)\} = \frac{1}{1+x}\{1 + x\tanh nx\} =$$

$$= 1 + \frac{x}{1+x}\{\tanh nx - 1\} \leq 1$$

and for $p \geq 2$

$$w_p(x)L_{n,1}\left(\frac{1}{w_p(t)};x\right) = \frac{1}{1+x^p}\{1 + L_{n,1}(t^p;x)\} \leq$$

$$\leq 1+\sum_{k=1}^{p-1}\xi_{p,k}\frac{x^k}{1+x^p}+\sum_{k=1}^{\left[\frac{p+1}{2}\right]}\eta_{p,k}\frac{x^{2k-1}}{n^{p+1-2k}}|\tanh nx-1|.$$

But for every $r\in N$, $n\in N$ and $x\geq 0$ we have

$$0<|\tanh nx-1|x^r=\frac{2x^r}{e^{2nx}+1}\leq \ddot{a}2^{1-r}r!n^{-r}.$$

Hence, for $x\geq 0$ and $n\in N$, we get

$$w_p(x)L_{n,1}\left(\frac{1}{w_p(t)};x\right)\leq 1+\sum_{k=1}^{p-1}\xi_{p,k}+\sum_{k=1}^{\left[\frac{p+1}{2}\right]}\eta_{p,k}2^{-2k}(2k-1)!n^{-p}\leq M_p,$$

which implies (13) for $i=1$.
The proof of (13) for $i=2$ is analogous by Lemma 1.

Let $i=3$ or $i=4$. From (12) it follows

$$0\leq S(nx)\leq 1 \quad \text{and} \quad 0<T(nx)\leq 1 \quad \text{for all} \quad x\geq 0\text{and} \quad n\in N. \tag{14}$$

Using these inequalities and Lemma 2 and arguing as in the case $i=1$, we immediately obtain (13) for $i=3,4$.

From (1), (2), (5) - (10) and Lemma 3 we derive the following

Lemma 4. *For every fixed $p\in N_0$ there exists a positive constant M_p such that for every $f\in C_{1,p}$ and for all $n\in N$, $1\leq i\leq 4$ one has*

$$\|L_{n,i}(f(t);\cdot)\|_{1,p}\leq M_p\|f\|_{1,p}.$$

This fact and (5) - (10) show that $L_{n,i}$ is an operator from the space $C_{1,p}$ into $C_{1,p}$, $p\in N_0$.

Lemma 5. *Let $q>0, r>q$ and let n_0 be a fixed natural number such that*

$$n_0>1\left(1n\frac{r}{q}\right)^{-1}. \tag{15}$$

Then there exists a positive constant M_q such that for all $n>n_0$ one has

$$\left\|L_{n,i}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r}\leq M_q, \quad 1\leq i\leq 4. \tag{16}$$

Proof. From (5) - (10) we get for all $n\in N$ and $x\in R_0$

$$\begin{aligned}
L_{n,1}(e^{qt};x) &= \frac{\cosh\left(e^{\frac{q}{n}}nx\right)}{\cosh nx},\\
L_{n,2}(e^{qt};x) &= \frac{n}{2q}\left(e^{\frac{2q}{n}}-1\right)L_{n,1}(e^{qt},x),\\
L_{n,3}(e^{qt};x) &= \frac{1+\sinh\left(e^{\frac{q}{n}}nx\right)}{1+\sinh nx},\\
L_{n,4}(e^{qt};x) &= \frac{n}{2q}\left(e^{\frac{2q}{n}}-1\right)L_{n,3}(e^{qt};x)+\left\{1-\frac{n}{2q}\left(e^{\frac{2q}{n}}-1\right)\right\}\frac{1}{1+\sinh nx}.
\end{aligned}$$

For a given $q>0$ denote by

$$q_n := n\left(e^{\frac{q}{n}} - 1\right), \qquad n \in N. \tag{17}$$

The sequence (q_n) is decreasing and

$$q < q_n < qe^{\frac{q}{n}} \le qe^q \quad \text{for} \quad n > n_0.$$

If $r>q$ and n_0 is a fixed integer given by (15), then $r > qe^{\frac{q}{n_0}} > q_{n_0} > q_n$ for $n>n_0$. From above we get for every $r>q$ and $n>n_0, x \in R_0$

$$v_r(x)L_{n,1}\left(\frac{1}{v_q(t)};x\right) = e^{-rx}L_{n,1}(e^{-qt};x) = e^{-rx}\frac{\cosh\left(e^{\frac{q}{n}}nx\right)}{\cosh nx} \le 2e^{(q_n-r)x} \le 2,$$

which implies

$$\left\|L_{n,1}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} \le 2 \qquad \text{for} \qquad n>n_0. \tag{18}$$

Since $0 \le e^x - 1 \le xe^x$ for $x \ge 0$, we have by (18)

$$\left\|L_{n,2}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} \le e^{2q}\left\|L_{n,1}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} \le 2e^{2q}$$

for all $n>n_0$ and $r>q$.

Similarly, from above we get for $r>q, n>n_0$ and $x \ge 0$

$$v_r(x)L_{n,3}\left(\frac{1}{v_q(t)};x\right) = e^{-rx}L_{n,3}(e^{qt};x) \le$$

$$\le 1 + e^{-rx}\frac{\sinh\left(e^{\frac{q}{n}}nx\right)}{1+\sinh nx} \le 1 + e^{(q_n-r)x} \le 2,$$

which yields

$$\left\|L_{n,3}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} \le 2 \qquad \text{for} \qquad n>n_0. \tag{19}$$

For $q>0$ and $n \in N$ we have

$$0 < e^{\frac{2q}{n}} - 1 \le \frac{2q}{n}e^{\frac{2q}{n}}, \qquad \left|1 - \frac{n}{2q}\left(e^{\frac{2q}{n}} - 1\right)\right| \le \frac{2q}{n}e^{\frac{2q}{n}}.$$

Using these inequalities and (19), we get for $r>q$ and $n>n_0$

$$\left\|L_{n,4}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} \le e^{2q}\left\|L_{n,3}\left(\frac{1}{v_q(t)};\cdot\right)\right\|_{2,r} + 2qe^{2q} \le 2(1+q)e^{2q}.$$

Thus the proof of (16) is completed.

Using Lemma 5 we easily obtain

Lemma 6. *Suppose that $f \in C_{2,q}$ for some $q > 0, r > q$ and let n_0 be given by (15). Then*

$$\|L_{n,i}(f;\cdot)\|_{2,r} \leq M_i \|f\|_{2,q}, \qquad i = 1,2,3,4,$$

for all $n > n_0$, where $M_1 = M_3 = 2, M_2 = 2e^{2q}$ and $M_4 = 2(1+q)e^{2q}$.
This proves that $L_{n,i}$ is an operator from a given $C_{2,q}$ into $C_{2,r}, r > q > 0$.

Using the mathematical induction for $s \in N_0$, we can prove the following lemma on derivatives of the order s.

Lemma 7. *For every $s \in N_0$ there exist a finitely many real numbers $\alpha_{2s,j}, \alpha_{2s+1,j}, 0 \leq j \leq s$ and $\beta_{2s,j}, \beta_{2s+1,j}$, $0 \leq j \leq 2s$, depending only on s and j such that for each $n \in N$ and $x \geq 0$ one has*

$$\frac{d^{2s}}{dx^{2s}}\left(\frac{1}{\cosh nx}\right) = \frac{n^{2s}}{\cosh nx}\sum_{j=0}^{s} \alpha_{2s,j}(\tanh nx)^{2j},$$

$$\frac{d^{2s+1}}{dx^{2s+1}}\left(\frac{1}{\cosh nx}\right) = \frac{n^{2s+1}}{\cosh nx}\sum_{j=0}^{s} \alpha_{2s+1,j}(\tanh nx)^{2j+1},$$

$$\frac{d^{2s}}{dx^{2s}}\left(\frac{1}{1+\sinh nx}\right) = \frac{n^{2s}}{(1+\sinh nx)^{2s+1}}\sum_{j=0}^{2s} \beta_{2s,j}(\sinh nx)^{j},$$

$$\frac{d^{2s+1}}{dx^{2s+1}}\left(\frac{1}{1+\sinh nx}\right) = \frac{n^{2s+1}}{(1+\sinh nx)^{2s+2}}\sum_{j=0}^{2s} \beta_{2s,j+1}(\sinh nx)^{j}.$$

From above and by (12) and (14) we immediately obtain the following

Corollary 1. *For every fixed $s \in N_0$ there exists a positive constant M_s such that for all $x \geq 0$ and $n \in N$ one has*

$$\left|\left(\frac{1}{\cosh nx}\right)^{(s)}\right| \leq M_s \frac{n^s}{\cosh nx}, \tag{20}$$

$$\left|\left(\frac{1}{1+\sinh nx}\right)^{(s)}\right| \leq M_s \frac{n^s}{1+\sinh nx}. \tag{21}$$

Lemma 8. *For every fixed $s \in N_0$ and $p \in N_0$ there exists a positive constant $M_{p,s}$ such that for all $n \in N$ one has*

$$\sup_{x \in R_0} w_p(x) \sum_{k=0}^{\infty} \left|a_{n,k}^{(s)}(x)\right| \frac{1}{w_p\left(\frac{2k}{n}\right)} \leq M_{p,s} \cdot n^s, \tag{22}$$

$$\sup_{x \in R_0} w_p(x) \sum_{k=0}^{\infty} \left|b_{n,k}^{(s)}(x)\right| \frac{1}{w_p\left(\frac{2k+1}{n}\right)} \leq M_{p,s} : n^s. \tag{23}$$

Proof. The inequalities (22) and (23 for $s = 0$ are given in Lemma 3. Let $s \geq 1$. By (9), (10), (20) and (21),

$$
\begin{aligned}
\left|a_{n,k}^{(s)}(x)\right| &= \left|\sum_{j=0}^{s}\binom{s}{j}\left(\frac{1}{\cosh nx}\right)^{(s-j)}\left(\frac{(nx)^{2k}}{(2k)!}\right)^{(j)}\right| \\
&\leq M_s \frac{n^s}{\cosh nx}\sum_{j=0}^{s}\frac{1}{n^j}\left(\frac{(nx)^{2k}}{(2k)!}\right)^{(j)}, \\
\left|b_{n,k}^{(s)}(x)\right| &= \left|\sum_{j=0}^{s}\binom{s}{j}\left(\frac{1}{1+\sinh nx}\right)^{(s-j)}\left(\frac{(nx)^{2k+1}}{(2k+1)!}\right)^{(j)}\right| \\
&\leq M_s \frac{n^s}{1+\sinh nx}\sum_{j=0}^{s}\frac{1}{n^j}\left(\frac{(nx)^{2k+1}}{(2k+1)!}\right)^{(j)},
\end{aligned}
\tag{24}
$$

for all $n \in N$, $x \in R_0$ and $k \in N_0$. Hence

$$
w_p(x)\sum_{k=0}^{\infty}\left|a_{n,k}^{(s)}(x)\right|\frac{1}{w_p\left(\frac{2k}{n}\right)} \leq
$$

$$
\leq M_s \frac{n^s w_p(x)}{\cosh nx}\sum_{j=0}^{s}\sum_{k\geq\frac{j}{2}}\frac{(nx)^{2k-j}}{(2k-j)!}\cdot\frac{1}{w_p\left(\frac{2k}{n}\right)}.
$$

But by (1) and (5) we have for $j = 2m, m \in N$

$$
\begin{aligned}
\sum_{k\geq\frac{j}{2}}\frac{(nx)^{2k-j}}{(2k-j)}\frac{1}{w_p\left(\frac{2k}{n}\right)} &= \sum_{k=0}^{\infty}\frac{(nx)^{2k}}{(2k)!}\frac{1}{w_p\left(\frac{2k+2m}{n}\right)} \leq \\
&\leq (2m+1)^p(\cosh nx)L_{n,1}\left(\frac{1}{w_p(t)};x\right).
\end{aligned}
$$

If $j = 2m+1, m \in N_0$, then, by (1) and (5) - (10), follows for $x \geq 0$ and $n \in N$

$$
\begin{aligned}
\sum_{k\geq\frac{j}{2}}\frac{(nx)^{2k-j}}{(2k-j)}\frac{1}{w_p\left(\frac{2k}{n}\right)} &= \sum_{k=m+1}^{\infty}\frac{(nx)^{2k-2m-1}}{(2k-2m-1)!}\frac{1}{w_p\left(\frac{2k}{n}\right)} = \\
&= \sum_{k=0}^{\infty}\frac{(nx)^{2k+1}}{(2k+1)!}\frac{1}{w_p\left(\frac{2k+2m+2}{n}\right)} \\
&\leq (2m+3)^p(1+\sinh nx)L_{n,3}\left(\frac{1}{w_p(t)};x\right).
\end{aligned}
$$

Consequently,

$$
w_p(x)\sum_{k=0}^{\infty}\left|a_{n,k}^{(s)}(x)\right|\frac{1}{w_p\left(\frac{2k}{n}\right)} \leq
$$

$$\leq M_{p,s}n^s\left\{\left\|L_{n,1}\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{1,p}+\frac{1+\sinh nx}{\cosh nx}\left\|L_{n,3}\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{1,p}\right\},$$

which by Lemma 3 and the inequality

$$0<\frac{1+\sinh nx}{\cosh nx}\leq 2 \qquad \text{for} \qquad x\in R_0, n\in N,$$

gives the desired assertion (22).

Similarly, using (24), we obtain (23).

Analogously, using Lemma 5 and Corollary 1, we can prove the following

Lemma 9. *For every fixed $s\in N_0$ and $r>q>0$ there exists a positive constant $M_{q,r,s}$ such that*

$$\sup_{x\in R_0} v_r(x)\sum_{k=0}^{\infty}\left|a_{n,k}^{(s)}(x)\right|\frac{1}{v_q\left(\frac{2k}{n}\right)}\leq M_{q,r,s}\cdot n^s, \tag{25}$$

$$\sup_{x\in R_0} v_r(x)\sum_{k=0}^{\infty}\left|b_{n,k}^{(s)}(x)\right|\frac{1}{v_q\left(\frac{2k+1}{n}\right)}\leq M_{q,r,s}\cdot n^s, \tag{26}$$

for all $n>n_0$, where n_0 is given by (15).

3. The Berstein inequality

In this section we shall give an inequalities of the type (14) and (19) for derivates $L_{n,i}^{(s)}(f,\cdot)$, $s\in N$, called the Bernstein inequalities for the operators $L_{n,i}(f,\cdot)$ ([7]).

Theorem 1. *For every fixed $p\in N_0$ and $s\in N_0$ there exists a positive constant $M_{p,s}$, depending only on p and s, such that for every $f\in C_{1,p}$ and for all $n\in N$ and $1\leq i\leq 4$ one has*

$$\left\|L_{n,i}^{(s)}(f;\cdot)\right\|_{1,p}\leq M_{p,s}n^s\|f\|_{1,p}. \tag{27}$$

Proof. The inequality (27) for $s=0$ is given in Lemma 4.

Let $i=1$ and $s\geq 1$. From (5) and (9) we get for every fixed $n\in N$ and $x\geq 0$

$$\left|\frac{d^s}{dx^s}L_{n,1}(f;x)\right|\leq\sum_{k=0}^{\infty}|a_{n,k}^{(s)}(x)|\left|f\left(\frac{2k}{n}\right)\right|$$

$$\leq\|f\|_{1,p}\sum_{k=0}^{\infty}|a_{n,k}^{(s)}(x)|\frac{1}{w_p\left(\frac{2k}{n}\right)},$$

which by (2) and (22) immediately yields (27) for $i=1$. The proof for $i=2,3,4$ is analogous, but we use (21) - (23).

Arguing as the proof of Theorem 1 and using (25) and (26) we easily obtain

Theorem 2. *Suppose that q,r,s and n_0 are a fixed numbers such that $r>q>0$, $s\in N_0$ and n_0 satisfies (15). Then there exists a positive constant $M^*\equiv M_{q,r,s,n_0}$, depending only on q,r,s,n_0, such that for every $f\in C_{2,q}$, $1\leq i\leq 4$ and for all $n>n_0$ one has*

$$\left\|L_{n,i}^{(s)}(f;\cdot)\right\|_{2,r}\leq M^*n^s\|f\|_{2,q}. \tag{28}$$

From Theorems 1 and 2 we derive the following two corollaries.

Corollary 2. *$L_{n,i}, n \in N, 1 \le i \le 4$, is a linear positive operator from $C_{1,p}$ into $C^{\infty}_{1,p}$ for every $p \in N_0$.*

Corollary 3. *$L_{n,i}, n \in N, 1 \le i \le 4$, is a linear positive operator from $C_{2,q}$ into $C^{\infty}_{2,r}$ for $r > q > 0$, provided that $n > n_0$, where n_0 satisfies (15).*

2. THE OPERATORS $L_{m,n,i}$ FOR FUNCTIONS OF TWO VARIABLES

1. Notation

In this section we shall introduce an analogues of definitions given in §1.

1.1. Let $R_0^2 := \{(x,y) : x, y \in R_0\}$ and let for a fixed $p_1, p_2 \in N_0$

$$w_{p_1,p_2}(x,y) := w_{p_1}(x)w_{p_2}(y), \qquad (x,y) \in R_0^2, \tag{29}$$

where $w_p(\cdot)$ is defined by (1).
Denote by C_{1,p_1,p_2} the space of real-valued functions f defined on R_0^2 such that $w_{p_1,p_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2. Let the norm in C_{1,p_1,p_2} be defined by

$$\|f\|_{1,p_1,p_2} := \sup_{(x,y)\in R_0^2} w_{p_1,p_2}(x,y)|f(x,y)|. \tag{30}$$

1.2. Applying (3) we define for a fixed $q_1, q_2 > 0$ the weight function

$$v_{q_1,q_2}(x,y) := v_{q_1}(x) \cdot v_{q_2}(y) \qquad (x,y) \in R_0^2. \tag{31}$$

Let C_{2,q_1,q_2} be the space of real - valued functions f defined on R_0^2 such that $v_{q_1,q_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2. Let for $f \in C_{2,q_1,q_2}$

$$\|f\|_{2,q_1,q_2} := \sup_{(x,y)\in R_0^2} v_{q_1,q_2}(x,y)|f(x,y)|. \tag{32}$$

For a fixed $s \in N$ and $p_1, p_2 \in N_0$ we denote by

$$C^s_{1,p_1,p_2} := \left\{ f \in C_{1,p_1,p_2} : \frac{\partial^{j+k} f}{\partial x^j \, \partial y^k} \in C_{1,p_1,p_2}, \quad 0 \le j+k \le s \right\}$$

and analogously we define C^s_{2,q_1,q_2} with $q_1, q_2 > 0$ and $s \in N$.

1.3. In the spaces $C_{1,p_1,p_2}, p_1, p_2 \in N_0$ and $C_{2,q_1,q_2}, q_1, q_2 > 0$, we define the following operators $L_{m,n,i}, m, n \in N$, $i = 1,2,3,4$,

$$L_{m,n,1}(f;x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)f\left(\frac{2j}{m}, \frac{2k}{n}\right), \tag{33}$$

$$L_{m,n,2}(f;x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)\frac{mn}{4}\int_{\frac{2j}{m}}^{\frac{2j+2}{m}}\int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(u,v)dudv, \tag{34}$$

$$L_{m,n,3}(f;x,y) := B_{m,n}(x,y)f(0,0) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y)f\left(\frac{2j+1}{m},\frac{2k+1}{n}\right), \tag{35}$$

$$L_{m,n,4}(f;x,y) := B_{m,n}(x,y)f(0,0)+$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y)\frac{mn}{4}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}}\int_{\frac{2k+1}{n}}^{\frac{2k+1}{n}} f(u,v)dudv, \tag{36}$$

$(x,y) \in R_0^2$, where $a_{m,j}(\cdot), b_{m,j}(\cdot)$ are given by (9), (10) and

$$B_{m,n}(x,y) := \frac{1+\sinh mx + \sinh ny}{(1+\sinh mx)(1+\sinh ny)}. \tag{37}$$

From (33) - (37) and (9), (10) it follows that

$$L_{m,n,i}(1,x,y) = 1, \qquad (x,y) \in R_0^2, \qquad m,n \in N \quad \text{and} \quad 1 \le i \le 4. \tag{38}$$

Moreover, we see that $L_{m,n,i}$ are linear positive operators. We shall show that $L_{m,n,i}$, $m,n \in N$, $1 \le i \le 4$ act from the space C_{1,p_1,p_2}, $p_1,p_2 \in N_0$, into C_{1,p_1,p_2}. Lemma 13 shows that $L_{m,n,i}$ act from $C_{2,q_1,q_2}, q_1,q_2 > 0$, into C_{2,r_1,r_2} with every $r_1 > q_1, r_2 > q_2$ provided that $m > m_0$ and $n > n_0$, where m_0, n_0 are natural numbers such that

$$m_0 > q_1\left(1n\frac{r_1}{q_1}\right)^{-1}, \qquad n_0 > q_2\left(1n\frac{r_2}{q_2}\right)^{-1}. \tag{39}$$

2. Auxiliary results

Using the results given in §1 and the above definitions, we shall prove

Lemma 10. *For every $p_1,p_2 \in N_0$ there exist a positive constant M_{p_1,p_2} such that for all $m,n \in N$ and $1 \le i \le 4$ holds*

$$\left\|L_{m,n,i}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1,p_1,p_2} \le M_{p_1,p_2}. \tag{40}$$

Proof. From (29), (30), (33) - (38) and (1), (2), (5) - (10) we derive the inequality

$$L_{m,n,i}\left(\frac{1}{w_{p_1,p_2}(t,z)};x,y\right) \le L_{m,i}\left(\frac{1}{w_{p_1}(t)};x\right)L_{n,i}\left(\frac{1}{w_{p_2}(z)};y\right) \tag{41}$$

for all $m,n \in N, (x,y) \in R_0^2$ and $1 \le i \le 4$, which implies

$$\left\|L_{m,n,i}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1,p_1,p_2} \le \left\|L_{m,i}\left(\frac{1}{w_{p_1}(t)};\cdot\right)\right\|_{1,p_1}\left\|L_{n,i}\left(\frac{1}{w_{p_2}(z)};\cdot\right)\right\|_{1,p_2}$$

Further (40) follows by Lemma 3.

Using Lemma 10 we immediately obtain

Lemma 11. *Let $f \in C_{1,p_1,p_2}$ for some $p_1, p_2 \in N_0$. Then there exists a positive constant M_{p_1,p_2} such that*

$$\|L_{m,n,i}(f;\cdot,\cdot)\|_{1,p_1,p_2} \leq M_{p_1,p_2}\|f\|_{1,p_1,p_2} \tag{42}$$

for all $m, n \in N$ and $1 \leq i \leq 4$.

This fact and (33) - (37) prove that $L_{m,n,i}$ is a linear positive operator from C_{1,p_1,p_2}, $p_1, p_2 \in N_0$, into C_{1,p_1,p_2}.

Similarly as Lemma 10, using Lemma 5, (31), (32) and (41), we can prove the following two lemmas.

Lemma 12. *For every fixed $r_1 > q_1 > 0, r_2 > q_2 > 0$ there exists a positive constant M_{q_1,q_2,r_1,r_2} such that*

$$|L_{m,n,i}(f;\cdot,\cdot)\|_{1,r_1,r_2} \leq M_{q_1,q_2,r_1,r_2} \tag{43}$$

for all $m > m_0, n > n_0$ and $1 \leq i \leq 4$, where m_0 and n_0 are given by (39).

Lemma 13. *Suppose that $f \in C_{2,q_1,q_2}$ for some $q_1, q_2 > 0$ and $r_1 > q_1, r_2 > q_2$. Then there exists a positive constant M_{q_1,q_2,r_1,r_2} such that*

$$\|L_{m,n,i}(f,\cdot,\cdot)\|_{2,r_1,r_2} \leq M_{q_1,q_2,r_1,r_2}\|f\|_{2,q_1,q_2} \tag{44}$$

for all $m > m_0, n > n_0$ and $1 \leq i \leq 4$, where m_0, n_0 are given by (39). Hence we see that $L_{m,n,i}$, $m > m_0, n > n_0$ is a linear positive operator from C_{2,q_1,q_2} into C_{2,r_1,r_2} with $r_1 > q_1$, $r_2 > q_2$.

3. Theorems

Now we shall give an analogues of Theorems 1 and 2.

Theorem 3. *For every fixed $p_1, p_2 \in N_0$ and $s_1, s_2 \in N_0$ there exists a positive constant $M^* \equiv M_{p_1,p_2,s_1,s_2}$ such that for every $f \in C_{1,p_1,p_2}$ and for all $m, n \in N$ and $1 \leq i \leq 4$ one has*

$$\left\|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}}L_{m,n,i}(f;x,y)\right\|_{1,p_1,p_2} \leq M^* m^{s_1} n^{s_2}\|f\|_{1,p_1,p_2}. \tag{45}$$

Proof. If $s_1 = s_2 = 0$, then (45) is identical to (42). Let $i = 1$. Then by (33) and (29, (30) we have

$$\left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}}L_{m,n,1}(f;x,y)\right| \leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{m,j}^{(s_1)}(x)||a_{n,k}^{(s_2)}(y)|\left|f\left(\frac{2j}{m},\frac{2k}{n}\right)\right| \leq$$

$$\leq \|f\|_{1,p_1,p_2}\left(\sum_{j=0}^{\infty}|a_{m,j}^{(s_1)}(x)|\frac{1}{w_{p_1}\left(\frac{2j}{m}\right)}\right)\left(\sum_{k=0}^{\infty}|a_{n,k}^{(s_2)}(y)|\frac{1}{w_{p_2}\left(\frac{2k}{n}\right)}\right)$$

for all $m, n \in N$ and $(x,y) \in R_0^2$. Now by (29) and Lemma 8, we get

$$w_{p_1,p_2}(x,y)\left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}}L_{m,n,i}(f;x,y)\right| \leq M_{p_1,p_2,s_1,s_2}m^{s_1}n^{s_2}\|f\|_{1,p_1,p_2}$$

for all $(x,y) \in R_0^2$ and $m,n \in N$, which implies (45) for $i=1$.

The proof of (45) for $i=2$ is identical by (34) and Lemma 8.

Let $i=3$. Then by (35), (29) and (30) we have for all $s_1, s_2 \in N_0$, $(x,y) \in R_0^2$ and $m,n \in N$

$$\left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}} L_{m,n,3}(f;x,y)\right| \le \left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}} B_{m,n}(x,y)\right| |f(0,0)| + \tag{46}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |b_{m,j}^{(s_1)}(x)||b_{n,k}^{(s_2)}(y)| \left|f\left(\frac{2j+1}{m}, \frac{2k+1}{n}\right)\right| \le$$

$$\le \|f\|_{1,p_1,p_2} \left\{ \left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}} B_{m,n}(x,y)\right| + \right.$$

$$\left. + \left(\sum_{j=0}^{\infty} |b_{m,j}^{(s_1)}(x)| \frac{1}{w_{p_1}\left(\frac{2j+1}{m}\right)}\right) \left(\sum_{k=0}^{\infty} |b_{n,k}^{(s_2)}(y)| \frac{1}{w_{p_2}\left(\frac{2k+1}{n}\right)}\right) \right\}.$$

From (37) it follows that

$$B_{m,n}(x,y) = \frac{1}{1+\sinh ny} + \frac{1}{1+\sinh mx} - \frac{1}{(1+\sinh mx)(1+\sinh ny)}, \tag{47}$$

which by (21) implies

$$\left|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\,\partial y^{s_2}} B_{m,n}(x,y)\right| = \left|\left(\frac{1}{1+\sinh mx}\right)^{(s_1)}\right| \left|\left(\frac{1}{1+\sinh ny}\right)^{(s_2)}\right| \le$$

$$\le M_{s_1,s_2} \frac{m^{s_1} n^{s_2}}{(1+\sinh mx)(1+\sinh ny)} \le M_{s_1,s_2} m^{s_1} n^{s_2}$$

for all $(x,y) \in R_0^2$, $m,n \in N$ and $s_1, s_2 \in N$.

If $s_1 \cdot s_2 = 0$, then from (47) or (37) we get

$$|B_{m,n}(x,y)| \le 1, \qquad (x,y) \in R_0^2, \qquad m,n \in N,$$

and by (21)

$$\left|\frac{\partial^{s_1}}{\partial x^{s_1}} B_{m,n}(x,y)\right| = \frac{\sinh ny}{1+\sinh ny} \left|\left(\frac{1}{1+\sinh mx}\right)^{(s_1)}\right| \le M_{s_1} m^{s_1},$$

$$\left|\frac{\partial^{s_2}}{\partial y^{s_2}} B_{m,n}(x,y)\right| = \frac{\sinh mx}{1+\sinh mx} \left|\left(\frac{1}{1+\sinh ny}\right)^{(s_2)}\right| \le M_{s_2} n^{s_2},$$

for $(x,y) \in R_0^2$, $m,n \in N$.

Using these inequalities and Lemma 9 to (46), we easily obtain the desired Bernstein inequality (45) for $i=3$.

The proof of (45) for $i = 4$ is analogous to $i = 3$. Thus the proof is finished.

Arguing as in the proof of Theorem 3 and using (43), (44) and Lemma 9, we can prove

Theorem 4. *Let $r_1 > q_1 > 0, r_2 > q_2 > 0$, $s_1, s_2 \in N_0$ and let $f \in C_{2,q_1,q_2}$. Then there exists a positive constant M^* depending only on $q_1, q_2, r_1, r_2, s_1, s_2$ such that*

$$\left\| \frac{\partial^{s_1+s_2}}{\partial x^{s_1}\, \partial y^{s_2}} L_{m,n,i}(f;x,y) \right\|_{2,r_1,r_2} \leq M^* m^{s_1} n^{s_2} \|f\|_{2,q_1,q_2},$$

for all $1 \leq i \leq 4$, $m > m_0$ and $n > n_0$ where m_0 and n_0 are natural numbers satisfying conditions (39).

Theorems 3 and 4 imply

Corollary 4. *1^o)$L_{m,n,i}, m,n \in N, i = 1,2,3,4,$ acts from the space C_{1,p_1,p_2} into C^{∞}_{1,p_1,p_2}, $p_1, p_2 \in N_0$.*

2^o)$L_{m,n,i}, i = 1,2,3,4,$ acts from the space C_{2,q_1,q_2} into C^{∞}_{2,r_1,r_2} with every $r_1 > q_1 > 0$ and $r_2 > q_2 > 0$, provided that $m > m_0$ and $n > n_0$, where m_0, n_0 are given by (39).

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