DEFORMATIONS OF LEGENDRE CURVES
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1. INTRODUCTION

In contact geometry $\mathbb{R}^3$ with the standard Darboux form $\eta = \frac{1}{2}(dz - ydx)$ and Sasakian metric $g = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta$ is a central example. Sectional curvatures of plane sections containing the characteristic vector field $2 \frac{\partial}{\partial z}$ are equal to $+1$ and sectional curvatures of planes orthogonal to the characteristic vector field are equal to $-3$; for this reason we denote this Sasakian manifold by $\mathbb{R}^3(-3)$. This is also the contact structure on the Heisenberg group, $\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R}$ $\cong \mathbb{R}^3$, both $\eta$ and $g$ being left invariant. In this paper we first study deformations of Legendre curves in $\mathbb{R}^3(-3)$ in the direction of the principal normal, especially 2-minimal curves. In particular we show that 2-minimal Legendre curves arise from 2-minimal curves in the Euclidean plane which were characterized in [4]. For deformations of curves in Euclidean 3-space in the direction of the principal normal see [5].

Our result has an application to the theory of 2-minimal curves in the Euclidean plane. In [4] it was shown that closed 2-minimal curves have self-intersections and we show here that moreover the algebraic area of a closed 2-minimal planar curve in zero.

To prove our main result we need a lemma on Bessel functions which may be of independent interest and we devote Section 4 of the paper to this lemma.

In the last section of this paper we consider deformations of curves in a general $K$-contact manifold in the direction of the characteristic vector field. Critical curves for this variational problem are the so-called $C$-loxodromes [6].

2. CONTACT MANIFOLDS

By a contact manifold we mean a $C^\infty$ manifold $M^{2n+1}$ together with a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given $\eta$ there exists a unique vector field $\xi$, such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$, called the characteristic vector field. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^{n} y^i dx^i$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle (i.e. the bundle of $2n$-planes annihilated by $\eta$) is as far from being integrable as possible. In particular the maximum dimension of an integral submanifold is only $n$. From the Darboux theorem it is clear that $n$-dimensional integral submanifolds exist, namely, those given by $x^i =$ constant, $z =$ constant. A 1-dimensional integral submanifold is called a Legendre curve, especially to avoid confusion with an integral curve of the vector field $\xi$.

A Riemannian metric $g$ is an associated metric to a contact structure $\eta$ if there exists a
tensor field $\phi$ of type $(1,1)$ satisfying
\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y).
\]

We refer to $(\eta, g)$ or $(\phi, \xi, \eta, g)$ as a contact metric structure. If $\xi$ is a Killing vector field with respect to $g$, the contact metric structure is called a $K$-contact structure. It is well known that on a $K$-contact manifold
\[
\nabla X \xi = -\phi X,
\]
where $\nabla$ denotes the Levi-Civita connection of $g$. The space $\mathbb{R}^3(-3)$ discussed above is $K$-contact. For a general reference to the ideas of this section see [2].

In the space $\mathbb{R}^3(-3)$ discussed in the introduction, closed Legendre curves have an interesting elementary property which we now state.

Area Property of Closed Legendre Curves. The projection $\gamma^*$ of a closed Legendre curve $\gamma$ in $\mathbb{R}^3(-3)$ to the xy-plane must have self-intersections; moreover the algebraic area enclosed in zero.

Since $dz - ydx = 0$ along $\gamma$, this follows from the elementary formula for the area enclosed by a curve given by Green’s theorem,
\[
0 = -\int_\gamma dz = \int_{\gamma^*} -ydx = \text{area},
\]
the area being $+$ for $\gamma^*$ traversed counterclockwise and $-$ for clockwise.

One of the results of [1] is the following.

Theorem. The curvature of a Legendre curve in $\mathbb{R}^3(-3)$ is equal to twice the curvature of its projection to the xy-plane with respect to the Euclidean metric.

3. $k$-DEFORMATIONS AND $k$-MINIMALITY

The theory of $k$-deformations, $k$-minimality and $k$-stability was developed in [4] and we briefly review this theory here. Let $M$ be a compact Riemannian manifold and $\Delta$ the Laplacian acting on the space $C^\infty(M)$ of $C^\infty$ functions on $M$. Define a metric on $C^\infty(M)$ by $(f, g) = \int_M fg dA$ where $dA$ is the volume form on $M$. It is well known that $\Delta$ is a self-adjoint operator which has an infinite discrete sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots / \lambda^\infty + \infty$. For each $i \in \mathbb{N}$ the eigenspace $V_i$ of $\lambda_i$ is finite dimensional; $V_0$ is 1-dimensional and consists of constant functions. The eigenspaces are mutually orthogonal and their sum is dense in $C^\infty(M)$. Therefore one can make a spectral decomposition $f = f_0 + \sum_{i=1}^\infty f_i$, for each real $C^\infty$ function $f$ on $M$, where $f_0$ is a constant and $\Delta f_i = \lambda f_i$ for $i > 0$. The set $T(f) = \{ i \in \mathbb{N}_0 | f_i \neq 0 \}$ is called the type of $f$ and $f$ is of finite type if $T(f)$ is a finite set.

The subject of the study in [4] was compact oriented hypersurfaces $x : M^n \rightarrow N^{n+1}$ isometrically immersed in a Riemannian manifold $N^{n+1}$. For a unit vector field $\xi$, usually normal, defined on $M$, define a deformation by
\[
\exp_{x(p)} tf(p)\xi(p), \quad p \in M, \quad t \in (-\epsilon, \epsilon)
\]
and in [4] the area functional for these deformations was studied. Here we are concerned with closed curves \( \gamma : [0, L] \to M \) in a Riemannian manifold \( M \) parametrized by arc length and we study the length integral \( L(t) \) under various deformations.

For each \( q \in \mathbb{N}_0 \), let \( \mathcal{F}_q \) be the class of all deformations (in direction \( \zeta \)) associated to functions \( f \in \sum_{i \geq q} V_i \). Clearly \( \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \). A deformation in \( \mathcal{F}_k \) is called a \( k \)-deformation. A closed curve \( \gamma \) is said to be \( k \)-minimal if \( L'(0) = 0 \) for all deformations in \( \mathcal{F}_k \). If \( \gamma \) is \( k \)-minimal, we say that \( \gamma \) is \( \ell \)-stable, \( \ell \geq k \), if \( L''(0) \geq 0 \) for all deformations in \( \mathcal{F}_\ell \). One of the results of [4] is that every compact \( k \)-minimal hypersurface is \( q \)-stable for some \( q \geq k \); it was also shown that a \( k \)-minimal plane curve is \( k \)-stable.

Consider a closed plane curve of length \( 2\pi \). The Laplacian is just \(-\frac{d^2}{dx^2}\), the eigenvalues are \( \lambda_n = n^2 \) and a basis of the corresponding eigenspace is given by \( \{ \cos ns, \sin ns \} \). By Lemma 4.1 of [4], a closed plane curve is \( k \)-minimal if and only if its curvature \( \kappa \) is of finite type \(< k \); in particular

\[
\kappa(s) = a_0 + \sum_{n=1}^{k-1} \{ a_n \cos ns + b_n \sin ns \}.
\]

We now state the following result from [4].

**Theorem.** For each zero \( j_{\ell, m} \) of the Bessel function \( J_\ell \) of order \( \ell \), the curve \( \gamma_{\ell, m} \) defined by

\[
\gamma_{\ell, m} = \left( \int_0^s \cos(\ell u + j_{\ell, m} \sin u) du, \int_0^s \sin(\ell u + j_{\ell, m} \sin u) du \right)
\]

is a closed \( 2 \)-minimal curve. Conversely up to rigid motions every \( 2 \)-minimal plane curve can be obtained in this way.

4. A LEMMA ON BESSEL FUNCTIONS

In this section we prove a formula involving Bessel functions which is not found in the treatise of Watson [7] and seems to be new.

**Lemma.** \( \sum_{m=1}^{\infty} \frac{1}{m} (J_{\ell - m}^2(x) - J_{\ell + m}^2(x)) = J_\ell(x) \sum_{k=0}^{\ell-1} \frac{2(-1)^k \ell}{k!(\ell-k)!} \frac{1}{x^{\ell-k}} J_k(x) \).

**Proof.** That the series on the left converges for all \( x \) follows from Watson p. 31. We will use the following well known properties of Bessel functions

\[
J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \quad [7, p. 17] \tag{4.1}
\]

\[
J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x), \quad [7, p. 17] \tag{4.2}
\]

\[
J_0^2(x) + 2 \sum_{r=1}^{\infty} J_r^2(x) = 1, \quad [7, p. 31] \tag{4.3}
\]

\[
J_{-n}(x) = (-1)^n J_n(x), \quad [7, p. 43] \tag{4.4}
\]

Set \( a_{\ell, k} = \frac{2(-1)^k \ell}{k!(\ell-k)!} \) for \( 0 \leq k < \ell \) and \( a_{\ell, k} = 0 \) for \( k < 0 \). The following are immediate or easy to prove

\[
a_{\ell, \ell-1} = 2\ell \tag{4.5}
\]
\[ a_{\ell,k} = a_{\ell-1,k-1} + 2(\ell - k - 1)a_{\ell-1,k}, \quad (4.6) \]

\[ a_{\ell-1,k-1} - 2ka_{\ell-1,k} - a_{\ell-2,k-2} + 2ka_{\ell-2,k-1} = 0, \quad 0 \leq k < \ell - 1. \quad (4.7) \]

for \( \ell = 0 \) there is nothing to prove in the formula of the Lemma. For \( \ell = 1 \), (4.1), (4.2) and (4.3) yield

\[
\sum_{m=1}^{\infty} \frac{1}{m} (J_{m-1}^2(x) - J_{m+1}^2(x)) = \sum_{m=1}^{\infty} \frac{4m}{m^2} J_m(x) J'_m(x) - \frac{1}{x} \left( \sum_{m=1}^{\infty} J_m^2(x) \right)' \\
= \frac{2}{x} \left( \frac{1}{2} (1 - J_0^2(x)) \right)' = -\frac{2}{x} J_0(x) J'_0(x) \\
= -\frac{2}{x} J_0(x) J_1(x).
\]

For \( \ell = 2 \), proceeding in the same manner we have

\[
\sum_{m=1}^{\infty} \frac{1}{m} (J_{m-2}^2(x) - J_{m+2}^2(x)) = \sum_{m=1}^{\infty} \frac{1}{m} (J_{m-2}^2(x) - J_m^2(x) + J_m^2(x) - J_{m+2}^2(x)) \\
= \sum_{m=1}^{\infty} \left( \frac{4(m-1)}{mx} J_{m-1}(x) J'_{m-1}(x) + \frac{4(m+1)}{mx} J_{m+1}(x) J'_{m+1}(x) \right) \\
= \frac{2}{x} \left( \sum_{m=1}^{\infty} (J_{m-1}^2(x) + J_{m+1}^2(x)) \right)' - \frac{2}{x} \left( \sum_{m=1}^{\infty} (J_{m-1}^2(x) - J_{m+1}^2(x)) \right)' \\
= \frac{2}{x} (J_0^2(x) + \frac{1}{2} (1 - J_0^2(x)) + \frac{1}{2} (1 - J_0^2(x)) - J_0^2(x))' - \frac{2}{x} \left( \frac{2}{x} J_0(x) J_1(x) \right)' \\
= -\frac{4}{x^2} J_0(x) J'_0(x) + \frac{4}{x^3} J_0(x) J_1(x) + \frac{4}{x^2} J_1^2(x) - \frac{4}{x^2} J_0(x) J'_1(x) \\
= \left( \frac{4}{x^2} J_0(x) + \frac{4}{x} J_1(x) \right) (-J'_0(x) + \frac{1}{x} J_1(x)) \\
= \left( \frac{4}{x^2} J_0(x) + \frac{4}{x} J_1(x) \right) J_2(x).
\]

Now we come to the induction step of the proof for \( \ell \geq 3 \).

\[
\sum_{m=1}^{\infty} \frac{1}{m} (J_{m-\ell}^2(x) - J_{m+\ell}^2(x)) = \sum_{m=1}^{\infty} \frac{1}{m} (J_{m-\ell}^2(x) - J_{m-\ell+2}(x) + J_{m-\ell+2}^2(x) \\
- J_{m+\ell-2}(x) + J_{m+\ell-2}(x) - J_{m+\ell}(x)) \\
= \sum_{m=1}^{\infty} \left( \frac{4(m - \ell + 1)}{mx} J_{m-\ell+1}(x) J'_{m-\ell+1}(x) + \frac{4(m + \ell - 1)}{mx} J_{m+\ell-1}(x) J'_{m+\ell-1}(x) \right)
\]
Deformations of Legendre curves

\[ + \left( \sum_{k=0}^{\ell-3} a_{\ell-2,k} x^{\ell-2-k} J_k(x) \right) J_{\ell-2}(x) \]

\[ = \frac{2}{x} \left( \sum_{m=1}^{\infty} j_{m-\ell+1}(x) + j_{m+\ell-1}(x) \right)' - \frac{2(\ell - 1)}{x} \left( \sum_{m=1}^{\infty} \frac{1}{m} (j_{m-\ell+1}(x) - j_{m+\ell-1}(x)) \right)' \]

\[ + \left( \sum_{k=0}^{\ell-3} a_{\ell-2,k} x^{\ell-2-k} J_k(x) \right) J_{\ell-2}(x) \]

\[ = \frac{2}{x} \left( 1 - j_{\ell-1}(x) \right)' - \frac{2(\ell - 1)}{x} \left( \sum_{k=0}^{\ell-2} a_{\ell-1,k} x^{\ell-1-k} J_k(x) \right) J_{\ell-1}(x) \]

\[ + \frac{2(\ell - 1)}{x} \left( \sum_{k=0}^{\ell-2} \frac{a_{\ell-1,k}}{x^{\ell-1-k}} J_k(x) \right) J_{\ell-1}(x) \]

\[ = -\frac{2}{x} J_{\ell-1}(x)(J_{\ell-2}(x) - J_{\ell}(x)) + \frac{2(\ell - 1)}{x} \left( \sum_{k=0}^{\ell-2} \frac{(\ell - 1 - k)a_{\ell-1,k}}{x^{\ell-k}} J_k(x) \right) J_{\ell-1}(x) \]

\[ - \frac{2(\ell - 1)}{x} \left( \sum_{k=0}^{\ell-2} a_{\ell-1,k} x^{\ell-1-k} J_k(x) \right) J_{\ell-1}(x) \]

\[ + \frac{\ell-2}{x^2} 2kae_{\ell-2,k-1} J_k(x) J_{\ell-2}(x) - \frac{\ell-1}{x^2} a_{\ell-2,k-2} J_k(x) J_{\ell-2}(x) \]

Reindexing and using (4.5) this becomes

\[ -\frac{2}{x} J_{\ell-1}(x)(J_{\ell-2}(x) - J_{\ell}(x)) - \frac{2(\ell - 2)}{x} J_{\ell-1}(x) J_{\ell-2}(x) \]

\[ + \sum_{k=0}^{\ell-2} \frac{(kae_{\ell-2,k-1} - a_{\ell-2,k-2})}{x^{\ell-k}} J_k(x) J_{\ell-2}(x) \]

\[ + \left( \sum_{k=0}^{\ell-2} \frac{(\ell - 1 - k)a_{\ell-1,k}}{x^{\ell-k}} J_k(x) \right)(J_{\ell-2}(x) + J_{\ell}(x)) \]

\[ - \left( \sum_{k=0}^{\ell-2} \frac{1}{x^{\ell-k}} J_k(x) \right)(J_{\ell-2}(x) - J_{\ell}(x)) \]

\[ - (\ell - 1)(\sum_{k=0}^{\ell-2} \frac{a_{\ell-1,k}}{x^{\ell-k}} J_k(x))(J_{\ell-2}(x) - J_{\ell}(x)) \]

Using (4.1) on the next to last line and reindexing again, this line is

\[ \sum_{k=0}^{\ell-1} \frac{a_{\ell-1,k-1}}{x^{\ell-k}} J_k(x)(J_{\ell-2}(x) + J_{\ell}(x)) - \sum_{k=0}^{\ell-2} \frac{kae_{\ell-1,k}}{x^{\ell-k}} J_k(x)(J_{\ell-2}(x) + J_{\ell}(x)) \]
Now looking at the coefficients of \( J_\ell(x) \) and \( J_{\ell-2}(x) \), applying (4.6) and (4.7), and using (4.5) one more time gives the result.

5. DEFORMATION OF LEGENDRE CURVES IN DIRECTION OF PRINCIPAL NORMAL

Let \( \gamma : [0, L] \to \mathbb{R}^3(-3) \) be a closed Legendre curve parametrized by arc length in the space \( \mathbb{R}^3(-3) \). Differentiating \( \eta(\gamma') = 0 \) along \( \gamma \) we see from (2.1) that \( \nabla_{\gamma'} \gamma' \) is orthogonal to \( \xi \) and hence that \( \nabla_{\gamma'} \gamma' \) is in the direction \( \pm \phi \gamma' \). Thus

\[
\nabla_{\gamma'} \gamma' = \pm \kappa \phi \gamma'
\]

where \( \kappa \geq 0 \) is the curvature and \( \pm \phi \gamma' \) the principal normal.

Now consider a deformation of \( \gamma \) in the direction of the principal normal,

\[
\gamma_t(s) = \exp_{\gamma(s)} t\phi(s) \phi \gamma'(s)
\]

and the length

\[
L(t) = \int_0^t g(\gamma'_t, \gamma'_t)^{1/2} ds.
\]

Computing \( L'(0) \) in the usual manner we have

\[
L'(0) = -\int_0^t f g(\phi \gamma', \nabla_{\gamma'} \gamma') ds.
\]

Using the orthonormal basis \( e = 2\frac{\partial}{\partial y}, \phi e = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \xi = 2\frac{\partial}{\partial z} \), we have \( \phi \gamma' = -\frac{1}{2} x' e + \frac{1}{2} y' \phi e \) and hence

\[
g(\phi \gamma', \nabla_{\gamma'} \gamma') = \frac{1}{4}(y'x'' - x'y'' - (x'^2 + y'^2)(z' - yx'))
\]

\[
= \frac{1}{4}(y'x'' - x'y'')
\]

since \( \eta(\gamma') = \frac{1}{2}(z' - yx') = 0 \) for a Legendre curve. Thus in view of the theorem in Section 2, Legendre k-minimal curves in \( \mathbb{R}^3(-3) \) arise from k-minimal curves in the xy-plane \([0, 2\pi] \to E^2 \) by \( z = \int_0^s y dx \), \( s \) being arc length on the plane curve. The condition for Legendre k-minimal curves becomes

\[
\frac{1}{4}(y'x'' - x'y'') = a_0 + \sum_{n=1}^{k-1} a_n \cos ns + b_n \sin ns.
\]

If the plane curve is closed, the Legendre curve is closed if \( z(2\pi) = 0 \).

Thus 0-minimal curves correspond to lines in the plane. Integration of \( y dx \) gives a parabola as the Legendre curve; this parabola is a geodesic in \( \mathbb{R}^3(-3) \) but clearly there are no closed 0-minimal Legendre curves.
Since \( x'^2 + y'^2 = 4 \) for a Legendre curve, if \( y'x'' - x'y'' = \text{const.} \), the curve in the \( xy \)-plane is a circle. Thus from the area property of closed Legendre curves given in Section 2, there are no closed 1-minimal Legendre curves. Since some closed 3-minimal curves in the plane (see e.g. Fig. 7 of [4]) fail to satisfy this property we cannot expect a general result. For 2-minimal curves we have the following theorem.

**Theorem 1.** Every closed 2-minimal curve in the plane gives rise to a closed 2-minimal Legendre curve \( \gamma \) in \( \mathbb{R}^3(-3) \) by integration of \( z' = yx' \) and conversely.

**Proof.** Closed 2-minimal curves in the plane were described explicitly by the theorem stated in Section 3 and we have just noted the correspondence between \( k \)-minimal curves in the plane and \( k \)-minimal Legendre curves in \( \mathbb{R}^3(-3) \). Thus it remains to prove the closure of the Legendre curve \( \gamma \), i.e. to show that

\[
z(2\pi) = \int_0^{2\pi} \cos(\ell v + j_{\ell, m} \sin v) \int_0^v \sin(\ell u + j_{\ell, m} \sin u) \, du \, dv = 0.
\]

Set

\[
f_1(v) = \cos(\ell v + j_{\ell, m} \sin v), \quad f_2(v) = \int_0^v \sin(\ell u + j_{\ell, m} \sin u) \, du
\]

each of which is a periodic function of period \( 2\pi \). Now consider the Fourier expansions of \( f_1 \) and \( f_2 \),

\[
f_1(v) = A_0 + \sum_{m=1}^{\infty} A_m \cos mv + B_m \sin mv
\]

\[
f_2(v) = A_0^* + \sum_{m=1}^{\infty} A_m^* \cos mv + B_m^* \sin mv
\]

and we must show that \( A_0 A_0^* + \sum_{m=1}^{\infty} A_m A_m^* + \sum_{m=1}^{\infty} B_m B_m^* = 0 \). Now since \( j_{\ell, m} \) is a zero of \( J_\ell \), it follows from ([7], p. 19) that \( A_0 = \frac{1}{2\pi} \int_0^{2\pi} \cos(\ell v + j_{\ell, m} \sin v) \, dv = 0 \). Moreover \( B_m = \frac{1}{\pi} \int_0^{2\pi} f_1(v) \sin mv \, dv = 0 \) as is easily seen by shifting the interval to \([-\pi, \pi]\) and noting that the integrand is an odd function. Thus we must show that \( \sum_{m=1}^{\infty} A_m A_m^* = 0 \).

First of all by [7], p. 19,

\[
A_m = \frac{1}{\pi} \int_0^{2\pi} \cos(\ell v + j_{\ell, m} \sin v) \cos mv \, dv
\]

\[
= \frac{1}{\pi} \left\{ \int_0^{2\pi} \cos((m - \ell) v - j_{\ell, m} \sin v) \, dv + \int_0^{2\pi} \cos((m + \ell) v + j_{\ell, m} \sin v) \, dv \right\}
\]

\[
= 2 \{ J_{m-\ell}(j_{\ell, m}) + J_{m+\ell}(-j_{\ell, m}) \}.
\]
Secondly

\[ A_m^* = \frac{1}{\pi} \int_0^{2\pi} \left( \int_0^\nu \sin(\ell u + j_{\ell,m} \sin u) du \right) \cos mv dv \]

\[ = -\frac{1}{m\pi} \int_0^{2\pi} \sin(\ell v + j_{\ell,m} \sin v) \sin mv dv \]

\[ = -\frac{1}{mn\pi} \left\{ \int_0^{2\pi} \cos((m - \ell)v - j_{\ell,m} \sin v) dv - \int_0^{2\pi} \cos((m + \ell)v + j_{\ell,m} \sin v) dv \right\} \]

\[ = -\frac{2}{m} \{ J_{m-\ell}(j_{\ell,m}) - J_{m+\ell}(-j_{\ell,m}) \}. \]

Thus the proof reduces to showing that

\[ \sum_{m=1}^{\infty} \frac{4}{m} \{ J_{m-\ell}(j_{\ell,m}) - J_{m+\ell}(-j_{\ell,m}) \} = 0 \]

but since \( j_{\ell,m} \) is zero of \( J_{\ell} \), this is consequence of the lemma of Section 4.

We have just seen that the theory of 2-minimal curves in the plane contributes to the theory of Legendre curves and we now show that our result on 2-minimal Legendre curves contributes to the theory of 2-minimal curve in the plane. In [4] it was shown that every closed 2-minimal curve in the plane has a line of symmetry and a point of self-intersection, the proof of the latter assertion being somewhat extensive. Now we see from the area property of closed Legendre curves that by projecting a 2-minimal Legendre curve in \( \mathbb{R}^3(-3) \) to the \( xy \)-plane, the self-intersection is immediate and that the algebraic area vanishes. Thus we have the following result.

**Theorem 2.** Every closed 2-minimal curve in \( E^2 \) has a point of self-intersection and algebraic area zero.

In particular we reproduce the following six figures from [4] which illustrate the area property in an interesting and attractive manner.

![Figure 1](image1.png)

\( k = f_{0,1} \cos(s) \)

![Figure 2](image2.png)

\( k = f_{0,2} \cos(s) \)
6. \( \xi \)-DEFORMATIONS

In this section we consider deformations of curves in the direction of the characteristic vector field \( \xi \) in a \( K \)-contact manifold \( M^{2n+1} \), we call such a deformation a \( \xi \)-deformation. For a curve \( \gamma : [0, L] \to M^{2n+1} \) set
\[
\gamma_t(s) = \exp_{\gamma(s)} tf(s)\xi(\gamma(s))
\]
and consider the length integral \( L(t) = \int_0^L g(\gamma'_t, \gamma'_t)^{1/2} ds \). Calculating \( L'(0) \) now yields
\[
L'(0) = -\int_0^L fg(\xi, \nabla_{\gamma_t'} \gamma')ds.
\]
Therefore by virtue of (2.1), \( L'(0) = 0 \) for every \( C^{\infty} \) function \( f \) if and only if
\[
g(\xi, \nabla_{\gamma_t'} \gamma') = \gamma' g(\xi, \gamma') = 0,
\]
i.e. the angle $\theta$ between $\xi$ and $\gamma'$ is constant along the curve; such curves were called 
**C-loxodromes** by Tachibana and Tashiro [6]. Note that even in the space $\mathbb{R}^3(-3)$ a closed 
C-loxodrome which is not a Legendre curve ($\theta = \pi/2$) is possible. For example 

$$
\gamma = \left( \sin \frac{2s}{\sqrt{5}}, \cos \frac{2s}{\sqrt{5}}, \frac{1}{4} \sin \frac{4s}{\sqrt{5}} \right)
$$

is a closed C-loxodrome in $\mathbb{R}^3(-3)$ for which $\theta \neq 0, \frac{\pi}{2}$.

Computing the second variation in the usual manner and noting that the spectrum of the 
Laplacian for closed curves of length $L$ is $\{ (2\pi k/L)^2 | k \in \mathbb{N} \}$ we have 

$$
L''(0) = \int_0^L f'2(1 - \eta(\gamma')^2) - f^2 \eta(\gamma')^2 ds
\geq \int_0^L \left( \frac{2\pi k}{L} \right)^2 (1 - \eta(\gamma')^2) - \eta(\gamma')^2 f^2 ds
= \frac{(2\pi k)^2 (1 - \eta(\gamma')^2)}{L^2} - L^2 \eta(\gamma')^2 \int_0^L f^2 ds.
$$

Thus we have the following result.

**Proposition.** A Legendre curve in a K-contact manifold is 0-stable and a C-loxodrome is 
$\ell$-stable for some $\ell$.

For the question of 1-minimal curves under $\xi$-deformations, $\gamma'g(\xi, \gamma') = \text{const} \neq 0$ implies 
$\cos \theta = As + B$ and hence there are no closed 1-minimal curves for $\xi$-deformations.

For 2-minimal curves, one expects closed 2-minimal curves to exist. The condition of 
2-minimality is 

$$
\gamma'(\xi, \gamma') = a_0 + a_1 \cos \left( \frac{2\pi s}{L} \right) + b_1 \sin \left( \frac{2\pi s}{L} \right).
$$

In $\mathbb{R}^3(-3)$ this becomes 

$$
\frac{1}{2} (\xi' - yx') = a_0 + a_1 \cos \left( \frac{2\pi s}{L} \right) + b_1 \sin \left( \frac{2\pi s}{L} \right).
$$

So the problem is to choose $x(s), y(s)$ periodic such that 

$$
z = \int yy' ds - a_1 \frac{L^2}{2\pi^2} \cos \left( \frac{2\pi s}{L} \right) - b_1 \frac{L^2}{2\pi^2} \sin \left( \frac{2\pi s}{L} \right) + cs + d
$$

is periodic and the arc length condition 

$$
\frac{1}{4} (\xi'^2 + y'^2 + (\xi' - yx')^2) = 1
$$

is satisfied. For example the vertical circle 

$$x = 2 \sin s, \quad y = 0, \quad z = 2 \cos s$$
easily satisfies these conditions.

The above situation is analogous to studying variations of curves in a fixed direction $a$ in Euclidean space. If $T$ is the unit tangent field the $k$-minimality condition is

$$(T \cdot a)' = a_0 + \sum_{n=1}^{k-1} a_n \cos \left( \frac{2\pi n}{L} s \right) + b_n \sin \left( \frac{2\pi n}{L} s \right).$$

Thus 0-minimal curves would be the generalized helices but they are not closed; 1-minimality means $T \cdot a$ is linear in $s$ and hence the curve is not closed. For 2-minimal curves for deformations in the direction of the $z$-axis with $a_0 = 0$, $z'' = a_1 \cos \frac{2\pi}{L} s + b_1 \sin \frac{2\pi}{L} s$. Integrating and taking the first constant of integration to be zero,

$$z = -\frac{a_1 L^2}{4\pi^2} \cos \left( \frac{2\pi}{L} s \right) - \frac{b_1 L^2}{4\pi^2} \sin \left( \frac{2\pi}{L} s \right) + c$$

with $x(s), y(s)$ subject only to being periodic and $x'' + y'' + z'' = 1$.

Deformations of this type for all directions give a variational characterization of curves of finite type [3].
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