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4-DIMENSIONAL PROJECTIVE PLANES WITH A 3-DIMENSIONAL ELATION GROUP

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Abstract. *We consider 4-dimensional flexible topological projective planes with the following property: There exists a line W such that the elation group with respect to W is 3-dimensional. We prove that besides known planes no further planes with this property exist.*
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1. INTRODUCTION

Let Σ be the group of continuous collineations of a 4-dimensional compact flexible projective plane (P, L), denote by Δ the connected component of the identity and let N be the nilradical of Δ. By [2, 4, 15] we may suppose that Δ is a 6-dimensional solvable Lie group fixing some flag $v \in W$. Since the cases $\dim N \geq 5$ and $\dim N \leq 3$ are settled in [5] and [11], respectively, we make the assumption $\dim N = 4$ from now on. There are three nilpotent 4-dimensional real Lie algebras,

$$R^4, \langle R^3, \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix} \rangle, \langle R^3, \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & 0 \end{pmatrix} \rangle.$$

The first case leads to translation planes, one shift plane or the planes with two fixed points and two fixed lines [12, 13, 8, 14]. The third algebra is excluded by H. Klein [10], therefore we concentrate here on the second Lie algebra which is isomorphic to $Nil \times R$, where $Nil := < y, v, n | [n, y] = v >$. Note that $Nil \times R$ has the one-dimensional commutator algebra $< v >$ and the 2-dimensional center $< u, v >$.

Let μ_v, μ_W be the minimal dimension of orbits of Δ on $L_v \setminus \{W\}$ and on $W \setminus \{v\}$, respectively, then there are (up to duality) the following 6 cases of orbit types:

$$(\mu_v, \mu_W) = (0,0), (0,1), (0,2), (1,1), (1,2), (2,2).$$

The first four cases have been settled already [14, 4, 7, 9], therefore only the two orbit cases (1,2) and (2,2) are left.

Denote by τ and σ the dimension of the group of translations with axis W and of the group of elations with center v. We may suppose that both dimensions are ≤ 3 since all flexible translation planes are classified. In the present note we make the assumption that at least one of the two numbers τ and σ is equal to 3, which means that the plane has (up to duality) a 3-dimensional translation group. The study of such planes was begun in [1] and continued in [20]. We get the result that besides planes already known no further planes of this type exist.

In brief we consider the following three assumptions:

- The maximal nilpotent normal subgroup of Δ is $Nil \times R$.
- The orbit type is (1,2) or (2,2) (up to duality).
- The plane has (up to duality) a 3-dimensional translation group ($\tau = 3$).

We recall the following two lemmas that are frequently used:
Lemma on quadrangles [17]: If $\varphi \in \Delta$ fixes a quadrangle, i.e. four points, no three of which are collinear, then $\varphi = 1$.
Lemma on free stabilizers [2]: Let Δ be a 3-dimensional connected solvable Lie group acting transitively on the plane R^2. Then either the stabilizer $\Delta_x, x \in R^2$, fixes some further point $x' \neq x$ or the commutator group Δ' acts transitively on R^2.

Note that by [16, 6] the solvable connected Lie groups that act transitively on R^2 are known. The non-affine actions have the lines $x = const.$ as sets of imprimitivity. The action on the space of these lines may be an L_2-action or an R-action.

2. FIXED TRANSLATION GROUP ORBIT

We make the assumption that at least one of the numbers τ and σ is equal to 3 and by dualizing we take $\tau = 3$. So we have a projective plane with a 3-dimensional translation group as studied in [1]. All orbits of the translation group R^3 on $P\backslash W$ are homeomorphic to R^3 and the space Ξ of these orbits is homeomorphic to R coordinatized by the parameter x. We considered in [1] the following two possibilities: either the group Δ fixes an element of Ξ, say $x = 0$, or Δ acts transitively on Ξ. Thus we have the following three cases:

1. Fixed translation group orbit,
2. Δ is transitive on Ξ, N is not,
3. N is transitive on Ξ.

Let us begin with the first case. The fixed orbit determines the orbit number $\mu_v = 1$ and since we consider only the two orbit types (1,2) and (2,2), the other orbit number has to be $\mu_W = 2$. This means that Δ acts transitively on $W\backslash\{v\}$. Denote the fixed orbit by $F = \{(0, y, u, v) | y, u, v \in R\}$. Since the translation group R^3 acts regularly on the fixed orbit F, the group Δ is the semidirect product of R^3 with the stabilizer Δ_0.

Proposition. The Lie algebra of Δ_0 is generated by the following three endomorphisms:

$$\begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & \beta \end{pmatrix}, \beta \neq 1, \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 0 & \alpha \end{pmatrix}, \alpha \neq 0.$$

Here the first endomorphism generates shears with respect to the axis $0 \vee v$, which means $\sigma = 3$.

Proof. The Lie group $Nil \times R$ has a 2-dimensional center C. The one-dimensional stabilizer N_0 fixes the orbit 0^C pointwise and by the lemma on quadrangles it follows $0^C = 0 \vee v$. Therefore the one-dimensional stabilizer N_0 consists of shears with respect to the axis $0 \vee v$ and induces the first endomorphism (after a suitable conjugation).

Next we note that the transitive action of Δ_0 on $W\backslash\{v\}$ is effective. For a kernel of dimension 1 would mean that we have a plane with 3-dimensional translation group of type: "fixed orbit, with homologies", and all such planes are classified in [1,20]. Since the commutator of Δ_0

is 1-dimensional (otherwise $dimN \geq 5$) we know from the lemma on free stabilizers that the stabilizer of Δ_0 on a point $w \in W\backslash\{v\}$ fixes further points on $W\backslash\{v\}$. Remind the notation of coordinates $[\xi, \eta]$ at infinity [1]. In this notation the action of the shears on $W\backslash v$ has the form $[\xi, \eta] \to [\xi, \eta + s], s \in R$, and Σ_0 consists of lower triangular 3×3-matrices

$$\begin{pmatrix} a & & \\ d & b & \\ f & e & c \end{pmatrix}.$$

The induced action of Δ on the ξ-coordinate is like R (and not L_2) since the dimension of the nilradical is only 4. This means that the triangular matrices of Δ_0 have all $a = b$.

The one-parameter subgroup of Δ_0 which fixes the point $[0,0] \in W\backslash\{v\}$ has $d = f = 0$ for all its elements. If a were identically 0 then we would get a contradiction to the lemma on quadrangles. Therefore this one-parameter subgroup is generated by an endomorphism of the form

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & \gamma & \beta \end{pmatrix}.$$

Here we may assume that $(\gamma, \beta) \neq (0, 1)$ since the case "with homologies" is already settled.

As a third generating one-parameter group we choose a group which shifts in ξ-direction on $W\backslash\{v\}$. Responsible for those shifts is the position (2,1) in the lower triangular matrix. Hence we may suppose that the generating endomorphism has a 1 at that position. Subtracting suitable multiples of the former two endomorphisms we may assume 0 at the positions (1,1), (2,2) and (3,1). This gives the endomorphism

$$\begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & \delta & \alpha \end{pmatrix}.$$

Here we may suppose that $\alpha \neq 0$ otherwise we would get a 5-dimensional nilradical.

By commuting the second and third endomorphism we get a matrix which must have a 0 at the position (3,2) since the nilradical has dimension 4. This leads to the equation

$$\delta(\beta - 1) - \gamma\alpha = 0.$$

If $\beta = 1$ then the assumption $(\gamma, \beta) \neq (0, 1)$ implies $\gamma \neq 0$ and we get $\alpha = 0$ a contradiction. Therefore $\beta \neq 1$ and by conjugation we can bring γ to 0. From the equation then follows $\delta = 0$.

Proposition. The action of Δ on F has a unique globalization to

$$P\backslash W = \{(x, y, u, v) | x, y, u, v \in R\}.$$

This global action is the semidirect product of the translation group T with

$$\Delta_0 = \langle \left\{ \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ n & 0 & 1 & \\ 0 & n & 0 & 1 \end{pmatrix} | n \in R \right\}, \left\{ \begin{pmatrix} e^{(2-\beta)r} & & & \\ & e^r & & \\ & & e^r & \\ & & & e^{\beta r} \end{pmatrix} | r \in R \right\},$$

$$\left\{ \begin{pmatrix} e^{-\alpha t} & & & \\ & 1 & & \\ & t & 1 & \\ & & & e^{\alpha t} \end{pmatrix} | t \in R, \right\} \rangle.$$

Proof. Since the group of elations with center v is 3-dimensional, the set of elation axes in $\mathcal{L}_v \backslash \{W\}$ corresponds to the 1-dimensional subspaces of $E = \{(n,u,v)|n,u,v \in R\}$ represented by $x = \frac{u}{n}, y = \frac{v}{n}, n \neq 0$. Therefore the horizontal action can be calculated by letting Δ act via conjugation on the group E of elations:

$$\left\{ \begin{pmatrix} e^r & & \\ & e^r & \\ & & e^{\beta r} \end{pmatrix} | r \in R \right\} \quad \textit{defines} \quad : (x,y) \to (e^{(2-\beta)r}x, e^r y),$$

$$\left\{ \begin{pmatrix} 1 & & \\ t & 1 & \\ 0 & 0 & e^{\alpha t} \end{pmatrix} | t \in R \right\} \quad \textit{defines} \quad : (x,y) \to (e^{-\alpha t}x, y).$$

Now let us glue together the two half spaces on $x \geq 0$ and $x \leq 0$ along F (defined by $x = 0$). The one-parameter stabilizer $\Delta_{0,[0,0]}$ has three orbits in $0 \vee [0,0]$ defined by $y = 0$: the point 0 and the two one-dimensional orbits for $x > 0$ and $x < 0$. Since the horizontial action is classical, these three orbits are glued together in the standard way. We may choose the topology on $P \backslash W$ in such a way that this one-dimensional curve coincides with the x-axis. By applying the translation group we get then the unique globalization.

Proposition. This transformation group cannot act as a collineation group of a projective plane.

Proof. The 2-dimensional subgroup of Δ_0 fixing $(x,y) = (1,0)$ is:

$$\langle \left\{ \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ n & 0 & 1 & \\ 0 & n & 0 & 1 \end{pmatrix} | n \in R \right\}, \left\{ \begin{pmatrix} 1 & & & \\ & e^r & & \\ & re^r & e^r & \\ & & & e^{2r} \end{pmatrix} | r \in R \right\} \rangle$$

The action of this group on $W \backslash \{v\}$ is

$$[\xi, \eta] \to [\xi + r, e^r \eta], [\xi, \eta] \to [\xi, \eta + n]$$

and this action is transitive. The induced action on the vertical line $Y_{(1,0)}$ is

$$(1,0,u,v) \to (1,0,e^r u, e^{2r} v), (1,0,u,v) \to (1,0,u,v+n)$$

and this action is not transitive. Since the perspectivity $0 : Y_{(0,1)} \to W \backslash \{v\}$ in the plane is a bijection, we have constructed a contradiction.

3. Δ IS TRANSITIVE ON Ξ, N IS NOT

We now suppose that Δ acts transitively on $P \backslash W$ but N fixes some translation group orbit, say $x = 0$. Similarly to the last section we see that N is the semidirect product of the translation group $T = \{(y,u,v) | y,u,v \in R\}$ with the group $\left\{ \begin{pmatrix} 1 & & \\ 0 & 1 & \\ n & 0 & 1 \end{pmatrix} | n \in R \right\}$ of shears. Note that we have also a 3-dimensional group of elations with center v:

$$E = \left\{ \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ u & 0 & 1 & \\ v & n & 0 & 1 \end{pmatrix} | u,v,n \in R \right\}.$$

Proposition. The linear action of Δ_0 on the translation group $T = \{(y,u,v) | y,u,v \in R\}$ is generated by the following two endomorphism

$$n = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}, x_1 = \begin{pmatrix} a_1 & & \\ c_1 & a_1 & \\ 0 & e_1 & b_1 \end{pmatrix}, c_1 \neq 0.$$

Proof. Let us first show that Δ induces on $W \backslash \{v\}$ a 3-dimensional transitive action with a one-dimensional commutator. The kernel $\Delta_{[W]}$ of this action contains T and is at least 3-dimensional. If the kernel were at least 4-dimensional then the plane would be a translation plane, but all flexible translation planes are known. We may assume that the action is transitive since a fixed element would lead to the planes in [7] and a one-dimensional orbit would mean the dual situation of the last section. Since the points of $W \backslash \{v\}$ correspond to one-dimensional subspaces of T, the action of $\Delta / \Delta_{[W]}$ on $W \backslash \{v\}$ is an affine action. The induced action on the ξ-coordinate on $W \backslash \{v\}$ is similar to R (and not L_2) since $dimN = 4$. This implies that the commutator of $\Delta / \Delta_{[W]}$ has dimension 1.

By duality the same holds for the induced action of Δ on $L_v \backslash \{W\}$, the so called horizontal action. By the lemma on free stabilizers both actions have the following property: the stabilizer on one element fixes further elements. Using the lemma on quadrangles this implies that the 2-dimensional stabilizer Δ_0 is transitive on $W \backslash \{v\}$. This means $c_1 \neq 0$ in the endomorphism above. Since the action on the ξ-coordinate is similar to R, the two entries at position (1,1) and (2,2) of the lower triangular matrix are equal. Dually we have

Proposition. The linear action of $\Delta_l, [0,0] \in L, L \neq W$ on the elation group $E = \{(n,u,v) | n,u,v \in R\}$ is generated by the two endomorphisms

$$-y = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} a_2 & & \\ c_2 & a_2 & \\ 0 & e_2 & b_2 \end{pmatrix}, c_2 \neq 0.$$

To describe the Lie algebra $\mathcal{L}(\Delta)$ we calculate the action of adx_1 and of adx_2 on the nilradical $N = \{(v,n,y,u) | v,n,y,u \in R\}$

$$adx_1 = \begin{pmatrix} b_1 & & & e_1 \\ & b_1 - a_1 & & \\ & & a_1 & \\ & & c_1 & a_1 \end{pmatrix}, adx_2 = \begin{pmatrix} b_2 & & & e_2 \\ & a_2 & & \\ & & b_2 - a_2 & \\ & c_2 & & a_2 \end{pmatrix}.$$

The commutator of these endomorphisms is

$$\begin{pmatrix} 0 & e_1 c_2 & -e_2 c_1 & e_1(a_2 - b_2) - e_2(a_1 - b_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c_2(2a_1 - b_1) & c_1(b_2 - 2a_2) & 0 \end{pmatrix}.$$

Since $adx_2\, adx_1 - adx_1 adx_2 = ad[x_1, x_2]$ and $[x_1, x_2] \in N$, this commutator must have zero at all positions $\neq$ (1,2), (1,3). Using $c_1 \neq 0$ and $c_2 \neq 0$ this implies $b_1 = 2a_1, b_2 = 2a_2$. Here, $a_1 \neq 0$ (similarly $a_2 \neq 0$), for otherwise

$$\langle T, \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ c_1 & 0 & \\ & e_1 & 0 \end{pmatrix} \rangle$$

would be a 5-dimensional nilpotent ideal. Changing to $\frac{x_1}{a_1}$ we may suppose $a_1 = 1$, $b_1 = 2$ and similarly $a_2 = 1$, $b_2 = 2$. From this we get $e_1 = e_2 := e$ and we have the two simplified endomorphisms

$$adx_1 = \begin{pmatrix} 2 & & & e \\ & 1 & & \\ & & 1 & \\ & & c_1 & 1 \end{pmatrix}, adx_2 = \begin{pmatrix} 2 & & & e \\ & 1 & & \\ & & 1 & \\ & c_2 & & 1 \end{pmatrix}.$$

We now see that $< x_1 - x_2, N >$ is a 5-dimensional nilpotent ideal of $\mathcal{L}(\Delta)$, a contradiction to our assumption.

4. N IS TRANSITIVE ON Ξ

In this case the nilradical N acts transitively on $P\backslash W$. The 3-dimensional translation group T fixes each orbit $x \in \Xi$ and the fourth parameter of N is transitive on Ξ. The group Δ is the semidirect product of N and the stabilizer Δ_0.

4.1. The nilradical N

Proposition. We can assume by conjugation that the fourth one-parameter group acts on T in one of the following ways (described by the generating endomorphism):

$$v = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, s = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}.$$

Note that these three groups induce on $W\backslash\{v\}$ the following actions: vertical shifting in direction of η, horizontal shifting in the direction of ξ and shears with respect to the $\eta - axis$.

Proof. The fourth endomorphism has the form

$$\begin{pmatrix} 0 & & \\ a & 0 & \\ c & b & 0 \end{pmatrix},$$

where at least one of the numbers a, b and c is $\neq 0$. Let us assume first that $ab \neq 0$. Then, up to conjugation, we may suppose $a = b = 1$ and $c = 0$. This endomorphism generates the one-parameter group

$$\left\{ \begin{pmatrix} 1 & & \\ t & 1 & \\ \frac{t^2}{2} & t & 1 \end{pmatrix} | t \in R \right\},$$

which induces on $W\backslash\{v\}$ the following action:

$$(\xi, \eta)^t = (\xi + t, \eta + t\xi + \frac{t^2}{2}).$$

Since N / T is a normal subgroup of Δ / T, the orbits of this one-parameter group on $W\backslash\{v\}$ are permuted by Δ. If one of these orbits would be invariant, then the induced (affine) action of Δ on W could only be 2-dimensional, a contradiction. Therefore Δ acts transitively on the set of these orbits and the affine action of Δ on $W\backslash\{v\}$ has the form

$$\{(\xi, \eta) \mapsto (r\xi + m, r^2\eta + n) | r > 0, m, n \in R\}.$$

This group has a 2-dimensional commutator, which implies that the nilradical of Δ is 5-dimensional, a contradiction.

It follows that we need only consider the following three possibilities:

$$\begin{pmatrix} 0 & & \\ 1 & 0 & \\ c & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ 0 & 0 & \\ c & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}.$$

In the first two cases conjugation with

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -c & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & & \\ c & 1 & \\ 0 & 0 & 0 \end{pmatrix}$$

brings c to 0.

Since the orbit case (2,0) is already settled in [7], we need only consider the orbit types (2,2) and (2,1), in other words, the action of Δ on $W\backslash\{v\}$ is either transitive (2) or it has a one-dimensional orbit (1).

4.2. The collineation group Δ

Proposition. For the action of Δ on the translation group T via conjugation there are at most the following possibilities (described by generating endomorphisms):

$$2v: \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & \gamma \end{pmatrix}, \gamma \neq 1,$$

$$2h: \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \delta & \\ & & 1 \end{pmatrix}, \delta \neq 1,$$

$$1v,1: \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \beta & & \\ & 0 & \\ & & 1 \end{pmatrix},$$

$$1v,2: \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \beta & & \\ & 0 & \\ & 1 & 0 \end{pmatrix},$$

$$1s: \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \alpha & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1-\alpha & \\ 1 & & 1 \end{pmatrix}, \alpha \neq 1,$$

$$1h: \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \beta & & \\ & 0 & \\ & & 1 \end{pmatrix}, \alpha, \beta \neq 1.$$

Proof.

$2v$: The action of Δ on the set of parallels of the η-axis on $W\backslash\{v\}$ is an R-action since $\dim N = 4$. Therefore the lower triangular matrices for the endomoprhisms have equal entries at the position (1,1) and (2,2). Hence we may assume that the endomorphisms generating a shift group in ξ-direction and the stabilizer on [0,0] have form

$$X = \begin{pmatrix} a & & \\ 1 & a & \\ & e & c \end{pmatrix}, Y = \begin{pmatrix} \alpha & & \\ & \alpha & \\ & \epsilon & \gamma \end{pmatrix}.$$

Since the group induced on $W\backslash\{v\}$ is 3-dimensional with a one-dimensional commutator, the stabilizer on [0,0] fixes further points on $W\backslash\{v\}$. So it is either the group of shears with respect to the η-axis ($\gamma = \alpha, \epsilon \neq 0$) or up to conjugation it fixes the ξ-axis elementwise ($\epsilon = 0, \gamma \neq \alpha$). Note that we may exclude a group of homotheties which would lead to a translation plane by the theorem on homologies and elations [19, 61.20]. In the first case we get the Lie algebra *Nil* which has

$$commutator = center, \ generated\ by \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}.$$

This implies $a = c$ and $\alpha = \gamma$ and hence $dimN = 5$, a contradiction.

In the second case $L_2 \times R$ we may switch to $\alpha = 1, \gamma \neq 1$. Commuting X and Y gives the entry $e(1 - \gamma)$ at the position (3,2) and all other elements of the commutator are zero. Since $dimN = 4$ it follows $e = 0$. Changing from X to $X - aY$ brings X to

$$\begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & c' \end{pmatrix}$$

and by conjugating suitably we get

$$\begin{pmatrix} 0 & & \\ c' & 0 & \\ & & c' \end{pmatrix}.$$

Dividing now by c' we get the endomorphisms of the proposition.

$2s$: If the fourth endomorphism is

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}$$

then we commute it with an endomorphism belonging to a shift group in ξ-direction

$$\begin{pmatrix} a & & \\ 1 & b & \\ & e & c \end{pmatrix}$$

and get

$$\begin{pmatrix} 0 & & \\ 0 & 0 & \\ -1 & c-b & 0 \end{pmatrix}.$$

This is a contradiction since the commutator algebra of a solvable Lie algebra is contained in the nilradical.

$2h$: The N- orbits on $W\backslash\{v\}$ are the parallels of the ξ-axis and they are permuted under Σ. The parallels of the η-axis are permuted and since in η-direction there is only the group R (because of $\dim N = 4$) the induced action on $W\backslash\{v\}$ is uniquely determined and isomorphic to $L_2 \times R$. The shift group in η-direction is induced by an endomorphism

$$\begin{pmatrix} a & & \\ & a & \\ 1 & & a \end{pmatrix}, a \neq 0$$

where we may put $a = 1$ up to conjugation and multiplication by a constant. The stabilizer on [0,0] fixes all points of the η-axis and comes from an endomorphism

$$\begin{pmatrix} 1 & & \\ & \delta & \\ & & 1 \end{pmatrix}, \delta \neq 1.$$

$1v$: The one-dimensional orbit on $W\backslash\{v\}$ has the form ξ = const., say $\xi = 0$. Since the η-axis is fixed, all 3×3-matrices have zero at position (2,1). The 2-dimensional subgroup of Δ which fixes $[0,0] \in W\backslash\{v\}$ and a point $p \in P\backslash W$, say the origin, is commutative since $\dim N = 4$. Therefore the action of Δ_0 on $0 \vee v$ is either

$$\{(u,v) \to (au, dv) : a, d \geq 0\} \quad or \quad \{(u,v) \to (au, du + av) : a \geq 0, d \in R\}.$$

This leads to the two cases $1v, 1$ and $1v, 2$ as stated in the proposition.

$1s$: Since

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}$$

generates on $W\backslash\{v\}$ the group

$$\{[\xi,\eta] \to [\xi,\eta+s\xi], s \in R\},$$

the one-dimensional orbit must be a vertical, say $\xi = 0$. Since $dimN = 4$ the induced action on this orbit is only R (not L_2) and by these conditions the induced affine action on $W\backslash\{v\}$ is uniquely determined:

$$\{[\xi,\eta] \to [\alpha\xi, s\xi+\eta+l], a > 0, s, l \in R\}.$$

The 3×3 matrix which induces the group

$$\{[\xi,\eta] \to [a\xi,\eta], a > 0\}$$

has the form

$$\begin{pmatrix} \gamma & & \\ & \alpha & \\ & & \gamma \end{pmatrix}, \gamma \neq 0.$$

Here, $\gamma \neq 0$ otherwise the lemma on quadrangles would lead to a contradiction. So this endomorphism may be assumed to be

$$\begin{pmatrix} 1 & & \\ & \alpha & \\ & & 1 \end{pmatrix}, \alpha \neq 1.$$

The 3×3 endomorphism which generates the group $\{[\xi,\eta] \to [\xi,\eta+l], l \in R\}$ on $W\backslash\{v\}$ is

$$\begin{pmatrix} \beta & & \\ & \beta & \\ 1 & & \beta \end{pmatrix}, \beta \neq 0$$

or, by conjugation

$$\begin{pmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{pmatrix}.$$

Subtracting the previous endomorphism we get the proposition for this case.

$1h$: The one-dimensional orbit on $W\backslash\{v\}$ has the form $\eta = \text{const.}$, say $\eta = 0$. Since the coordinate lines $\xi = \text{const.}$ are permuted, the induced action on $W\backslash\{v\}$ is the group $L_2 \times R$ acting as

$$\{[\xi,\eta] \to [a\xi+r, b\eta], a, b > 0, r \in R\}.$$

The endomorphisms of T now follow.

4.3. Trying to construct the projective planes

We shall prove these transformation groups act as a collineation group of a projective plane. Let us begin with the $2v$ case.

Case $2v$:

We may take the following action of Σ on the affine space $P\backslash W$ coordinatized by $R^4 = \{(x,y,u,v)\,|x,y,u,v \in R\}$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & x & 0 & 1 \end{pmatrix} |x \in R\right\}, \left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^s \end{pmatrix} |s \in R\right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e^t & \\ & & & e^{\gamma t} \end{pmatrix} |t \in R, \gamma \neq 1\right\}, where \quad N = \left\{\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ y & & 1 & \\ v & x & & 1 \end{pmatrix} |x,y,y,v \in R\right\}.$$

Hence we get the regular action of N on itself:

$$(\bar{x},\bar{y},\bar{u},\bar{v})^{(x,y,u,v)} = (x+\bar{x}, y+\bar{y}, u+\bar{u}, v+\bar{v}+x\bar{y}).$$

Conjugation by the $s-$ and $t-$parameter groups gives the isotropy groups:
$(x,y,u,v)^s = (e^s x, y, u+sy, e^s v)$,
$(x,y,u,v)^t = (e^{(\gamma-1)t}x, e^t y, e^t u, e^{\gamma t} v)$.

This plane has a 3-dimensional translation group $R^3 = \{(y,u,v)|y,y,v \in R\}$ and by [1] the points $p \in W\backslash\{v\}$ correspond to the 1-dimensional subspaces $(y,u,v)R, y \neq 0$, of R^3. We describe these subspaces by $(y,u,v)R = (1, \frac{u}{y}, \frac{v}{y})R$ and introduce coordinates $[\xi,\eta], \xi = \frac{u}{y}, \eta = \frac{v}{y}$ on $W\backslash\{v\}$. The action of Σ on $W\backslash\{v\}$ corresponds to the action on the set of 1-dimensional subspaces $(y,u,v)R, y \neq 0$, of R^3 induced by conjugation. Thus we have the following actions of the $x-$, $s-$ and $t-$parameter groups on $W\backslash\{v\}$:

i) action of x-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [\xi, \eta + x]$,
ii) action of s-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [\xi + s, e^s\eta]$,
iii) action of t-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [\xi, e^{(\gamma-1)t}\eta], \gamma \neq 1$.

Construction of the geometry

Let L be the line joining $(x,y,u,v) = (0,0,0,0)$ and $[\xi,\eta] = [0,0]$. Then Σ_L is a 2-dimensional group containing the t-parameter group and the translation group in the direction [0,0].

Let $w_-, z_-, w_+, z_+ \in R$ be such that $(-1, 0, w_-, z_-) \in L$ and $(1, 0, w_+, z_+) \in L$, then L contains also the t-orbits of these two points:

$$\{(-e^{(\gamma-1)t}, 0, w_-e^t, z_-e^{\gamma t})|t \in R\} \cup \{(e^{(\gamma-1)t}, 0, w_+, e^t, z_+e^{\gamma t})|t \in R\}.$$

Using now the translation gorup in the direction $[\xi, \eta] = [0, 0]$, we get the affine part of L: $L = \{(x, y, f(x), g(x))|x, y \in R\}$, where

$$f(x) = \begin{cases} w_-(-x)^{\frac{1}{\gamma-1}} & : \quad x \le 0 \\ w_+x^{\frac{1}{\gamma-1}} & : \quad x \ge 0 \end{cases}, g(x) = \begin{cases} z_-(-x)^{\frac{\gamma}{\gamma-1}} & : \quad x \le 0 \\ z_+x^{\frac{\gamma}{\gamma-1}} & : \quad x \ge 0 \end{cases}.$$

In order to get the other lines, we first apply the shift map t (i.e. x-parameter group): $(x, y, f(x), g(x))^{-t} = (x - t, y, f(x), g(x) - ty)$ and get the lines $L^t = \{(x, y, f(x + t), g(x + t) - ty)|x, y \in R\}$.
We apply the s-parameter group to L^t and get the points

$$(e^s x, y, f(x + t) + sy, e^s(g(x + t) - ty)).$$

Substituting $\eta = e^s x$ and denoting η by x again, we get the lines

$$L^{t,s} = \{(x, y, f(e^{-s}x + t) + sy, e^s[g(e^{-s}x + t) - ty])|x, y \in R\}.$$

The corresponding parallels through the origin (0,0,0,0) are

$$L_0^{t,s} = \{(x, y, f(e^{-s}x + t) - f(t) + sy, e^s(g(e^{-s}x + t) - g(t) - ty))|x, y \in R\}.$$

Then the set of lines $\{L_0^{t,s}|t, s \in R\} \cup \{(0, 0, u, v)|u, v \in R\}$ is the pencil of lines through the origin.

We now intersect the lines of this pencil with the vertical line at $(x, y) = (1, 0)$. This leads to the map

$$h : R^2 \to R^2 : (s, t) \to (f(e^{-s} + t) - f(t), e^s(g(e^{-s} + t) - g(t)),$$

and this map would be a homeomorphism if the plane existed. But since $h(s, 0) = (f(e^{-s}),$ $e^s g\ (e^{-s})) = e^{\frac{1}{\gamma-1}-s}(w_+, z_+) = e^{\alpha s}(w_+, z_+)$, $\alpha \neq 0$ the map h is not proper, a contradiction.

Case $2h$: Here we get the following action of Σ on the affine space $P \backslash W$ coordinatized by $R^4 = \{(x, y, u, v)|x, y, u, v \in R\}$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{pmatrix}|x \in R\right\}, \left\{\begin{pmatrix} 1 & & & \\ & e^s & & \\ & & e^s & \\ & se^s & & e^s \end{pmatrix}|s \in R\right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & e^t & & \\ & & e^{\delta t} & \\ & & & e^t \end{pmatrix} \middle| t \in R, \delta \neq 1 \right\}, \text{where} \quad N = \left\{\begin{pmatrix} 1 & & & \\ y & 1 & & \\ u & x & 1 & \\ v & & & 1 \end{pmatrix} \middle| x, y, u, v \in R \right\}.$$

Hence we get the regular action of N on itself:

$$(\bar{x}, \bar{y}, \bar{u}, \bar{v})^{(x,y,u,v)} = (x + \bar{x}, y + \bar{y}, u + \bar{u} + x\bar{y}, v + \bar{v}).$$

Conjugation by the s- and t- parameter groups gives the isotropy groups:
$(x, y, u, v)^s = (x, e^s y, e^s u, e^s v + se^s y)$,
$(x, y, u, v)^t = (e^{(\delta-1)t}x, e^t y, e^{\delta t}u, e^t v)$.
Similarly to the case $2v$ we have the following actions of x, s and t-parameter groups on $W\backslash\{v\}$ by conjugation, coordinatized by $[\xi, \eta]$, $\xi = \frac{u}{y}, \eta = \frac{v}{y}$
i) action of x-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [\xi + x, \eta]$,
ii) action of s-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [\xi, \eta + s]$,
iii) action of t-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [e^{(\delta-1)t}\xi, \eta]$.

Construction of the geometry

Let L be the line joining $(x, y, u, v) = (0,0,0,0)$ and $[\xi, \eta] = [0,0]$. Then Σ_L is a 2-dimensional group containing the t-parameter group and the translation group in the direction [0,0].

Let $w_-, z_-, w_+, z_+ \in R$ be such that $(-1, 0, w_-, z_-) \in L$ and $(1, 0, w_+, z_+) \in L$. Then similarly to case $2v$, we get the affine part of L:
$L = \{(x, y, f(x), g(x)) | x, y \in R\}$, where

$$f(x) = \begin{cases} w_-(-x)^{\frac{\delta}{\delta-1}} & : \quad x \leq 0 \\ w_+ x^{\frac{\delta}{\delta-1}} & : \quad x \geq 0 \end{cases}, g(x) = \begin{cases} z_-(-x)^{\frac{1}{\delta-1}} & : \quad x \leq 0 \\ z_+ x^{\frac{1}{\delta-1}} & : \quad x \geq 0 \end{cases}.$$

Applying to the shift map $x = -t$ and the s-parameter group we get the following lines:

$$L^{t,s} = \{(x, y, e^s f(x + t) - ty, e^s g(x + t) + sy) | x, y \in R\}.$$

The corresponding parallels through the origin (0,0,0,0) are

$$L_0^{t,s} = \{(x, y, e^s[f(x + t) - f(t) + te^{-s}y], e^s[g(x + t) - g(t)] + sy) | x, y \in R\}.$$

Then the set of lines $\{L_0^{t,s} | t, s \in R\} \cup \{(0, 0, u, v) | u, v \in R\}$ is the pencil of lines through the origin.

We now try to join the origin (0,0,0,0) with the points $\{(1, 0, u, v) | u, v \in R\}$ of the vertical line at $(x, y) = (1, 0)$. This leads to the map $h : R^2 \to R^2 : (s, t) \to (e^s[f(1 + t) - f(t)], e^s[(g(1 + t) - g(t)])$. But since $h(s, 0) = (e^s f(1), e^s g(1)) = e^s(w_+, z_+)$, the map h is not proper and cannot be a homomorphism.

Case $1v, 1$: This case lead to the following action of Σ on the affine space $P\backslash W$ coordinatized by $R^4 = \{(x,y,u,v)|x,y,u,v \in R\}$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \middle| x \in R\right\}, \left\{\begin{pmatrix} 1 & & & \\ & e^{\alpha s} & & \\ & & e^{s} & \\ & & & 1 \end{pmatrix} \middle| s \in R\right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & e^{\beta t} & & \\ & & 1 & \\ & & & e^{t} \end{pmatrix} \middle| t \in R, \right\}, where \quad N = \left\{\begin{pmatrix} 1 & & & \\ y & 1 & & \\ u & & 1 & \\ v & x & & 1 \end{pmatrix} \middle| x,y,u,v \in R\right\}.$$

We get the regular action of N on itself:

$$(\bar{x},\bar{y},\bar{u},\bar{v})^{(x,y,u,v))} = (x+\bar{x}, y+\bar{y}, u+\bar{u}, v+\bar{v}, x\bar{y}).$$

Conjugation by the s- and t-parameter groups gives the isotropy groups:
$(x,y,u,v)^s = (e^{-\alpha s}x, e^{\alpha s}y, e^s u, v)$,
$(x,y,u,v)^t = (e^{(1-\beta)t}x, e^{\beta t}y, u, e^t v)$.
We have the following actions of $x-$, $s-$ and $t-$parameter groups on $W\backslash\{v\}$ by conjugation, coordinatized by $[\xi,\eta]$, $\xi = \frac{u}{y}, \eta = \frac{v}{y}$:
i) action of x-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [\xi, \eta + x]$,
ii) action of s-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [e^{(1-\alpha)s}\xi, e^{-\alpha s}\eta]$,
iii) action of t-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [e^{-\beta t}\xi, e^{(1-\beta)t}\eta]$.
Note that in this case the action of Σ on $W\backslash\{v\}$ has a 1-dimensional orbit, that is, the coordinate line $\xi = 0$ is invariant under the action of Σ on $W\backslash\{v\}$.
Here, $\alpha \neq 0$, otherwise the s-parameter group would lead to a contradiction to the lemma on quadrangles. The one-parameter group defined by $s = \frac{1-\beta}{\alpha}t$ fixes each point $(x,0,0,0) \in P\backslash W$ and each point $[0,\eta] \in W$, a contradiction to the lemma on quadrangles. Therefore there exists no plane in case $1v, 1$.

Case $1v, 2$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \middle| x \in R\right\}, \left\{\begin{pmatrix} 1 & & & \\ & e^{\alpha s} & & \\ & & e^{s} & \\ & & & e^{s} \end{pmatrix} \middle| s \in R\right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & e^{\beta t} & & \\ & & 1 & \\ & & t & 1 \end{pmatrix} \middle| t \in R \right\}, where \quad N = \left\{\begin{pmatrix} 1 & & & \\ y & 1 & & \\ u & & 1 & \\ v & x & & 1 \end{pmatrix} \middle| x, y, u, v \in R \right\}.$$

The regular action of N on itself is:

$$(\bar{x}, \bar{y}, \bar{u}, \bar{v})^{(x,y,u,v)} = (x + \bar{x}, y + \bar{y}, u + \bar{u}, v + \bar{v} + x\bar{y}).$$

Conjugation by the s- and t-parameter groups gives the isotropy groups:
$(x, y, u, v)^s = (e^{(1-\alpha)s}x, e^{\alpha s}u, e^s u, e^s v)$,
$(x, y, u, v)^t = (e^{-\beta t}x, e^{\beta t}y, u, tu + v)$.
We have the following actions of $x-$, $s-$ and $t-$parameter groups on $W\backslash\{v\}$ by conjugation, coordinatized by $[\xi, \eta]$, $\xi = \frac{u}{y}, \eta = \frac{v}{y}$:
i) action of x-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [\xi, \eta + x]$,
ii) action of s-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [e^{(1-\alpha)s}\xi, e^{(1-\alpha)s}\eta]$,
iii) action of t-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [e^{-\beta t}\xi, te^{-\beta t}\xi + e^{-\beta t}\eta]$.
Note that in this case the action of Σ on $W\backslash\{v\}$ has a 1-dimensional orbit, that is, the coordinate line $\xi = 0$ is invariant under the action of Σ on $W\backslash\{v\}$. The 2-dimensional stabilizer $\Sigma_{(0,0,0,0)}$ fixes also the point [0,0] at infinity and consists of the $s-$ and the $t-$parameter subgroups. This stabilizer fixes also the line $(0,0,0,0) \vee [0,0]$ and the one-dimensional curve in it which is defined by $y = 0$. Now the one-dimensional subgroup of the stabilizer defined by $(1 - \alpha)s = -\beta t$ acts trivially on the parameter x and fixes therefore every point of this one-dimensional curve. It also fixes each point $[0, \eta], \eta \in R$ at infinity, a contradiction to the lemma on quadrangles. We have thus shown that the case $1v, 2$ does not lead to a projective plane.

Case $1s$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{pmatrix} \middle| x \in R \right\}, \left\{\begin{pmatrix} 1 & & & \\ & e^{s} & & \\ & & e^{\alpha s} & \\ & & & e^{s} \end{pmatrix} \middle| s \in R, \alpha \neq 1 \right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e^{(1-\alpha)t} & \\ & t & & 1 \end{pmatrix} \middle| t \in R, \alpha \neq 1 \right\}, where \quad N = \left\{\begin{pmatrix} 1 & & & \\ y & 1 & & \\ u & & 1 & \\ v & & x & 1 \end{pmatrix} \middle| x, y, u, v \in R \right\}.$$

The regular action of N on itself is:

$$(\bar{x},\bar{y},\bar{u},\bar{v})^{(x,y,u,v)} = (x+\bar{x}, y+\bar{y}, u+\bar{u}, v+\bar{v}+x\bar{u}).$$

Conjugation by the s and t-parameter groups gives the isotropy groups:
$(x,y,u,v)^s = (e^{(1-\alpha)s}x, e^s y, e^{\alpha s}u, e^s v)$,
$(x,y,u,v)^t = (e^{-(1-\alpha)t}x, y, e^{(1-\alpha)t}u, ty+v)$.
We have the following actions (by conjugation) of the $x-$, $s-$ and $t-$parameter groups on $W\backslash\{v\}$, coordinatized by $[\xi,\eta]$, $\xi = \frac{u}{y}, \eta = \frac{v}{y}$:
i) action of x-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [\xi, \eta+\xi x]$,
ii) action of s-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [e^{(\alpha-1)s}\xi, \eta]$,
iii) action of the t-parameter group on $W\backslash\{v\} : [\xi,\eta] \to [e^{(1-\alpha)t}\xi, \eta+t]$.
By combining the s- and the t-parameter groups we get the following one-parameter group: $(x,y,u,v) \to (x, e^r y, e^r u, re^r y + e^r v)$, $r \in R$ inducing on W the action:

$$[\xi,\eta] \to [\xi, \eta + r] r \in R.$$

This group fixes the points (0,0,0,0) and (1,0,0,0), their joining line and its intersection with W. This is a contradiction since the r-parameter group acts freely on W.

Case $1h$: Σ is generated by the translation group R^3 and the following three linear one-parameter groups:

$$\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{pmatrix} | x \in R\right\}, \left\{\begin{pmatrix} 1 & & & \\ & e^{\alpha s} & & \\ & & e^s & \\ & & & 1 \end{pmatrix} | s \in R, \alpha \neq 1\right\},$$

$$\left\{\begin{pmatrix} 1 & & & \\ & e^{\beta t} & & \\ & & 1 & \\ & & & e^t \end{pmatrix} | t \in R, \beta \neq 1\right\}, \textit{where} \quad N = \left\{\begin{pmatrix} 1 & & & \\ y & 1 & & \\ u & x & 1 & \\ v & & & 1 \end{pmatrix} | x,y,u,v \in R\right\}.$$

The regular action of N on itself is:

$$(\bar{x},\bar{y},\bar{u},\bar{v})^{(x,y,u,v)} = (x+\bar{x}, y+\bar{y}, u+\bar{u}+x\bar{y}, v+\bar{v}).$$

Conjugation by the s- and t-parameter groups gives the isotropy groups:
$(x,y,u,v)^s = (e^{(1-\alpha)s}x, e^{\alpha s}y, e^s u, v)$,
$(x,y,u,v)^t = (e^{-\beta t}x, e^{\beta t}y, u, e^t v)$.
We have the following actions of the x-, s- and t-parameter groups on $W\backslash\{v\}$, coordinatized by $[\xi,\eta]$, $\xi = \frac{u}{y}, \eta = \frac{v}{y}$:

i) action of the x-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [\xi + x, \eta]$,
ii) action of the s-parameter group on $W\backslash\{v\} : [\xi, \eta] \to [e^{(1-\alpha)s}\xi, e^{-\alpha s}\eta]$,
iii) action of the t-parameter group on $W\backslash\{v\} : [\xi, \eta \to [e^{-\beta t}\xi, e^{(1-\beta)t}\eta]$.
Note that in this case the action of Σ on $W\backslash\{v\}$ has a 1-dimensional orbit, that is, the coordinate line $\eta = 0$ is invariant under the action of Σ on $W\backslash\{v\}$.

Now let us combine that s- and the t-parameter group in such a way that the action on the x-coordinate will be trivial. For this we take the product of the map s and the map $t = \frac{(1-\alpha)s}{\beta}$. This leads to the one-parameter group

$$(x, y, u, v) \to (x, e^r y, e^s u, e^{\frac{(1-\alpha)s}{\beta}} v), r \in R.$$

Here we may assume $\beta \neq 0$ for otherwise the t-parameter group would give a contradiction to the lemma on quadrangles. The r-action on $W\backslash\{v\}$ is:

$$[\xi, \eta] \to [\xi, f(\alpha, \beta, s)\eta].$$

The r-parameter group fixes the point v at infinity, each point $(x, 0, 0, 0), x \in R$ and each point $[\xi, 0]$ on W. This is a contradiction to the lemma on quadrangles, hence there exists no plane in this case.

REFERENCES

[1] D. BETTEN, *4-dimensionale projektive Eben mit 3-dimensionaler Translationsgruppe*, Geom. Dedicata 16, 179-193, 1984.

[2] D. BETTEN, *4-dimensional compact projective planes with a 7-dimensional collineation group*, Geom. Dedicata 36, 151-170, 1990.

[3] D. BETTEN, *4-dimensional compact projective planes with a nilpotent collineation group*, Mitt. Math. Ges. Hamburg 12, 741-749, 1991.

[4] D. BETTEN, *Orbits in 4-dimensional compact projective planes*, Journal of Geometry 42, 30-40, 1991.

[5] D. BETTEN, *4-dimensional compact projective planes with a 5-dimensional nilradical*, Geom. Dedicata 58, 259-289, 1995.

[6] D. BETTEN, M. FORST, *Transitive Wirkungen auf Flächen - Effektive Lie-Algebren-Paare der Codimension 2*, Math. Seminar Univ. Kiel, 1977.

[7] D. BETTEN, H. KLEIN, *4-dimensional compact projective planes with two fixed points (1994)*, J. of Geom. 55, 31-56, 1996.

[8] D. BETTEN, N. KNARR, *Rotationsflächen-Ebenen*, Abh. Math. Sem. Univ. Hamburg 57, 227-234, 1987.

[9] D. BETTEN, B. POLSTER, *4-dimensional compact projective planes of orbit type Δ:(1,1)*, to appear.

[10] H. KLEIN, *4-dimensional projective planes with collineation group $N_{6,28}$*, Results Math. 28, 100-116, 1995.

[11] H. KLEIN, *4-dimensional projective planes with a small nilradical*, Geom. Ded. 58, 53-62, 1995.

[12] N. KNARR, *Topologische Differenzenflächenebenen*, Diplomarbeit, Kiel, 1983.

[13] N. KNARR, *Topologische Differenzenflächenebenen mit nichtkommutativer Standgruppe*, Dissertation, Kiel, 1986.

[14] N. KNARR, *4-dimensionale projektive Ebenen vom Lenz-Barlotti-Typ II.2*, Results in Math. 12, 134-147, 1987.

[15] R. LÖWEN, *Four-dimensional compact projective planes with a nonsolvable automorphism group*, Geom. Ded. 36, 225-234, 1990.

[16] G.D. MOSTOW, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math. 52, 606-636, 1950.

[17] H.R. SALZMANN, *Kollineationsgruppen kompakter vier-dimensionaler Ebenen*, Math. Z. 117, 112-124, 1970.

[18] H.R. SALZMANN, *Kollineationsgruppen kompakter 4-dimensionaler Ebenen II*, Math. Z. 121, 104-110, 1971.

[19] H.R. SALZMANN, D. BETTEN, T. GRUNDHÖFER, H. HÄHL, R. LÖWEN, M. STROPPEL, *Compact projective planes*, de Gruyter Berlin, 1995.

[20] C. WEIGAND, *Konstruktion topologischer projektiver Ebenen, die keine Translationsebenen sind*, Mitt. Math. Sem. Giessen 177, 1987.

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